Density of balanced 3-partite graphs without 3-cycles or 4-cycles

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Abstract

Let C_k be a cycle of order k, where $k \geqslant 3$. Let $\mathrm{ex}(n, n, n, \{C_3, C_4\})$ be the maximum number of edges in a balanced 3-partite graph whose vertex set consists of three parts, each has n vertices that has no subgraph isomorphic to C_3 or C_4. We construct dense balanced 3-partite graphs without 3-cycles or 4-cycles and show that $\mathrm{ex}(n, n, n, \{C_3, C_4\}) \geqslant (\frac{6\sqrt{2}-8}{(\sqrt{2}-1)^{3/2}} + o(1))n^{3/2}$.

Mathematics Subject Classifications: 05C15, 05C35, 05C38

1 Introduction

Let $\mathcal{F}$ be a family of graphs, we say that a graph G is $\mathcal{F}$-free if it contains no member of $\mathcal{F}$ as a subgraph. The Turán number of $\mathcal{F}$, denoted $\mathrm{ex}(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. If $\mathcal{F} = \{F\}$, we denote the Turán number by

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$ex(n, F)$. A long standing problem of Erdős [4, 5] is to determine the asymptotics for the corresponding extremal function $\mathrm{ex}(n, \{C_3, C_4\})$. Erdős [5] conjectured that

$$\mathrm{ex}(n, \{C_3, C_4\}) = \left(\frac{1}{2\sqrt{2}} + o(1)\right) n^{3/2}.$$

The lower bound followed from the bipartite incidence graph of a projective plane with $\frac{n}{2}$ points. However, Allen, Keevash, Sudakov and Verstraëte [1] made the opposite conjecture to Erdős':

$$\liminf_{n\to\infty} \frac{\mathrm{ex}(n, \{C_3, C_4\})}{\mathrm{ex}_{\chi\leqslant 2}(n, C_4)} > 1,$$

where $\mathrm{ex}_{\chi\leqslant 2}(n, C_4)$ is the maximum number of edges in a bipartite n-vertex C_4-free graph and $\mathrm{ex}_{\chi\leqslant 2}(n, C_4) = \big(\frac{1}{2\sqrt{2}} + o(1)\big)n^{3/2}$ (See, e.g. [8]). Garnick, Kwong and Lazebnik [6] determined the exact values of $\mathrm{ex}(n, \{C_3, C_4\})$ for all $n \leqslant 24$. More exact values of $\mathrm{ex}(n, \{C_3, C_4\})$ for $25 \leqslant n \leqslant 30$ were determined by Garnick and Njeuwejaar [7]. The best upper bound for this problem is the following: $\mathrm{ex}(n, \{C_3, C_4\}) \leqslant \frac{n\sqrt{n-1}}{2}$ (See, e.g. [3, 6]).

For a family $\mathcal{F}$ of graphs, we use $\mathrm{ex}(n, n, n, \mathcal{F})$ to denote the maximum number of edges in balanced 3-partite graphs on partition classes of size n, which are $\mathcal{F}$-free. If $\mathcal{F}$ consists of just one graph F, we denote $\mathrm{ex}(n, n, n, \{F\})$ by $\mathrm{ex}(n, n, n, F)$. Recently, the authors [10] constructed a balanced 3-partite graph on partition classes of size n, which is C_4-free and has $\big(\frac{3}{\sqrt{2}} + o(1)\big)n^{3/2}$ edges. Combining the upper bound from [12], they proved that

$$\mathrm{ex}(n, n, n, C_4) = \left(\frac{3}{\sqrt{2}} + o(1)\right) n^{3/2}.$$

In this paper, based on the construction in [10], we will construct a balanced 3-partite graph on partition classes of size n with $\big(\frac{6\sqrt{2}-8}{(\sqrt{2}-1)^{3/2}} + o(1)\big)n^{3/2}$ edges, which is $\{C_3, C_4\}$-free. Thus, for all sufficiently large n,

$$1.82 < \frac{6\sqrt{2}-8}{(\sqrt{2}-1)^{3/2}} + o(1) \leqslant \frac{\mathrm{ex}(n, n, n, \{C_3, C_4\})}{n^{3/2}} \leqslant \frac{3}{\sqrt{2}} + o(1) < 2.122.$$

The upper bound follows from $\mathrm{ex}(n, n, n, \{C_3, C_4\}) \leqslant \mathrm{ex}(n, n, n, C_4)$. It would be an interesting problem to determine the asymptotic behavior of $\mathrm{ex}(n, n, n, \{C_3, C_4\})$.

