A combinatorial characterization of extremal generalized hexagons

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Abstract

A finite generalized $2d$-gon of order (s,t) with $d \in \{2,3,4\}$ and $s \neq 1$ is called *extremal* if t attains its maximal possible value s^{e_d}, where $e_2 = e_4 = 2$ and $e_3 = 3$. The problem of finding combinatorial conditions that are both necessary and sufficient for a finite generalized $2d$-gon of order (s,t) to be extremal has so far only been solved for the generalized quadrangles. In this paper, we obtain a solution for the generalized hexagons. We also obtain a related combinatorial characterization for extremal regular near hexagons.

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1 Introduction

Generalized polygons were introduced by Jacques Tits [23] in 1959, and have ever since been widely studied [19, 25]. They form an important class of point-line geometries that include the projective planes. Many of the known examples arise from classical groups or groups of Lie type ($G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$), and they naturally arise in extremal graph theory as those point-line geometries whose incidence graphs have diameter n and girth $2n$ for some integer $n \geqslant 2$. If the latter holds, then the generalized polygon is called a generalized n-gon. The generalized 3-gons are precisely the projective planes. While generalized 2-gons are trivial structures, quite the opposite is the case for generalized n-gons with $n \geqslant 3$ which have a rich structure on their own, see [19, 22, 25].

Many of the known generalized polygons have an order (s,t), meaning that every line is incident with precisely $s+1$ points and every point is incident with exactly $t+1$ lines. In fact, it can be shown, see e.g. [25, Corollary 1.5.3], that every generalized polygon with only vertices of degree at least three in its incidence graph has an order.

All generalized n-gons discussed in this paper are finite and have an order (s,t). One of the most important results in the theory of generalized polygons is the Feit-Higman theorem [9] which states that apart from ordinary n-gons every such generalized polygon must have n equal to either 2, 3, 4, 6, 8 or 12, with the case $n = 12$ only occurring when at least one of s, t is equal to 1.

In the case that $n = 2d$ is equal to 4, 6 or 8, restrictions on the numbers s and t can be found in the literature in the form of divisibility conditions, inequalities or certain numbers that need to be squares. Higman [13, 14] proved that $t \leqslant s^2$ for any finite generalized quadrangle or octagon of order (s,t), $s \neq 1$, and Haemers and Roos [10, 12] proved that $t \leqslant s^3$ for any finite generalized hexagon of order (s,t), $s \neq 1$. A finite generalized $2d$-gon of order (s,t) with $d \in \{2,3,4\}$ and $s \neq 1$ is called *extremal* if t attains its maximal possible value s^{e_d}, where $e_2 = e_4 = 2$ and $e_3 = 3$.

For each $d \in \{2,3,4\}$, there are known examples of extremal generalized $2d$-gons. The generalized quadrangles associated with the classical groups of type $PGO_6^-(q)$ are examples of extremal generalized quadrangles, but other examples exist which are related to so-called flocks of quadratic cones. There is only one known family of extremal generalized hexagons consisting of the so-called dual twisted triality hexagons. These are related to groups of type ${}^3D_4(q)$ and have order (q,q^3) for some prime power q. There is also one family of extremal generalized octagons known consisting of the so-called Ree-Tits octagons. These are related to groups of type ${}^2F_4(q)$ and have order (q,q^2), where $q = 2^h$ with h odd. Both families were discovered by Jacques Tits [23, 24]. The question whether these are the only extremal generalized hexagons and octagons remains as of today one of the most important open problems in the theory of generalized polygons.

Another problem in the theory of generalized polygons that has attracted attention is the characterization of extremal generalized $2d$-gons by means of combinatorial properties. In other words, can one give necessary and sufficient combinatorial conditions that would force a finite generalized $2d$-gon of order (s,t) to be extremal. In such a combinatorial characterization, it is preferable that the involved combinatorial properties do not contain any reference to the parameters s and t.

A solution of this problem was already obtained in the 70's for the case of generalized quadrangles not long after the inequality $t \leqslant s^2$ was found, see [4]. So far, no solution has been obtained for the generalized hexagons, despite the problem being open for several years now, see e.g. [15, p. 10] and [16, p. 88] for early references to this problem. In the present paper, we obtain a solution of this problem within the framework of regular near hexagons. Along our way, we also obtain a related combinatorial characterization for the so-called extremal regular near hexagons.

A *near polygon* is essentially a point-line geometry having the property that for every point-line pair (p,L), there exists a unique point on L that is nearest to p with respect to the distance function $\mathrm{d}(\cdot,\cdot)$ in the collinearity graph. If this collinearity graph has diameter 3, then the near polygon is also called a *near hexagon*. Two points at maximal distance 3 will be called *opposite*. A finite near hexagon is said to be *regular with parameters* (s,t,t_2) if it has order (s,t) and if every two points at distance 2 have precisely t_2+1 common neighbours. The finite generalized hexagons of order (s,t) are precisely the regular near hexagons with parameters $(s,t,0)$. The collinearity graphs of regular near polygons provide one of the main families of distance-regular graphs, see [2, Chapter 6]. The Haemers-Roos inequality for generalized hexagons can be generalized to regular near hexagons.

Proposition 1. *Let $\mathcal{S}$ be a regular near hexagon with parameters (s,t,t_2), where $s>1$. Then $t \leqslant s^3+t_2(s^2-s+1)$.*

The inequality in Proposition 1 is called the *Mathon bound* or *Haemers-Mathon bound*. A proof of it can be found in [3, 11, 17, 18]. In the present paper, we obtain another proof of the Haemers-Mathon bound as a by-product of our main results.

We call a regular near hexagon with parameters (s,t,t_2) *extremal* if $s \neq 1$ and $t = s^3+t_2(s^2-s+1)$. Besides the dual twisted triality hexagons and Hermitian dual polar spaces of rank 3, there are two other known examples of extremal regular near hexagons, one with parameters $(2,2,14)$ related to the Mathieu group M_{24}, and another one with parameters $(2,1,11)$ related to M_{12}. Both were discovered in [21]. The collinearity graphs of extremal generalized hexagons are so-called Q-polynomial distance-regular graphs, an important class of distance-regular graphs [2]. In fact, it is the case that any distance-regular graph of diameter 3 whose parameters are compatible with those of an extremal regular near hexagon must be the collinearity graph of such an extremal regular near hexagon. We refer to [8] for more details about this.

The characterization result for extremal generalized hexagons we obtain is somewhat similar to the known characterization result for extremal generalized quadrangles. We first mention this characterization result before proceeding to the case of generalized hexagons.

With a *triad* in a generalized quadrangle we mean a set $T=\{x,y,z\}$ of three mutual noncollinear points. A *center* of T is defined as a point collinear with x, y and z.

Proposition 2 ([4, 19])**.** *Suppose $\mathcal{Q}$ is a finite generalized quadrangle of order (s,t). Then the triads of $\mathcal{Q}$ have a constant number of centers if and only if either $s=1$, $t=1$ or $t=s^2$. In particular, a finite generalized quadrangle of order (s,t) with $s,t \geqslant 2$ is extremal if and only if the triads have a constant number of centers.*

From [1, 4, 19], we also know that every triad in a finite generalized quadrangle of order (s,s^2), $s \neq 1$ has exactly $s+1$ centers.

