Pattern-functions, statistics, and shallow permutations

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Abstract

We study relationships between permutation statistics and pattern-functions, counting the number of times particular patterns occur in a permutation. This allows us to write several familiar statistics as linear combinations of pattern counts, both in terms of a permutation and in terms of its image under the fundamental bijection. We use these enumerations to resolve the question of characterizing so-called "shallow" permutations, whose depth (equivalently, disarray/displacement) is minimal with respect to length and reflection length. We present this characterization in several ways, including vincular patterns, mesh patterns, and a new object that we call "arrow patterns." Furthermore, we specialize to characterizing and enumerating shallow involutions and shallow cycles, encountering the Motzkin and large Schröder numbers, respectively.

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1 Introduction

Much of the research about permutation patterns has focused on understanding (characterizing, enumerating, and so on) the permutations that avoid given sets of patterns. In this work, we take the study of permutation patterns in a different direction, enumerating the occurrences of a pattern in a permutation (that is, not just classifying zero/nonzero), and we demonstrate the utility of this refinement of the standard pattern containment question. The general idea of this perspective has also been taken in [2, 3, 5, 9, 10, 14, 15, 18, 19], but our approach is rather different from the majority of

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those works. In particular, we explore relationships between certain permutation statistics and the number of times particular patterns occur in a permutation. This appreciation for the quantity of pattern occurrences extends recent work [1, 4, 21, 24] and suggests that further analyses in this vein may be similarly fruitful.

Definition 1. Following terminology from [1], a *pattern-function* is a function that can be written as a linear combination of pattern counts. A pattern-function whose patterns contain no more than d symbols is a *d-function*.

We demonstrate the value of pattern-functions by using them to calculate the statistics studied by Diaconis and Graham in [8]. This will involve the permutation and, in some cases, a detour via the fundamental bijection. After establishing this, we use those pattern-functions to answer an open question of Petersen and the second author about so-called "shallow" permutations [17] (equivalently, [8, Question 3]), and we go on to examine special cases of that result more closely.

We will use this type of pattern counting to calculate established permutation statistics, namely by writing those statistics as linear combinations of certain pattern counts. Our attention in this paper is focused on the four permutation statistics discussed by Diaconis and Graham [8]: length

$$\ell_S(\sigma),$$

reflection length

$$\ell_T(\sigma),$$

Spearman's disarray (also called *total displacement* by Knuth [13])

$$\mathrm{dis}(\sigma) = \sum_{i=1}^{n} |\sigma_i - i|, \tag{1}$$

and what we call the *variance* of a permutation

$$V(\sigma) = \sum_{i=1}^{n} (\sigma_i - i)^2. \tag{2}$$

We will write each of these four statistics as linear combinations of pattern counts.

The disarray/displacement statistic dis from Equation (1) is related to the *depth* of permutation, studied by Petersen and the second author [17]:

$$\mathrm{dp}(\sigma) = \sum_{\sigma_i > i} (\sigma_i - i).$$

In particular, $\mathrm{dis}(\sigma) = 2\,\mathrm{dp}(\sigma)$. In [17], they show that

$$\mathrm{dp}(\sigma) \geqslant \frac{\ell_S(\sigma) + \ell_T(\sigma)}{2}$$

for all σ, and they ask for which permutations this is an equality; i.e., which permutations have minimal depth relative to their length and reflection length. Note that this is equivalent to [8, Question 3].

Definition 2. A permutation σ is *shallow* if its depth is equal to $(\ell_S(\sigma) + \ell_T(\sigma))/2$.

We will answer that question generally using a new type of pattern containment that we call *arrow patterns*, which in our particular case can also be phrased in terms of mesh patterns. In doing so, we give a bijection between shallow involutions and circles with non-intersecting chords (counted by the Motzkin numbers), and a bijection between shallow cycles and separable permutations (counted by the large Schröder numbers). We also prove that a permutation is shallow if and only if its image under the fundamental bijection avoids the vincular patterns $\underline{5}\,\underline{24}\,\underline{13}$, $\underline{4}\,\underline{25}\,\underline{13}$, and $\underline{31}\,\underline{42}$ (Theorem 23).

The paper is organized as follows. In Section 2, we introduce terminology and notation that we will use throughout the work. We will also make a few simple observations that recast the familiar permutation statistics *descents* and *length* in terms of pattern counts, as motivation of the work to come. Section 3 uses pattern-functions to calculate the reflection length, variance, and disarray/displacement of a permutation. These are addressed in Corollary 12, Theorems 14, and Theorem 15 respectively. The variance calculation is the first to rely on the ingenuity of this pattern-function approach. Reflection length and disarray/displacement are calculated via pattern-functions applied to the permutation's image under the fundamental bijection, which will motivate the work in Section 4. There, we introduce arrow patterns and construct a pattern-function for ℓ_S applied to a permutation's image under the fundamental bijection. We use this to characterize shallow permutations in Theorem 23. We then apply this in Section 5 to characterize and count shallow involutions and cycles, now without requiring arrow patterns, in Corollaries 25 and 31, respectively. We conclude with suggestions for further research.

2 Definitions and context

Let $\mathfrak{S}_n$ denote the set of permutations of $[1, n] := \{1, \ldots, n\}$. A permutation $\sigma \in \mathfrak{S}_n$ can be described in many ways, and we will be using two of these options. The first, *one-line notation*, writes $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ as a word where $\sigma_i := \sigma(i)$ is the image of i under the map σ. The second, *cycle notation*, writes σ as a product of disjoint cycles, each of which describes an orbit of σ acting on $[1, n]$. For example, one cycle can be written as $(\sigma^t(1)\ \sigma^{t+1}(1)\ \cdots\ \sigma^{-1}(1)\ 1\ \sigma(1)\ \cdots\ \sigma^{t-1}(1))$ for any t. One-line notation is the framework for studying permutation patterns, which we will define below, and cycle notation will be relevant because our work will make extensive use of the so-called *fundamental bijection* between permutations written in one-line notation and permutations written in cycle notation.

Example 3. The permutation $421365 \in \mathfrak{S}_6$, written in one-line notation, is written in cycle notation as $(143)(2)(56)$. There are $3 \cdot 1 \cdot 2 \cdot 3! = 36$ ways to write this permutation in cycle notation, including $(2)(431)(65)$ and $(56)(314)(2)$, and so on.

The fundamental bijection (see [20]) starts with a permutation σ written in a "standard representation" of cycle notation, and maps this bijectively to a permutation $\Phi(\sigma)$ in one-line notation. The standard representation of a permutation σ writes each cycle with its largest element in the leftmost position, and writes the cycles from left to right in increasing order of those largest elements. The standard representation of the permutation in Example 3 is (2)(431)(65). The permutation $\Phi(\sigma)$ is obtained by erasing the cycle demarcations, so $\Phi(421365) = 243165$. This process is invertible by noting that every cycle in σ starts with a left-to-right maximum of $\Phi(\sigma)$. We will sometimes abuse terminology and refer to σ's image under the fundamental bijection as "the fundamental bijection" of σ.

Two sequences s and t over ordered sets are *isomorphic* if, for all i and j, we have $s_i \leqslant s_j$ if and only if $t_i \leqslant t_j$. We write $s \sim t$ to denote that s is isomorphic to t.

