A linear hypergraph extension of Turán's Theorem

Guorong Gao*

School of Mathematical Sciences
University of Science and Technology of China
Hefei, Anhui, China

guoronggao@yeah.net

An Chang†

Center for Discrete Mathematics and Theoretical Computer Science
Fuzhou University
Fuzhou, Fujian, China

anchang@fzu.edu.cn.

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Abstract

An r-uniform hypergraph is linear if every two edges intersect in at most one vertex. Given a family of r-uniform hypergraphs $\mathcal{F}$, the linear Turán number $\mathrm{ex}_r^{lin}(n, \mathcal{F})$ is the maximum number of edges of a linear r-uniform hypergraph on n vertices that does not contain any member of $\mathcal{F}$ as a subgraph.

Let K_l be a complete graph with l vertices and $r \geqslant 2$. The r-expansion of K_l is the r-graph K_l^+ obtained from K_l by enlarging each edge of K_l with a vertex set of size $r-2$ disjoint from $V(K_l)$ such that distinct edges of K_l are enlarged by disjoint sets. Let $T_2(n, l)$ be the Turán graph, i.e., almost balanced complete l-partite graph with n vertices. When $l \geqslant r \geqslant 3$ and n is sufficiently large, we prove the following extension of Turán's Theorem

$$\mathrm{ex}_r^{lin}\left(n, K_{l+1}^+\right) \leqslant \frac{|T_2(n, l)|}{\binom{r}{2}},$$

with equality holds if and only if there exist almost balanced l-partite r-graphs such that each pair of vertices from distinct parts are contained in one hyperedge exactly. Moreover, some results on linear Turán number of general configurations are also presented.

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1 Introduction

The r-uniform hypergraph (or r-graph) $H = (V(H), E(H))$ consists of a set $V(H)$ of vertices and a set $E(H)$ of edges, where each edge is an r-element subset of $V(H)$. A graph is a 2-uniform hypergraph, and the 3-uniform hypergraphs are called triple systems. Given a hypergraph H and a family of hypergraphs $\mathcal{F}$, we say H is $\mathcal{F}$-*free* if H does not contain any member of $\mathcal{F}$ as a subgraph. The Turán number, denoted by $\mathrm{ex}_r(n, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free r-graph on n vertices. The $\mathcal{F}$-free r-graphs with n vertices and $\mathrm{ex}_r(n, \mathcal{F})$ edges is called extremal hypergraphs of $\mathcal{F}$.

To obtain the r-graph $T_r(n, l)$, $l \geqslant r$, partition n vertices into l almost equal parts (that is, of sizes $\lfloor \frac{n}{l} \rfloor$ or $\lceil \frac{n}{l} \rceil$) and take all those edges which intersect every part in at most one vertex. Let $t_r(n, l)$ be the number of edges of the $T_r(n, l)$. Denote by K_l the complete graph with l vertices. The classical Turán's Theorem is

Theorem 1. *([25]) Fix $l \geqslant 2$. Then*

$$\mathrm{ex}_2\left(n, K_{l+1}\right) = t_2(n, l).$$

Moreover, equality is achieved only by the Turán graph $T_2(n, l)$.

A dozen years ago, Mubayi [19] and Pikhurko [21] gave a hypergraph extension of Turán's theorem. Given a graph F and positive integer $r \geqslant 3$, the r-*expansion* of F is the r-graph F^+ obtained from F by enlarging each edge of F with a vertex set of size $r-2$ disjoint from $V(F)$ such that distinct edges of F are enlarged by disjoint sets. Mubayi [19] proved that if $l \geqslant r$ is fixed, then $ex_r\left(n, K_{l+1}^+\right) = t_r(n, l) + o\left(n^r\right)$ and Pikhurko [21] improved this to an exact result for n sufficiently large.

Theorem 2. *([19, 21]) Fix $l \geqslant r \geqslant 2$ and let n be sufficiently large. Then*

$$\mathrm{ex}_r\left(n, K_{l+1}^+\right) = t_r(n, l).$$

Moreover, equality is achieved only by the r-graph $T_r(n, l)$.

There is a vast amount of literature on the Turán problem in graphs and hypergraphs. We refer the reader to the surveys of recent results [10, 17, 20].