Suppose now that $\mathcal{S}$ is a finite generalized hexagon of order (s,t). For every point u of $\mathcal{S}$ and every $i \in \{0,1,2,3\}$, let $\Gamma_i(u)$ denote the set of points at distance i from u. With a *triad* in $\mathcal{S}$ we mean a set $T=\{x,y,z\}$ of three points satisfying $\mathrm{d}(x,y)=\mathrm{d}(z,y)=3$ and $\mathrm{d}(x,z)=2$. If we denote by u the unique neighbour of x and z, then $\mathrm{d}(u,y) \in \{2,3\}$. The triad T is said to be of *type I* or *II* depending on whether $\mathrm{d}(u,y)$ is 2 or 3. With a *center* of T, we mean a point of $\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)$. By [8, Corollary 3.5], we know that if $\mathcal{S}$ is an extremal generalized hexagon of order (s,s^3), $s \neq 1$, then each triad of type I has precisely s^2+1 centers, and each triad of type II has precisely s^2+s+1 centers. By

Theorem 3.8 of [7], we also know that $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ for any three mutually opposite points x, y and z of $\mathcal{S}$. These properties form the basis of our characterization result. Indeed, let us consider the following properties in a generalized hexagon $\mathcal{S}$:

($P1$) The triads of type I have a constant number of centers.

($P2$) The triads of type II have a constant number of centers.

($P3$) $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ for any three mutually opposite points x, y and z.

The following result that we will prove in this paper can be regarded as the equivalent of Proposition 2 for generalized hexagons.

Theorem 3. *The properties ($P1$), ($P2$) and ($P3$) hold in a finite generalized hexagon of order (s,t) if and only if $s = 1$, $t = 1$ or $t = s^3$. In particular, a finite generalized hexagon of order (s,t) with $s,t \geqslant 2$ is extremal if and only if the properties ($P1$), ($P2$) and ($P3$) hold.*

In fact, the characterization result we obtain is somewhat more general. In (P1) and (P2), we may restrict to triads containing two given opposite points x and y, and in (P3) to points z at distance 3 from these two given points x and y. We also obtain a related combinatorial characterization for extremal regular near hexagons (Proposition 10). One of the combinatorial properties involved in this characterization still involves the parameter s, but not the parameters t_2 and t.

We also note that if $t = 1$, then there are no triads of type I and so condition (P1) is then void (and so satisfied by convention). Similarly, if $s = 1$ then there are no triads of type II and condition (P2) is void.

We have verified with the aid of a computer whether some of the small generalized hexagons of order (s,t) with $s,t \geqslant 2$ and $t \neq s^3$ satisfy (P1), (P2) and/or (P3). Among those that we checked ($H(2)$, $H(2)^D$, $H(3) \cong H(3)^D$, $H(4)$ and $H(4)^D$), we verified that the hexagons $H(2)^D$, $H(4)^D$ satisfy (P1), the hexagons $H(2)$, $H(2)^D$, $H(3)$ and $H(4)$ satisfy (P2) and none of them satisfy (P3). So, assuming $s,t \geqslant 2$, there are non-extremal generalized hexagons that satisfy (P1) and (P2) (but not (P3)), namely $H(2)^D$. We have used here the notation $H(q)$ to denote the split-Cayley hexagon of order (q,q), and $H(q)^D$ denotes the point-line dual of $H(q)$, see [25].

Remarks. (1) Ronan [20, Remark 3.2] already showed that the triads of type I in dual twisted triality hexagons of order (q,q^3) have q^2+1 centers.

(2) The first record of the Haemers-Roos inequality $t \leqslant s^3$, namely Willem Haemers' PhD theses [10] from 1979, already mentions combinatorial properties that must be satisfied by any extremal generalized hexagon (page 58, Theorem 5.2.6), but does not show that these are also sufficient. In fact, at present it is still an open problem whether these properties are also sufficient.

(3) From Neumaier [18, Theorem 1.1] (see also [2, Theorem 2.3.2]), it follows that a generalized hexagon or octagon is extremal if and only if certain numbers $q(a,b,c)$ are 0 for all points a, b and c. These numbers $q(a,b,c)$ are algebraic expressions involving triple intersection numbers. For certain choices of the points a, b and c, the condition $q(a,b,c)=0$ is however equivalent with $t=s^2$ (for generalized octagons) or $t=s^3$ (for generalized hexagons). So, this result does not provide a combinatorial characterization: it does not provide sufficient combinatorial properties (besides the worthless $t=s^2$ or $t=s^3$) to conclude that the generalized polygon is extremal. The combinatorial condition $q(a,b,c)=0$ moreover involves both parameters s and t.

(4) The distance-regular collinearity graph of an extremal generalized $2d$-gon, $d \in \{2,3,4\}$, has the property that one of its so-called Krein parameters is zero (see [2]). It holds more generally that if a Krein parameter in a distance-regular graph is zero, then certain relationships exist between the triple intersection numbers (see again [18, Theorem 1.1] or [2, Theorem 2.3.2]). Sometimes, these relationships can be used to prove the nonexistence of the distance-regular graphs under consideration, see e.g. [5].

2 An identity in regular near hexagons

Suppose $\mathcal{S}$ is a regular near hexagon with parameters (s,t,t_2) and point set $\mathcal{P}$. If x and y are two points, then $\mathrm{d}(x,y)$ denotes the distance between x and y in the collinearity graph of $\mathcal{S}$. For every $i \in \{0,1,2,3\}$, let k_i denote the constant number of points at distance i from a given point. Then $k_0=1$, $k_1=s(t+1)$, $k_2=\frac{s^2t(t+1)}{t_2+1}$ and $k_3=\frac{s^3t(t-t_2)}{t_2+1}$. The total number v of points of $\mathcal{S}$ is equal to $k_0+k_1+k_2+k_3=(s+1)(1+st+\frac{s^2t(t-t_2)}{t_2+1})$. There exist constants p_{ij}^k with $i,j,k \in \{0,1,2,3\}$ such that if x and y are two points at distance k from each other, then $|\Gamma_i(x)\cap\Gamma_j(y)|=p_{ij}^k$. This property is a special case of a more general result on distance-regular graphs, see [2, §4.1]. The theory of distance-regular graphs or elementary counting also yields that if $i,j,k \in \{0,1,2,3\}$, then $p_{ij}^k=p_{ji}^k$ and