Definition 4. A permutation $\sigma \in \mathfrak{S}_n$ *contains* a *π-pattern* if a subsequence of the one-line notation of σ is isomorphic to the one-line notation of π. Such a subsequence is an *occurrence* of the pattern pattern π. Otherwise σ *avoids* the pattern π.

Example 5. The permutation $421365 \in \mathfrak{S}_6$ contains four occurrences of the pattern 123: $236, 235, 136, 135$. The permutation avoids the pattern 1234.

The relationship between a permutation and a pattern is typically studied as a binary question: contain or avoid? To recognize the value of a more granular analysis, we introduce the following terminology and notation. The latter is meant to mimic how coefficients are extracted from polynomials via, for example, $[x^3]f(x)$.

Definition 6. The *count* of a pattern π in σ is the number of occurrences of π in σ. This is denoted $[\pi](\sigma)$.

In other words, the count of π in σ is positive if and only if σ contains a π-pattern. Example 5 showed that $[123](421365) = 4$ and $[1234](421365) = 0$.

Many specializations of pattern containment exist in the literature, including vincular, bivincular, barred, and mesh (for references to these and broader pattern-related topics, see [12]). Of these, the ones primarily used in this work are vincular patterns, developed in [1], in which portions of the pattern might be required to be bonded together. A necessary kludge to extend vincular patterns, called arrow patterns will be introduced in Section 4, but we note at the end of that section that the two particular arrow patterns we employ can be rewritten as mesh patterns.

Definition 7. A *vincular* pattern is a permutation, in which consecutive symbols may be bonded together. A permutation $\sigma \in \mathfrak{S}_n$ contains a vincular pattern if a subsequence of the one-line notation of σ occurs in the appropriate relative order, with bonded numbers appearing consecutively in σ. The bonds will be indicated with underbrackets; that is, if a segment α of the pattern is required to appear consecutively, then we will write $\underline{\alpha}$.

We will typically include an underbracket on single numbers that are not bonded to their neighbors. We do this for aesthetic reasons, and one could certainly omit it if desired.

Example 8. Consider the permutation 421365. The vincular pattern $\underline{12}\,\underline{3}$ occurs twice, as 136 and 135, and there is one occurrence of $\underline{123}$, as 136.

When discussing an occurrence of a vincular pattern, we will often indicate the required bonds as a sort of reminder. So in the permutation 421365, we could reference the occurrence $\underline{13}\,\underline{6} \sim \underline{12}\,\underline{3}$. The idea of *count* from Definition 6 carries over to the setting of vincular patterns. For example, $[\underline{12}\,\underline{3}](421365) = 2$ and $[\underline{123}](421365) = 1$.

We can now make our first characterization (and a very easy one, at that) of a classical permutation statistic in terms of pattern counts.

Observation 9. *For any permutation σ, the number of descents in σ is equal to $[\underline{21}](\sigma)$.*

Our goal is to use pattern counts to enumerate a range of phenomena. The notation that we employ for this is similar to that introduced in Definition 6, treating pattern counts as operators and summing them as needed. A simple example of this is that for any $\sigma \in \mathfrak{S}_n$, values appearing in consecutive positions either increase or decrease. There are $n-1$ pairs of consecutive positions, and so we have the pattern-function

$$[\underline{12}](\sigma) + [\underline{21}](\sigma) = n - 1.$$

Because this is true for all $\sigma \in \mathfrak{S}_n$, we will write the general fact as

$$\left([\underline{12}] + [\underline{21}]\right)(\sigma) = n - 1,$$

or, when the meaning of σ is clear, simply as

$$[\underline{12}] + [\underline{21}] = n - 1. \tag{3}$$

The set $\mathfrak{S}_n$ is a Coxeter group, and the length and reflection length of a permutation give a notion of the permutation's complexity as a Coxeter group element. More precisely, let S be the set of simple reflections (adjacent transpositions) and let T be the set of all reflections (conjugates of the simple reflections). Then the *length* of a permutation σ, denoted $\ell_S(\sigma)$, is the minimal number of elements of S needed to form a product equalling σ and the *reflection length* of σ, denoted $\ell_T(\sigma)$, is the minimal number of elements of T needed to form a product equalling σ. That is,

$$\begin{aligned}
\ell_S(\sigma) &= \min\{k : \sigma = s_1 \cdots s_k \text{ for } s_i \in S\} \text{ and} \\
\ell_T(\sigma) &= \min\{k : \sigma = t_1 \cdots t_k \text{ for } t_i \in T\}.
\end{aligned}$$

It is well known that the number of inversions equals a permutation's length, which we can now write as follows.

Observation 10. $\ell_S(\sigma) = [\underline{2}\,\underline{1}](\sigma)$

3 Enumerations via pattern counts

Diaconis and Graham consider four statistics: $\ell_S(\sigma)$, $\ell_T(\sigma)$, $\mathrm{dis}(\sigma)$, and the variance $V(\sigma)$. Observation 10 translated the first of these into the language of patterns, and our goal is to do similarly for the other metrics.

Reflection length, like length, can be written as a pattern-function. Unfortunately, its formulation is less clean, requiring a sum whose terms depends on n. We note at the end of the section why reflection length is not a d-function for any finite d. To start, we use a result of Brändén and Claesson to write the number of cycles in a permutation in terms of pattern counts in its image under the fundamental bijection.

Proposition 11 (cf. [4, Proposition 3])**.** *The number of left-to-right maxima in a permutation σ is*

$$\sum_{k\geqslant 1}(-1)^{k-1}\sum_{\substack{\pi\in\mathfrak{S}_k\\ \pi(k)=1}}[\pi](\sigma).$$

Because the sum of a permutation's reflection length and number of cycles is equal to its size, Proposition 11 allows reflection length to be written in terms of pattern counts in the fundamental bijection.

Corollary 12. *For $\sigma\in\mathfrak{S}_n$, let $\Phi(\sigma)$ be the fundamental bijection of σ. Then*

$$\ell_T(\sigma)=n-\sum_{k\geqslant 1}(-1)^{k-1}\sum_{\substack{\pi\in\mathfrak{S}_k\\ \pi(k)=1}}[\pi](\Phi(\sigma)).$$

Compared to length and reflection length, calculating a pattern-function for variance is a much more interesting challenge. To do so, we first characterize variance in terms of the inversion set of a permutation. Our proof uses some handy properties of summations, but the result can also be proved inductively or by a geometric argument.