In this paper, we focus on the Turán problem in linear hypergraphs. An r-graph is *linear* if every two edges have at most one common vertex. Similar to the Turán number, given a family of r-graphs $\mathcal{F}$, the linear Turán number is the maximum number of edges of an $\mathcal{F}$-free linear r-graph on n vertices. We denote it by $\mathrm{ex}_r^{lin}(n, \mathcal{F})$ and simply write $\mathrm{ex}_r^{lin}(n, F)$ instead of $\mathrm{ex}_r^{lin}(n, \{F\})$ when $\mathcal{F} = \{F\}$.

Interestingly, the linear Turán problem is closely related to the function $f_r(n, v, e)$, where $f_r(n, v, e)$ is the maximum number of edges in an n-vertex r-uniform hypergraph not carrying e edges on v vertices, where $r \geqslant 3$. The study of $f_r(n, v, e)$ was initiated by Brown, Erdős, and Sós [1] in 1970's. In one of the classical results in extremal combinatorics, Ruzsa and Szemerédi[22] showed that $n^{2-o(1)} \leqslant f_3(n, 6, 3) = o(n^2)$. This result was extended by Erdős, Frankl and Rödl [5] to $n^{2-o(1)} \leqslant f_r(n, 3r-3, 3) = o(n^2)$.

Let C_k be a cycle of length k and C_k^+ be the r-expansion of C_k. It is easy to see that $f_r(n, 3r-3, 3) = \mathrm{ex}_r^{lin}(n, C_3^+)$ for sufficiently large n. Thus $n^{2-o(1)} \leqslant \mathrm{ex}_r^{lin}(n, C_3^+) = o(n^2)$. Lazebnik and Verstraëte [18] prove that $\mathrm{ex}_3^{lin}(n, \{C_3^+, C_4^+\}) = \frac{1}{6}n^{3/2} + o(n^{3/2})$ and it implies that $f_3(n, 8, 4) = \frac{1}{6}n^{3/2} + o(n^{3/2})$. In recent decades, the linear Turán problem has attracted considerable attention and there are many interesting new results: see, e.g., [9, 11, 12, 13, 14, 15, 18, 24].

Define $TD_r(n, l)$ be the linear r-graph with n vertices partitioning into l almost equal parts, and edges intersecting every part in at most one vertex such that each pair of vertices from distinct parts are contained in one edge exactly. One should note that the linear r-graph $TD_r(n, l)$ is equivalent to the so-called group divisible design in the design theory. Given a triple (r, n, l), determining the existence of $TD_r(n, l)$ is still a very open problem in design theory. It can be found in [2] that the $TD_r(n, r)$ exists for sufficiently large n and r divides n. For $l|n$, Hanani [16] proved that $TD_3(n, l)$ exists if and only if: (1) $l \geqslant 3$, (2) $\frac{(l-1)n}{l} \equiv 0 \ (mod\ 2)$, (3) $\frac{(l-1)n^2}{l} \equiv 0 \ (mod\ 6)$. We refer the reader to [2, 26] for more knowledge about the group divisible design.

Usually, the linear r-graph $TD_r(n, l)$ is not unique when it exists. In addition, it is easy to see that $|TD_r(n, l)| = t_2(n, l)/\binom{r}{2}$. The following theorem is a linear hypergraph extension of the Turán's theorem.

Theorem 3. *Fix $l \geqslant r \geqslant 3$ and let n be sufficiently large. Then*

$$\mathrm{ex}_r^{lin}\left(n, K_{l+1}^+\right) \leqslant \frac{t_2(n, l)}{\binom{r}{2}}.$$

Moreover, if $TD_r(n, l)$ exists, then equality holds and the extremal hypergraph is $TD_r(n, l)$.

Gao, Chang and Hou [13] gave a spectral version of Theorem 3 when $l = r$. They prove that if an n-vertex linear r-graph H is K_{r+1}^+-free, then the spectral radius of the adjacency tensor of H is no more than n/r. Clearly, Theorem 3 implies that $\mathrm{ex}_r^{lin}\left(n, K_{l+1}^+\right) \leqslant \mathrm{ex}_2(n, K_{l+1})/\binom{r}{2}$ for $l \geqslant r \geqslant 3$ and sufficiently large n. However, given an arbitrary graph F, the inequality $\mathrm{ex}_r^{lin}(n, F^+) \leqslant \mathrm{ex}_2(n, F)/\binom{r}{2}$ does not generally hold. We give an example in the following theorem.

