Congruences for Consecutive Coefficients of Gaussian Polynomials with Crank Statistics

Dennis Eichhorn

Department Of Mathematics
University of California, Irvine
Irvine, California 92697, U.S.A.

deichhor@math.uci.edu

Lydia Engle

University of Minnesota
Minneapolis, Minnesota 55455, U.S.A.

engle258@umn.edu

Brandt Kronholm

Department of Mathematics
University of Texas Rio Grande Valley
Edinburg, Texas 78539-2999, U.S.A.

brandt.kronholm@utrgv.edu

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Abstract

In this paper, we establish infinite families of congruences in consecutive arithmetic progressions modulo any odd prime ℓ for the function $p(n, m, N)$, which enumerates the partitions of n into at most m parts with no part larger than N. We also treat the function $p(n, m, (a, b])$, which bounds the largest part above and below, and obtain similar infinite families of congruences.

For $m \leqslant 4$ and $\ell = 3$, simple combinatorial statistics called "cranks" witness these congruences. We prove this analytically for $m = 4$, and then both analytically and combinatorially for $m = 3$. Our combinatorial proof relies upon explicit dissections of convex lattice polygons.

Mathematics Subject Classifications: 05A17, 11P82, 11P83

This paper is dedicated to the memory of Freeman Dyson, his contributions to the theory of partitions, and his contributions to science in general.

1 Introduction and Main Theorems

1.1 Partition Congruences

In 1919, Ramanujan [12] observed and proved the following congruences in arithmetic progressions for the ordinary partition function,

$$\begin{aligned} p\big(5n+4\big) &\equiv 0 \pmod{5}, \\ p\big(7n+5\big) &\equiv 0 \pmod{7}, \\ \text{and} \quad p\big(11n+6\big) &\equiv 0 \pmod{11}. \end{aligned} \tag{1}$$

Fifty years later, A. O. L. Atkin showed that $p(n)$ enjoys many more congruences in arithmetic progressions [3]. For example,

$$p(206839n + 2623) \equiv 0 \pmod{17}. \tag{2}$$

In 2000, Ono [10] showed that congruences like (2) are individual instances of infinitely many such congruences, by proving that for any prime $\ell \geqslant 5$, there exists integers A, B such that

$$p\big(An + B\big) \equiv 0 \pmod{\ell}. \tag{3}$$

Recently, the third author established several infinite families of congruences for the function enumerating partitions of n into *exactly* m parts [7, 8, 9]. In this paper, we redefine $p(n, m)$ to be the number of partitions of n into *at most* m parts. Since $p(n+m, m)$ is equal to the number of partitions of n into exactly m parts, we can readily translate the results in [7, 8, 9] into our current notation. Below, we cite one of the aforementioned families, where the congruences occur in intervals of consecutive arithmetic progressions of the form $p\big(An + B, m\big) \equiv 0 \pmod{\ell}$, where B can take on any value in a particular interval.

We require a definition.

Definition 1. We write $\operatorname{lcm}(m)$ to indicate the least common multiple of the numbers from 1 through m.

Theorem 2. *[8] Let ℓ be an odd prime and suppose $2 \leqslant m \leqslant \ell + 1$. Then for each $1 \leqslant t \leqslant \binom{m+1}{2} - 1$. we have*

$$p\big(\ell \operatorname{lcm}(m)\, k - t, m\big) \equiv 0 \pmod{\ell} \tag{4}$$

holds for all $k \geqslant 0$.

Example 3. Setting $\ell = 3$ and $m = 3$ we have $p(18k - t, 3) \equiv 0 \pmod{3}$ for $1 \leqslant t < 5$.

In this paper, we consider several refinements of Theorem 2.

Our first refinement treats the number of partitions of n into at most m parts with largest part at most N, denoted $p\big(n, m, N\big)$. This theorem shows that adding certain bounds on the largest part of our partitions preserves many of the congruences in Theorem 2.

Theorem 4. *Let ℓ be an odd prime and suppose $2 \leqslant m \leqslant \ell+1$, $1 \leqslant s \leqslant m$, and $j \geqslant 1$. Set $A = \ell \operatorname{lcm}(m)$ and $C = \ell \operatorname{lcm}(m-1)$. Then for each $\frac{s(s-1)}{2} < t < ms - \frac{s(s-1)}{2}$, the congruence*

$$p\big(Ak - t, m, Cj - s\big) \equiv 0 \pmod{\ell} \tag{5}$$

holds for all $k \geqslant 1$.

We add a second level of refinement by treating $p\big(n, m, (a, b]\big)$, the number of partitions of n with at most m parts, largest part greater than a but at most b. Notice that

$$p\big(n, m, (a, b]\big) = p\big(n, m, b\big) - p\big(n, m, a\big). \tag{6}$$

To simplify our notation, we define I_j to be $(j-1, j]$ throughout the paper, $CI_j = (C(j-1), Cj]$ to be the dilation of I_j by C, and for any interval $I = (a, b]$, $I - s = (a - s, b - s]$ to be the translation of I by $-s$.

Corollary 5. *Let ℓ be an odd prime and suppose $2 \leqslant m \leqslant \ell+1$, $1 \leqslant s \leqslant m$, and $j \geqslant 1$. Set $A = \ell \operatorname{lcm}(m)$ and $C = \ell \operatorname{lcm}(m-1)$. Then for each $\frac{s(s-1)}{2} < t < ms - \frac{s(s-1)}{2}$, the congruence*

$$p\big(Ak - t, m, CI_j - s\big) \equiv 0 \pmod{\ell} \tag{7}$$

holds for all $k \geqslant 1$.

Examples of Theorem 4 and Corollary 5 can be found in Section 2.1.

In the following subsection, we discuss witnesses for these congruences which demonstrate a way in which these new refinements and the original Theorem 2 can be seen by directly examining the combinatorics of the associated sets of partitions.

1.2 Combinatorial Witnesses for Partition Congruences

We recall the definition of an integer partition.

Definition 6. A partition λ of a positive integer n is a finite nonincreasing sum of positive integers
$\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. We refer to λ_i as the i^{th} part of the partition.

In 1944, Freeman Dyson [5] called for *direct* proofs of Ramanujan's congruences that show how the sets of partitions enumerated in (1) can be divided into five, seven, and eleven equinumerous subclasses, respectively. He remarked,

> "...it is unsatisfactory to receive no concrete idea of how the division is to be made. We require a proof which will not appeal to generating functions, but will demonstrate by cross-examination of the partitions themselves...

Dyson conjectured that a very simple statistic on partitions called the "rank" of a partition, the largest part minus the number of parts, witnesses this division when considered modulo 5 and 7. Dyson denoted the number of partitions of n whose rank is congruent to r modulo ℓ by $N\big(r,\ell,n\big)$, and so he wrote his conjecture as

$$\begin{aligned} N\big(0,5,5n+4\big) &= N\big(1,5,5n+4\big) = N\big(2,5,5n+4\big) \\ &= N\big(3,5,5n+4\big) = N\big(4,5,5n+4\big) \end{aligned}$$

and

$$N\big(0,7,7n+5\big) = N\big(1,7,7n+5\big) = \cdots = N\big(6,7,7n+5\big).$$

Using analytic methods, Atkin and Swinnerton-Dyer [3] proved Dyson's conjecture. However, a combinatorial proof that the rank witnesses Ramanujan's first two congruences remains elusive. Dyson further hypothesized the existence of a different statistic, called the "crank", that would witness Ramanujan's congruence modulo 11 in the same way. In 1988, Andrews and Garvan [2] found a crank that not only witnessed Ramanujan's congruence modulo 11, but also witnessed Ramanujan's congruences modulo 5 and 7 with a new division into 5 and 7 classes, respectively. However, in both cases, the proofs were analytic, and they did not employ a cross-examination of the partitions themselves as Dyson had hoped.

In most of the literature, the Andrews-Garvan crank is referred to as "*the* crank." In addition, we may refer to any statistic on partitions (especially one that witnesses divisibilities) that is not Dyson's rank as "*a* crank." In Theorem 7 below, we consider the congruences modulo 3 in both Theorem 4 and Corollary 5, and we find that there is a simple crank that witnesses these congruences. Remarkably, this crank allows us to give a direct combinatorial proof of some of those congruences by cross-examination of the partitions themselves, in the way Dyson had imagined that his original conjecture would be treated.

When we have designated a crank other than Dyson's rank or the Andrews-Garvan crank, we define
$M'\big(r,n,m\big)$ to be the number of partitions of n into at most m parts with crank value r, and $M'\big(r,n,m,N\big)$ to be the number of those partitions that have no part larger than N. We further define $M'\big(r,\ell,n,m\big)$ to be the number of partitions of n into at most m parts with crank value r modulo ℓ, $M'\big(r,\ell,n,m,N\big)$ to be the number of those partitions that have no part larger than N, and $M'\big(r,\ell,n,m,(a,b]\big)$ to be the number of those partitions with largest part confined to the interval $(a,b]$. For each r between 0 and $\ell-1$, we refer to the sets of partitions counted by each of $M'\big(r,\ell,n,m\big), M'\big(r,\ell,n,m,N\big)$, and $M'\big(r,\ell,n,m,(a,b]\big)$ as the r^{th} *crank class modulo* ℓ of the partitions counted by $p\big(n,m\big), p\big(n,m,N\big)$, and $p\big(n,m,(a,b]\big)$, respectively.

Theorem 7. *For $\ell=3$, the second part of the partition is a crank witnessing the congruences of Theorem 4 and Corollary 5 when $m\in\{2,3\}$. For $\ell=3$ and $m=4$, if $n\leqslant 2N$, the second part of the partition is a crank witnessing the congruences of Theorem 4 and Corollary 5, whereas if $n>2N$, the third part of the partition is a crank witnessing those congruences.*

In cases where a k^{th} part of a partition with less than k parts is referenced, the k^{th} part is to be interpreted as zero.

In Section 3, we restate Theorem 7 with additional details, and we then prove the theorem case by case according to m, the maximum number of parts. The proof for $m = 2$ is direct and requires minimal background. For $m = 3$, we offer two proofs; the first is a purely combinatorial realization where we treat partitions as integer lattice points, while the second proof uses generating functions (q-series) to produce closed formulas for $M'\big(r, 3, n, 3, N\big)$. A highlight of the combinatorial proof is that we work up from the smaller sets $\mathcal{M}'\big(r, 3, n, 3, (a, b]\big)$ to the larger sets $\mathcal{M}'\big(r, 3, n, 3, N\big)$ and then $\mathcal{M}'\big(r, 3, n, 3\big)$, which is the opposite order in which we treat these in the analytic proof. The case $m = 4$ is treated with the same q-series procedure as $m = 3$.

2 Examples, Background, and Proofs of Theorem 4 and Corollary 5

2.1 Examples of Theorem 4 and Corollary 5

With many parameters, Theorem 4 and Corollary 5 are quite general, and can be difficult to parse. The examples below illustrate how the results of Theorem 4 and Corollary 5 change as we vary the upper bound on the size of the parts in our partitions.

Example 8. Set $\ell = 5$, $m = 5$, and $j = 4$. By Theorem 4, we have:

- For $s = 1$, the following congruences in an interval of four $(0 < t < 5)$ consecutive arithmetic progressions within the Gaussian polynomial $\left[{239+5 \atop 5}\right] = \sum_{n=0}^{1195} p\big(n, 5, 239\big)q^n$

$$\begin{aligned}
p\big(296, 5, 239\big) &\equiv p\big(596, 5, 239\big) \equiv p\big(896, 5, 239\big) \equiv 0 \pmod 5\\
p\big(297, 5, 239\big) &\equiv p\big(597, 5, 239\big) \equiv p\big(897, 5, 239\big) \equiv 0 \pmod 5\\
p\big(298, 5, 239\big) &\equiv p\big(598, 5, 239\big) \equiv p\big(898, 5, 239\big) \equiv 0 \pmod 5\\
p\big(299, 5, 239\big) &\equiv p\big(599, 5, 239\big) \equiv p\big(899, 5, 239\big) \equiv 0 \pmod 5).
\end{aligned}$$

- Lowering the upper bound on the part sizes by one, we have a new Gaussian polynomial with even more congruences.