Lemma 13. *For any permutation $\sigma\in\mathfrak{S}_n$,*

$$V(\sigma)=2\sum_{(i,j)\in\mathrm{Inv}(\sigma)}(\sigma_i-\sigma_j).$$

Proof. Because σ is a permutation, we can write

$$V(\sigma)=\sum_{i=1}^{n}(\sigma_i-i)^2=\sum_{i=1}^{n}\left(\sigma_i^2-2i\sigma_i+i^2\right)=2\sum_{i=1}^{n}i^2-2\sum_{i=1}^{n}i\sigma_i.$$

To simplify the right-hand side of the proposition, we note that

$$x-y+|x-y|=\begin{cases}2(x-y) & \text{if } x>y, \text{ and}\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\begin{aligned}
2\sum_{(i,j)\in\mathrm{Inv}(\sigma)}(\sigma_i-\sigma_j) &= \sum_{i<j}(\sigma_i-\sigma_j+|\sigma_i-\sigma_j|)\\
&= \sum_{i<j}\sigma_i-\sum_{i<j}\sigma_j+\sum_{i<j}|\sigma_i-\sigma_j|\\
&= \sum_{i=1}^{n}\sigma_i(n-i)-\sum_{i=1}^{n}\sigma_i(i-1)+\sum_{i<j}|\sigma_i-\sigma_j|\\
&= (n+1)\sum_{i=1}^{n}i-2\sum_{i=1}^{n}i\sigma_i+\sum_{i<j}|\sigma_i-\sigma_j|. \qquad (4)
\end{aligned}$$

The sets $\{\{i,j\}:1\leqslant i<j\leqslant n\}$ and $\{\{\sigma_i,\sigma_j\}:1\leqslant i<j\leqslant n\}$ are equal, and we can partition the $\binom{n}{2}$ pairs $\{x,y\}$ in either of them by the difference $|x-y|$. Therefore

$$\sum_{i<j}|\sigma_i-\sigma_j|=\sum_{i<j}(j-i)=\sum_{\delta=1}^{n}\delta(n-\delta)=n\sum_{\delta=1}^{n}\delta-\sum_{\delta=1}^{n}\delta^2. \qquad (5)$$

Combining Equations (4) and (5) yields

$$2\sum_{(i,j)\in\mathrm{Inv}(\sigma)}(\sigma_i-\sigma_j)=(2n+1)\sum_{i=1}^{n}i-2\sum_{i=1}^{n}i\sigma_i-\sum_{i=1}^{n}i^2.$$

Summation identities for $\sum i$ and $\sum i^2$ complete the proof. $\square$

We are now ready to write the variance $V(\sigma)$ as a 3-function.

Theorem 14. $V(\sigma)=2([21]+[231]+[312]+[321])(\sigma)$.

Proof. Lemma 13 gave

$$V(\sigma)=2\sum_{(i,j)\in\mathrm{Inv}(\sigma)}(\sigma_i-\sigma_j),$$

and we will show that this sum is equal to

$$([21]+[231]+[312]+[321])(\sigma).$$

Inversions are counted by [21]. For each inversion (i,j), there are $\sigma_i-\sigma_j-1$ numbers a such that $\sigma_j<a<\sigma_i$. Each of those numbers is counted exactly once by the sum $([21]+[231]+[312]+[321])(\sigma)$. More specifically, the position of 2 relative to 3 and 1 in the pattern corresponds to the value $\sigma^{-1}(a)$ relative to i and j. $\square$

Similar to reflection length (and unlike length and variance), our pattern-function for the disarray/displacement of a permutation is in terms of the permutation's image under the fundamental bijection.

Theorem 15. *For any permutation σ, we have* $\operatorname{dis}(\sigma) = 2\left([\underline{21}] + [\underline{2}\,\underline{31}] + [\underline{31}\,\underline{2}]\right)(\Phi(\sigma))$.

Proof. For ease of notation, set $\phi := \Phi(\sigma)$. We will actually prove an equivalent statement, that the depth $\operatorname{dp}(\sigma)$ is equal to

$$\left([\underline{21}] + [\underline{2}\,\underline{31}] + [\underline{31}\,\underline{2}]\right)(\phi).$$

The fundamental bijection guarantees that if $\phi_j > \phi_{j+1}$ then ϕ_j and ϕ_{j+1} are in the same cycle, and $\sigma_{\phi_j} = \phi_{j+1}$.

Consider the standard cycle representation of σ. If $i > \sigma_i$, then σ_i is not the largest element in its cycle and hence σ_i is not written first in the cycle in standard form. Let j be such that $i = \phi_{j-1}$, and hence $\sigma_i = \phi_j$. Then we can write

$$i - \sigma_i = \phi_{j-1} - \phi_j = 1 + \#\{x : \phi_j < x < \phi_{j-1}\}. \tag{6}$$

Because $\sum_i (\sigma_i - i) = \sum_{i \neq \sigma_i} (\sigma_i - i) = 0$, we have

$$\operatorname{dp}(\sigma) = \sum_{\sigma_i > i} (\sigma_i - i) = -\sum_{i > \sigma_i} (\sigma_i - i) = \sum_{i > \sigma_i} (i - \sigma_i).$$

Consider the rightmost expression in Equation (6). The 1 in that sum identifies the $\underline{21}$-pattern formed by $\phi_{j-1}\phi_j = i\sigma_i$ in ϕ. Now consider the locations of all $\{x : \phi_j < x < \phi_{j-1}\}$ in the one-line notation for ϕ. If x appears to the left of ϕ_{j-1}, then the values $\{x, \phi_{j-1}, \phi_j\}$ form a $\underline{2}\,\underline{31}$-pattern in ϕ. Otherwise those letters form a $\underline{31}\,\underline{2}$-pattern in ϕ. Thus

$$\sum_{i > \sigma_i} (i - \sigma_i) = \left([\underline{21}] + [\underline{2}\,\underline{31}] + [\underline{31}\,\underline{2}]\right)(\phi). \quad \square$$

We close this section with an application of these formulas. Using [1, Proposition 4] and the linearity of expectation, the pattern-function formulation of each of these statistics allow us to systematically determine their expected value. For example, Observation 10 gives us $E[\ell_S] = \frac{n^2-n}{4}$, Theorem 14 gives us $E[V] = \frac{n^3-n}{6}$, and Theorem 15 gives us $E[\operatorname{dis}] = 2E[\operatorname{dp}] = \frac{n^2-1}{3}$. We could similarly find a polynomial in n for the expected value of any finite pattern-function. A more exciting example, from Corollary 12, first requires a summation formula.

Proposition 16 ([11, §6.4])**.** *For $n > 0$, we have*

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k},$$

the nth harmonic number H_n.

From this we see $E[\ell_T] = n - H_n$ which is equivalent to the classic result that the expected number of cycles in a uniformly selected permutation is the nth harmonic number. Furthermore, there cannot be a finite length pattern-function for either reflection length or the number of cycles, since their expected value is not a polynomial in n.

4 Characterizing shallow permutations

In order to address the question of [17] to characterize shallow permutations, it is necessary to consider length, reflection length, and depth simultaneously. Thus we want the pattern-functions for those statistics to be comparable; namely, all in terms of the same object. While Observation 10 computes length as a simple pattern-function of the permutation, Corollary 12 writes reflection length as a pattern-function of the fundamental bijection, as does Theorem 15 for depth. Thus it is necessary to recalculate length in terms of pattern counts in the fundamental bijection, which we do in Theorem 21.

The expected values of length and depth are polynomial in n. The expected value of reflection length is not, so neither is the expected value of $\mathrm{dp}(\sigma) - (\ell_S(\sigma) + \ell_T(\sigma))/2$. We conclude that there is no finite pattern-function for this difference using only vincular patterns. Thus we introduce a new type of pattern that simultaneously captures some information from σ and some from $\Phi(\sigma)$. This is a generalization of vincular patterns directly applied to $\Phi(\sigma)$, which we call *arrow patterns*. At the end of the section we briefly mention how to rewrite the arrow patterns we use in this result as a linear combination of mesh patterns.