- $p_{ij}^0=\delta_{ij}k_i$, with $k_0=1$, $k_1=s(t+1)$, $k_2=\frac{s^2t(t+1)}{t_2+1}$ and $k_3=\frac{s^3t(t-t_2)}{t_2+1}$;
- $p_{00}^1=p_{02}^1=p_{03}^1=p_{13}^1=0$, $p_{01}^1=1$, $p_{11}^1=s-1$, $p_{12}^1=st$, $p_{22}^1=s(s-1)t$, $p_{23}^1=\frac{s^2t(t-t_2)}{t_2+1}$ and $p_{33}^1=\frac{s^2(s-1)t(t-t_2)}{t_2+1}$;
- $p_{00}^2=p_{01}^2=p_{03}^2=0$, $p_{02}^2=1$, $p_{11}^2=t_2+1$, $p_{12}^2=(s-1)(t_2+1)$, $p_{13}^2=s(t-t_2)$, $p_{22}^2=(s-1)^2t_2-s+\frac{st(t+1)}{t_2+1}$, $p_{23}^2=\frac{s(s-1)(t-t_2)(t+t_2+1)}{t_2+1}$ and $p_{33}^2=\frac{s(t-t_2)}{t_2+1}((s^2-s+1)t-s(t_2+1))$;
- $p_{00}^3=p_{01}^3=p_{02}^3=p_{11}^3=0$, $p_{03}^3=1$, $p_{12}^3=t+1$, $p_{13}^3=(s-1)(t+1)$, $p_{22}^3=\frac{(s-1)(t+1)(t+t_2+1)}{t_2+1}$, $p_{23}^3=\frac{t+1}{t_2+1}((s^2-s+1)t-s(t_2+1))$ and $p_{33}^3=\frac{(s^2+1)(s-1)t(t+1)}{t_2+1}-(s^3-1)t$.

Let $p_1,p_2,\dots,p_v$ be an ordering of the points of $\mathcal{S}$, and let M be the real symmetric $v\times v$ matrix whose entries are defined as follows:

$$M_{ij} := (-\frac{1}{s})^{d(p_i,p_j)}, \qquad \forall i,j \in \{1,2,\dots,v\}.$$

By [3, §(i)] or [18, Remark 2.2], we then know that $M^2=\alpha M$, where

$$\alpha = k_0+\frac{k_1}{s^2}+\frac{k_2}{s^4}+\frac{k_3}{s^6} = \frac{s+1}{s^3}(s^2+st+\frac{t(t-t_2)}{t_2+1}).$$

The matrix $\frac{1}{\alpha}M$ is in fact just the minimal idempotent corresponding to the eigenvalue $-(t+1)$ of the distance-regular collinearity graph of $\mathcal{S}$. In fact, the equality $M^2 = \alpha M$ can also be obtained as follows. Using the above values of the p^k_{ij}'s, we verify that $\alpha(-\frac{1}{s})^k = \sum_{i,j=0}^{3} p^k_{ij}(-\frac{1}{s})^{i+j}$ for every $k \in \{0, 1, 2, 3\}$, i.e. $\alpha M_{xy} = \sum_{z\in\mathcal{P}} M_{xz}M_{zy} = (M^2)_{xy}$ for all $x, y \in \mathcal{P}$.

For every set U of points of $\mathcal{S}$, let χ_U denote its characteristic vector, i.e. χ_U is the row matrix of dimensions $1 \times v$ whose i-th entry with $i \in \{1, 2, \ldots, v\}$ is equal to 1 if $p_i \in U$ and equal to 0 otherwise. If U is a singleton $\{u\}$, then we denote χ_U also by χ_u. We often identify a (1×1) matrix with its unique entry.

Now, let x and y be two opposite points of $\mathcal{S}$ and define

$$A_x := \Gamma_1(x) \cap \Gamma_2(y), \qquad A_y := \Gamma_1(y) \cap \Gamma_2(x).$$

As any line through one of x, y contains a unique point at distance 2 from the other, we know that $|A_x| = |A_y| = t + 1$. We distinguish seven possibilities for a point z of $\mathcal{S}$:

(1) $z = x$ or $z = y$;

(2) $z \in A_x$ or $z \in A_y$;

(3) $z \in \Gamma_1(x) \setminus A_x$ or $z \in \Gamma_1(y) \setminus A_y$;

(4) $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is contained on a line joining a point of A_x with a point of A_y;

(5) $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is not contained on a line joining a point of A_x with a point of A_y;

(6) $z \in \Gamma_2(x) \cap \Gamma_3(y)$ or $z \in \Gamma_3(x) \cap \Gamma_2(y)$;

(7) $z \in \Gamma_3(x) \cap \Gamma_3(y)$.

Let Z denote the set of all points that have distance three from one of x, y and distance at least two from the other, i.e. $Z = (\Gamma_2(x) \cap \Gamma_3(y)) \cup (\Gamma_3(x) \cap \Gamma_2(y)) \cup (\Gamma_3(x) \cap \Gamma_3(y))$.

Lemma 4. (1) *We have $t \geqslant t_2 + 1$.*

(2) *If $s \geqslant 2$, then the sets $\Gamma_2(x) \cap \Gamma_3(y)$, $\Gamma_3(x) \cap \Gamma_2(y)$ and $\Gamma_3(x) \cap \Gamma_3(y)$ are nonempty. If $s = 1$ and $t > t_2 + 1$, then $\Gamma_3(x) \cap \Gamma_3(y) = \emptyset$ and $\Gamma_2(x) \cap \Gamma_3(y) \neq \emptyset \neq \Gamma_3(x) \cap \Gamma_2(y)$. If $s = 1$ and $t = t_2 + 1$, then $Z = \emptyset$.*

Proof. (1) Let $z \in \Gamma_1(x) \cap \Gamma_2(y)$. Through z, there are $t_2 + 1$ lines containing a point of $\Gamma_1(y)$ and all these lines are distinct from xz, implying that $t + 1 \geqslant (t_2 + 1) + 1$.

(2) Suppose $s \geqslant 2$. Then take $z \in \Gamma_1(x) \setminus A_x$ and consider a line L through z distinct from xz. As $\mathrm{d}(z, y) = 3$, there exists a unique point u on L at distance 2 from y. If w is a point of L distinct from z and u, then $\mathrm{d}(x, w) = 2$ and $\mathrm{d}(w, y) = 3$, implying that $\Gamma_2(x) \cap \Gamma_3(y) \neq \emptyset$. In a similar way, one proves that $\Gamma_3(x) \cap \Gamma_2(y) \neq \emptyset$. Now, there are $t_2 + 1$ lines through w containing a point of $\Gamma_1(x)$. As $t + 1 > t_2 + 1$, there exists a

line L' through w not containing a point of $\Gamma_1(x)$. This line L' contains a unique point $u' \in \Gamma_2(y)$. If w' is a point of L' distinct from w and u', then $w' \in \Gamma_3(x) \cap \Gamma_3(y)$, showing that also $\Gamma_3(x) \cap \Gamma_3(y) \neq \emptyset$.