  For $s = 2$, we have seven $(1 < t < 9)$ consecutive arithmetic progressions within the Gaussian polynomial $\left[{238+5 \atop 5}\right] = \sum_{n=0}^{1190} p\big(n, 5, 238\big)q^n$.

$$\begin{aligned}
p\big(292, 5, 238\big) &\equiv p\big(592, 5, 238\big) \equiv p\big(892, 5, 238\big) \equiv 0 \pmod 5\\
p\big(293, 5, 238\big) &\equiv p\big(593, 5, 238\big) \equiv p\big(893, 5, 238\big) \equiv 0 \pmod 5\\
p\big(294, 5, 238\big) &\equiv p\big(594, 5, 238\big) \equiv p\big(894, 5, 238\big) \equiv 0 \pmod 5\\
p\big(295, 5, 238\big) &\equiv p\big(595, 5, 238\big) \equiv p\big(895, 5, 238\big) \equiv 0 \pmod 5\\
p\big(296, 5, 238\big) &\equiv p\big(596, 5, 238\big) \equiv p\big(896, 5, 238\big) \equiv 0 \pmod 5\\
p\big(297, 5, 238\big) &\equiv p\big(597, 5, 238\big) \equiv p\big(897, 5, 238\big) \equiv 0 \pmod 5\\
p\big(298, 5, 238\big) &\equiv p\big(598, 5, 238\big) \equiv p\big(898, 5, 238\big) \equiv 0 \pmod 5).
\end{aligned}$$

- Lowering the upper bound on the part sizes by one more, we have yet another Gaussian polynomial with even more congruences.

  For $s = 3$, we have eight $(3 < t < 12)$ consecutive arithmetic progressions for the coefficients of the Gaussian polynomial $\left[{237+5 \atop 5}\right] = \sum_{n=0}^{1185} p\big(n, 5, 237\big)q^n$,

$$\begin{aligned}
p\big(289, 5, 237\big) &\equiv p\big(589, 5, 237\big) \equiv p\big(889, 5, 237\big) \equiv 0 \pmod 5\\
p\big(290, 5, 237\big) &\equiv p\big(590, 5, 237\big) \equiv p\big(890, 5, 237\big) \equiv 0 \pmod 5\\
p\big(291, 5, 237\big) &\equiv p\big(591, 5, 237\big) \equiv p\big(891, 5, 237\big) \equiv 0 \pmod 5\\
p\big(292, 5, 237\big) &\equiv p\big(592, 5, 237\big) \equiv p\big(892, 5, 237\big) \equiv 0 \pmod 5\\
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p\big(294, 5, 237\big) &\equiv p\big(594, 5, 237\big) \equiv p\big(894, 5, 237\big) \equiv 0 \pmod 5\\
p\big(295, 5, 237\big) &\equiv p\big(595, 5, 237\big) \equiv p\big(895, 5, 237\big) \equiv 0 \pmod 5\\
p\big(296, 5, 237\big) &\equiv p\big(596, 5, 237\big) \equiv p\big(896, 5, 237\big) \equiv 0 \pmod 5.
\end{aligned}$$

We now demonstrate Corollary 5 in Example 9 below. This highlights that many of the congruences for $p(n, m)$ in Theorem 2 can be doubly refined as congruences for $p\big(n, m, (a, b]\big)$. We also vary the upper and lower bounds on the size of the parts in our partitions to demonstrate how a single congruence for $p(n, m)$ can be refined into collections of congruences for $p\big(n, m, (a, b]\big)$ in several different ways.

Example 9. Set $\ell = 5$, $m = 5$, $k = 1$, and $t = 6$, so that $p\big(294, 5\big) \equiv 0 \pmod 5$. Corollary 5 reveals many intervals $(a, b]$ for which $p\big(294, 5, (a, b]\big) \equiv 0 \pmod 5$. Varying the parameter $s = 2, 3, 4$ gives us three distinct sums equal to $p\big(294, 5\big)$ where each summand $p\big(294, 5, (a, b]\big) = p\big(294, 5, 60I_j - s\big)$ is also a multiple of 5.

- $s = 2$

$$\begin{aligned}
p\big(294, 5\big) &= \sum_{j\geqslant 1} p\big(294, 5, 60I_j - 2\big)\\
&= p\big(294, 5, (-2, 58]\,\big) + p\big(294, 5, (58, 118]\,\big) + p\big(294, 5, (118, 178]\,\big)\\
&\quad + p\big(294, 5, (178, 238]\,\big) + p\big(294, 5, (238, 298]\,\big)\\
&= 0 + 1{,}069{,}755 + 1{,}432{,}910 + 342{,}485 + 23{,}160\\
&= 2{,}868{,}310 \equiv 0 \pmod 5.
\end{aligned}$$

- $s = 3$

$$\begin{aligned}
p\big(294, 5\big) &= \sum_{j\geqslant 1} p\big(294, 5, 60I_j - 3\big)\\
&= p\big(294, 5, (-3, 57]\,\big) + p\big(294, 5, (57, 117]\,\big) + p\big(294, 5, (117, 177]\,\big)\\
&\quad + p\big(294, 5, (177, 237]\,\big) + p\big(294, 5, (237, 297]\,\big)\\
&= 0 + 1{,}034{,}725 + 1{,}455{,}640 + 353{,}210 + 24{,}735\\
&= 2{,}868{,}310 \equiv 0 \pmod 5.
\end{aligned}$$

- $s = 4$

$$
\begin{aligned}
p\big(294, 5\big) &= \sum_{j\geqslant 1} p\big(294, 5, 60I_j - 4\big) \\
&= p\big(294, 5, (-4, 56]\,\big) + p\big(294, 5, (56, 116]\,\big) + p\big(294, 5, (116, 176]\,\big) \\
&\quad + p\big(294, 5, (176, 236]\,\big) + p\big(294, 5, (236, 296]\,\big) \\
&= 0 + 999{,}650 + 1{,}478{,}115 + 364{,}160 + 26{,}385 \\
&= 2{,}868{,}310 \equiv 0 \pmod 5.
\end{aligned}
$$

2.2 Background Material for Theorem 4 and Corollary 5

We use the standard q-rising factorial notation throughout,

$$(a;q)_d = \prod_{i=1}^{d} (1 - aq^{i-1}).$$

It is well-known that the generating function for $p\big(n, m\big)$ is given by

$$\sum_{n=0}^{\infty} p\big(n, m\big) q^n = \frac{1}{(q;q)_m}.$$

Gaussian polynomials, denoted by $\left[{N+m \atop m}\right]$, are generating functions for $p\big(n, m, N\big)$.

$$\sum_{n=0}^{mN} p\big(n, m, N\big) q^n = \begin{bmatrix} N+m \\ m \end{bmatrix} = \frac{(q;q)_{N+m}}{(q;q)_m (q;q)_N} = \frac{(q^{N+1};q)_m}{(q;q)_m}. \tag{8}$$

Gaussian polynomials are reciprocal polynomials of degree mN.

Lemma 10. *[1] We have*

$$(z;q)_m = \sum_{h=0}^{m} (-1)^h \begin{bmatrix} m \\ h \end{bmatrix} q^{h(h-1)/2} z^h. \tag{9}$$

2.3 Proof of Theorem 4 and Corollary 5

We are now ready to prove Theorem 4.

Proof of Theorem 4. Let ℓ be an odd prime, and for $2 \leqslant m \leqslant \ell + 1$, consider

$$
\begin{aligned}
\sum_{n=0}^{mN} p(n, m, N) q^n &= \begin{bmatrix} N+m \\ m \end{bmatrix} = \frac{(q^{N+1};q)_m}{(q;q)_m} = \sum_{h=0}^{m} \frac{(-1)^h}{(q;q)_m} \begin{bmatrix} m \\ h \end{bmatrix} q^{hN + h(h+1)/2} \\
&= \sum_{h=0}^{m} \frac{(-1)^h q^{hN + h(h+1)/2}}{(q;q)_{m-h} (q;q)_h},
\end{aligned} \tag{10}
$$

by the definition of the Gaussian polynomial and an application of Lemma 10 with $z = q^{N+1}$.

Let $N = \ell \operatorname{lcm}(m-1)\, j - s$. We now prove our theorem by showing that the desired congruences in arithmetic progressions hold for each individual term of the sum in (10). For $h = 0$, the summand in (10) simplifies to $1/(q;q)_m$. By Theorem 2, we have that for every $n = \ell \operatorname{lcm}(m)\, k - t$, the summand in (10) is $0 \pmod{\ell}$ for $1 \leqslant t \leqslant \binom{m+1}{2} - 1$. Similarly, for $h = m$, the summand in (10) simplifies to $(-1)^m q^{m\ell \operatorname{lcm}(m-1)\, j - ms + m(m+1)/2}/(q;q)_m$. By Theorem 2, we have that for every $n = \ell \operatorname{lcm}(m)\, k - t$, the summand in (10) is then $0 \pmod{\ell}$ for $ms - \binom{m+1}{2} < t < ms$. Thus the sum of the $h = 0$ and $h = m$ terms of the sum in (10) is $0 \pmod{\ell}$ for $0 < t < ms$ if $s \leqslant (m+1)/2$, and is $0 \pmod{\ell}$ for $ms - \binom{m+1}{2} < t < \binom{m+1}{2}$ if $s > (m+1)/2$.

For $1 \leqslant h \leqslant m-1$, we now also show congruences in an interval of consecutive arithmetic progressions for the summands in (10). In these cases, our arithmetic progressions have a much smaller common difference, and they fill a more narrow interval. We rewrite the summand in (10) as

$$\frac{E_h(q) q^{h(\ell \operatorname{lcm}(m-1)\, j - s) + h(h+1)/2}}{(1 - q^{\operatorname{lcm}(m-1)})^\ell} \equiv \frac{E_h(q) q^{h\ell \operatorname{lcm}(m-1)\, j + h(h+1)/2 - hs}}{(1 - q^{\ell \operatorname{lcm}(m-1)})}$$
$$\equiv E_h(q) q^{h(h+1)/2 - hs} g\left(q^{\ell \operatorname{lcm}(m-1)}\right) \pmod{\ell}, \quad (11)$$

for some function g, where $E_h(q) = \frac{(-1)^h (1 - q^{\operatorname{lcm}(m-1)})^\ell}{(q;q)_{m-h}(q;q)_h}$. Notice that when $m \leqslant \ell$, each of the m factors in the denominator of $E_h(q)$ divides $(1 - q^{\operatorname{lcm}(m-1)})$, so in fact $E_h(q)$ is a polynomial. In the case $m = \ell + 1$, $E_h(q) \equiv \frac{(-1)^h (1 - q^{\operatorname{lcm}(\ell)})^{\ell - 1} (1 - q^{\operatorname{lcm}(\ell - 1)})^\ell}{(1-q)^2 (q^2;q)_{\ell - h} (q^2;q)_{h-1}} \pmod{\ell}$. We then have that $(1-q)^2$ divides $(1 - q^{\operatorname{lcm}(\ell-1)})^\ell$ and each of the $\ell - 1$ factors in $(q^2;q)_{\ell-h}(q^2;q)_{h-1}$ divide $(1 - q^{\operatorname{lcm}(\ell)})$, so again $E_h(q)$ is a polynomial. In either case, the degree of the polynomial $E_h(q)$ is $\ell \operatorname{lcm}(m-1) + h(m-h) - m(m+1)/2$, which is always strictly less than $\ell \operatorname{lcm}(m-1)$. Thus the power series expansion of the right-hand side of (11) only has terms where the exponent of q is congruent to r modulo $\ell \operatorname{lcm}(m-1)$ for $r \in \{h(h+1-2s)/2, \ldots \ldots, \ell \operatorname{lcm}(m-1) - h(h-1-2(m-s))/2 - m(m+1)/2\}$. Taking the union of these sets from $h = 1$ to $m-1$, we have

$$\bigcup_{h=1}^{m-1} \left\{ \frac{h(h+1-2s)}{2}, \ldots, \ell \operatorname{lcm}(m-1) - \frac{h(h-1-2(m-s))}{2} - \frac{m(m+1)}{2} \right\}$$
$$= \left\{ -\frac{s(s-1)}{2}, \ldots, \ell \operatorname{lcm}(m-1) + \frac{s(s-1)}{2} - ms \right\}$$

This means that when $s(s-1)/2 < t < ms - s(s-1)/2$, exponents of the form ℓ $\operatorname{lcm}(m-1)\, k' - t$ do not appear in the power series expansion of any summand in (11) for $h \neq 0, m$. In particular, for t in that same range, exponents of the form $\ell \operatorname{lcm}(m)\, k - t$ do not appear in the power series expansion of any summand in (11) for $h = 0, m$.