Definition 17. An *arrow pattern* α in $\mathfrak{S}_k$ consists of

- sets of integers $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_h\}$, and $C = \{c_1, \dots, c_h\}$ where $A \cup B \cup C = [1, k]$,
- a string $\nu = a_1 \cdots a_m$, in which some consecutive symbols may be bonded together (this resembles a vincular pattern, although A may not equal $[1, m]$), and
- a (possibly empty) collection of h arrows $\{b_i \to c_i : i = 1, \dots, h\}$.

Let $\tau \in \mathfrak{S}_r$ be a permutation, and define σ so that $\Phi(\sigma) = \tau$. Let $\omega := x_{a_1} x_{a_2} \cdots x_{a_m} = \tau_{t_1} \cdots \tau_{t_m}$ be a substring of the one-line notation for τ. The subsequence ω is an *occurrence* of the arrow pattern α in τ if

- the substring ω is order isomorphic to ν,
- if a_j and a_{j+1} are bonded in ν, then x_{a_j} and $x_{a_{j+1}}$ are adjacent in τ (that is, $t_{j+1} = t_j + 1$), and
- there exists $X = \{x_1 < \cdots < x_k\} \subseteq [1, r]$ such that $\{x_{a_1}, x_{a_2}, \dots, x_{a_m}\} \subseteq X$ and, for every arrow $b_i \to c_i$, we have $\sigma(x_{b_i}) = x_{c_i}$ for $1 \leqslant i \leqslant h$.

Note that an *occurrence* of an arrow pattern refers to an occurrence of the underlying vincular pattern, subject to the additional constraints imposed by arrows. Generally speaking, arrow patterns are ripe for further study. In particular, they could be poised to bridge the divide between cycle notation and one-line notation, enabling the application of pattern techniques and results to a substantially broader range of questions. For this paper we will be concerned with patterns which contain a single arrow. We will also always have one end of the arrow contained in the vincular portion.

Example 18. The permutation $\tau = 63248175$ has $\sigma = \Phi^{-1}(\tau) = 74268351$. The permutation τ has two occurrences of the arrow pattern $\underset{1\to2}{\underline{12}}$; namely, 24 and 17. The occurrence 24 matches $\underline{12}$ because $\sigma(2) = 4$ letting $X = \{2 < 4\}$. Similarly, 17 matches $\underline{12}$ where $X = \{1 < 7\}$ and $\sigma(1) = 7$. Although 48 matches $\underline{12}$, it is not an occurrence of the given arrow pattern because $\sigma(4) = 6 \neq 8$. The subsequence 48 does, however, match $\underset{1\to2}{\underline{13}}$ where $X = \{4 < 6 < 8\}$ and $\sigma(4) = 6$.

Arrow patterns offer many pattern coincidences stemming from the presence of the fundamental bijection in the definition, and we highlight some identities that we will use subsequently. Sometimes, arrow patterns might simplify to a vincular pattern, due to properties of the fundamental bijection. For example, with a descent ab, we would not need to write $a \to b$, so we have coincidences like

$$\underset{2\to1}{\underline{21}} = \underline{21} \quad \text{and} \quad \underset{2\to1}{\underline{2}\,\underline{43}} = \underline{21}\,\underline{43}\,.$$

Sometimes, the arrow expressions imply bonds, as with

$$\underset{1\to2}{\underline{1}\,\underline{2}} = \underset{1\to2}{\underline{12}},$$

but we may need to keep the arrow itself. For example, the identity permutation, which is its own image under the fundamental bijection, contains $\underline{12}$ but avoids the arrow patterns listed above. Some equalities are subtle, such as

$$[\underset{1\to2}{\underline{1}\,\underline{3}}] = [\underset{1\to2}{\underline{2}\,\underline{3}}] \quad \text{and} \quad [\underset{1\to2}{\underline{1}\,\underline{43}}] = [\underset{1\to2}{\underline{2}\,\underline{43}}], \tag{7}$$

which result from knowing that the 1 and 2 come before the $\underline{3}$ and $\underline{43}$ respectively. Note in these last cases that the pattern counts are equal, but the patterns themselves are not.

We now present a finite pattern-function for reflection length using arrow patterns.

Proposition 19. $\ell_T(\sigma) = \left([\underline{21}] + [\underset{1\to2}{\underline{12}}] \right) (\Phi(\sigma))$.

Proof. Set $\phi := \Phi(\sigma)$, and suppose that $\sigma \in \mathfrak{S}_n$ has c cycles. It is well-known that $\ell_T(\sigma) = n - c$. There are $c-1$ indices i such that ϕ_i and ϕ_{i+1} are in distinct cycles. Thus there are $(n-1)-(c-1) = \ell_T(\sigma)$ indices i such that $\sigma(\phi_i) = \phi_{i+1}$. Assume $\sigma(\phi_i) = \phi_{i+1}$ and let $x := \phi_i$ and $y := \phi_{i+1}$. There are two cases: either $x > y$ in which case $\underline{xy} \sim \underline{21}$, or $x < y$ and so $\underline{xy} \sim \underset{1\to2}{\underline{12}}$. $\square$

The pattern-function for depth that arises from Theorem 15 is not ideal for comparing with the pattern-function for length that we will develop below in Theorem 21. To address this, we introduce an alternative pattern-function for depth, using arrow patterns.

Theorem 20. $\mathrm{dp}(\sigma) = \ell_T(\sigma) + \left([\underline{2}\,\underline{31}] + [\underline{41}\,\underline{32}] + [\underline{31}\,\underline{42}] + [\underset{1\to4}{\underline{1}\,\underline{23}}] + [\underset{2\to4}{\underline{2}\,\underline{13}}] \right) (\Phi(\sigma))$.

Proof. Throughout this proof, let all vincular pattern counts be applied to the fundamental bijection $\Phi(\sigma)$. Given a $\underline{x}\,\underline{y} \sim \underline{1}\,\underline{2}$ occurrence in $\Phi(\sigma)$ there are five possible relative values for $\sigma(x)$ and four possible relative values for $\sigma^{-1}(x)$. Taken together, these yield

$$[\underline{1}\,\underline{2}] = [\underline{21}\,\underline{3}] + \underset{1\to1}{[\underline{1}\,\underline{2}]} + \underset{1\to2}{[\underline{1}\,\underline{3}]} + \underset{1\to3}{[\underline{1}\,\underline{2}]} + \underset{1\to2}{[\underline{12}]} = \underset{1\to2}{[\underline{2}\,\underline{3}]} + \underset{1\to1}{[\underline{1}\,\underline{2}]} + [\underline{21}\,\underline{3}] + [\underline{31}\,\underline{2}].$$

By subtracting common terms and using Equation (7), this reduces to $\underset{1\to3}{[\underline{1}\,\underline{2}]} + \underset{1\to2}{[\underline{12}]} = [\underline{31}\,\underline{2}]$.