2 Results

We first give the following lemma. We use it to count the number of triangles in the graph we constructed.

Lemma 1. *Let $p \geqslant 5$ be a prime number and m be an integer such that $\frac{p}{3} \leqslant m \leqslant \frac{p-1}{2}$. Let $A_m = \{1, 2, \ldots, m\}$. Then the number of solutions of the equation $x + y + z = p$ with $(x, y, z) \in A_m^3$ is $\frac{(3m-p+2)(3m-p+1)}{2}$.*

Proof. We can enumerate all the solutions as follows. Suppose that (x, y, z) is such a solution. It must be that $p - 2m \leqslant x, y, z \leqslant m$. For a fixed x such that $p - 2m \leqslant x \leqslant m$, all the solutions (x, y, z) are

$$(x, m, p - m - x), (x, m - 1, p - m - x + 1), \ldots, (x, p - m - x, m).$$

That is, there are exactly $2m - p + x + 1$ solutions (x, y, z) for a fixed x. Hence, the total number of solutions is

$$\sum_{x=p-2m}^{m} (2m - p + x + 1) = \frac{(3m - p + 2)(3m - p + 1)}{2}. \qquad \square$$

Now we have our main result.

Theorem 2. *Let $p \geqslant 5$ be a prime number. Then we have*

$$\mathrm{ex}(n, n, n, \{C_3, C_4\}) \geqslant \left(\frac{6\sqrt{2} - 8}{(\sqrt{2} - 1)^{3/2}} + o(1) \right) n^{3/2},$$

where $n = \lfloor (\sqrt{2} - 1)p \rfloor p$.

Proof. Let α be a real number with $\frac{1}{3} \leqslant \alpha \leqslant \frac{1}{2}$. Let $R_{\alpha,p} \subset \mathbb{F}_p$ be the set of multiplicative inverses of the elements of $\{1, 2, \ldots, \lfloor \alpha p \rfloor\} \subset \mathbb{F}_p$. We construct a balanced 3-partite graph $G_{\alpha,p}$ as follows. Let V_1, V_2 and V_3 be disjoint copies of $R_{\alpha,p} \times \mathbb{F}_p$. Let $G_{\alpha,p}$ be the graph whose vertex set is $V_1 \cup V_2 \cup V_3$, where for all $(a, x) \in V_1, (b, y) \in V_2$ and $(c, z) \in V_3$,

- (a, x) is adjacent to (b, y) iff $ab = x - y$,
- (b, y) is adjacent to (c, z) iff $bc = y - z$,
- (c, z) is adjacent to (a, x) iff $ca = z - x$.

One may check that $G_{\alpha,p}$ is $2|R_{\alpha,p}|$-regular. Let $n = |V_1| = |V_2| = |V_3| = \lfloor \alpha p \rfloor p$. Then we have $|E(G)| = 3n|R_{\alpha,p}| = (3\alpha^{1/2} + o(1))n^{3/2}$. Notice that for any $a, b \in R_{\alpha,p}$, we have $a^{-1} + b^{-1} \neq 0$ and then $a + b = (a^{-1} + b^{-1})ab \neq 0$. The similar argument in [10] shows that $G_{\alpha,p}$ is C_4-free. For completeness we include the argument here.

Claim 1 $G_{\alpha,p}$ is C_4-free.

Proof of Claim 1 We just need to show that for any two vertices in V_i, say $i = 1$, they have at most one common neighbor. Let $(a_1, x_1), (a_2, x_2) \in V_1$ and $N((a_1, x_1)) \cap N((a_2, x_2)) = U$. We will show that $|U| \leqslant 1$.

Suppose $|U| \geqslant 2$. We first consider the case $|U \cap V_2| \geqslant 2$ or $|U \cap V_3| \geqslant 2$, say $|U \cap V_2| \geqslant 2$. Let $(b_1, y_1), (b_2, y_2) \in U \cap V_2$. Then $a_i b_j = x_i - y_j$ for $i = 1, 2$ and $j = 1, 2$. Then we have $(a_1 - a_2)b_j = x_1 - x_2$ and $(b_1 - b_2)a_j = y_2 - y_1$ for $j = 1, 2$. If $a_1 = a_2$, then $x_1 = x_2$, a contradiction. If $a_1 \neq a_2$, then $b_1 = b_2$. Since $(b_1 - b_2)a_j = y_2 - y_1$ for $j = 1, 2$, we have $y_1 = y_2$, a contradiction.