Now, for any $1 \leqslant s \leqslant m$, we see that summing (10) over all h, the theorem follows.

Corollary 5 follows immediately from (6). □

3 Combinatorial Witnesses and the Proof of Theorem 7

The proof of Theorem 7 comes in three cases depending on $m \in \{2, 3, 4\}$. For the case $m = 2$, the proof is direct and requires almost no background information. For the case $m = 3$, we supply two proofs, one combinatorial and another analytic, in Sections 3.1 and 3.4. We offer an analytic proof in Section 3.5 for the case $m = 4$ that follows the very same procedure as the case $m = 3$.

We restate Theorem 7 with additional details.

Theorem 7. *For $\ell = 3$, the second part of the partition is a crank witnessing the congruences of Theorem 4 and Corollary 5 when $m \in \{2, 3\}$. In other words,* (12)-(17) *hold.*

When $m = 2$, for $r \in \{0, 1, 2\}$, $j, k \geqslant 1$, and the ordered pairs $(s, t) \in \{(1, 1), (2, 2)\}$, we have

$$M'\big(r, 3, 6k - t, 2, 3I_j - s\big) = \frac{p\big(6k - t, 2, 3I_j - s\big)}{3}, \tag{12}$$

$$M'\big(r, 3, 6k - t, 2, 3j - s\big) = \frac{p\big(6k - t, 2, 3j - s\big)}{3}, \tag{13}$$

$$\text{and} \qquad M'\big(r, 3, 6k - t, 2\big) = \frac{p\big(6k - t, 2\big)}{3}. \tag{14}$$

When $m = 3$ we have for $r \in \{0, 1, 2\}$, $j, k \geqslant 1$, and the ordered pairs $(s, t) \in \{(1, 1), (1, 2), (2, 2), (2, 3), (2, 4), (3, 4), (3, 5)\}$, we have

$$M'\big(r, 3, 18k - t, 3, 6I_j - s\big) = \frac{p\big(18k - t, 3, 6I_j - s\big)}{3}, \tag{15}$$

$$M'\big(r, 3, 18k - t, 3, 6j - s\big) = \frac{p\big(18k - t, 3, 6j - s\big)}{3}, \tag{16}$$

$$\text{and} \qquad M'\big(r, 3, 18k - t, 3\big) = \frac{p\big(18k - t, 3\big)}{3}. \tag{17}$$

For the case $\ell = 3$ and $m = 4$, when $36k - t \leqslant 2N$, a crank witnessing the congruences of Theorem 4 and Corollary 5 is second part modulo 3, and when $36k - t > 2N$, a crank witnessing the congruences of Theorem 4 and Corollary 5 is third part modulo 3. In other words, (18)-(20) *hold.*

When $m = 4$ and for $r \in \{0, 1, 2\}$, $j, k \geqslant 1$, and the ordered pairs $(s, t) \in \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (4, 7), (4, 8), (4, 9)\}$, we have

$$M'\big(r, 3, 36k - t, 4, 18I_j - s\big) = \frac{p\big(36k - t, 4, 18I_j - s\big)}{3}, \tag{18}$$

$$M'\big(r, 3, 36k - t, 4, 18j - s\big) = \frac{p\big(36k - t, 4, 18j - s\big)}{3}, \tag{19}$$

$$\text{and} \qquad M'\big(r, 3, 36k - t, 4\big) = \frac{p\big(36k - t, 4\big)}{3}. \tag{20}$$

Remark 11. In Theorem 7, when $m = 4$, the reader may be surprised to see a crank that is defined piecewise as the second part when $n \leqslant 2N$ and the third part when $n > 2N$. This piecewise definition is a byproduct of the combinatorics underlying the well-known symmetry $p(n, m, N) = p(mN - n, m, N)$. Specifically, by taking the complement of Ferrers diagrams inside an $m \times N$ rectangle, we have that the i^{th} part of such a partition of n is equal to N minus the $(m-i)^{\text{th}}$ part of a partition of $mN - n$.

We prove the case $m = 2$ of Theorem 7 below.

Proof of Theorem 7 for $m = 2$. We begin by proving (12). Let $(s, t) \in \{(1, 1), (2, 2)\}$.

When $3j - s < \lceil (6k-t)/2 \rceil + 3$ or $3j - s > 6k - t$, there are no partitions counted by $p\big(6k - t, 2, 3I_j - s\big)$. For $\lceil (6k - t)/2\rceil + 3 \leqslant 3j - s \leqslant 6k - t$, there are three partitions counted by $p\big(6k - t, 2, 3I_j - s\big)$, and they are

$$\begin{aligned} &(3j - s) + (6k - 3j), \\ &(3j - s - 1) + (6k - 3j + 1), \quad \text{and} \\ &(3j - s - 2) + (6k - 3j + 2). \end{aligned}$$

We see that the second parts of these three partitions form a complete residue system modulo 3, and therefore

$$\begin{aligned} M'\big(0, 3, 6k - t, 2, 3I_j - s\big) &= M'\big(1, 3, 6k - t, 2, 3I_j - s\big) \\ &= M'\big(2, 3, 6k - t, 2, 3I_j - s\big) = 1. \end{aligned}$$

Now (13) follows from (12) since

$$M'\big(r, 3, 6k - t, 2, 3J - s\big) = \sum_{j=1}^{J} M'\big(r, 3, 6k - t, 2, 3I_j - s\big)$$

for each $r \in \{0, 1, 2\}$. Finally, (14) follows from (13) by setting $j = 2k$. □

In the case $m = 2$, we note that the first part is also a crank witnessing the congruences of Theorem 4 and Corollary 5, which again follows by the combinatorics underlying the symmetry $p(n, m, N) = p(mN - n, m, N)$.

3.1 Integer Lattices and a Combinatorial/Bijective Proof of the case $m = 3$ of Theorem 7

In this section, we give a direct proof of the case $m = 3$ of Theorem 7. To do this, we treat partitions into at most three parts as vectors in $\mathbb{Z}^3$. We define the sets of partitions counted by each of $p\big(n, m\big), p\big(n, m, N\big)$, and $p\big(n, m, (a, b]\big)$ to be $\mathcal{P}\big(n, m\big), \mathcal{P}\big(n, m, N\big)$, and $\mathcal{P}\big(n, m, (a, b]\big)$, respectively. We then construct five triplets of vectors such that each triplet contains one partition from each of the three possible crank classes determined by λ_2 modulo 3. Then, we give an explicit covering of $\mathcal{P}\big(18k - t, 3, 6I_j - s\big)$ with translations of our five triplets, such that the translated triplets are disjoint.

We treat a partition λ of $n = \lambda_1 + \lambda_2 + \lambda_3$ as an integer vector $\lambda = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \in \mathbb{Z}^3$ so that the set of partitions of n into at most three parts becomes

$$\mathcal{P}\big(n,3\big) = \left\{ \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \in \mathbb{Z}^3 \mid \lambda_1 + \lambda_2 + \lambda_3 = n, \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant 0 \right\}. \tag{21}$$

For example, in Figure 1, we see the set $\mathcal{P}(51,3)$.

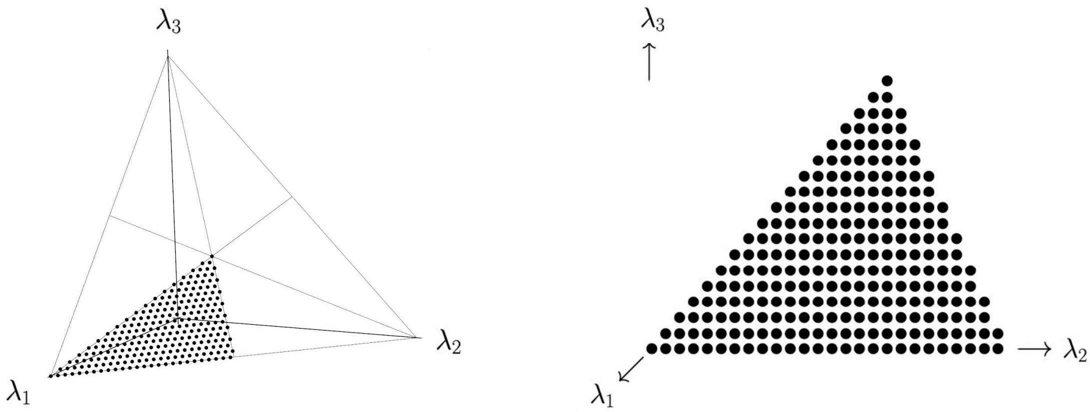


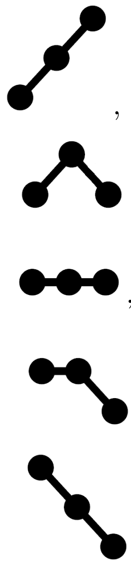
Figure 1: Two views of the set $\mathcal{P}\big(51,3\big) \subset \mathbb{Z}^3$. On the left we include the ambient large equilateral triangle $\begin{pmatrix}51\\ 0\\ 0\end{pmatrix}$, $\begin{pmatrix}0\\ 51\\ 0\end{pmatrix}$, $\begin{pmatrix}0\\ 0\\ 51\end{pmatrix}$ and its medians. On the right we display a planar version of $\mathcal{P}\big(51,3\big)$. The node on the bottom left is the partition $\begin{pmatrix}51\\ 0\\ 0\end{pmatrix}$, the bottom right is $\begin{pmatrix}26\\ 25\\ 0\end{pmatrix}$, and the node at the top is $\begin{pmatrix}17\\ 17\\ 17\end{pmatrix}$.

Remark 12. Notice that when a set of partitions is displayed as in the right side of Figure 1, for each partition λ, the crank value λ_2 corresponds to the apparent horizontal location of the lattice point/partition on the page.

Combinatorial proof of Theorem 7 for $m = 3$. For the crank λ_2, we begin by choosing triplets of integer lattice points that each span all three crank classes modulo 3. We then cover $\mathcal{P}\big(18k - t, 3, 6I_j - s\big)$ with disjoint translations of these triplets, demonstrating that that those partitions are equally distributed among the three crank classes.

The five lattice point triplets are

$$
\begin{aligned}
A &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} \right\}, \\
B &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \right\}, \\
C &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \right\}, \\
D &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}, \text{ and} \\
E &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \right\}.
\end{aligned}
\tag{22}
$$



In Figure 2, we give the cover for $\mathcal{P}(87,3)$ by translations of the triplets A, B, C, D and E. Note that triplet of the second coordinates in each of A, B, C, D and E form a complete residue system modulo 3. In other words, once translated, each triplet of partitions spans the three crank classes modulo 3.