Suppose $s = 1$. Then $\mathcal{S}$ can be regarded as a bipartite graph, implying that $\Gamma_3(x) \cap \Gamma_3(y) = \emptyset$. By symmetry, it suffices to show that $\Gamma_2(x) \cap \Gamma_3(y) = \emptyset$ if and only if $t = t_2 + 1$. Note that every point $z \in \Gamma_2(x) \cap \Gamma_3(y)$ is collinear with a point of A_x. Now, through each point $z' \in A_x$, there are $t_2 + 1$ lines containing a point of A_y, one line containing the point x and $t - t_2 - 1$ other lines. If K is one of these $t - t_2 - 1$ other lines, then the unique point of $K \setminus \{z'\}$ belongs to $\Gamma_2(x) \cap \Gamma_3(y)$, and every point of $\Gamma_2(x) \cap \Gamma_3(y)$ is obtained in this way. We conclude that $\Gamma_2(x) \cap \Gamma_3(y) = \emptyset$ if and only if $t = t_2 + 1$. $\square$

We show that there exists a unique regular near hexagon $\mathcal{S}$ with parameters (s, t, t_2), where $s = 1$ and $t = t_2+1$. Let ∞ be a distinguished point of $\mathcal{S}$. Then $|\Gamma_1(\infty)| = |\Gamma_2(\infty)| = t_2+2$ and $|\Gamma_3(\infty)| = 1$. We denote the unique point in $\Gamma_3(\infty)$ by $\overline{\infty}$. Every point $a \in \Gamma_1(\infty)$ is collinear with t points in $\Gamma_2(\infty)$ and so is noncollinear with a unique point $\bar{a} \in \Gamma_2(\infty)$. This implies that $\mathcal{S}$ is uniquely determined, up to isomorphism. If we identify the points in $\Gamma_1(\infty)$ with the elements of $\{1, 2, \ldots, t_2 + 2\}$ and if i, j are arbitrary elements of $\{1, 2, \ldots, t_2 + 2\}$, then we have the following adjacencies in $\mathcal{S}$: $\infty \sim i$, $\overline{j} \sim \overline{\infty}$ and $i \sim \overline{j}$ if and only if $i \neq j$.

Lemma 5. (1) *If $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is contained on a line joining a point of A_x with a point of A_y, then this line is unique.*

(2) *Let $z \in \Gamma_2(x) \cap \Gamma_2(y)$. If z is collinear with a point z' of A_x, then zz' is the unique line through z connecting a point of A_x with a point of A_y.*

(3) *Let $z \in \Gamma_2(x)$. Let A denote the set of all points of A_x which are on one of the t_2+1 lines through x that contain a point collinear with z, and put $A' := A_x \setminus A$. Then every point of A has distance* 1 *or* 2 *from z and every point of A' has distance* 3 *from z.*

(4) *Let $z \in \Gamma_1(x) \setminus A_x$. Then there are $t_2 + 1$ points in A_y at distance* 2 *from z and the remaining $t - t_2$ points in A_y lie at distance* 3 *from z.*

Proof. (1) Suppose $z \in x_1y_1$ with $x_1 \in A_x$ and $y_1 \in A_y$ two collinear points, and $z \in x_2y_2$ with $x_2 \in A_x \setminus \{x_1\}$ and $y_2 \in A_y \setminus \{y_1\}$ two collinear points. The line x_1y_1 has at least three points, implying that every line of $\mathcal{S}$ is incident with at least three points. As x_1 and x_2 are two common neighbours of x and z, there is by Shult and Yanushka [21, Proposition 2.5] a unique quad[1] Q_1 containing these points. Similarly, since y_1 and y_2 are two common neighbours of z and y, there is a unique quad Q_2 containing these points. As the lines x_1y_1 and x_2y_2 are contained in Q_1 and Q_2, these quads should coincide by [21, Proposition 2.5]. But that is impossible as the points $x \in Q_1$ and $y \in Q_2$ lie at distance 3 from each other.

[1]A quad is a convex subspace on which the induced subgeometry is a generalized quadrangle.

(2) The line zz' contains two points at distance 2 from y, namely z and z', and therefore contains a unique point z'' collinear with y. As x, z', z'', y is a shortest path connecting x and y, we necessarily have $z'' \in A_y$. So, zz' is a line through z connecting a point of A_x with a point of A_y. By (1), there is at most one line through z having that property.

(3) Every point of A is collinear with a point of $\Gamma_1(x) \cap \Gamma_1(z)$ and lies therefore at distance 1 or 2 from z. Suppose $z' \in A'$ lies at distance at most 2 from z. As the line xz' contains two points at distance at most 2 from z, namely x and z', it contains a unique point collinear with z, in contradiction with the fact that the line xz' is not included in the collection of $t_2 + 1$ lines through x meeting $\Gamma_1(z)$.

(4) As $\mathrm{d}(z, y) = 3$, every point of A_y lies at distance 2 or 3 from z. So, it suffices to count those at distance 2 from z. The line xz meets A_x in a point z'. The t_2+1 neighbours of z' and y lie in A_y and have distance 2 from z. Conversely, suppose $z'' \in A_y$ has distance 2 from z. Since $\mathrm{d}(x, z'') = \mathrm{d}(z, z'') = 2$, the line xz contains a unique point collinear with z''. As this point has distance at most 2 from y, it coincides with z', implying that z'' is one of the $t_2 + 1$ neighbours of z' and y. $\square$

For every $z \in Z$, we define a number $N(x, y, z)$ in the following way:

- if $z \in \Gamma_2(x) \cap \Gamma_3(y)$, then

$$N(x, y, z) := s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| - (s+1)(s+t_2) - 1;$$

- if $z \in \Gamma_3(x) \cap \Gamma_2(y)$, then

$$N(x, y, z) := N(y, x, z);$$

- if $z \in \Gamma_3(x) \cap \Gamma_3(y)$, then

$$N(x, y, z) := |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| - |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|.$$

For every point z of $\mathcal{S}$, we also define

$$N_z := \chi_z \cdot M \cdot \Big(s(s + t_2 + 1) \cdot (\chi_x - \chi_y) + \chi_{A_x} - \chi_{A_y}\Big)^T \in \mathbb{Q}.$$

Lemma 6. (1) *If $z = x$, then $N_z = \frac{s+1}{s^2} \cdot (s^3 + t_2(s^2 - s + 1) - t)$. If $z = y$, then $N_z = -\frac{s+1}{s^2} \cdot (s^3 + t_2(s^2 - s + 1) - t)$.*

(2) *If $z \in A_x$, then $N_z = -\frac{s+1}{s^3} \cdot (s^3 + t_2(s^2 - s + 1) - t)$. If $z \in A_y$, then $N_z = \frac{s+1}{s^3} \cdot (s^3 + t_2(s^2 - s + 1) - t)$.*

(3) *If $z \in \Gamma_1(x) \setminus A_x$, then $N_z = -\frac{s+1}{s^3} \cdot (s^3 + t_2(s^2 - s + 1) - t)$. If $z \in \Gamma_1(y) \setminus A_y$, then $N_z = \frac{s+1}{s^3} \cdot (s^3 + t_2(s^2 - s + 1) - t)$.*

(4) *If $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is contained on a line joining a point of A_x with a point of A_y, then $N_z = 0$.*

(5) *If* $z \in \Gamma_2(x) \cap \Gamma_2(y)$ *is not contained on a line joining a point of* A_x *with a point of* A_y*, then* $N_z = 0$.

(6) *If* $z \in \Gamma_2(x) \cap \Gamma_3(y)$*, then* $N_z = -\frac{s+1}{s^3} \cdot N(x,y,z)$. *If* $z \in \Gamma_3(x) \cap \Gamma_2(y)$*, then* $N_z = \frac{s+1}{s^3} \cdot N(x,y,z)$.