Now we consider that case $|U \cap V_2| = 1$ and $|U \cap V_3| = 1$. Assume $(b, y) \in U \cap V_2$ and $(c, z) \in U \cap V_3$. Then we have $a_i b = x_i - y$ and $a_i c = z - x_i$ for $i = 1, 2$. Thus $a_1(b+c) = z - y$ and $a_2(b+c) = z - y$. By the definition of $R_{\alpha,p}$, we have $b + c \neq 0$, thus $a_1 = a_2$ and then $x_1 = x_2$, a contradiction. □

Recall that our goal is to construct a $\{C_3, C_4\}$-free balanced 3-partite graph. We will delete some edges from $G_{\alpha,p}$ to make it C_3-free. Now we compute the number of triangles of $G_{\alpha,p}$. Suppose that $u_1u_2u_3u_1$ is a triangle of $G_{\alpha,p}$, where $u_1 = (a, x) \in V_1$, $u_2 = (b, y) \in V_2$ and $u_3 = (c, z) \in V_3$. By the construction of G, we have $ab = x - y$, $bc = y - z$ and $ca = z - x$. Thus, we have $ab + bc + ca = 0$. Hence, $abc(a^{-1} + b^{-1} + c^{-1}) = 0$, and we have $a^{-1} + b^{-1} + c^{-1} = 0$. This means that given a triple (a, b, c) with $a^{-1} + b^{-1} + c^{-1} = 0$ and $a, b, c \in R_{\alpha,p}$, there exist exactly p triangles $u_1u_2u_3u_1$ with $u_1 = (a, x) \in V_1$, $u_2 = (b, y) \in V_2$ and $u_3 = (c, z) \in V_3$ for some $x, y, z \in \mathbb{F}_p$. By Lemma 1, the number of solutions of the equation $a^{-1} + b^{-1} + c^{-1} = 0$ with $(a, b, c) \in R^3_{\alpha,p}$ is $\frac{(3\lfloor \alpha p \rfloor - p+2)(3\lfloor \alpha p \rfloor - p+1)}{2}$. Thus the number of triangles in $G_{\alpha,p}$ is

$$\frac{(3\lfloor \alpha p \rfloor - p + 2)(3\lfloor \alpha p \rfloor - p + 1)}{2} \times p = \left(\frac{(3\alpha - 1)^2}{2\alpha^{3/2}} + o(1) \right) n^{3/2}.$$

Now we obtain a $\{C_3, C_4\}$-free subgraph $G'_{\alpha,p}$ of $G_{\alpha,p}$ by deleting one edge from each triangle of $G_{\alpha,p}$. Also, we have $|E(G'_{\alpha,p})| = \big(3\alpha^{1/2} - \frac{(3\alpha-1)^2}{2\alpha^{3/2}} + o(1)\big)n^{3/2}$. Let $f(\alpha) = 3\alpha^{1/2} - \frac{(3\alpha-1)^2}{2\alpha^{3/2}}$, where $\frac{1}{3} \leqslant \alpha \leqslant \frac{1}{2}$. By direct calculation, we have

$$f_{\max} = f(\sqrt{2} - 1) = \frac{6\sqrt{2} - 8}{(\sqrt{2} - 1)^{3/2}}.$$

By the definition of $\mathrm{ex}(n, n, n, \{C_3, C_4\})$, the result holds. □

A result of Baker, Harman and Pintz [2] implies that for every large integer m, there exists a prime p satisfying $(1 - o(1))m \leqslant p \leqslant m$. By Theorem 2 and a standard density of primes argument, we have

$$\mathrm{ex}(n, n, n, \{C_3, C_4\}) \geqslant \left(\frac{6\sqrt{2} - 8}{(\sqrt{2} - 1)^{3/2}} + o(1) \right) n^{3/2}.$$

Remark The graph $G_{\alpha,p}$ is a union of three C_4-free bipartite graphs that has appeared in the literatures (See, e.g. [9, 13]). Also, one can use a trick of the proof of Theorem 5 case (ii) in [11] to extend the construction to prime powers $q = p^r$, where $p \geqslant 5$ is a prime number and r is a positive integer. Let μ be a primitive element of $\mathbb{F}_q$. The key to the argument is that each element $a \in \mathbb{F}_q$ can be written in a unique form $a = \sum_{i=0}^{r-1} a_i \mu^i$, where $a_i \in \mathbb{F}_p$. Let $R_{\alpha,q} \subset \mathbb{F}_q$ be the set of multiplicative inverses of the elements of $\{a \in \mathbb{F}_q : a_{r-1} \in \{1, 2, \ldots, \lfloor \alpha p \rfloor\}\}$. Then the same construction with partite sets $R_{\alpha,q} \times \mathbb{F}_q$ satisfies our conclusion.

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