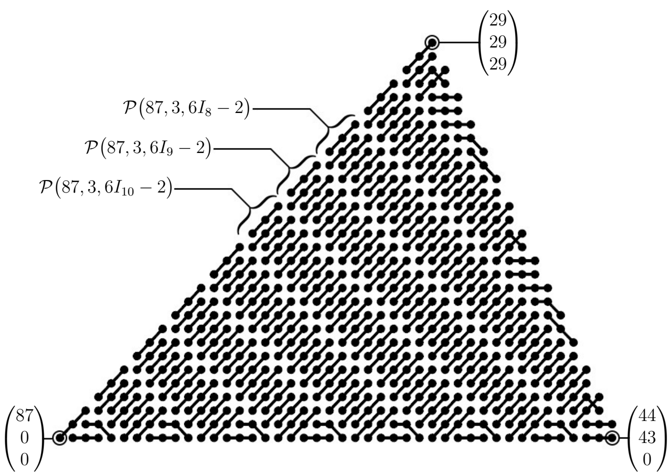


Figure 2: $\mathcal{P}(87,3) = \bigcup_{j=6}^{15} \mathcal{P}(87,3,6I_j - 2)$, with each set of the form $\mathcal{P}(87,3,6I_j - 2)$ covered by translations of triplets described in (22). We indicate a few individual partitions and sets $\mathcal{P}(87,3,6I_j - 2)$. The set $\mathcal{P}(87,3,6I_8 - 2) = \mathcal{P}(87,3,(40,46])$ is in the first regime and consists of 42 triplets, while the sets $\mathcal{P}(87,3,6I_9 - 2) = \mathcal{P}(87,3,(46,52])$ and $\mathcal{P}(87,3,6I_{10} - 2) = \mathcal{P}(87,3,(52,58])$ are in the second regime and consist of 39 and 33 triplets, respectively.

We consider the sets $\mathcal{P}\big(18k - t, 3, 6I_j - s\big)$ in two separate regimes depending on j. The first regime is defined by $k+1 \leqslant j \leqslant \lceil \frac{3k}{2} \rceil$. We require translations of all five triplets A, B, C, D, E to cover the sets $\mathcal{P}\big(18k - t, 3, 6I_j - s\big)$ in this regime. This cover is given in Table 1 and is followed by a discussion and an example in Figure 3. The second regime is defined by $\lceil \frac{3k}{2} \rceil + 1 \leqslant j \leqslant 3k$. We require translations of only the three triplets A, C, and D to cover the sets $\mathcal{P}\big(18k - t, 3, 6I_j - s\big)$ in this regime. This cover is given in Table 2 and is followed by a discussion and an example in Figure **??**.

Table 1: $\mathcal{P}\big(18k - t, 3, 6I_j - s\big)$ for $k+1 \leqslant j \leqslant \lceil \frac{3k}{2} \rceil$. Let $j' = (j - k - 1)$.

Triplet	Translations	Conditions
A	$\vec{a_1}(x) = \begin{pmatrix} 6k + 6j' + 6 - s \\ 6k + 6j' - s - x \\ 6k - 12j' - 6 + 2s - t + x \end{pmatrix}$	$0 \leqslant x \leqslant 9j' + \lfloor \frac{t-3s}{2} \rfloor + 3$
	$\vec{a_2}(y) = \begin{pmatrix} 6k + 6j' + 5 - s \\ 6k + 6j' - 1 - s - y \\ 6k - 12j' + 2s - 4 - t + y \end{pmatrix}$	$0 \leqslant y \leqslant 9j' + \lceil \frac{t-3s}{2} \rceil + 1$
B	$\begin{pmatrix} 6k + 6j' + 5 - s \\ 6k + 6j' - s \\ 6k - 12j' - 5 + 2s - t \end{pmatrix}$	
C	$\vec{c_1} = \begin{pmatrix} 6k + 6j' + 5 - s \\ 6k + 6j' + 1 - s \\ 6k - 12j' - 6 + 2s - t \end{pmatrix}$	unless $j = \frac{3k+1}{2}$, in which case, for pairs (s,t), C is translated by
	$\vec{c_2} = \begin{pmatrix} 6k + 6j' + 6 - s \\ 6k + 6j' + 1 - s \\ 6k - 12j' - 7 + 2s - t \end{pmatrix}$	$\begin{cases} \vec{c_1},\ \vec{c_2},\ \vec{c_3} & \text{for } (2,2),(3,4) \\ \vec{c_1},\ \vec{c_2} & \text{for } (1,1),(2,3),(3,5) \\ \vec{c_1} & \text{for } (1,2),(2,4) \end{cases}$
	$\vec{c_3} = \begin{pmatrix} 6k + 6j' + 6 - s \\ 6k + 6j' + 2 - s \\ 6k - 12j' - 8 + 2s - t \end{pmatrix}$	
D	$\begin{pmatrix} 6k + 6j' + 6 - s \\ 6k + 6j' + 3 - s \\ 6k - 12j' - 3 + 2s - t \end{pmatrix}$	unless $j = \frac{3k+1}{2}$, in which case D is not translated at all
E	$\begin{pmatrix} 6k + 6j' + 6 - s \\ 6k + 6j' + 4 - s \\ 6k - 12j' - 4 + 2s - t \end{pmatrix}$	unless $j = \frac{3k+1}{2}$, in which case E is not translated at all

We begin with a discussion of "first regime" coverings in general. To see that Table 1 does indeed describe a disjoint covering of the partitions in the first regime, we begin by setting $j' = (j - k - 1)$, so that the first regime is the union of the sets $\mathcal{P}\big(18k - t, 3, 6I_{j'+k+1} - s\big)$ over $0 \leqslant j' \leqslant \lceil \frac{3k}{2} \rceil - k - 1$. From Table 1, we see that for $0 \leqslant j' < \lceil \frac{3k}{2} \rceil - k - 1$, $\mathcal{P}\big(18k - t, 3, 6I_{j'+k+1} - s\big)$ is covered by $18j' + t - 3s + 6$ copies of A, three copies of C, one copy of D, and one copy of E. Observe that this covers all of the relevant partitions, starting with E translated by $\begin{pmatrix} 6k+6j'+6-s \\ 6k+6j'+4-s \\ 6k-12j'-4+2s-t \end{pmatrix}$, and ending with A translated by $\overrightarrow{a_1}\big(9j'+\lfloor\frac{t-3s}{2}\rfloor+3\big) = \begin{pmatrix} 6k+6j'+6-s \\ 6k-3j'-\frac{t-s}{2}-3 \\ 6k-3j'-\frac{t-s}{2}-3 \end{pmatrix}$ if $t-s$ is even or by $\overrightarrow{a_2}\big(9j'+\lceil\frac{t-3s}{2}\rceil+1\big) = \begin{pmatrix} 6k+6j'+5-s \\ 6k-3j'-\frac{t-s-1}{2}-3 \\ 6k-3j'-\frac{t-s-1}{2}-3 \end{pmatrix}$ if $t-s$ is odd. Notice that the aforementioned starting translation of E covers the three partitions that have their third parts as small as possible within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$ and their first parts as large as possible within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$, while the final translation of A covers the partitions within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$ where the second and third part are equal.

When $j' = \lceil\frac{3k}{2}\rceil-k-1$, the set of possible configurations of triplets that cover $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$ is more complicated, and we break the analysis into four cases. In the first case, when k is even, in the same configuration described above, we have $18j'+t-3s+6$ copies of A, three copies of C, one copy of D, and one copy of E. In the second case, when k is odd and $(s,t)\in\{(1,2),(2,4)\}$, we have $18j'+t-3s+6$ copies of A and one copy of C. Notice that in the second case, the translation of C covers the three partitions that have their third parts as small as possible within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$, while again the final translation of A covers the partitions within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$ where the second and third part are equal. In the third case, when k is odd and $(s,t)\in\{(1,1),(2,3),(3,5)\}$, we have $18j'+t-3s+6$ copies of A and two copies of C. Notice that in the third case, the second translation of C covers the three partitions that have their third parts as small as possible within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$, while again the final translation of A covers the partitions within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$ where the second and third parts are equal. In fourth case, when k is odd and $(s,t)\in\{(2,2),(3,4)\}$, we have $18j'+t-3s+6$ copies of A and three copies of C. Notice that in the fourth case, the third translation of C covers the three partitions that have their third parts as small as possible within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$, while again the final translation of A covers the partitions within $\mathcal{P}\big(18k-t,3,6I_{j'+k+1}-s\big)$ where the second and third parts are equal. Hence, Table 1 describes a disjoint covering of the partitions in the first regime.

We now discuss "second regime" coverings in general. To see that Table 2 describes a disjoint covering of the partitions in the second regime, we begin by noting that for $\lceil\frac{3k}{2}\rceil+1 \leqslant j \leqslant 3k$, the set $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ is covered by $18k-6j+s-t+2$ copies of A, one copy of C, and one copy of D. Observe that this covers all of the relevant partitions, starting with C and D translated by $\begin{pmatrix} 6j-2-s \\ 18k-6j+2+s-t \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 6j-4-s \\ 18k-6j+3+s-t \\ 1 \end{pmatrix}$, respectively. The covering of the set $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ ends immediately if $j=3k$ and $t\geqslant s+2$, and otherwise it ends with the aforementioned

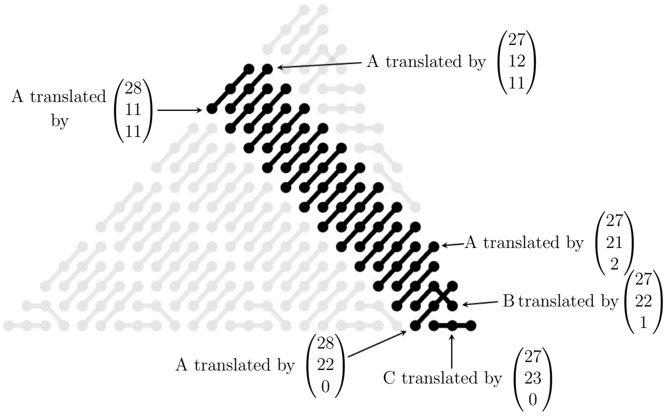


Figure 3: In this figure we have highlighted the set $\mathcal{P}\big(50,3,6I_5-2\big)=\mathcal{P}\big(50,3,(22,28]\big)$ within $\mathcal{P}\big(50,3\big)$. This set has parameter values $k=3, t=4$, $s=2$, and $j=5$, and since $3+1\leqslant 5\leqslant \lceil 3(3)/2\rceil$, we are
in the first regime, described by Table 1. A is translated by $\overrightarrow{a_1}(0),\overrightarrow{a_1}(1),\ldots,\overrightarrow{a_1}(11)$ beginning with $\begin{pmatrix}28\\22\\0\end{pmatrix}$ at the bottom right and ending with $\begin{pmatrix}28\\11\\11\end{pmatrix}$, and again by $\overrightarrow{a_2}(0),\overrightarrow{a_2}(1),\ldots,\overrightarrow{a_2}(9)$ beginning with $\begin{pmatrix}27\\21\\2\end{pmatrix}$ and ending with $\begin{pmatrix}27\\12\\11\end{pmatrix}$ near the top left. B is translated once by $\begin{pmatrix}27\\22\\1\end{pmatrix}$, and C is translated once by $\begin{pmatrix}27\\23\\0\end{pmatrix}$.

$18k-6j+s-t+2$ translations of A. The last translation of A is $\overrightarrow{a_4}(9k-3j+\lfloor\frac{s-t}{2}\rfloor)=\begin{pmatrix}6j-s\\9k-3j+\frac{s-t}{2}\\9k-3j+\frac{s-t}{2}\end{pmatrix}$ if $s-t$ is even or $\overrightarrow{a_3}(9k-3j+\lceil\frac{s-t}{2}\rceil)=\begin{pmatrix}6j-1-s\\9k-3j+\frac{s-t+1}{2}\\9k-3j+\frac{s-t+1}{2}\end{pmatrix}$ if $s-t$ is odd. Notice that the aforementioned starting translations of C and D cover the partitions $\left\{\begin{pmatrix}6j-2-s\\18k-6j+2+s-t\\0\end{pmatrix},\begin{pmatrix}6j-3-s\\18k-6j+3+s-t\\0\end{pmatrix},\begin{pmatrix}6j-4-s\\18k-6j+4+s-t\\0\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}6j-4-s\\18k-6j+3+s-t\\1\end{pmatrix},\begin{pmatrix}6j-5-s\\18k-6j+4+s-t\\1\end{pmatrix}\begin{pmatrix}6j-5-s\\18k-6j+5+s-t\\0\end{pmatrix}\right\}$, respectively, while the final translation of A covers the partitions within $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ where the second and third part are equal. Hence, Table 2 describes a disjoint covering of the partitions in the second regime.