Let $K_l(s_1, \ldots, s_l)$ be the complete l-partite graph with class sizes $s_1, \ldots, s_l$ and write $K_l(1, 2) = K_l(1, 2, \ldots, 2)$ for short. Simonovits [23] shows that for $l \geqslant 2$ and sufficiently large n,

$$\mathrm{ex}_2(n, K_{l+1}(1, 2)) = t_2(n, l) + \frac{n}{2} - \begin{cases} \frac{1}{2}(n - l \cdot \lfloor \frac{n}{l} \rfloor) & \text{if } \lfloor \frac{n}{l} \rfloor \text{ is even,} \\ \frac{1}{2}(l \cdot \lfloor \frac{n}{l} \rfloor + l - n) & \text{if } \lfloor \frac{n}{l} \rfloor \text{ is odd.} \end{cases}$$

The following result implies that $\mathrm{ex}_3^{lin}(n, K_{l+1}^+(1, 2)) > \frac{1}{3}\,\mathrm{ex}_2(n, K_{l+1}(1, 2))$ for sufficiently large n.

Theorem 4. *Fix $l \geqslant 3$. Let n be sufficiently large and $6l|n$. Then*

$$\mathrm{ex}_3^{lin}(n, K_{l+1}^+(1, 2)) = \frac{1}{3}t_2(n, l) + \frac{n}{3}.$$

Given a graph F, let $\chi(F)$ be the chromatic number of F. Although $\mathrm{ex}_r^{lin}(n,F^+) \leqslant \mathrm{ex}_2(n,F)/\binom{r}{2}$ does not generally hold, we prove that it holds asymptotically when $r=3$ and $\chi(F) \geqslant 4$.

Theorem 5. *Let F be a graph with $\chi(F) \geqslant 4$. Then*

$$\mathrm{ex}_3^{lin}(n,F^+) = \frac{1}{3}\,\mathrm{ex}_2(n,F) + o(n^2).$$

Note that Theorem 5 holds for graphs with chromatic number at least 4. However, for graphs with small chromatic number, we have

Theorem 6. *Let $r \geqslant 3$ and F be a graph with $\chi(F) \leqslant r$. Then*

$$\mathrm{ex}_r^{lin}(n,F^+) = o(n^2).$$

Note that $\mathrm{ex}_r^{lin}(n,C_3^+) = o(n^2)$ can be deduced from Theorem 6. Since $f_r(n,3r-3,3) = \mathrm{ex}_r^{lin}(n,C_3^+)$ for sufficiently large n, Theorem 6 can be regarded as an extension of Ruzsa, Szemerédi's result [22] and Erdős, Frankl, Rödl's result [5]. Let W_k be the wheel graph obtained from the joint of a vertex and a cycle of length k. Mubayi and Verstraëte [20] raise a problem that whether $\mathrm{ex}_3(n,W_{2k}^+) = O(n^2)$ or not? It is still an open problem even for W_4. Since $\chi(W_{2k}) = 3$, by Theorem 6, we can immediately obtain the following corollary.

Corollary 7. *For integers $k \geqslant 2$ and $r \geqslant 3$, we have*

$$\mathrm{ex}_r^{lin}(n,W_{2k}^+) = o(n^2).$$

The rest of this paper is organized as follows. In next Section, we give some preliminaries. We present the proof of Theorem 3 in the Section 3. Theorem 4 is proven in Section 4. Finally, we prove Theorem 5 and Theorem 6 in the Section 5.

2 Preliminaries

Before presenting our main results, we need to introduce some basic notations and known results.

For a set V, let $\binom{V}{r}$ be the set of r-element subsets of V. Let H be an r-graph. For any vertex set $S \subset V(H)$, the subgraph of H induced by S is the r-graph with vertex set S and edge set $\{e \in E(H) : e \subset S\}$, denoted by $H[S]$. The *degree* of S in H is $d_H(S) = |\{e \in E(H) : S \subset e\}|$, and we write $d_H(v)$ if $S=\{v\}$. We use $\Delta(H)$ and $\delta(H)$ to denote the maximal degree and minimal degree of a vertex of H, respectively. The *shadow graph* of H is the graph with vertex set $V(H)$ and edge set $\left\{S \in \binom{V}{2} : d_H(S) \geqslant 1\right\}$, denoted by ∂H.

Given an r-graph F and a positive integer t, the *t-blow-up* $F(t)$ is an r-graph obtained by replacing each vertex of F by t copies of itself and each edge by corresponding complete r-partite r-graph of these copies.

For convenience, the pair $\{a, b\}$ and the triple $\{a, b, c\}$ are sometimes referred to as ab and abc, respectively.

Next we will introduce some known results which will be used to prove our results.