(7) *If* $z \in \Gamma_3(x) \cap \Gamma_3(y)$*, then* $N_z = \frac{s+1}{s^3} \cdot N(x,y,z)$.

Proof. For every $i \in \{0,1,2,3\}$, let P_i [Q_i, R_i, resp. S_i] denote the number of points in $\{x\}$ [A_x, A_y, resp. $\{y\}$] at distance i from z. Then

$$N_z := s(s+t_2+1) \cdot \Big(\sum_{i=0}^{3} P_i \cdot (-\frac{1}{s})^i - \sum_{i=0}^{3} S_i \cdot (-\frac{1}{s})^i \Big) + \sum_{i=0}^{3} Q_i \cdot (-\frac{1}{s})^i - \sum_{i=0}^{3} R_i \cdot (-\frac{1}{s})^i.$$

So, N_z can be computed once we know the values of the P_i's, Q_i's, R_i's and S_i's. These values readily follow from Lemma 5. We mention them below, omitting those that are always zero.

(1) If $z = x$, then we have $P_0 = 1$, $Q_1 = t+1$, $R_2 = t+1$ and $S_3 = 1$. If $z = y$, then we have $P_3 = 1$, $Q_2 = t+1$, $R_1 = t+1$ and $S_0 = 1$.

(2) If $z \in A_x$, then $P_1 = 1$, $Q_0 = 1$, $Q_2 = t$, $R_1 = t_2 + 1$, $R_3 = t - t_2$ and $S_2 = 1$. If $z \in A_y$, then $P_2 = 1$, $Q_1 = t_2 + 1$, $Q_3 = t - t_2$, $R_0 = 1$, $R_2 = t$ and $S_1 = 1$.

(3) If $z \in \Gamma_1(x) \setminus A_x$, then $P_1 = 1$, $Q_1 = 1$, $Q_2 = t$, $R_2 = t_2 + 1$, $R_3 = t - t_2$ and $S_3 = 1$. If $z \in \Gamma_1(y) \setminus A_y$, then $P_3 = 1$, $Q_2 = t_2 + 1$, $Q_3 = t - t_2$, $R_1 = 1$, $R_2 = t$ and $S_1 = 1$.

(4) If $z \in \Gamma_2(x) \cap \Gamma_2(y)$ lies on a line connecting a point of A_x with a point of A_y, then $P_2 = 1$, $Q_1 = 1$, $Q_2 = t_2$, $Q_3 = t - t_2$, $R_1 = 1$, $R_2 = t_2$, $R_3 = t - t_2$ and $S_2 = 1$.

(5) If $z \in \Gamma_2(x) \cap \Gamma_2(y)$ is not contained in a line joining a point of A_x with a point of A_y, then $P_2 = 1$, $Q_2 = t_2 + 1$, $Q_3 = t - t_2$, $R_2 = t_2 + 1$, $R_3 = t - t_2$ and $S_2 = 1$.

(6) If $z \in \Gamma_2(x) \cap \Gamma_3(y)$, then $P_2 = 1$, $Q_1 = |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)|$, $Q_2 = t_2 + 1 - |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)|$, $Q_3 = t - t_2$, $R_2 = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$, $R_3 = t + 1 - |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ and $S_3 = 1$. If $z \in \Gamma_3(x) \cap \Gamma_2(y)$, then $P_3 = 1$, $Q_2 = |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$, $Q_3 = t + 1 - |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$, $R_1 = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_1(z)|$, $R_2 = t_2 + 1 - |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_1(z)|$, $R_3 = t - t_2$ and $S_2 = 1$.

(7) If $z \in \Gamma_3(x) \cap \Gamma_3(y)$, then $P_3 = 1$, $Q_2 = |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$, $Q_3 = t + 1 - |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$, $R_2 = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$, $R_3 = t + 1 - |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ and $S_3 = 1$. □

Proposition 7. *We have*

$$\sum_{z \in Z} N(x,y,z)^2 = 2 \cdot \Big(s^3 + t_2(s^2 - s + 1) - t \Big) \cdot \Omega,$$

where

$$\Omega := \Big(s^3 + t_2(s^2 - s + 1) - t \Big) \cdot \Big(\frac{t(t - t_2)}{t_2 + 1} - s \Big) + \Big(s^2 + st_2 - t_2 - 1 \Big) \cdot \Big(s^2 + st + \frac{t(t - t_2)}{t_2 + 1} \Big).$$

Proof. Putting $\eta := s(s+t_2+1)\cdot(\chi_x - \chi_y) + \chi_{A_x} - \chi_{A_y}$ and using $M^2 = \alpha M$, we find

$$\sum_{z\in\mathcal{P}} N_z^2 = \sum_{z\in\mathcal{P}} (\chi_z \cdot M \cdot \eta^T)^2 = \eta\cdot M\cdot M\cdot \eta^T = \alpha\cdot\eta\cdot M\cdot\eta^T$$

$$= \alpha\cdot\Big(s(s+t_2+1)\cdot(N_x - N_y) + \sum_{z\in A_x} N_z - \sum_{z\in A_y} N_z\Big).$$

Taking into account Lemma 6, this equality becomes

$$\Big(2\cdot(\frac{s+1}{s^2})^2 + 2(t+1)\cdot(\frac{s+1}{s^3})^2 + 2(s-1)(t+1)\cdot(\frac{s+1}{s^3})^2\Big)\cdot(s^3+t_2(s^2-s+1)-t)^2$$

$$+\frac{(s+1)^2}{s^6}\cdot\sum_{z\in Z} N(x,y,z)^2 = \frac{s+1}{s^3}\Big(s^2+st+\frac{t(t-t_2)}{t_2+1}\Big)\cdot$$

$$\Big(s(s+1+t_2)\cdot\frac{2(s+1)(s^3+t_2(s^2-s+1)-t)}{s^2} + \frac{2(t+1)(s+1)(t-s^3-t_2(s^2-s+1))}{s^3}\Big),$$

which simplifies to $\sum_{z\in Z} N(x,y,z)^2 = 2\cdot\Big(s^3+t_2(s^2-s+1)-t\Big)\cdot\Omega$. □

3 Characterization of extremal regular near hexagons

We continue with the notation of Section 2. In particular, $\mathcal{S}$ is a regular near hexagon with parameters (s,t,t_2) and x, y are two opposite points of $\mathcal{S}$. We put

$$Z_1 := \Gamma_2(x)\cap\Gamma_3(y), \qquad Z_2 := \Gamma_3(x)\cap\Gamma_2(y).$$

With exception of Proposition 10, we assume here that $s>1$ or $t>t_2+1$. Then

$$|Z_1| = |Z_2| = p^3_{23} = \frac{(t+1)((s^2-s+1)t - s(t_2+1))}{t_2+1} > 0. \qquad (1)$$

Lemma 8. *We have*

$$\sum_{z\in Z_1} |\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)| = s(t+1)(t-t_2-1).$$

Proof. We count in two ways the pairs (u,z), where $z\in Z_1$ and $u\in\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)$. The number of such pairs is obviously $\sum_{z\in Z_1}|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)|$.