Thus, we have shown that Tables 1 and 2 describe for each j a covering of $\mathcal{P}\big(18k-$

Table 2: $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ for $\lceil\frac{3k}{2}\rceil+1\leqslant j\leqslant 3k$.

Triplet	Translations	Conditions
A	$\overrightarrow{a_3}(x)=\begin{pmatrix}6j-1-s\\18k-6j+1+s-t-x\\x\end{pmatrix}$	$0\leqslant x\leqslant 9k-3j+\lceil\frac{s-t}{2}\rceil$
	$\overrightarrow{a_4}(y)=\begin{pmatrix}6j-s\\18k-6j+s-t-y\\y\end{pmatrix}$	$0\leqslant y\leqslant 9k-3j+\lfloor\frac{s-t}{2}\rfloor$
C	$\begin{pmatrix}6j-2-s\\18k-6j+2+s-t\\0\end{pmatrix}$	
D	$\begin{pmatrix}6j-4-s\\18k-6j+3+s-t\\1\end{pmatrix}$	

$t,3,6I_j-s\big)$ by disjoint translations of the lattice point triplets A,B,C,D, and E. Therefore, since the triplet of second coordinates in each of A,B,C,D and E forms a complete residue system modulo 3, the partitions in $\mathcal{P}\big(18k-t,3,6I_j-s\big)$ are equally distributed among the three crank classes modulo 3, making the *second part* of the partition a crank witnessing the congruences of Theorem 4 and Corollary 5 for $\ell=m=3$. $\square$

3.2 Formulas for $p\big(18k-t,3,6I_j-s\big)$, $p\big(18k-t,3,6j-s\big)$, and $p\big(18k-t,3\big)$

A byproduct of the combinatorial proof of the case $m=3$ of Theorem 7 is that it allows us to establish formulas for $p\big(18k-t,3,6I_j-s\big)$, $p\big(18k-t,3,6j-s\big)$, and $p\big(18k-t,3\big)$.

Examining Table 1 and Table 2, we have

$$p\big(18k-t,3,6I_j-s\big)=\begin{cases}0 & \text{for } j<k+1\\ 3\big(18j-18k-6-3s+t\big) & \text{for } k+1\leqslant j<\lceil\frac{3k}{2}\rceil\\ 3\big(18j-18k-10-s\big) & \text{for } j=\frac{3k+1}{2}\\ \quad =27k-3(1+s) & \\ 3\big(18k-6j+4+s-t\big) & \text{for } \lceil\frac{3k}{2}\rceil<j\leqslant 3k\\ 0 & \text{for } j>3k.\end{cases} \tag{23}$$

Summing the appropriate values from (23), we have proved the following proposition.

Proposition 13. *For integers j,k and the ordered pairs*

$(s,t) \in \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$, *one has*

$$p\big(18k-t,3,6j-s\big) \\ = \begin{cases} 0 & \textit{for } j < k+1 \\ 3(k-j)(9k-9j-3+3s-t) & \textit{for } k+1 \leqslant j < \lceil \frac{3k}{2} \rceil \\ 3(k-j)(9k-9j-3+3s-t)-3(1+s) & \textit{for } j = \frac{3k+1}{2} \\ 3\big(j(18k+s-t+1)-3j^2+k(-18k-3s+2t)\big) & \textit{for } \lceil \frac{3k}{2} \rceil < j < 3k \\ 27k^2+3k(t-3) \;=\; p\big(18k-t,3\big) & \textit{for } j \geqslant 3k. \end{cases} \tag{24}$$

Note that each of the five expressions in (24) is a multiple of 3.

3.3 An Interlude Prior to the Analytic Proofs for the Cases $m \in \{3,4\}$ of Theorem 7 and a Definition

In Sections 3.4 and 3.5, we give analytic proofs of the cases $m \in \{3,4\}$ of Theorem 7. We follow a procedure for producing formulas detailed in [4] for producing formulas for $p\big(n,m,N\big)$ to similarly establish formulas for each crank class $M'\big(r,3,n,3,N\big)$ for $r \in \{0,1,2\}$ and observe that these three formulas are the same. The procedure begins with a generating function and the end result is a collection of polynomial formulas for $M'\big(r,3,n,3,N\big)$ for all r and n. The resulting collection of formulas is called a "quasipolynomial".

Definition 14. A function $f(n)$ is a *quasipolynomial* if there exist d polynomials $f_0(n),\ldots,f_{d-1}(n)$ such that

$$f(n) = \begin{cases} f_0(n) & \text{if } n \equiv 0 \pmod{d} \\ f_1(n) & \text{if } n \equiv 1 \pmod{d} \\ \quad\vdots & \qquad\vdots \\ f_{d-1}(n) & \text{if } n \equiv d-1 \pmod{d} \end{cases}$$

for all $n \in \mathbb{Z}$. The polynomials f_i are called the *constituents* of the quasipolynomial f and the number of them, d, is the *period* of f.

The method used to generate such quasipolynomials obligates us to adhere to a strict interpretation of binomial coefficients for the constituents of $p(n,m,N)$. For a and b natural numbers, when $a < b$ then $\binom{a}{b} = 0$, and when $a \geqslant b$, then $\binom{a}{b} = \frac{a!}{b!(a-b)!}$.

A quasipolynomial consisting of 36 constituents describing $p\big(n,3,N\big)$ for all n and N can be found in the Appendix of [4]. Applying some arithmetic to these constituents, it is possible to express $p\big(18k-t,3,6j-s\big)$ for all $k,j \in \mathbb{Z}_{\geqslant 0}$ for the seven ordered pairs determined by the hypotheses of Theorem 4, $(s,t) \in \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),$

$(3,5)\}$, with one formula.

$$\begin{aligned} p\bigl(18k-t,3,6j-s\bigr) &= (6-t)\tbinom{3k+1}{2} + t\tbinom{3k}{2} - 3(s-t+4)\tbinom{3k-j+1}{2} \\ &\quad - 3(6-(s-t+4))\tbinom{3k-j}{2} + 3(2s-t+2)\tbinom{3k-2j+1}{2} \\ &\quad + 3(6-(2s-t+2))\tbinom{3k-2j}{2} - (t-3s)\tbinom{3k-3j+1}{2} \\ &\quad - (6-(t-3s))\tbinom{3k-3j}{2}. \end{aligned} \tag{25}$$

Compare (25) to (24) and note that each term in (25) is a multiple of 3.

From the analytic proof of the case $m=3$ of Theorem 7 we obtain a similarly condensed formula for the relevant functions $M'\bigl(r,3,18k-t,3,6j-s\bigr)$ for s and t listed in the same seven ordered pairs above. We do not attempt to condense the formulas for the case $m=4$.

3.4 Analytic Proof of the case $m=3$ of Theorem 7

For each $r \in \{0,1,2\}$, we produce a quasipolynomial formula for $M'(r,3,18k-t,3,18j-s+6x)$ for $x \in \{0,1,2\}$ and the ordered pairs $(s,t) \in \{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$ below. Then by inspection, we find the following constituents to be equal:

$$\begin{aligned} M'(0,3,18k-t,3,18j-s+6x) &= M'(1,3,18k-t,3,18j-s+6x) \\ &= M'(2,3,18k-t,3,18j-s+6x). \end{aligned} \tag{26}$$

Because the sequence $\{6j-s\}_{j\geqslant 0}$ is comprised of the three subsequences $\{18j-s, 18j-s+6, 18j-s+12\}_{j\geqslant 0}$, (16) follows.

Analytic proof of Theorem 7 for $m=3$. Our first goal is to establish a generating function for the partitions of n into at most 3 parts, no part larger than N, with crank value r, where the crank value is determined by the *second part*.

Combinatorial arguments produce the generating function:

$$\begin{aligned} f(z,q) &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} M'(r,n,3,N) z^r q^n \\ &= \sum_{j=0}^{N} \left(q^j + q^{j+1} + \cdots + q^N\right) z^j q^j \left(1+q+\cdots+q^j\right). \end{aligned} \tag{27}$$

We rewrite (27) as the following rational function.

$$\begin{aligned} f(z,q) &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} M'(r,n,3,N) z^r q^n \\ &= \frac{1-q^{N+1}-zq+zq^{N+4}+z^{N+2}q^{2N+3}-z^{N+2}q^{3N+6}-z^{N+3}q^{2N+6}+z^{N+3}q^{3N+7}}{(1-q)(1-zq)(1-zq^2)(1-zq^3)}. \end{aligned} \tag{28}$$

We multiply the far right side of (28) by $E(z,q)/E(z,q)$ where

$$E(z,q) = \sum_{i=0}^{17} q^i \times \sum_{i=0}^{17} (zq)^i \times \sum_{i=0}^{8} \left(zq^2\right)^i \times \sum_{i=0}^{5} \left(zq^3\right)^i.$$

We note that the polynomial $E(z,q)$ is constructed specifically so that $E(z,q) \times (1-q)(1-zq)(1-zq^2)(1-zq^3) = (1-q^{18})(1-z^{18}q^{18})(1-z^9q^{18})(1-z^6q^{18})$. Please see [4] for more details including a generalization of $E(q)$. Now,

$$\begin{aligned} \frac{E(z,q)}{E(z,q)} \times \frac{1-q^{N+1}-zq+zq^{N+4}+z^{N+2}q^{2N+3}-z^{N+2}q^{3N+6}-z^{N+3}q^{2N+6}+z^{N+3}q^{3N+7}}{(1-q)(1-zq)(1-zq^2)(1-zq^3)} \\ = \frac{A(z,q)}{(1-q^{18})(1-z^{18}q^{18})(1-z^9q^{18})(1-z^6q^{18})}, \quad (29) \end{aligned}$$

where we write $A(z,q)$ for the numerator on the left side of (29).

So that we may analyze $M'(r,3,n,3,N)$, the r^{th} crank classes modulo 3, we now dissect $A(z,q)$ as a sum of three polynomials organized by their powers of z taken modulo 3: $A(z,q) = A_0(z^3,q) + zA_1(z^3,q) + z^2A_2(z^3,q)$. Using this dissection and replacing z by $\zeta = e^{2\pi i/3}$, we have

$$f(\zeta,q) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} M'(r,3,n,3,N)\zeta^r q^n = \frac{A_0(1,q)+\zeta A_1(1,q)+\zeta^2 A_2(1,q)}{(1-q^{18})^4}. \quad (30)$$

We may then express the right side of (30) as

$$\left(A_0(q)+\zeta A_1(q)+\zeta^2 A_2(q)\right) \times \sum_{k\geqslant 0} \binom{k+3}{3} q^{18k}, \quad (31)$$

since

$$\frac{1}{(1-q)^b} = \sum_{a\geqslant 0} \binom{a+b-1}{b-1} q^a.$$

Hence,

$$\begin{aligned} f(\zeta,q) &= \sum_{n\geqslant 0}\sum_{r=0}^{2} M'(r,3,n,3,N)\zeta^r q^n \\ &= \sum_{n\geqslant 0} M'(0,3,n,3,N)q^n + \sum_{n\geqslant 0} M'(1,3,n,3,N)\zeta q^n + \sum_{n\geqslant 0} M'(2,3,n,3,N)\zeta^2 q^n \end{aligned}$$

$$= A_0(q) \times \sum_{k\geqslant 0}\binom{k+3}{3}q^{18k} + \zeta A_1(q) \times \sum_{k\geqslant 0}\binom{k+3}{3}q^{18k} + \zeta^2 A_2(q) \times \sum_{k\geqslant 0}\binom{k+3}{3}q^{18k}. \quad (32)$$

Multiplying and collecting like terms from each of the series in the right side of (32), we are able to build three period 18 quasipolynomials.