An r-graph H is l-partite if $V(H)$ can be partitioned into l parts such that every edge in H intersects each part in at most one vertex. An l-partite r-graph is complete if we take all those edges which intersect every part in at most one vertex.

Lemma 8. *([4]) Let $r \geqslant 2$ and H be an r-partite r-graph. Then*

$$\mathrm{ex}_r(n, H) = o(n^r).$$

Lemma 9. *([6, 8]) Let F be a graph with $\chi(F) \geqslant 2$. Then*

$$\mathrm{ex}_2(n, F) = (1 - \frac{1}{\chi(F) - 1} + o(1))\binom{n}{2}.$$

Let $K_{l+1}(1, t_1, t_2, \dots, t_l)$ be the complete $(l+1)$-partite graph with class sizes $1 \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_l$. Fix l and t, let

$$g(n) = \begin{cases} 0 & \text{if } t \text{ is odd,} \\ \frac{1}{2}(l \cdot \lfloor \frac{n}{l} \rfloor - n) & \text{if } t \text{ is even and } \lfloor \frac{n}{l} \rfloor \text{ is even,} \\ \frac{1}{2}(n - l \cdot \lfloor \frac{n}{l} \rfloor - l) & \text{if } t \text{ is even and } \lfloor \frac{n}{l} \rfloor \text{ is odd.} \end{cases}$$

Lemma 10. *([23, 7]) Fix positive integers $l \geqslant 2$ and $1 \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_l$. For sufficiently large n,*

$$\mathrm{ex}_2(n, K_{l+1}(1, t_1, \dots, t_l)) = t_2(n, l) + \frac{(t_1 - 1)n}{2} + g(n).$$

Moreover, if G is a $K_{l+1}(1, t_1, \dots, t_l)$-free graph with n vertices and $t_2(n, l) + \frac{(t_1-1)n}{2} + g(n)$ edges, then $V(G)$ can partition into $V_1 \cup V_2 \dots \cup V_l$ such that $\Delta(G[V_i]) \leqslant t_1 - 1$.

Lemma 11. *([5]) Let F be a graph with f vertices. For every $\varepsilon > 0$, there exists $\delta = \delta(f, \varepsilon) > 0$ such that if one has to delete at least εn^2 edges from a graph G to make it F-free, then G has at least δn^f copies of F.*

3 Proof of Theorem 3

Let H be a linear r-graph. If the vertex pair $ab \subset V(H)$ is contained in some edges of H, then ab is exactly contained in one edge of H. We use h_{ab} to denote the edge of H containing ab.

Lemma 12. *Fix integers $r \geqslant 3$, $s \geqslant 2$ and $t \geqslant r^2 s^3$. Let H be a linear r-graph and F be a graph with s vertices. If the shadow graph $\partial(H)$ contains a copy of $F(t)$, then H contains a copy of F^+.*

Proof. Let H be a linear r-graph and $F(t) \subset \partial(H)$. Our goal is to find a copy of F^+ in H. Let X be the set of vertices in F^+ which correspond to the vertices in F and $Y = V(F^+)\backslash X$. We will use the following steps to find the vertex sets X and Y in H.

We take a copy of $F(t)$ in $\partial(H)$. Let $V(F) = \{v_1, v_2, \dots, v_s\}$ and $V(F(t)) = \bigcup_{1\leqslant i\leqslant s} V_i$, where V_i is the t copies of the vertex v_i, $1 \leqslant i \leqslant s$. Let $X_1 = \emptyset$ and $Y_1 = \emptyset$. At the first step, choose $x_1 \in V_1$ and $x_2 \in V_2$ and let $X_2 = X_1 \cup \{x_1, x_2\}$. If x_1x_2 is not an edge of $F(t)$, then we let $Y_2 = Y_1$. If x_1x_2 is an edge of $F(t)$, then we let $Y_2 = Y_1 \cup (h_{x_1x_2}\backslash\{x_1, x_2\})$.