Proposition 19 yields

$$[\underline{31}\,\underline{2}] = \ell_T(\sigma) + \underset{1\to3}{[\underline{1}\,\underline{2}]} - [\underline{21}]. \tag{8}$$

Similarly, we can analyze $[\underline{1}\,\underline{32}]$ to get

$$[\underline{21}\,\underline{43}] + \underset{1\to1}{[\underline{1}\,\underline{32}]} + \underset{1\to2}{[\underline{1}\,\underline{43}]} + \underset{1\to3}{[\underline{1}\,\underline{42}]} + \underset{1\to4}{[\underline{1}\,\underline{32}]} + \underset{1\to3}{[\underline{132}]} = \underset{1\to2}{[\underline{2}\,\underline{43}]} + \underset{1\to1}{[\underline{1}\,\underline{32}]} + [\underline{21}\,\underline{43}] + [\underline{31}\,\underline{42}] + [\underline{41}\,\underline{32}],$$

and Equation (7) allows us to reduce this to

$$\underset{1\to3}{[\underline{1}\,\underline{42}]} + \underset{1\to4}{[\underline{1}\,\underline{32}]} + \underset{1\to3}{[\underline{132}]} = [\underline{31}\,\underline{42}] + [\underline{41}\,\underline{32}]. \tag{9}$$

Considering the element prior to the "2" in the arrow pattern $\underset{1\to3}{[\underline{1}\,\underline{2}]}$ produces the identity

$$\underset{1\to3}{[\underline{1}\,\underline{2}]} = \underset{2\to4}{[\underline{2}\,\underline{13}]} + \underset{1\to4}{[\underline{1}\,\underline{23}]} + \underset{1\to4}{[\underline{1}\,\underline{32}]} + \underset{1\to3}{[\underline{1}\,\underline{42}]} + \underset{1\to3}{[\underline{132}]}$$

which combines with Equation (9) to yield

$$\underset{1\to3}{[\underline{1}\,\underline{2}]} = [\underline{31}\,\underline{42}] + [\underline{41}\,\underline{32}] + \underset{2\to4}{[\underline{2}\,\underline{13}]} + \underset{1\to4}{[\underline{1}\,\underline{23}]}$$

The result now follows from Equation (8) and Proposition 15. $\square$

We can now calculate the length of an arbitrary permutation by enumerating arrow pattern occurrences in its image under the fundamental bijection. The benefit of this, as opposed to the straightforward statement in Observation 10, is that it can be easily compared with Theorem 20, and hence will enable us to address shallow permutations.

Theorem 21. *Let $\sigma \in \mathfrak{S}_n$ be an arbitrary permutation. Then*

$$\ell_S(\sigma) = \ell_T(\sigma) + 2\left([\underline{2}\,\underline{31}] + [\underline{41}\,\underline{32}] + \underset{1\to4}{[\underline{1}\,\underline{23}]}\right)(\Phi(\sigma)) \tag{10}$$

Proof. We prove the claim by inducting on $\ell_T(\sigma)$. If $\ell_T(\sigma) = 0$ then $\sigma = e$, the identity permutation. The theorem holds in this case because $\Phi(e) = e$ and all quantities in the statement of the theorem would be 0.

Assume Equation (10) holds for all permutations with reflection length less than some $k > 0$, and we will prove that it also holds when $\ell_T(\sigma) = k$. We start by defining:

$$\begin{aligned} h &:= \min\{x : \sigma_x \neq x\}, \\ i &:= (\sigma^{-1})_h, \\ r &:= \sigma_h, \text{ and} \\ \sigma' &:= (r\ h)\sigma. \end{aligned}$$

Note that minimality of h means $h < i, r$. The one-line notations of σ and σ' differ only in the positions of r and h, with

$$\sigma = 123(h-1)r\cdots h\cdots \quad \text{and} \quad \sigma' = 123(h-1)h\cdots r\cdots .$$

In standard cycle representation, the h is removed from its cycle in σ and inserted after $(1)(2)\cdots(h-1)$ as an additional fixed point in σ':

$$\sigma = (1)\cdots(h-1)\cdots(\cdots\ i\ h\ r\cdots)\cdots \quad \text{and} \quad \sigma' = (1)\cdots(h-1)(h)\cdots(\cdots\ i\ r\cdots)\cdots .$$

Let $K := \{k : h < k < i \text{ and } r > \sigma_k > h\}$. Consider the inversion set $\mathrm{Inv}(\sigma')$ and define

$$\mathrm{Inv}^+(\sigma') := \{(a, j) \in \mathrm{Inv}(\sigma') : a \neq i\} \sqcup \{(h, j) : (i, j) \in \mathrm{Inv}(\sigma')\},$$

where $|\,\mathrm{Inv}^+(\sigma')| = |\,\mathrm{Inv}(\sigma')|$ and $\mathrm{Inv}^+(\sigma') \subseteq \mathrm{Inv}(\sigma)$. The inversion set $\mathrm{Inv}(\sigma)$ can be written as the disjoint union

$$\mathrm{Inv}(\sigma) = \mathrm{Inv}^+(\sigma') \sqcup \{(h, k), (k, i) : k \in K\} \sqcup \{(h, i)\},$$

which implies that

$$\ell_S(\sigma) = \ell_S(\sigma') + 2|K| + 1.$$

Because a permutation's reflection length and number of cycles sum to its size, and because σ' has one more cycle than σ, we have $\ell_T(\sigma) = \ell_T(\sigma') + 1$. Set

$$P := \{\underline{2}\,\underline{31}, \underline{41}\,\underline{32}, \underset{1\to 4}{\underline{1}\,\underline{23}}\},$$

and call an occurrence of any of these three patterns a *P-pattern*. For a permutation π, we write $P(\pi)$ to denote all P-patterns in π. For the remainder of this proof, set $\phi := \Phi(\sigma)$ and $\phi' := \Phi(\sigma')$. Therefore by the inductive hypothesis it suffices to show that $|K|$ is equal to

$$|P(\phi)| - |P(\phi')|. \tag{11}$$

Recall the definition of h, meaning that the first $h - 1$ values in ϕ and ϕ' are fixed. Thus if h appears in any P-pattern in either permutation, then h must be the smallest value in the occurrence.

We will define an injection $\chi : P(\phi') \to P(\phi)$, with $X := P(\phi) \setminus \chi(P(\phi'))$. We will then construct a bijection $\rho : X \to K$ to complete the proof. The map χ is defined as follows:

Case 0: If $p \in P(\phi')$ and $p \in P(\phi)$, meaning that p does not use the letter h, then $\chi(p) := p$. Otherwise, either a bonded group or the arrow of p in ϕ' gets "interrupted" by the h in ϕ, which can occur in five different ways.

Case 1: An occurrence of $\underline{2}\,\underline{31}$ appears in ϕ' but not in ϕ if and only if it is $\underline{x}\,\underline{ir} \in P(\phi')$. Thus $\underline{x}\,\underline{ihr} \sim \underline{3}\,\underline{412}$ in ϕ, and we set $\chi(\underline{x}\,\underline{ir}) := \underline{x}\,\underline{ih}$.

Case 2: If $\underline{ir}\,\underline{xy} \sim \underline{41}\,\underline{32} \in P(\phi')$ then $\underline{ihr}\,\underline{xy} \sim \underline{512}\,\underline{43}$ in ϕ. Set $\chi(\underline{ir}\,\underline{xy}) := \underline{ih}\,\underline{xy}$.

Case 3: If $\underline{xy}\,\underline{ir} \sim \underline{41}\,\underline{32} \in P(\phi')$ then $\underline{xy}\,\underline{ihr} \sim \underline{52}\,\underline{413}$ in ϕ. Set $\chi(\underline{xy}\,\underline{ir}) := \underline{y}\,\underline{ih} \sim \underline{2}\,\underline{31}$.