Example 15. Setting $(s,t)=(2,4)$, with $x=2$, so that $n=18k=4$ and $N=18j+10$, we compute the constituent $M'\big(1,3,18k-4,3,18j+10\big)$.

$$\sum_{k\geqslant 1} M'\big(1,3,18k-4,3,18j+10\big)\zeta q^{18k-4} = \sum_{k\geqslant 1} M'\big(1,3,18(k-1)+14,3,18j+10\big)\zeta q^{18k-4}$$
$$= \zeta\left(8q^{14}+2q^{32}-10q^{50}-2q^{18j+14}-\cdots+10q^{54j+86}\right)\times\sum_{k\geqslant 1}\binom{k+2}{3}q^{18k}. \quad (33)$$

Hence, we arrive at

$$\begin{aligned} M'(1,3,18k-4,3,18j+10) &= 8\tbinom{k+2}{3}+2\tbinom{k+1}{3}-10\tbinom{k}{3}-2\tbinom{k+2-j}{3}-36\tbinom{k+2-(j+1)}{3} \\ &\quad +24\tbinom{k+2-(j+2)}{3}+14\tbinom{k+2-(j+3)}{3}+10\tbinom{k+2-(2j+1)}{3} \\ &\quad +30\tbinom{k+2-(2j+2)}{3}-36\tbinom{k+2-(2j+3)}{3}-4\tbinom{k+2-(2j+4)}{3} \\ &\quad -8\tbinom{k+2-(3j+2)}{3}-2\tbinom{k+2-(3j+3)}{3}+10\tbinom{k+2-(3j+4)}{3}. \end{aligned} \quad (34)$$

By examining the three quasipolynomials for $M'(r,3,n,3,18j+6x-s)$ for $r=0,1,2$ we are able to show that for $x=0,1,2$ and the ordered pairs

$$(s,t)\in\{(1,1),(1,2),(2,2),(2,3),(2,4),(3,4),(3,5)\}$$

the following constituents are equal:

$n=18k-t,\ N=18j-s$

$$M'(0,3,18k-t,3,18j-s) = M'(1,3,18k-t,3,18j-s) = M'(2,3,18k-t,3,18j-s)$$
$$= -(t-12)\binom{k+2}{3}+2(t-3)\binom{k+1}{3}-(t+6)\binom{k}{3}$$
$$+3(t-s-10)\binom{k-j+2}{3}-6(t-s-1)\binom{k-j+1}{3}+3(t-s+8)\binom{k-j}{3}$$
$$-3(t-2s-8)\binom{k-2j+2}{3}+6(t-2s+1)\binom{k-2j+1}{3}-3(t-2s+10)\binom{k-2j}{3}$$
$$+(t-3s-6)\binom{k-3j+2}{3}-2(t-3s+3)\binom{k-3j+1}{3}+(t-3s+12)\binom{k-3j}{3}$$
$$=\frac{p\big(18k-4,3,18j-s\big)}{3} \quad (35)$$

$n=18k-t,\ N=18j-s+6$

$$M'(0,3,18k-t,3,18j-s+6) = M'(1,3,18k-t,3,18j-s+6)$$
$$= M'(2,3,18k-t,3,18j-s+6)$$
$$= -(t-12)\binom{k+2}{3}+2(t-3)\binom{k+1}{3}-(t+6)\binom{k}{3}$$

$$
\begin{aligned}
&+2(t-s-7)\binom{k-j+2}{3}-3(t-s+8)\binom{k-j+1}{3}\\
&\qquad\qquad+36\binom{k-j}{3}+(t-s+2)\binom{k-j-1}{3}\\
&-(t-2s-2)\binom{k-2j+2}{3}+36\binom{k-2j+1}{3}\\
&\qquad\qquad+3(t-2s-8)\binom{k-2j}{3}-2(t-2s+7)\binom{k-2j-1}{3}\\
&+(t-3s-6)\binom{k-3j+1}{3}-2(t-3s+3)\binom{k-3j}{3}+(t-3s+12)\binom{k-3j-1}{3}\\
&\qquad\qquad=\frac{p\big(18k-3,3,18j-s+6\big)}{3} \qquad (36)
\end{aligned}
$$

$n = 18k - t,\ N = 18j - s + 12$

$$
\begin{aligned}
&M'(0,3,18k-t,3,18j-s+12) = M'(1,3,18k-t,3,18j-s+12)\\
&\qquad\qquad= M'(2,3,18k-t,3,18j-s+12)\\
&= -(t-12)\binom{k+2}{3}+2(t-3)\binom{k+1}{3}-(t+6)\binom{k}{3}\\
&\quad+(t-s-4)\binom{k-j+2}{3}-36\binom{k-j+1}{3}\\
&\qquad\qquad-3(t-s-10)\binom{k-j}{3}+2(t-s+5)\binom{k-j-1}{3}\\
&\quad-2(t-2s-5)\binom{k-2j+1}{3}+3(t-2s+10)\binom{k-2j}{3}\\
&\qquad\qquad-36\binom{k+2j-1}{3}-(t-2s+4)\binom{k-2j-2}{3}\\
&\quad+(t-3s-6)\binom{k-3j}{3}-2(t-3s+3)\binom{k-3j-1}{3}+(t-3s+12)\binom{k-3j-2}{3}\\
&\qquad\qquad=\frac{p\big(18k-4,3,18j-s+12\big)}{3} \qquad (37)
\end{aligned}
$$

Thus, (35), (36), and (37) together, amount to an analytic proof of the case $m = 3$ of Theorem 7. □

3.5 Analytic Proof of the case $m = 4$ from Theorem 7

In the case $m = 4$, since the crank is defined differently depending on whether or not $n \leqslant 2N$, we require the following proposition to deduce the truth of Theorem 7 in the case $n > 2N$ from the case when $n \leqslant 2N$.

Proposition 16. *If the coefficient of $z^r q^n$ in $f(z,q)$ is the number of partitions of n with largest part at most N, number of parts at most m, and $\lambda_a = r$, then the coefficient of $z^r q^{mN-n}$ in $f(z,q)$ is the number of partitions of n with largest part at most N, number of parts at most m, and $\lambda_{m+1-a} = N - r$.*

Proposition 16 follows by the reasoning given in Remark 11.

Analytic proof of Theorem 7 for $m = 4$. We follow the same procedure here that was done for the case $m = 3$ in Section 3.4, however, we now have 16 ordered pairs (s,t) meeting the hypotheses of Theorem 4: $(s,t) \in \{(1,1),(1,2),(1,3),(2,2),(2,3),(2,4),(2,5),$ $(2,6),(3,4),(3,5),(3,6),(3,7),(3,8),(4,7),(4,8),(4,9)\}$.

Our first goal is to establish a generating function for the partitions of n into at most 4 parts, no part larger than N, with crank value r, where the crank value is determined by the second part of the partition for $n \leqslant 2N$. Combinatorial arguments give the generating function

$$f(z,q) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} M'(r,3,n,4,N) z^r q^n = \sum_{j=0}^{N} \left(q^j + q^{j+1} + \cdots + q^N\right) z^j q^j \begin{bmatrix} j+2 \\ 2 \end{bmatrix}. \tag{38}$$

Summing in a way that yields four rational functions, each with four factors in the denominator, we have

$$\begin{aligned} f(z,q) = {} & \frac{-q^{N+1}\left(1-(zq)^{N+1}\right)}{(1-zq)(1-q)^2(1-q^2)} + \frac{\left(1+q^{N+2}+q^{N+3}\right)\left(1-(zq^2)^{N+1}\right)}{(1-zq^2)(1-q)^2(1-q^2)} \\ & - \frac{q\left(1+q+q^{N+3}\right)\left(1-(zq^3)^{N+1}\right)}{(1-zq^3)(1-q)^2(1-q^2)} + \frac{q^3\left(1-(zq^4)^{N+1}\right)}{(1-zq^4)(1-q)^2(1-q^2)}. \end{aligned} \tag{39}$$

Let $E_i(z,q) = (1 - z^{36/i} q^{36})/(1 - zq^i)$ for $i \in \{1,2,3,4\}$. Then,

$$\begin{aligned} & f(z,q) \\ & = \frac{E_1(z,q)}{E_1(z,q)} \times \frac{-q^{N+1}\left(1-(zq)^{N+1}\right)}{(1-zq)(1-q)^2(1-q^2)} + \frac{E_2(z,q)}{E_2(z,q)} \times \frac{\left(1+q^{N+2}+q^{N+3}\right)\left(1-(zq^2)^{N+1}\right)}{(1-zq^2)(1-q)^2(1-q^2)} \\ & + \frac{E_3(z,q)}{E_3(z,q)} \times \frac{-q\left(1+q+q^{N+3}\right)\left(1-(zq^3)^{N+1}\right)}{(1-zq^3)(1-q)^2(1-q^2)} + \frac{E_4(z,q)}{E_4(z,q)} \times \frac{q^3\left(1-(zq^4)^{N+1}\right)}{(1-zq^4)(1-q)^2(1-q^2)} \\ & = \frac{A(z,q)}{(1-z^{36}q^{36})(1-q^{36})^3} + \frac{B(z,q)}{(1-z^{18}q^{36})(1-q^{36})^3} \\ & \quad + \frac{C(z,q)}{(1-z^{12}q^{36})(1-q^{36})^3} + \frac{D(z,q)}{(1-z^{9}q^{36})(1-q^{36})^3}. \end{aligned} \tag{40}$$

So that we may analyze $M'(r,3,n,4,N)$, the r^{th} crank classes modulo 3, we now dissect $A(z,q)$, $B(z,q)$, $C(z,q)$, and $D(z,q)$ as sums of three polynomials organized by their powers of z taken modulo 3:

$$\begin{aligned} A(z,q) &= A_0(z^3,q) + zA_1(z^3,q) + z^2A_2(z^3,q),\\ B(z,q) &= B_0(z^3,q) + zB_1(z^3,q) + z^2B_2(z^3,q),\\ C(z,q) &= C_0(z^3,q) + zC_1(z^3,q) + z^2C_2(z^3,q), \text{ and}\\ D(z,q) &= D_0(z^3,q) + zD_1(z^3,q) + z^2D_2(z^3,q). \end{aligned}$$

Using this dissection and replacing z by $\zeta = e^{2\pi i/3}$, we have

$$\begin{aligned} f(\zeta,q) &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} M'(r,3,n,4,N)\zeta^r q^n\\ &= \frac{A_0(1,q)+\zeta A_1(1,q)+\zeta^2 A_2(1,q)}{(1-q^{36})^4} + \frac{B_0(1,q)+\zeta B_1(1,q)+\zeta^2 B_2(1,q)}{(1-q^{36})^4}\\ &\quad + \frac{C_0(1,q)+\zeta C_1(1,q)+\zeta^2 C_2(1,q)}{(1-q^{36})^4} + \frac{D_0(1,q)+\zeta D_1(1,q)+\zeta^2 D_2(1,q)}{(1-q^{36})^4}. \end{aligned} \tag{41}$$

We now dissect the right side of (41) into three products:

$$(A_0(q)+B_0(q)+C_0(q)+D_0(q)) \times \sum_{k\geqslant 0}\binom{k+3}{3} q^{36k},$$

$$\zeta\left(A_1(q)+B_1(q)+C_1(q)+D_1(q)\right) \times \sum_{k\geqslant 0}\binom{k+3}{3} q^{36k}, \text{ and}$$

$$\zeta^2\left(A_2(q)+B_2(q)+C_2(q)+D_2(q)\right) \times \sum_{k\geqslant 0}\binom{k+3}{3} q^{36k}.$$

Similarly to the analytic proof for the case $m = 3$, we produce three quasipolynomials. The constituents corresponding to each of the 16 ordered pairs (s,t) are compiled in Appendix A.

We note that our quasipolynomials are in terms of $N = 36j - s$ and $N = 36j + 18 - s$ giving us 32 expressions in Appendix A. Taken in pairs they prove the result in (19). For example, the pair of constituents $M'\big(r,3,36k-7,4,36j-3\big)$ and $M'\big(r,3,36k-7,4,36j+15\big)$ in lines (64) and (65), show $M'\big(r,3,36k-7,4,18j-3\big) = p\big(36k-7,4,18j-3\big)/3$. For a given k, taking j large enough establishes (20), and finally, taking differences of the constituents for different values of j we obtain (18). Thus, Theorem 7 is proved for $n \leqslant 2N$.