At the second step, let $V_3' = V_3\backslash(X_2 \cup Y_2)$. Then $|V_3'| \geqslant t - r$. Let

$$B_3^{(1)} = \{v \in V_3' : \exists y \in X_2 \cup Y_2 \text{ and } y \neq x_1 \text{ such that } x_1yv \text{ is contained in an edge of } H\},$$

$$B_3^{(2)} = \{v \in V_3' : \exists y \in X_2 \cup Y_2 \text{ and } y \neq x_2 \text{ such that } x_2yv \text{ is contained in an edge of } H\}.$$

Since H is a linear r-graph, we have $|B_3^{(1)}| \leqslant (r-2)|X_2 \cup Y_2|$ and $|B_3^{(2)}| \leqslant (r-2)|X_2 \cup Y_2|$. Let $V_3'' = V_3'\backslash(B_3^{(1)} \cup B_3^{(2)})$. Then

$$\begin{aligned}|V_3''| &\geqslant t - r - 2(r-2)|X_2 \cup Y_2| \\ &\geqslant r^2s^3 - r - 2r(r-2) \\ &> 0.\end{aligned}$$

Choose an $x_3 \in V_3''$ and let $X_3 = X_2 \cup \{x_3\}$. If both x_1x_3 and x_2x_3 are not an edge of $F(t)$, then we let $Y_3 = Y_2$. If x_1x_3 is an edge and x_2x_3 is not an edge of $F(t)$, then we let $Y_3 = Y_2 \cup (h_{x_1x_3}\backslash\{x_1, x_3\})$. Since $x_3 \notin B_3^{(1)}$, it is easy to see $(h_{x_1x_3}\backslash\{x_1, x_3\}) \cap (X_2 \cup Y_2) = \emptyset$. If x_2x_3 is an edge and x_1x_3 is not an edge of $F(t)$, then we let $Y_3 = Y_2 \cup (h_{x_2x_3}\backslash\{x_2, x_3\})$. Since $x_3 \notin B_3^{(2)}$, it is easy to see $(h_{x_2x_3}\backslash\{x_2, x_3\}) \cap (X_2 \cup Y_2) = \emptyset$. If x_1x_3 and x_2x_3 are two edges of $F(t)$, then we let $Y_3 = Y_2 \cup (h_{x_1x_3}\backslash\{x_1, x_3\}) \cup (h_{x_2x_3}\backslash\{x_2, x_3\})$. Since H is linear, we have $(h_{x_1x_3}\backslash\{x_1, x_3\}) \cap (h_{x_2x_3}\backslash\{x_2, x_3\}) = \emptyset$. It is easy to see that $|X_3| = 3$ and $|Y_3| \leqslant 3(r-2)$.

At the k-th step, where $3 \leqslant k \leqslant s-1$, we will use the following method to add one vertex of V_{k+1} into X_k to obtain X_{k+1} and add at most $(r-2)k$ vertices into Y_k to obtain Y_{k+1}. Hence, $|X_k| = k$ and $|Y_k| \leqslant (r-2)\binom{k}{2}$. Let $V_{k+1}' = V_{k+1}\backslash(X_k \cup Y_k)$. Then $|V_{k+1}'| \geqslant t - |X_k \cup Y_k|$. For $x \in X_k$, let

$$B_{k+1}^{(x)} = \{v \in V_{k+1}' : \exists y \in X_k \cup Y_k \text{ and } y \neq x \text{ such that } xyv \text{ is contained in an edge of } H\}.$$

Since H is a linear r-graph, we have $|B_{k+1}^{(x)}| \leqslant (r-2)|X_k \cup Y_k|$. Let $B_{k+1} = \bigcup_{x\in X_k} B_{k+1}^{(x)}$ and $V_{k+1}'' = V_{k+1}'\backslash B_{k+1}$. Then

$$\begin{aligned}|V_{k+1}''| &\geqslant t - |X_k \cup Y_k| - k(r-2)|X_k \cup Y_k| \\ &\geqslant r^2s^3 - k - (r-2)\binom{k}{2} - k(r-2)k - k(r-2)^2\binom{k}{2} \\ &> 0.\end{aligned}$$

The last inequality holds since $r \geqslant 3$ and $s \geqslant k \geqslant 3$. Choose an $x_{k+1} \in V''_{k+1}$ and let $X_{k+1} = X_k \cup \{x_{k+1}\}$. For each $x \in X_k$, if xx_{k+1} is an edge of $F(t)$, then we add those vertices of $h_{xx_{k+1}} \backslash \{x, x_{k+1}\}$ into Y_k. We denote the obtained vertex set by Y_{k+1}.

Finally, at the $(s-1)$-th step, we obtain vertex sets X_s and Y_s. One can easily check that $F^+ \subseteq H[X_s \cup Y_s]$, as desired. □

We write $K_l(s_1, \ldots, s_r)$ for the complete l-partite graph with class sizes $s_1, \ldots, s_r$ and set for short $K_l(1,1,t) = K_l(1,1,t,\ldots,t)$. Note that we only need $|V_1| = |V_2| = 1$ in the proof of Lemma 12. Thus we have the following corollary.