Case 4: If $\underline{i}\,\underline{xy} \sim \underset{1\to 4}{\underline{1}\,\underline{23}} \in P(\phi')$ then h in ϕ interrupts the arrow. By definition of the fundamental bijection, $\underline{ih}\,\underline{xy} \sim \underset{1\to 5}{\underline{21}\,\underline{34}}$ in ϕ. Set $\chi(\underline{i}\,\underline{xy}) := \underline{h}\,\underline{xy} \sim \underset{1\to 4}{\underline{1}\,\underline{23}}$.

Case 5: Finally, if $\underline{x}\,\underline{ir} \sim \underset{1\to 4}{\underline{1}\,\underline{23}}$ then $\underline{x}\,\underline{ihr} \sim \underset{2\to 5}{\underline{2}\,\underline{314}}$ in ϕ. Set $\chi(\underline{x}\,\underline{ir}) := \underline{x}\,\underline{ih} \sim \underline{2}\,\underline{31}$.

A straightforward case analysis shows that the map χ is injective.

Consider an element $p \in X$, meaning that p is a P-pattern occurring in ϕ that is not in the image of χ. To avoid Case 0, the occurrence p must use h, and h must be minimal in the occurrence p, as discussed above. If $\underline{x}\,\underline{ih} \sim \underline{2}\,\underline{31} \in X$, then to avoid Cases 1, 3, and 5, we must have

$$\begin{aligned} & r > x, \\ & \text{if } r < i \text{ then } \sigma^{-1}(x) < i, \text{ and} \\ & \text{if } r \geqslant i \text{ then } \sigma_x < r. \end{aligned}$$

If $r < i$, then $h < \sigma^{-1}(x) < i$ and $r > x > h$, so set $\rho(\underline{x}\,\underline{ih}) := \sigma^{-1}(x)$. If $r \geqslant i$, then $h < x < i$ and $r > \sigma_x > h$, and we set $\rho(\underline{x}\,\underline{ih}) := x$. Every $k \in K$ with the position of k in ϕ coming before the position of h must fall into either one of these two options.

If $\underline{ih}\,\underline{xy} \sim \underline{41}\,\underline{32} \in X$, then $\sigma_x = y$ and to avoid Case 2 we must have $r > y$. Then $h < x < i$ and $r > y > h$, and we set $\rho(\underline{ih}\,\underline{xy}) := x$. If $\underline{h}\,\underline{xy} \sim \underset{1\to 4}{\underline{1}\,\underline{23}}$, then $\sigma_x = y$ and $\sigma_h = r > y$. To avoid Case 4 we must have $i > x$. Then $h < x < i$ and $r > y > h$, and we set $\rho(\underline{h}\,\underline{xy}) := x$. Every $k \in K$ with the position of k in ϕ coming after the position of h must fall into one of these two options. Inverting ρ is straightforward based on whether a given $k \in K$ appears before or after h. $\square$

Theorems 20 and 21 recover the inequality [17, Observation 2.2]:

$$\frac{\ell_T(\sigma) + \ell_S(\sigma)}{2} \leqslant \mathrm{dp}(\sigma) \leqslant \ell_S(\sigma),$$

or, in the language of [8], $\ell_T(\sigma) + \ell_S(\sigma) \leqslant \mathrm{dis}(\sigma) \leqslant 2\ell_S(\sigma)$. Furthermore, they show the following.

Corollary 22. *For any permutation σ,*

$$\mathrm{dp}(\sigma) = \frac{\ell_S(\sigma) + \ell_T(\sigma)}{2} + \left([\underline{31}\,\underline{42}] + [\underset{2\to4}{\underline{2}\,\underline{13}}] \right) (\Phi(\sigma)).$$

In particular, a permutation is shallow if and only if its image under the fundamental bijection avoids $\underline{31}\,\underline{42}$ and $\underset{2\to4}{\underline{2}\,\underline{13}}$.

We now can produce a characterization of shallow permutations that does not depend on arrow patterns.

Theorem 23. *A permutation $\sigma_n \in \mathfrak{S}_n$ is shallow if and only if its image, $\Phi(\sigma)$, under the fundamental bijection avoids*

$$\{\underline{5}\,\underline{24}\,\underline{13}, \underline{4}\,\underline{25}\,\underline{13}, \underline{31}\,\underline{42}\}.$$

Proof. From Corollary 22, a permutation $\sigma \in \mathfrak{S}_n$ is shallow if and only if $\Phi(\sigma)$ avoids

$$\{\underline{31}\,\underline{42}, \underset{2\to4}{\underline{2}\,\underline{13}}\}.$$

We will now show that containment (resp., avoidance) of the latter of these two patterns is equivalent to containment (resp., avoidance) of the patterns $\{\underline{5}\,\underline{24}\,\underline{13}, \underline{4}\,\underline{25}\,\underline{13}\}$.

For $\Phi(\sigma)$ to contain $\underline{x}\,\underline{yz} \sim \underset{2\to4}{\underline{2}\,\underline{13}}$, it must be that

$$y < x < z < \sigma(x).$$

Either $\sigma(x)$ appears to the right of x in the one-line notation for $\Phi(\sigma)$, meaning that $\sigma(x)$ is not the largest element in its cycle in σ, or $\sigma(x)$ appears to the left of x in the one-line notation for $\Phi(\sigma)$ because $\sigma(x)$ is the largest element in its cycle in σ. Consider first the former case. Then the elements $\{x, \sigma(x), y, z\}$ together with the largest element of x's cycle in σ form a

$$\underline{5}\,\underline{24}\,\underline{13}\text{-pattern}$$

in $\Phi(\sigma)$. In the latter case, we can make several observations. First, y (and z) is not in the same cycle as $\{x, \sigma(x)\}$, because x's cycle ends after x. Second, the largest element of another cycle appears immediately to the right of x in the one-line notation of $\Phi(\sigma)$, and it must be larger than $\sigma(x)$, by definition of the fundamental bijection. Therefore the elements $\{\sigma(x), x, y, z\}$ together with the element immediately after x in $\Phi(\sigma)$ form a

$$\underline{4}\,\underline{25}\,\underline{13}\text{-pattern}$$

in $\Phi(\sigma)$.

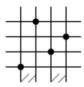
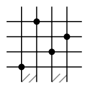
Now suppose that $\Phi(\sigma)$ contains $\underline{a}\,\underline{xb}\,\underline{yz} \sim \underline{5}\,\underline{24}\,\underline{13}$. Left-to-right maxima in $\Phi(\sigma)$ indicate the beginning of new cycles in σ, so we must have $b = \sigma(x)$. Therefore $\underline{x}\,\underline{yz} \sim \underset{2\to4}{\underline{2}\,\underline{13}}$.

If, on the other hand, $\Phi(\sigma)$ contains $\underline{a}\,\underline{xb}\,\underline{yz} \sim \underline{4}\,\underline{25}\,\underline{13}$, then either $\sigma(x) = b$, in which case we have $\underline{x}\,\underline{yz} \sim \underset{2\to4}{\underline{2}\,\underline{13}}$, or $\sigma(x) \neq b$. Suppose $\sigma(x) \neq b$, so that $\sigma(x)$ is the largest element in x's cycle in σ and it appears to the left of x in the one-line notation for $\Phi(\sigma)$. If a is in the same cycle as x, then $\sigma(x) \geqslant a$ and hence $\underline{x}\,\underline{yz} \sim \underset{2\to4}{\underline{2}\,\underline{13}}$. Otherwise, $\sigma(x)$ is the largest element in a cycle appearing to the right of the cycle containing a in the standard representation of σ, meaning once again that $\sigma(x) \geqslant a$ and so $\underline{x}\,\underline{yz} \sim \underset{2\to4}{\underline{2}\,\underline{13}}$.