For $n > 2N$, where the crank is the third part, we apply Proposition 16, and the other half of Theorem 7 follows. □

Example 17. Consider the Gaussian polynomial $\begin{bmatrix}232+4\\4\end{bmatrix}$. It can be shown that

$$\begin{aligned}p\big(36k-5,4,36j+16\big) &= 11\binom{3k+5}{3}+50\binom{3k+4}{3}+11\binom{3k+3}{3}\\ &-4\binom{3k+4-3j}{3}-131\binom{3k+3-3j}{3}-146\binom{3k+2-3j}{3}-7\binom{3k+1-3j}{3}\\ &+55\binom{3k+2-6j}{3}+286\binom{3k+1-6j}{3}+91\binom{3k-6j}{3}\\ &-2\binom{3k+1-9j}{3}-115\binom{3k-9j}{3}-160\binom{3k-1-9j}{3}-11\binom{3k-2-9j}{3}\\ &+6\binom{3k-1-12j}{3}+48\binom{3k-2-12j}{3}+18\binom{3k-3-12j}{3}. \qquad (42)\end{aligned}$$

We note that $N=232=36(6)+16$. With (42), we can compute the values $p\big(36k-5,4,232\big)$ for $0\leqslant k\leqslant 24$. For example, we set $k=7$ and $j=6$ and compute

$$p\big(283,4,232\big)=161616.$$

Setting $k=20$ and $j=6$, one may further compute

$$p\big(751,4,232\big)=41085.$$

Now, from Appendix A we examine (55) and compute the values of $M'\big(r,3,36k-5,4,36j+16\big)$ for $0\leqslant k\leqslant 24$.

$$\begin{aligned}&M'\big(r,3,36k-5,4,36j+16\big)\\ &=\begin{cases}107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3}-49\binom{k+2-j}{3} & \text{for } 36k-5\leqslant 72j+32\\ -1220\binom{k+1-j}{3}-1265\binom{k-j}{3}-58\binom{k-1-j}{3} & \text{(the crank is the second part)}\\ \\ 126\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+90\binom{4j-k+2}{3} & \text{for } 36k-5\geqslant 72j+32\\ -68\binom{3j-k+4}{3}-1309\binom{3j-k+3}{3}-1174\binom{3j-k+2}{3} & \text{(the crank is the third part)}\\ -41\binom{3j-k+1}{3}\end{cases} \qquad (55)\end{aligned}$$

Setting $k=7$ and $j=6$ in (55), we have

$$M'\big(r,3,283,4,232\big)=53872=\frac{161616}{3}=\frac{p\big(283,4,232\big)}{3}.$$

In this case, since $n=283\leqslant 464=2N$, the crank is the second part.

Setting $k=20$ and $j=6$, we compute

$$M'\big(r,3,751,4,232\big)=13695=\frac{41085}{3}=\frac{p\big(751,4,232\big)}{3}.$$

In this case, however, since $n=751>464=2N$, the crank is the third part.

4 Future Work

Numerical evidence suggests that the second part is a crank witnessing the congruences of Theorem 4 and Corollary 5 for larger values of m and ℓ than what Theorem 7 implies. For example, in Theorem 4 let $\ell = 7, m = 4, k = j = 5$ with $s = 1$ and $t = 6$, or let $\ell = 5, m = 4, k = j = 7$ with $s = 1$ and $t = 6$. In either case, we are considering partitions of 414 into at most $m = 4$ parts, each part no bigger than 209. Hence, Theorem 4 is doubly satisfied, both modulo 5 and modulo 7:

$$p\big(414, 4, 209\big) = 262{,}675 = 7 \times 37{,}525 = 5 \times 52{,}535 \equiv 0 \pmod{35}. \tag{43}$$

Furthermore, we find that

$$M'\big(r, 7, 414, 4, 209\big) = 37{,}525 = \frac{p\big(414, 4, 209\big)}{7} \tag{44}$$

$$M'\big(r, 5, 414, 4, 209\big) = 52{,}535 = \frac{p\big(414, 4, 209\big)}{5} \tag{45}$$

for all r. Thus the second part witnesses congruences from Theorem 4 in some cases where $\ell > 3$. This and other numerical evidence leads us to a conjecture.

Conjecture 18. Let ℓ be any odd prime. For $m \in \{2, 3\}$, the second part is a crank witnessing the congruences of Theorem 4 and Corollary 5 for all n. For $m > 3$, the second part is a crank witnessing the congruences of Theorem 4 and Corollary 5 for $n \leqslant mN/2$. For $n > mN/2$, the $(m-1)^{st}$ part is a crank witnessing the congruences of Theorem 4 and Corollary 5.

In contrast to Conjecture 18 about a single crank, in a forthcoming paper [6], it will be shown that for all odd primes ℓ, there are two fundamentally different cranks; the now familiar *second part*, and also *first part minus* $(\ell+1)^{st}$ *part*, both of which witness the congruences for partitions into at most m parts of Theorem 2. The crank *first part minus* $(\ell+1)^{st}$ *part* does not witness the congruences of Theorem 4 and Corollary 5 presented here.

Acknowledgments

A recently published result due to Dylan Pentland [11] establishes similar congruence properties for $p\big(n, m, N\big)$. However, Pentland's results do not coincide with the results presented here. It may be worthwhile to explore Pentland's methods with the goal of expanding the results of both papers. Also, the authors would like to thank George Andrews and the referee for input on earlier drafts of this paper.

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A Appendix: Constituents for $M'\big(r,3,36k-t,4,18j-s\big)$

There are 32 relevant constituents required to prove Theorem 7 for the case $m=4$. For $n\leqslant 2N$, the crank is the second part, and for $n>2N$, the crank is the third part.

A.1 $s=1$, $1\leqslant t\leqslant 3$

$$M'\big(r,3,36k-3,4,36j-1\big) = \begin{cases} 126\binom{k+2}{3}+432\binom{k+1}{3}+90\binom{k}{3} & \text{for } 36k-3\leqslant 72j-2 \\ -449\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-413\binom{k-j}{3} & \\ & \\ 147\binom{4j-k+2}{3}+426\binom{4j-k+1}{3}+75\binom{4j-k}{3} & \text{for } 36k-3>72j-2 \\ -527\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-347\binom{3j-k}{3} & \end{cases} \tag{46}$$

$$M'\big(r,3,36k-3,4,36j+17\big) = \begin{cases} 126\binom{k+2}{3}+432\binom{k+1}{3}+90\binom{k}{3}-58\binom{k+2-j}{3} & \text{for } 36k-3\leqslant 72j+34 \\ -1265\binom{k+1-j}{3}-1220\binom{k-j}{3}-49\binom{k-1-j}{3} & \\ & \\ 147\binom{4j-k+4}{3}+426\binom{4j-k+3}{3}+75\binom{4j-k+2}{3} & \text{for } 36k-3>72j+34 \\ -79\binom{3j-k+4}{3}-1352\binom{3j-k+3}{3}-1127\binom{3j-k+2}{3} & \\ -34\binom{3j-k+1}{3} & \end{cases} \tag{47}$$

$$M'\big(r,3,36k-2,4,36j-1\big) = \begin{cases} 137\binom{k+2}{3}+432\binom{k+1}{3}+83\binom{k}{3} & \text{for all } k \\ -487\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-379\binom{k-j}{3} & \end{cases} \tag{48}$$

$$M'\big(r,3,36k-2,4,36j+17\big) = \begin{cases} 137\binom{k+2}{3}+432\binom{k+1}{3}+83\binom{k}{3}-68\binom{k+2-j}{3} & \text{for all } k \\ -1309\binom{k+1-j}{3}-1174\binom{k-j}{3}-41\binom{k-1-j}{3} & \end{cases} \tag{49}$$

$$M'\big(r,3,36k-1,4,36j-1\big) = \begin{cases} 147\binom{k+2}{3}+426\binom{k+1}{3}+75\binom{k}{3} & \text{for } 36k-1\leqslant 72j-2 \\ -527\binom{k+2-j}{3}-1718\binom{k+1-j}{3}-347\binom{k-j}{3} & \\ & \\ 126\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+90\binom{4j-k}{3} & \text{for } 36k-1>72j-2 \\ -449\binom{3j-k+2}{3}-1730\binom{3j-k+1}{3}-413\binom{3j-k}{3} & \end{cases} \tag{50}$$

$$M'\big(r,3,36k-1,4,36j+17\big) = \begin{cases} 147\binom{k+2}{3}+426\binom{k+1}{3}+75\binom{k}{3}-79\binom{k+2-j}{3} & \text{for } 36k-1 \leqslant 72j+34 \\ -1352\binom{k+1-j}{3}-1127\binom{k-j}{3}-34\binom{k-1-j}{3} & \\ & \\ 126\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+90\binom{4j-k+2}{3} & \text{for } 36k-1 > 72j+34 \\ -58\binom{3j-k+4}{3}-1265\binom{3j-k+3}{3}-1220\binom{3j-k+2}{3} & \\ -49\binom{3j-k+1}{3} & \end{cases} \tag{51}$$

A.2 $s = 2,\ 2 \leqslant t \leqslant 6$

$$M'\big(r,3,36k-6,4,36j-2\big) = \begin{cases} 99\binom{k+2}{3}+432\binom{k+1}{3}+117\binom{k}{3} & \text{for } 36k-6 \leqslant 72j-4 \\ -379\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-487\binom{k-j}{3} & \\ & \\ 137\binom{4j-k+2}{3}+428\binom{4j-k+1}{3}+83\binom{4j-k}{3} & \text{for } 36k-6 > 72j-4 \\ -527\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-347\binom{3j-k}{3} & \end{cases} \tag{52}$$

$$M'\big(r,3,36k-6,4,36j+16\big) = \begin{cases} 99\binom{k+2}{3}+432\binom{k+1}{3}+117\binom{k}{3}-41\binom{k+2-j}{3} & \text{for } 36k-6 \leqslant 72j+32 \\ -1174\binom{k+1-j}{3}-1309\binom{k-j}{3}-68\binom{k-1-j}{3} & \\ & \\ 137\binom{4j-k+4}{3}+428\binom{4j-k+3}{3}+83\binom{4j-k+2}{3} & \text{for } 36k-6 > 72j+32 \\ -79\binom{3j-k+4}{3}-1352\binom{3j-k+3}{3}-1127\binom{3j-k+2}{3} & \\ -34\binom{3j-k+1}{3} & \end{cases} \tag{53}$$

$$M'\big(r,3,36k-5,4,36j-2\big) = \begin{cases} 107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3} & \text{for } 36k-5 \leqslant 72j-4 \\ -413\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-449\binom{k-j}{3} & \\ & \\ 126\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+90\binom{4j-k}{3} & \text{for } 36k-5 > 72j-4 \\ -487\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-379\binom{3j-k}{3} & \end{cases} \tag{54}$$

$$M'\big(r,3,36k-5,4,36j+16\big) = \begin{cases} 107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3}-49\binom{k+2-j}{3} & \text{for } 36k-5\leqslant 72j+32 \\ -1220\binom{k+1-j}{3}-1265\binom{k-j}{3}-58\binom{k-1-j}{3} & \\ & \\ 126\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+90\binom{4j-k+2}{3} & \text{for } 36k-5>72j+32 \\ -68\binom{3j-k+4}{3}-1309\binom{3j-k+3}{3}-1174\binom{3j-k+2}{3} & \\ -41\binom{3j-k+1}{3} & \end{cases} \tag{55}$$

$$M'\big(r,3,36k-4,4,36j-2\big) = \begin{cases} 117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3} & \text{for all } k \\ -449\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-413\binom{k-j}{3} & \end{cases} \tag{56}$$

$$M'\big(r,3,36k-4,4,36j+16\big) = \begin{cases} 117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3}-58\binom{k+2-j}{3} & \text{for all } k \\ -1256\binom{k+1-j}{3}-1220\binom{k-j}{3}-49\binom{k-1-j}{3} & \end{cases} \tag{57}$$