Corollary 13. *Given positive integers $l \geqslant r \geqslant 3$ and $t \geqslant r^2(l+1)^3$. Let H be a linear r-graph. If $\partial(H)$ contains a copy of $K_{l+1}(1,1,t)$, then H contains a copy of K^+_{l+1}.*

Combining Corollary 13 and Lemma 10, we can easily prove Theorem 3.

Proof of Theorem 3. Let us choose a constant integer $t \geqslant r^2(l+1)^3$. Let n be sufficiently large and H be a K^+_{l+1}-free linear r-graph with n vertices. Suppose that $|E(H)| > t_2(n,l)/\binom{r}{2}$. Then $|E(\partial(H))| > t_2(n,l)$. By Lemma 10, we have $K_{l+1}(1,1,t) \subset \partial(H)$. Then by Corollary 13, we have that H contains a copy of K^+_{l+1}, which is a contradiction. Thus we have $|E(H)| \leqslant t_2(n,l)/\binom{r}{2}$.

Now suppose that $|E(H)| = t_2(n,l)/\binom{r}{2}$ and H is not a $TD_r(n,l)$. It implies that $\partial(H) \ncong T_2(n,l)$. By Lemma 10, we can deduce that $T_2(n,l)$ is the unique extremal graph for $K_{l+1}(1,1,t)$. Since $|E(\partial(H))| = |E(H)| \times \binom{r}{2} = t_2(n,l)$ and $\partial(H) \ncong T_2(n,l)$, we have that $K_{l+1}(1,1,t) \subset \partial(H)$. Again by Corollary 13, we can deduce that H contains a copy of K^+_{l+1}, which is a contradiction. Thus, the equality holds if and only if H is a $TD_r(n,l)$. The proof is completed. □

4 Proof of Theorem 4

Let $K_{l+1}(1,3,t)$ be the complete $(l+1)$-partite graph with first part one vertex, the second part three vertices and the other parts t vertices. Let H be a linear 3-graph. For the vertex pair $ab \subset V(H)$, we still use h_{ab} to denote the edge of H containing ab.

Lemma 14. *Fix integers $l \geqslant 3$ and $t \geqslant 4(l+1)^3$. Let H be a linear 3-graph. If the shadow graph $\partial(H)$ contains a copy of $K_{l+1}(1,3,t)$, then H contains a copy of $K^+_{l+1}(1,2)$.*

Proof. Let H be a linear 3-graph and $K_{l+1}(1,3,t) \subset \partial(H)$. Our goal is to find a copy of $K^+_{l+1}(1,2)$ in H. Let X be the set of vertices in $K^+_{l+1}(1,2)$ which correspond to the vertices in $K_{l+1}(1,2)$, and $Y = V(K^+_{l+1}(1,2)) \backslash X$. We will use the following steps to find the vertex sets X and Y in H.

We take a copy of $K_{l+1}(1,3,t)$ in $\partial(H)$. For $3 \leqslant i \leqslant l+1$, let V_1, V_2 and V_i's be the $(l+1)$ parts of $K_{l+1}(1,3,t)$, where $V_1 = \{v_{11}\}$, $V_2 = \{v_{21}, v_{22}, v_{23}\}$ and $|V_i| = t$.

At the first step, let $x_{11} = v_{11}$, $x_{21} = v_{21}$. And then we let $X_1 = \{x_{11}, x_{21}\}$ and $Y_1 = h_{x_{11}x_{21}} \backslash \{x_{11}, x_{21}\}$. It is easy to see that $|Y_1| = 1$.

At the second step, let $V_2' = V_2 \backslash (\{v_{21}\} \cup Y_1)$. Since $|V_2| = 3$, we have that V_2' is not empty. Choose an $x_{22} \in V_2'$ and let $X_2 = X_1 \cup \{x_{22}\}$, $Y_2 = Y_1 \cup h_{x_{11}x_{22}} \backslash \{x_{11}, x_{22}\}$. Then $|X_2| = 3$ and $|Y_2| = 2$.