Therefore $\underset{2\to4}{\underline{2}\,\underline{13}}$-avoidance is equivalent to $\{\underline{5}\,\underline{24}\,\underline{13}, \underline{4}\,\underline{25}\,\underline{13}\}$-avoidance, completing the proof. $\square$

The arrow patterns provided above are novel, which might leave the reader dissatisfied. For those who prefer mesh patterns, we can also use those constructions to express these results.

Proposition 24.

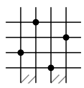
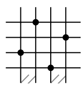
$$\underset{1\to4}{\underline{1}\,\underline{23}} =$$  $-$ 

Proof. Suppose $\underline{x}\,\underline{yz}$ is an occurrence of $\underset{1\to4}{\underline{1}\,\underline{23}}$ in τ, with $\Phi(\sigma) = \tau$. Then either $\sigma(x)$ follows x in the one-line notation of τ, or $\sigma(x)$ is at the beginning of x's cycle in σ and thus appears to the left of x in τ. In either case, let w be the value immediately after x in τ. By definition of the fundamental bijection, $w \geqslant \sigma(x)$, so we must have $\underline{xw}\,\underline{yz} \sim \underline{14}\,\underline{23}$.

Now suppose that $\underline{xw}\,\underline{yz} \sim \underline{14}\,\underline{23}$ but $\underline{x}\,\underline{yz}$ is not an occurrence of $\underset{1\to4}{\underline{1}\,\underline{23}}$. We must have $\sigma(x) < z$, so that $\sigma(x) < w$ and $\sigma(x)$ is the largest element of its cycle (appearing to the left of x in τ). Equivalently, there is no element greater than z to the left of w in the one-line notation for τ.

By set differences, then, the result is proved. $\square$

It can be shown similarly that

$$\underset{2\to4}{\underline{2}\,\underline{13}} =$$  $-$ 

These equivalences allow for mesh pattern formulations of Theorem 20 and 21.

5 Shallow Cycles and Involutions

In this section, we specialize our interest in shallow permutations to shallow cycles and shallow involutions.

Integer sequences celebrated throughout combinatorics turn up in the literature related to the length, depth, and reflection-length permutation statistics. For example in [17], Petersen and the second author show that the number of permutations $\sigma \in \mathfrak{S}_n$ which have $\ell_S(\sigma) = \ell_T(\sigma)$ is given by the Fibonacci numbers [16, A000045], and it is shown in [8] that the cases where $\ell_S(\sigma) = \mathrm{dp}(\sigma)$ are given by the Catalan numbers [16, A000108].

Here we will show that the shallow involutions are enumerated by the Motzkin numbers [16, A001006], via a bijection between shallow involutions and circles with non-intersecting chords. Then we classify shallow cycles and show they are enumerated by Schröder numbers [16, A006318], via a bijection with separable permutations.

Corollary 25. *An involution σ is shallow if and only if $\Phi(\sigma)$ avoids $\underline{31}\,\underline{42}$, and shallow involutions are counted by the Motzkin numbers.*

Proof. Let σ be an involution, meaning that its cycle form consists of 1- and 2-element cycles. Thus in $\Phi(\sigma)$, its image under the fundamental bijection, the descents occur between letters in the same 2-cycle of σ, and all other values are fixed points of σ. Moreover, the fundamental bijection $\Phi(\sigma)$ will have no $\underset{2\to 4}{\underline{2}\,\underline{13}}$ patterns because after the "4," we cannot have a consecutive "$\underline{13}$." Thus an involution σ is shallow if and only if $\Phi(\sigma)$ avoids the pattern $\underline{31}\,\underline{42}$.

Motzkin numbers count the number of ways to draw non-intersecting chords on a circle with n points. We can represent an involution $\sigma \in \mathfrak{S}_n$ as n points on a circle with chords connecting points in the same cycle in σ. To say that $\Phi(\sigma)$ avoids $\underline{31}\,\underline{42}$ is equivalent to saying that these chords are non-intersecting. □

We demonstrate this correspondence with two examples.

Example 26. The involution $(32)(4)(51)(7)(86) = 53241876$ is shallow: its depth is 7, its length is 11, its reflection length is 3, and $7 = (11 + 3)/2$. The involution $(32)(4)(61)(7)(85) = 63248175$ produces $\underline{61}\,\underline{85} \sim \underline{31}\,\underline{42}$ in $\Phi(63248175) = 32461785$, and so it is not shallow. Indeed, its depth is 9, its length is 13, and its reflection length is 3. As we see in Figure 1, the diagram for the first permutation (on the left) has no crossed chords, while the diagram for the second permutation (on the right) has a crossing.

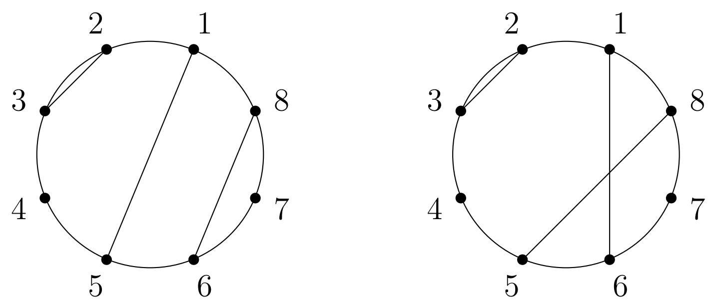


Figure 1: Involutions represented as circles with chords as discussed in Example 26. The involution on the left (with no crossed chords) is shallow, while the one on the right (with crossed chords) is not.

Having characterized and enumerated shallow involutions so succinctly, we now turn our attention to another special class of permutations: shallow cycles. To begin, we make some basic observations.

Observation 27. *For any cycle σ,*

$$[\underset{1\to 4}{\underline{1}\,\underline{23}}](\Phi(\sigma)) = [\underline{14}\,\underline{23}](\Phi(\sigma)) \quad \textit{and} \quad [\underset{2\to 4}{\underline{2}\,\underline{13}}](\Phi(\sigma)) = [\underline{24}\,\underline{13}](\Phi(\sigma)).$$

This allows for considerable simplification of the results from the previous section.

Corollary 28. *Let $\sigma \in \mathfrak{S}_n$ be a cycle. Then*

$$\begin{aligned}\ell_S(\sigma) &= n-1+2\left([\underline{2}\,\underline{31}]+[\underline{14}\,\underline{23}]+[\underline{41}\,\underline{32}]\right)(\Phi(\sigma)) \text{ and}\\ \mathrm{dp}(\sigma) &= n-1+\left([\underline{2}\,\underline{31}]+[\underline{14}\,\underline{23}]+[\underline{41}\,\underline{32}]+[\underline{24}\,\underline{13}]+[\underline{31}\,\underline{42}]\right)(\Phi(\sigma)),\\ &= \frac{\ell_S(\sigma)+\ell_T(\sigma)}{2}+\left([\underline{31}\,\underline{42}]+[\underline{24}\,\underline{13}]\right)(\Phi(\sigma)).\end{aligned}$$

In particular, a cycle τ is shallow if and only if $\Phi(\tau)$ avoids $\underline{31}\,\underline{42}$ and $\underline{24}\,\underline{13}$.