$$M'\big(r,3,36k-3,4,36j-2\big) = \begin{cases} 126\binom{k+2}{3}+432\binom{k+1}{3}+90\binom{k}{3} & \text{for } 36k-3\leqslant 72j-4 \\ -487\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-379\binom{k-j}{3} & \\ & \\ 107\binom{4j-k+2}{3}+434\binom{4j-k+1}{3}+107\binom{4j-k}{3} & \text{for } 36k-3>72j-4 \\ -413\binom{3j-k+2}{3}-1730\binom{3j-k+1}{3}-449\binom{3j-k}{3} & \end{cases} \tag{58}$$

$$M'\big(r,3,36k-3,4,36j+16\big) = \begin{cases} 126\binom{k+2}{3}+432\binom{k+1}{3}+90\binom{k}{3}-68\binom{k+2-j}{3} & \text{for } 36k-3\leqslant 72j+32 \\ -1309\binom{k+1-j}{3}-1174\binom{k-j}{3}-41\binom{k-1-j}{3} & \\ & \\ 107\binom{4j-k+4}{3}+434\binom{4j-k+3}{3}+107\binom{4j-k+2}{3} & \text{for } 36k-3>72j+32 \\ -49\binom{3j-k+4}{3}-1220\binom{3j-k+3}{3}-1265\binom{3j-k+2}{3} & \\ -58\binom{3j-k+1}{3} & \end{cases} \tag{59}$$

$$M'\big(r,3,36k-2,4,36j-2\big) = \begin{cases} 137\binom{k+2}{3}+428\binom{k+1}{3}+83\binom{k}{3} & \text{for } 36k-2\leqslant 72j-4 \\ -527\binom{k+2-j}{3}-1718\binom{k+1-j}{3}-347\binom{k-j}{3} & \\ & \\ 99\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+117\binom{4j-k}{3} & \text{for } 36k-2>72j-4 \\ -379\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-487\binom{3j-k}{3} & \end{cases} \tag{60}$$

$$M'\big(r,3,36k-2,4,36j+16\big) = \begin{cases} 137\binom{k+2}{3}+428\binom{k+1}{3}+83\binom{k}{3}-79\binom{k+2-j}{3} & \text{for } 36k-2\leqslant 72j+32 \\ -1352\binom{k+1-j}{3}-1127\binom{k-j}{3}-34\binom{k-1-j}{3} & \\ & \\ 99\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+117\binom{4j-k+2}{3} & \text{for } 36k-2>72j+32 \\ -41\binom{3j-k+4}{3}-1174\binom{3j-k+3}{3}-1309\binom{3j-k+2}{3} & \\ -68\binom{3j-k+1}{3} & \end{cases} \tag{61}$$

A.3 $s=3,\ 4\leqslant t\leqslant 8$

$$M'\big(r,3,36k-8,4,36j-3\big) = \begin{cases} 83\binom{k+2}{3}+428\binom{k+1}{3}+137\binom{k}{3} & \text{for } 36k-8\leqslant 72j-6 \\ -347\binom{k+2-j}{3}-1718\binom{k+1-j}{3}-527\binom{k-j}{3} & \\ & \\ 117\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+99\binom{4j-k}{3} & \text{for } 36k-8>72j-6 \\ -487\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-379\binom{3j-k}{3} & \end{cases} \tag{62}$$

$$M'\big(r,3,36k-8,4,36j+15\big) = \begin{cases} 83\binom{k+2}{3}+428\binom{k+1}{3}+137\binom{k}{3}-34\binom{k+2-j}{3} & \text{for } 36k-8\leqslant 72j+30 \\ -1127\binom{k+1-j}{3}-1352\binom{k-j}{3}-79\binom{k-1-j}{3} & \\ & \\ 117\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+99\binom{4j-k+2}{3} & \text{for } 36k-8>72j+30 \\ -68\binom{3j-k+4}{3}-1309\binom{3j-k+3}{3}-1174\binom{3j-k+2}{3} & \\ -41\binom{3j-k+1}{3} & \end{cases} \tag{63}$$

$$M'\big(r,3,36k-7,4,36j-3\big) = \begin{cases} 90\binom{k+2}{3}+432\binom{k+1}{3}+126\binom{k}{3} & \text{for } 36k-7\leqslant 72j-6 \\ -379\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-487\binom{k-j}{3} & \\ & \\ 107\binom{4j-k+2}{3}+434\binom{4j-k+1}{3}+107\binom{4j-k}{3} & \text{for } 36k-7>72j-6 \\ -449\binom{3j-k+2}{3}-1730\binom{3j-k+1}{3}-413\binom{3j-k}{3} & \end{cases} \tag{64}$$

$$M'\big(r,3,36k-7,4,36j+15\big) = \begin{cases} 90\binom{k+2}{3}+432\binom{k+1}{3}+126\binom{k}{3}-41\binom{k+2-j}{3} & \text{for } 36k-7\leqslant 72j+30 \\ -1174\binom{k+1-j}{3}-1309\binom{k-j}{3}-68\binom{k-1-j}{3} & \\ & \\ 107\binom{4j-k+4}{3}+434\binom{4j-k+3}{3}+107\binom{4j-k+2}{3} & \text{for } 36k-7>72j+30 \\ -58\binom{3j-k+4}{3}-1265\binom{3j-k+3}{3}-1220\binom{3j-k+2}{3} & \\ -49\binom{3j-k+1}{3} & \end{cases} \tag{65}$$

$$M'\big(r,3,36k-6,4,36j-3\big) = \begin{cases} 99\binom{k+2}{3}+432\binom{k+1}{3}+117\binom{k}{3} & \text{for all } k \\ -413\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-449\binom{k-j}{3} & \end{cases} \tag{66}$$

$$M'\big(r,3,36k-6,4,36j+15\big) = \begin{cases} 99\binom{k+2}{3}+432\binom{k+1}{3}+117\binom{k}{3}-49\binom{k+2-j}{3} & \text{for all } k \\ -1220\binom{k+1-j}{3}-1265\binom{k-j}{3}-58\binom{k-1-j}{3} & \end{cases} \tag{67}$$

$$M'\big(r,3,36k-5,4,36j-3\big) = \begin{cases} 107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3} & \text{for } 36k-5\leqslant 72j-6 \\ -449\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-413\binom{k-j}{3} & \\ & \\ 90\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+126\binom{4j-k}{3} & \text{for } 36k-5>72j-6 \\ -379\binom{3j-k+2}{3}-1726\binom{3j-k+1}{3}-487\binom{3j-k}{3} & \end{cases} \tag{68}$$

$$M'\big(r,3,36k-5,4,36j+15\big) = \begin{cases} 107\binom{k+2}{3}+434\binom{k+1}{3}+107\binom{k}{3}-58\binom{k+2-j}{3} & \text{for } 36k-5\leqslant 72j+30 \\ -1265\binom{k+1-j}{3}-1220\binom{k-j}{3}-49\binom{k-1-j}{3} & \\ & \\ 90\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+126\binom{4j-k+2}{3} & \text{for } 36k-5>72j+30 \\ -41\binom{3j-k+4}{3}-1174\binom{3j-k+3}{3}-1309\binom{3j-k+2}{3} & \\ -68\binom{3j-k+1}{3} & \end{cases} \tag{69}$$

$$M'\big(r,3,36k-4,4,36j-3\big) = \begin{cases} 117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3} & \text{for } 36k-4\leqslant 72j-6 \\ -487\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-379\binom{k-j}{3} & \\ & \\ 83\binom{4j-k+2}{3}+428\binom{4j-k+1}{3}+137\binom{4j-k}{3} & \text{for } 36k-4>72j-6 \\ -347\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-527\binom{3j-k}{3} & \end{cases} \tag{70}$$

$$M'\big(r,3,36k-4,4,36j+15\big) = \begin{cases} 117\binom{k+2}{3}+432\binom{k+1}{3}+99\binom{k}{3}-68\binom{k+2-j}{3} & \text{for } 36k-4\leqslant 72j+30 \\ -1309\binom{k+1-j}{3}-1174\binom{k-j}{3}-41\binom{k-1-j}{3} & \\ & \\ 83\binom{4j-k+4}{3}+428\binom{4j-k+3}{3}+137\binom{4j-k+2}{3} & \text{for } 36k-4>72j+30 \\ -34\binom{3j-k+4}{3}-1127\binom{3j-k+3}{3}-1352\binom{3j-k+2}{3} & \\ -79\binom{3j-k+1}{3} & \end{cases} \tag{71}$$

A.4 $s=4,\ 7\leqslant t\leqslant 9$

$$M'\big(r,3,36k-9,4,36j-4\big) = \begin{cases} 75\binom{k+2}{3}+426\binom{k+1}{3}+147\binom{k}{3} & \text{for } 36k-9\leqslant 72j-8 \\ -347\binom{k+2-j}{3}-1718\binom{k+1-j}{3}-527\binom{k-j}{3} & \\ & \\ 90\binom{4j-k+2}{3}+432\binom{4j-k+1}{3}+126\binom{4j-k}{3} & \text{for } 36k-9>72j-8 \\ -413\binom{3j-k+2}{3}-1730\binom{3j-k+1}{3}-449\binom{3j-k}{3} & \end{cases} \tag{72}$$

$$M'\big(r,3,36k-9,4,36j+14\big) = \begin{cases} 75\binom{k+2}{3}+426\binom{k+1}{3}+147\binom{k}{3}-34\binom{k+2-j}{3} & \text{for } 36k-9\leqslant 72j+28 \\ -1127\binom{k+1-j}{3}-1352\binom{k-j}{3}-79\binom{k-1-j}{3} & \\ & \\ 90\binom{4j-k+4}{3}+432\binom{4j-k+3}{3}+126\binom{4j-k+2}{3} & \text{for } 36k-9>72j+28 \\ -49\binom{3j-k+4}{3}-1220\binom{3j-k+3}{3}-1265\binom{3j-k+2}{3} & \\ -58\binom{3j-k+1}{3} & \end{cases} \tag{73}$$

$$M'\big(r,3,36k-8,4,36j-4\big) = \begin{cases} 83\binom{k+2}{3}+428\binom{k+1}{3}+137\binom{k}{3} & \text{for all } k \\ -379\binom{k+2-j}{3}-1726\binom{k+1-j}{3}-487\binom{k-j}{3} & \end{cases} \tag{74}$$

$$M'\big(r,3,36k-8,4,36j+14\big) = \begin{cases} 83\binom{k+2}{3}+428\binom{k+1}{3}+137\binom{k}{3}-41\binom{k+2-j}{3} & \text{for all } k \\ -1174\binom{k+1-j}{3}-1309\binom{k-j}{3}-68\binom{k-1-j}{3} & \end{cases} \tag{75}$$

$$M'\big(r,3,36k-7,4,36j-4\big) = \begin{cases} 90\binom{k+2}{3}+432\binom{k+1}{3}+126\binom{k}{3} & \text{for } 36k-7\leqslant 72j-8 \\ -413\binom{k+2-j}{3}-1730\binom{k+1-j}{3}-449\binom{k-j}{3} & \\ & \\ 75\binom{4j-k+2}{3}+426\binom{4j-k+1}{3}+147\binom{4j-k}{3} & \text{for } 36k-7>72j-8 \\ -347\binom{3j-k+2}{3}-1718\binom{3j-k+1}{3}-527\binom{3j-k}{3} & \end{cases} \tag{76}$$

$$M'\big(r,3,36k-7,4,36j+14\big) = \begin{cases} 90\binom{k+2}{3}+432\binom{k+1}{3}+126\binom{k}{3}-49\binom{k+2-j}{3} & \text{for } 36k-7\leqslant 72j+28 \\ -1220\binom{k+1-j}{3}-1265\binom{k-j}{3}-58\binom{k-1-j}{3} & \\ & \\ 75\binom{4j-k+4}{3}+426\binom{4j-k+3}{3}+147\binom{4j-k+2}{3} & \text{for } 36k-9>72j+28 \\ -34\binom{3j-k+4}{3}-1127\binom{3j-k+3}{3}-1352\binom{3j-k+2}{3} & \\ -79\binom{3j-k+1}{3} & \end{cases} \tag{77}$$