At the k-th step, where $3 \leqslant k \leqslant l+1$, we will use the following method to add two vertices of V_k into X_{k-1} to obtain X_k and add $4k-6$ vertices into Y_{k-1} to obtain Y_k. Hence, we have $|X_{k-1}| = 2k-3$ and $|Y_{k-1}| = 2(k-2)^2$. For $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant 2$, $\{i,j\} \neq \{1,2\}$, let $x_{ij} \in X_{k-1} \cap V_i$. That is, $X_{k-1} = \{x_{11}, x_{21}, x_{22}, \ldots, x_{(k-1)1}, x_{(k-1)2}\}$. For $x \in X_{k-1}$, let

$$B_{k1}^{(x)} = \{v \in V_k : \exists y \in X_{k-1} \cup Y_{k-1} \text{ and } y \neq x \text{ such that } xyv \text{ is an edge of } H\}.$$

Since H is a linear 3-graph, we have $|B_{k1}^{(x)}| \leqslant |X_{k-1} \cup Y_{k-1}|$. Let

$$B_{k1} = \bigcup_{x \in X_{k-1}} B_{k1}^{(x)} \quad \text{and} \quad V_k' = V_k \backslash B_{k1}.$$

Since $t \geqslant 4(l+1)^3$ and $k \leqslant l+1$, we have

$$\begin{aligned} |V_k'| &\geqslant |V_k| - |B_{k1}| \\ &\geqslant t - (2k-3) \cdot (2k-3+2(k-2)^2) \\ &> 4(l+1)^3 - 4l^3 \\ &> 0. \end{aligned}$$

Hence, V_k' is not empty. Choose an $x_{k1} \in V_k'$ and let $X_{k-1}' = X_{k-1} \cup \{x_{k1}\}$ and

$$Y_{k-1}' = \left(\bigcup_{x \in X_{k-1}} h_{x x_{k1}} \backslash \{x, x_{k1}\} \right) \cup Y_{k-1}.$$

Then $|X_{k-1}'| = 2k-2$ and $|Y_{k-1}'| = 2(k-2)^2 + 2k - 3$. And then for $x \in X_{k-1}$, we let

$$B_{k2}^{(x)} = \{v \in V_k : \exists y \in X_{k-1}' \cup Y_{k-1}' \text{ and } y \neq x \text{ such that } xyv \text{ is an edge of } H\}.$$

Since H is a linear 3-graph, we have $|B_{k2}^{(x)}| \leqslant |X_{k-1}' \cup Y_{k-1}'|$. Let

$$B_{k2} = \bigcup_{x \in X_{k-1}} B_{k2}^{(x)} \quad \text{and} \quad V_k'' = V_k \backslash B_{k2}.$$

Since $t \geqslant 4(l+1)^3$ and $k \leqslant l+1$, we have

$$\begin{aligned} |V_k''| &\geqslant |V_k| - |B_{k2}| \\ &\geqslant t - (2k-3) \cdot (2k-2+2(k-2)^2 + 2k-3) \\ &> 4(l+1)^3 - 4(l+1)^3 \\ &= 0. \end{aligned}$$

Thus we have that V_k'' is not empty. Choose an $x_{k2} \in V_k''$ and let $X_k = X_{k-1}' \cup \{x_{k2}\}$ and

$$Y_k = \left(\bigcup_{x \in X_{k-1}} h_{xx_{k2}} \backslash \{x, x_{k2}\} \right) \cup Y_{k-1}'.$$

Finally, at the $(l+1)$-th step, we obtain vertex sets X_{l+1} and Y_{l+1}. One can easily check that $K_{l+1}^{+}(1,2)$ is a subhypergraph of $H[X_{l+1} \cup Y_{l+1}]$, as desired. □

Combining Lemma 10 and Lemma 14, we can easily prove Theorem 4.

Proof of Theorem 4. Let us choose a constant integer $t \geqslant 4(l+1)^3$. Let n be sufficiently large and H be a $K_{l+1}^{+}(1,2)$-free linear 3-graph with n vertices. Suppose $|E(H)| > \frac{1}{3}t_2(n,l) + \frac{n}{3}$. Then $|E(\partial(H))| > t_2(n,l) + n$. By Lemma 10, we have that $K_{l+1}(1,3,t) \subset \partial(H)$. Hence, by Lemma 14, we deduce that H contains a copy of $K_{l+1}^{+}(1,2)$, which is a contradiction. Thus we have $|E(H)| \leqslant \frac{1}{3}t_2(n,l) + \frac{n}{3}$. That is, $\mathrm{ex}_3^{lin}(n, K_{l+1}^{+}(1,2)) \leqslant \frac{1}{3}t_2(n,l) + \frac{n}{3}$.