On the other hand, if (u,z) is such a pair, then $u\in\Gamma_1(x)\cap\Gamma_2(y)$. So, there are $|\Gamma_1(x)\cap\Gamma_2(y)| = t+1$ possibilities for u. Let $u\in\Gamma_1(x)\cap\Gamma_2(y)$ be given. There are t_2+1 lines through u containing a point of $\Gamma_1(y)$ and none of these lines contains a point of $\Gamma_3(y)$. The line ux does not contain points of $\Gamma_2(x)$. Note that each of the remaining $t-t_2-1$ lines L through u contains s points of $\Gamma_2(x)\cap\Gamma_3(y) = Z_1$, namely the s points of $L\setminus\{u\}$. So, for given u, there are $s(t-t_2-1)$ possibilities for z, and the total number of suitable pairs is also equal to $(t+1)s(t-t_2-1)$. □

Lemma 9. *We have*

$$\sum_{z\in Z_1} |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)| = \frac{t+1}{t_2+1}\Big((s-1)^2 t_2(t_2+1) + s(t+1)(t-t_2-1)\Big).$$

Proof. We count in two ways the triples (z,u,v), where $z\in Z_1$, $u\in\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)$ and $v\in\Gamma_1(z)\cap\Gamma_1(u)$. Note that for given $z\in Z_1$ and $u\in\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)$, there are t_2+1 possibilities for $v\in\Gamma_1(z)\cap\Gamma_1(u)$. So, the total number of such triples is equal to $(t_2+1)\cdot\sum_{z\in Z_1}|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$.

On the other hand, if (z,u,v) is a suitable triple, then $u\in\Gamma_2(x)\cap\Gamma_1(y)$, $v\in\Gamma_1(u)\cap\Gamma_2(y)$ and $z\in\Gamma_1(v)\cap\Gamma_2(u)\cap\Gamma_3(y)\cap\Gamma_2(x)$. Let u be one of the $t+1$ points of $\Gamma_2(x)\cap\Gamma_1(y)$. We then distinguish three possibilities for a point $v\in\Gamma_1(u)\cap\Gamma_2(y)$.

Suppose v is one of the t_2+1 points of $\Gamma_1(u)\cap\Gamma_1(x)$. Then no line through v containing a point of $\Gamma_1(y)$ contains a point of $\Gamma_3(y)$, and the line vx does not contain a point of $\Gamma_2(x)$. If L is one of the $t-t_2-1$ remaining lines through v, then L contains s points $z\in\Gamma_2(x)\cap\Gamma_3(y)\cap\Gamma_2(u)$, namely the points of $L\setminus\{v\}$. So, this case contributes $(t+1)(t_2+1)(t-t_2-1)s$ to the total number of triples.

Suppose v is one of the $(s-1)(t_2+1)$ points of $\Gamma_2(x)\cap\Gamma_1(u)$ on a line through u containing a point of $\Gamma_1(x)$. In order for a line through $v\in\Gamma_2(x)$ to contain a second point of $\Gamma_2(x)$, it should contain a point of $\Gamma_1(x)$ and so coincide with one of the t_2+1 lines through v containing a point of $\Gamma_1(x)$. However, one of these t_2+1 lines, namely uv, does not contain points of $\Gamma_2(u)$. If L is one of the remaining t_2 lines, then no point of L belongs to $\Gamma_1(y)$ (by Lemma 5(2)) and so each of the $s-1$ points of $\Gamma_2(x)\cap\Gamma_1(v)\cap L$ lies at distance 3 from y and at distance 2 from u. So, this case contributes $(t+1)(s-1)(t_2+1)t_2(s-1)$ to the total number of triples.

Suppose v is one of the $s(t-t_2-1)$ points on a line through u distinct from yu and not containing a point at distance 1 from x. Then $\mathrm{d}(x,v)=3$ and so any line L through v contains a unique point $z\in\Gamma_2(x)$. If we assume that L is not one of the t_2+1 lines through v containing a point of $\Gamma_1(y)$, then $\mathrm{d}(z,y)=3$, $\mathrm{d}(z,u)=2$. So, this case contributes $(t+1)s(t-t_2-1)(t-t_2)$ to the total number of triples.

The total number of triples is also equal to $(t+1)(t_2+1)(t-t_2-1)s+(t+1)(s-1)(t_2+1)t_2(s-1)+(t+1)s(t-t_2-1)(t-t_2) = (t+1)((s-1)^2t_2(t_2+1)+s(t+1)(t-t_2-1))$. The lemma follows from equating both quantities. ☐

Put

$$\gamma := \frac{s^2(t_2+1)(t-t_2-1)}{(s^2-s+1)t-s(t_2+1)} + \frac{(s-1)^2t_2(t_2+1)+s(t+1)(t-t_2-1)}{(s^2-s+1)t-s(t_2+1)}.$$

By (1) and Lemmas 8, 9, γ is the average value of the numbers

$$s\cdot|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)| + |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|,\ z\in Z_1.$$

Putting $N'(x,y,z) := s\cdot|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)| + |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)| - \gamma$ for every $z\in Z_1$, we thus have

$$\sum_{z\in Z_1} N'(x,y,z) = 0 \tag{2}$$

and

$$\begin{aligned} N(x,y,z) &= N'(x,y,z) + \gamma - (s+1)(s+t_2) - 1 \\ &= N'(x,y,z) - \frac{(s^3+t_2(s^2-s-1)-t)(st-t_2-1)}{(s^2-s+1)t-s(t_2+1)}. \end{aligned} \tag{3}$$

for every $z \in Z_1$. Using (1), (2) and (3), we have

$$\sum_{z\in Z_1} N(x,y,z)^2 = \sum_{z\in Z_1} N'(x,y,z)^2 + \frac{(s^3+t_2(s^2-s+1)-t)^2(st-t_2-1)^2(t+1)}{((s^2-s+1)t-s(t_2+1))(t_2+1)}.$$

If we define $N'(x,y,z) := s\cdot|\Gamma_1(y)\cap\Gamma_2(x)\cap\Gamma_1(z)| + |\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)| - \gamma$ for every $z \in Z_2$ and $N'(x,y,z) := N(x,y,z)$ for every $z \in \Gamma_3(x)\cap\Gamma_3(y)$, then similarly as above we have

$$\sum_{z\in Z_2} N(x,y,z)^2 = \sum_{z\in Z_2} N'(x,y,z)^2 + \frac{(s^3+t_2(s^2-s+1)-t)^2(st-t_2-1)^2(t+1)}{((s^2-s+1)t-s(t_2+1))(t_2+1)}.$$

Using Proposition 7, we thus find

$$\begin{aligned} \sum_{z\in Z} N'(x,y,z)^2 &= \sum_{x\in Z} N(x,y,z)^2 - \frac{2(t+1)(s^3+t_2(s^2-s+1)-t)^2(st-t_2-1)^2}{(t_2+1)((s^2-s+1)t-s(t_2+1))} \\ &= 2(s^3+t_2(s^2-s+1)-t)\Big(\Omega - \frac{(t+1)(s^3+t_2(s^2-s+1)-t)(st-t_2-1)^2}{(t_2+1)((s^2-s+1)t-s(t_2+1))}\Big) \\ &= \frac{2(s-1)(t+1)(t-t_2-1)(s^3+t_2(s^2-s+1)-t)(stt_2+st+t^2-t_2^2+t-t_2)}{(t_2+1)((s^2-s+1)t-s(t_2+1))}. \end{aligned} \tag{4}$$

As $t > t_2$, the number $stt_2+st+t^2-t_2^2+t-t_2$ is positive, and so the latter equation implies the Haemers-Mathon inequality mentioned in Proposition 1.