Proof. The sum of a permutation's reflection length and number of cycles is equal to its size. Let $\sigma \in \mathfrak{S}_n$ be a cycle, and so $\ell_T(\sigma) = n-1$. By Equation (10) and Observation 27 we have $\ell_S(\sigma) = n-1+2\left([\underline{2}\,\underline{31}]+[\underline{14}\,\underline{23}]+[\underline{41}\,\underline{32}]\right)(\Phi(\sigma))$. Thus

$$\frac{\ell_S(\sigma)+\ell_T(\sigma)}{2} = n-1+\left([\underline{2}\,\underline{31}]+[\underline{14}\,\underline{23}]+[\underline{41}\,\underline{32}]\right)(\Phi(\sigma)).$$

Using this, together with Theorem 20 and Observation 27, we have

$$\begin{aligned}\mathrm{dp}(\sigma) &= n-1+\left([\underline{2}\,\underline{31}]+[\underline{14}\,\underline{23}]+[\underline{41}\,\underline{32}]+[\underline{24}\,\underline{13}]+[\underline{31}\,\underline{42}]\right)(\Phi(\sigma))\\ &= \frac{\ell_S(\sigma)+\ell_T(\sigma)}{2}+\left([\underline{31}\,\underline{42}]+[\underline{24}\,\underline{13}]\right)(\Phi(\sigma)).\end{aligned}$$

□

To make a better characterization of shallow cycles, we note the following pattern coincidence.

Definition 29. For two sets of patterns, A and B, write $A \asymp B$ to indicate that A-avoidance is equivalent to B-avoidance; that is, that a permutation avoids all elements of A if and only if that permutation avoids all elements of B. When $A \asymp B$, we say that A and B are *coincident*.

Lemma 30. $\{3142, 2413\} \asymp \{\underline{31}\,\underline{42}, \underline{24}\,\underline{13}\}$.

Proof. We first show that $P := \{3142, 2413\}$ is coincident to $Q := \{\underline{31}\,\underline{4}\,\underline{2}, \underline{24}\,\underline{1}\,\underline{3}\}$. Certainly if a permutation avoids all P-patterns then it also avoids all Q-patterns. Now suppose that σ avoids all Q-patterns, but has a 3142-pattern in positions $i_1 < i_2 < i_3 < i_4$. Then it must be that $i_2 > i_1 + 1$. Let us choose i_1 to be maximal relative to i_2, and consider σ_{i_1+1}. To avoid a $\underline{31}\,\underline{4}\,\underline{2}$-pattern, we must have $\sigma_{i_1+1} > \sigma_{i_4}$. By maximality of i_1, we must also have $\sigma_{i_1+1} > \sigma_{i_3}$. But then σ has a $\underline{24}\,\underline{1}\,\underline{3}$-pattern in positions $\{i_1, i_1+1, i_2, i_3\}$, which is a contradiction. Thus σ must avoid 3142, and a similar argument shows that it avoids 2413. Hence $P \asymp Q$.

We will now show that Q is coincident to $R := \{\underline{31}\,\underline{42}, \underline{24}\,\underline{13}\}$, and the result will follow by transitivity of $\asymp$.

As before, if σ avoids all Q-patterns, then it necessarily avoids all R patterns. Now consider σ avoiding all R-patterns and suppose that σ has a $\underline{31}\,\underline{4}\,\underline{2}$-pattern in positions $i_1 < i_2 < i_3 < i_4$, with $i_2 = i_1 + 1$. Then we must have $i_4 > i_3 + 1$. Choose this occurrence of $\underline{31}\,\underline{4}\,\underline{2}$ so that $i_4 - i_3$ is minimal. Because of this minimality, and because σ avoids $\underline{31}\,\underline{42}$, we must have $\sigma_j < \sigma_2$ for $j \in \{i_3+1, i_4-1\}$. In other words, there is a $\underline{42}\,\underline{51}\,\underline{3}$-pattern in positions $\{i_1, i_2, i_3, i_3+1, i_4\}$ of σ, and a $\underline{42}\,\underline{5}\,\underline{13}$-pattern in positions $\{i_1, i_2, i_3, i_4-1, i_4\}$. Because σ avoids all R-patterns, we must have $\sigma_{i_3+1} < \sigma_{i_2+1} < \sigma_{i_4}$. If $\sigma_{i_2+1} < \sigma_{i_4-1}$, then σ will have a $\underline{31}\,\underline{4}\,\underline{2}$-pattern in positions $\{i_2, i_2+1, i_3, i_4-1\}$, violating minimality of $i_4 - i_3$. Thus $\sigma_{i_4-1} < \sigma_{i_2+1}$. In fact, a similar analysis shows that $\max\{\sigma_{i_3+1}, \sigma_{i_4-1}\} < \sigma_j < \sigma_{i_4}$ for all $j \in (i_2, i_3)$. But then σ has a $\underline{24}\,\underline{13}$-pattern in positions $\{i_3-1, i_3, i_4-1, i_4\}$, which is a contradiction. Thus σ must avoid $\underline{31}\,\underline{4}\,\underline{2}$, and a similar argument shows that it avoids $\underline{24}\,\underline{1}\,\underline{3}$. Hence $Q \asymp R$. $\square$

We can now characterize and enumerate shallow cycle permutations. An alternative proof of this result may also be deduced from [7, Theorem 1.1] together with [25, Theorem 1.1], the latter of which in preparation when our work was first posted to the arXiv.

Corollary 31. *There is a bijection between shallow cycle permutations in $\mathfrak{S}_n$ and separable permutations in $\mathfrak{S}_{n-1}$, and they are enumerated by the $(n-1)$st large Schröder number.*

Proof. Let $\sigma \in \mathfrak{S}_n$ be a cycle. Corollary 28 says that σ is shallow if and only if $([\underline{31}\,\underline{42}] + [\underline{24}\,\underline{13}])(\Phi(\sigma)) = 0$. By Lemma 30, this is equivalent to $\Phi(\sigma)$ avoiding both 3142 and 2413, meaning that $\Phi(\sigma)$ is *separable*. Because σ is a cycle, the leftmost letter in the one-line notation for $\Phi(\sigma)$ is n, and the rest of the word can be any separable permutation in $\mathfrak{S}_{n-1}$. These are enumerated by the large Schröder numbers [16, A006318]. $\square$

It is worth noting that the proof of this corollary goes beyond counting shallow n-cycles to clearly describe their structure. Let τ be the cycle $(n\ 1\ 2\ \cdots\ n-1)$ in $\mathfrak{S}_n$. A cycle $\sigma \in \mathfrak{S}_n$ is shallow if and only if there exists a separable permutation $\pi \in \mathfrak{S}_{n-1}$ such that $\sigma = \pi^{-1}\tau\pi$.

6 Directions for further research

The goal of this paper was to explore pattern-functions for permutation statistics and to demonstrate their utility by characterizing shallow permutations. Given the success of this approach, it seems likely that other quantities can also be written as pattern-functions, and perhaps other open questions can be similarly resolved. We have also used pattern-functions in this work to enumerate interesting classes of permutations. Those efforts, too, suggest a broader category of problems that can benefit from the perspective of pattern-functions.

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