For $6l|n$, we know from [2] that there exists a $TD_3(n,l)$. Clearly, $TD_3(n,l)$ is an l-partite 3-graph. For $1 \leqslant i \leqslant l$, let the vertex set V_i be the i-th part of $TD_3(n,l)$. It is easy to see $|V_i| = \frac{n}{l}$. Since $6l|n$, we can add $\frac{n}{3l}$ vertex-disjoint hyperedges to each V_i. We denote the obtained 3-graph by H_E. Then H_E has n vertices and $\frac{1}{3}t_2(n,l) + \frac{n}{3}$ edges. Next we will prove that H_E is $K_{l+1}^{+}(1,2)$-free.

Suppose that H_E contains a copy of $K_{l+1}^{+}(1,2)$. Let X be the set of vertices in $K_{l+1}^{+}(1,2)$ which correspond to the vertices in $K_{l+1}(1,2)$. Then $|X| = 2l+1$. Thus there are three vertices $v_1, v_2, v_3 \in X$ such that these vertices are contained in a V_i. Since $v_1, v_2, v_3 \in X$, at least two of v_1v_2, v_1v_3, v_2v_3 are edges of the $K_{l+1}(1,2)$. Moreover, by the construction of $K_{l+1}^{+}(1,2)$, these two edges of the $K_{l+1}(1,2)$ are contained in two different edges of the $K_{l+1}^{+}(1,2)$. But this contradicts to the construction of H_E. Hence, H_E is $K_{l+1}^{+}(1,2)$-free. Hence, for $6l|n$, we have $\mathrm{ex}_3^{lin}(n, K_{l+1}^{+}(1,2)) = \frac{1}{3}t_2(n,l) + \frac{n}{3}$. The proof is completed. □

5 Proof of Theorem 5 and Theorem 6

Proof of Theorem 5. Let F be a graph with $\chi(F) = l+1 \geqslant 4$. For $6l|n$, we know from [16] that there exists a $TD_3(n,l)$. Since $\partial(TD_3(n,l))$ is an l-partite graph and $\chi(F) = l+1$, we have that $\partial(TD_3(n,l))$ is F-free. Hence, $TD_3(n,l)$ is F^+-free. Since $|TD_3(n,l)| = \frac{1}{3}t_2(n,l)$, we have that

$$\mathrm{ex}_3^{lin}(n, F^+) \geqslant \frac{1}{3}t_2(n,l) + o(n^2).$$

By Lemma 9, we deduce that

$$\mathrm{ex}_3^{lin}(n, F^+) \geqslant \frac{1}{3}\,\mathrm{ex}_2(n,F) + o(n^2).$$

Let H be an F^+-free linear 3-graph with n vertices. By Lemma 12, we have that $\partial(H)$ is $F(t)$-free, where $t \geqslant 9(|V(F)|)^3$. Then by Lemma 9, we deduce that

$$|E(H)| = \frac{1}{3}|E(\partial(H))| \leqslant \frac{1}{3}\operatorname{ex}_2(n, F(t)) = \frac{1}{3}\operatorname{ex}_2(n, F) + o(n^2).$$

That is,

$$\operatorname{ex}_3^{lin}(n, F^+) \leqslant \frac{1}{3}\operatorname{ex}_2(n, F) + o(n^2).$$

Hence,

$$\operatorname{ex}_3^{lin}(n, F^+) = \frac{1}{3}\operatorname{ex}_2(n, F) + o(n^2).$$

The proof is completed. □

Proof of Theorem 6. Let F be a graph with $\chi(F) \leqslant r$ and H be an F^+-free linear r-graph with n vertices.

Suppose that H has cn^2 edges, where c is a positive constant number. Since H is linear, we have that $\partial(H)$ contains at least cn^2 copies of edge-disjoint K_r's. Hence, $\partial(H)$ has to delete at least cn^2 edges to make it K_r-free. By Lemma 11, there is a constant $\delta > 0$ such that $\partial(H)$ has at least δn^r copies of K_r's.

Now we construct an auxiliary r-graph H' as follows: The vertex set of H' is $V(H)$. If there is a $K_r \subseteq \partial(H)$, we let these r vertices of K_r be an edge of H'. Thus the number of edges of H' is at least δn^r. By Lemma 8, we deduce that H' contains a complete r-partite r-graph with each class of size $r^2(|V(F)|)^4$. It implies $\partial(H)$ contains a complete r-partite subgraph with each class of size $r^2(|V(F)|)^4$. Since $\chi(F) \leqslant r$, we have that $\partial(H)$ contains a copy of $F(t)$, where $t = r^2(|V(F)|)^3$. By Lemma 12, H contains a copy of F^+, a contradiction. Hence, we have $|E(H)| = o(n^2)$. This completes the proof. □

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