Consider the following properties for two opposite points x and y in a regular near hexagon with $s+1$ points on each line:

$(P_{1,2})_{xy}$ The numbers $s\cdot|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)| + |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$, $z \in Z_1$ are equal, as well as the numbers $s\cdot|\Gamma_1(y)\cap\Gamma_2(x)\cap\Gamma_1(z)| + |\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|$, $z \in Z_2$, necessarily to the same constant (namely γ).

$(P_3)_{xy}$ $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)| = |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$ if $z \in \Gamma_3(x)\cap\Gamma_3(y)$.

The following proposition makes mention of a family of near polygons, the so-called *Hamming near polygons*. These are near polygons whose collinearity graphs are cartesian products of a number of complete graphs. The cartesian product of three complete graphs of size $s+1$ is a regular near hexagon with parameters $(s,2,1)$.

Proposition 10. *Suppose x and y are two opposite points in a regular near hexagon with parameters (s,t,t_2). Then properties $(P_{1,2})_{xy}$ and $(P_3)_{xy}$ hold in $\mathcal{S}$ if and only if precisely one of the following hold:*

(1) *$s=1$, i.e. $\mathcal{S}$ is a thin near hexagon;*

(2) *$s\neq 1$, $t_2=0$ and $t=1$, i.e. $\mathcal{S}$ is a generalized hexagon of order $(s,1)$;*

(3) *$s\neq 1$, $t_2=1$ and $t=2$, i.e. $\mathcal{S}$ is Hamming near hexagon;*

(4) *$s\neq 1$ and $t=s^3+t_2(s^2-s+1)$, i.e. $\mathcal{S}$ is an extremal regular near hexagon.*

Proof. Equation (4) and Lemma 4(2) show that properties $(P_{1,2})_{xy}$ and $(P_3)_{xy}$ hold in $\mathcal{S}$ if and only if $s=1$, $t=t_2+1$ or $t=s^3+t_2(s^2-s+1)$.

Suppose now that $s\geqslant 2$ and $t=t_2+1$. If $t_2=0$, then $\mathcal{S}$ is a generalized hexagon, necessarily of order $(s,1)$. Suppose therefore that $t_2\geqslant 1$. By Shult and Yanushka [21, Proposition 2.5], we then know that every two points at distance 2 as well as any two distinct intersecting lines are contained in a unique quad (of order (s,t_2)). Now, consider a quad Q of order (s,t_2), a line L intersection Q in some singleton $\{x\}$ and a line K of Q through x. The unique quad Q' through K and L meets Q in the line K. As $t=t_2+1$ and there are t_2+1 lines in Q' through x, we see that $t_2+1=2$, i.e. $t_2=1$ and $t=2$. This in combination with the fact that every point of Q is contained in a unique line not in Q implies that $\mathcal{S}$ is a Hamming near hexagon, see e.g. Theorem 7.1 of [6]. $\square$

If $s=1$, then $\mathcal{S}$ can be regarded as a graph, and this graph is bipartite as $\mathcal{S}$ is a near polygon. If $z\in Z_1$, then $\Gamma_1(x)\cap\Gamma_1(z)\subseteq\Gamma_2(y)$ and $\Gamma_1(y)\subseteq\Gamma_2(x)\cap\Gamma_2(z)$, implying that the numbers $s\cdot|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)|+|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$ with $z\in Z_1$ are equal to $t+t_2+2$. Similarly, the numbers $s\cdot|\Gamma_1(y)\cap\Gamma_2(x)\cap\Gamma_1(z)|+|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|$ with $z\in Z_2$ are equal to $t+t_2+2$. If $z\in\Gamma_3(x)\cap\Gamma_3(y)$, then $\Gamma_1(x)\subseteq\Gamma_2(y)\cap\Gamma_2(z)$ and $\Gamma_1(y)\subseteq\Gamma_2(x)\cap\Gamma_2(z)$, implying that $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|=|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|=t+1$.

If $\mathcal{S}$ is a Hamming near hexagon, then the points of $\mathcal{S}$ are the elements of A^3, where A is a set of size $s+1$, such that the distance between points is given by the Hamming distance. From this it easily follows that if $z\in Z_1=\Gamma_2(x)\cap\Gamma_3(y)$, then $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)|=0$ and $|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|=1$, and if $z\in\Gamma_3(x)\cap\Gamma_3(y)$, then $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|=0$. So, the numbers $s\cdot|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)|+|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$ with $z\in Z_1$ and the numbers $s\cdot|\Gamma_1(y)\cap\Gamma_2(x)\cap\Gamma_1(z)|+|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|$ with $z\in Z_2$ are equal to $s\cdot 0+1=1$. If $z\in\Gamma_3(x)\cap\Gamma_3(y)$, then $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|=|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|=0$.

If $\mathcal{S}$ is an extremal regular near hexagon with parameters (s,t,t_2), $s\neq 1$, then Proposition 7 implies that the numbers $s\cdot|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)|+|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$ with $z\in Z_1$ and the numbers $s\cdot|\Gamma_1(y)\cap\Gamma_2(x)\cap\Gamma_1(z)|+|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|$ with $z\in Z_2$ are equal to $(s+1)(s+t_2)+1$. It seems not possible here to say something more specific about the numbers $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)|=|\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)|$, $z\in\Gamma_3(x)\cap\Gamma_3(y)$.

Lemma 11. *If $\mathcal{S}$ is a generalized hexagon of order $(s,1)$ and $z\in Z_1$, then $\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_1(z)=\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)=\emptyset$.*

Proof. Let u denote the unique neighbour of x and z. If $u \in \Gamma_2(y)$, then u would be incident with at least three lines, namely ux, uz and the unique line through u meeting $\Gamma_1(y)$. As this is impossible, we have $\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z) = \emptyset$.

Suppose $v \in \Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)$. Let L_1 and L_2 denote the unique lines through v meeting respectively $\Gamma_1(x)$ and $\Gamma_1(z)$. As there are only two lines through v and one of them is vy, we have $L_1 = L_2$. If the unique points w_1 and w_2 of $L_1 = L_2$ collinear with respectively x and z are distinct, then x and z would lie at distance 3 from each other. As this is not the case here, we have that w_1 and w_2 are equal, necessarily to u. But that is also impossible as u would then be incident with at least three lines, namely ux, uz and uv. So, $\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z) = \emptyset$. $\square$

If $\mathcal{S}$ is a generalized hexagon of order $(s, 1)$, then we thus have that the numbers $s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ with $z \in Z_1$ and the numbers $s \cdot |\Gamma_1(y) \cap \Gamma_2(x) \cap \Gamma_1(z)| + |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$ with $z \in Z_2$ are equal to $s \cdot 0 + 0 = 0$. If x, y and z are three opposite points of $\mathcal{S}$, then the equality $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$ follows from the following lemma.

Lemma 12. *If $\mathcal{S}$ is a generalized hexagon of order $(s, 1)$, and x, y, z are three opposite points of $\mathcal{S}$, then $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)|$ equals the number of lines at distance 1 from x, y and z. Depending on the choice of x, y and z, this number is 1 or 2 if $s = 2$ and 0, 1 or 2 if $s \geqslant 3$.*

Proof. Suppose $u \in \Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)$. Let K_1 and K_2 denote the unique lines though u meeting respectively $\Gamma_1(y)$ and $\Gamma_1(z)$. Then $K_1 \neq ux \neq K_2$. As there are only two lines through u, we have $K_1 = K_2$. Obviously, $K_1 = K_2$ is a line at distance 1 from x, y and z. Conversely, if K is a line at distance 1 from x, y and z, then the unique point in $\Gamma_1(x) \cap K$ is contained in $\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)$. This proves the first claim of the lemma.

Since $t = 1$, $\mathcal{S}$ is the flag-geometry of a projective plane π of order s, see [25, §1.6]. So, the points of $\mathcal{S}$ are the flags or incident point-line pairs of π and the lines of $\mathcal{S}$ are the points and lines of π, with incidence being reverse containment. Three opposite points x, y, z of $\mathcal{S}$ then correspond to three flags $\{x_1, L_1\}$, $\{x_2, L_2\}$, $\{x_3, L_3\}$ of π such that $x_i \notin L_j$ if $i \neq j$. A line at distance 1 from x, y, z is either a point x^* incident with L_1, L_2, L_3 or a line L^* incident with x_1, x_2, x_3. There is at most one choice for x^* and at most one choice for L^*. If $s = 2$, then at least one of x^*, L^* exists. $\square$

Remark: As mentioned in Section 1, a distance-regular graph of diameter 3 whose parameters are compatible with those of an extremal regular near hexagon is in fact the collinearity graph of an extremal regular near hexagon. So, Properties $(P_{1,2})_{xy}$ and $(P_3)_{xy}$ provide combinatorial properties for certain distance-regular graphs. In fact, in [8, Section 4], the nonexistence of certain distance-regular graphs was proved using this combinatorial information.

4 Characterization of extremal generalized hexagons

We continue with the notation introduced in the previous sections, but we suppose here that $t_2 = 0$. So, $\mathcal{S}$ is a generalized hexagon of order (s,t). Recall that x and y are two opposite points of $\mathcal{S}$. With exception of Proposition 17, we suppose here that $s,t > 1$. Put

$$Z_1 := \Gamma_2(x) \cap \Gamma_3(y), \qquad Z_1' := \{z \in Z_1 \,|\, \Gamma_1(x) \cap \Gamma_1(z) \cap \Gamma_2(y) \neq \emptyset\}, \qquad Z_1'' := Z_1 \setminus Z_1',$$

$$Z_2 := \Gamma_3(x) \cap \Gamma_2(y), \qquad Z_2' := \{z \in Z_2 \,|\, \Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_2(x) \neq \emptyset\}, \qquad Z_2'' := Z_2 \setminus Z_2'.$$

The triads of type I (respectively, type II) containing $\{x,y\}$ are then precisely the triples $\{x,y,z\}$ where $z \in Z_1' \cup Z_2'$ (respectively, $z \in Z_1'' \cup Z_2''$).

Lemma 13. *We have* $|Z_1| = |Z_2| = (t+1)((s^2-s+1)t-s)$, $|Z_1'| = |Z_2'| = (t+1)s(t-1)$ *and* $|Z_1''| = |Z_2''| = (t+1)(s-1)^2 t$.

Proof. We obviously have $|Z_1| = |Z_2| = p^3_{23} = (t+1)((s^2-s+1)t-s)$.

We count in two ways the number of pairs (z,u), where $z \in Z_1'$ and $u \in \Gamma_1(x) \cap \Gamma_1(z)$. As u is uniquely determined by z, there are $|Z_1'|$ such pairs. On the other hand, if (u,z) is such a pair, then $u \in \Gamma_1(x) \cap \Gamma_2(y)$ and so there are $t+1$ possible choices for u. For given $u \in \Gamma_1(x) \cap \Gamma_2(y)$, the point z cannot be contained on the line ux neither on the unique line through u containing a point of $\Gamma_1(y)$. If L is one of the remaining $t-1$ lines through u, then every point of $L \setminus \{u\}$ lies at distance 2 from x, at distance 3 from y and so belongs to Z_1'. The number $|Z_1'|$ of pairs is therefore also equal to $(t+1)s(t-1)$.

As Z_1 is the disjoint union of Z_1' and Z_1'', we have $|Z_1''| = |Z_1| - |Z_1'| = (t+1)(s-1)^2 t$. In a similar way, one also proves that $|Z_2'| = (t+1)s(t-1)$ and $|Z_2''| = (t+1)(s-1)^2 t$. $\square$

Lemma 14. *We have*

$$\sum_{z \in Z_1'} |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = |Z_1'| = (t+1)s(t-1), \qquad \sum_{z \in Z_1''} |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = 0.$$

Proof. If $z \in Z_1'$, then the unique point in $\Gamma_1(x) \cap \Gamma_1(z)$ lies at distance 2 from y. If $z \in Z_1''$, then the unique point in $\Gamma_1(x) \cap \Gamma_1(z)$ lies at distance 3 from y. $\square$

Lemma 15. *We have*

$$\sum_{z \in Z_1'} |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| = (t+1)(t-1)(s+t).$$

Proof. We count the number of quadruples (z,u,v,w), where $z \in Z_1'$, $u \in \Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)$, $v \in \Gamma_1(u) \cap \Gamma_1(z)$ and $w \in \Gamma_1(x) \cap \Gamma_1(z)$. As v is uniquely determined by u and z, and w is uniquely determined by x and z, the number of such quadruples is equal to $\sum_{z \in Z_1'} |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$.

On the other hand, if (z,u,v,w) is a suitable quadruple, then $u \in \Gamma_2(x) \cap \Gamma_1(y)$ and $v \in \Gamma_1(u) \cap \Gamma_2(y)$. As $u \in \Gamma_2(x) \cap \Gamma_1(y)$, there are $t+1 = |\Gamma_2(x) \cap \Gamma_1(y)|$ choices for u. We now have three possibilities for a point $v \in \Gamma_1(u) \cap \Gamma_2(v)$.

Suppose v is the unique neighbour of u and x. As $z \in \Gamma_1(v) \cap \Gamma_2(x) \cap \Gamma_3(y) \cap \Gamma_2(u)$, the suitable points z are the points of the form $L \setminus \{v\}$, where L is one of the $t-1$ lines through v distinct from vx and vu. So, this case distributes $(t+1)s(t-1)$ to the total number of quadruples. For each of these possibilities, we also have $w = v$.

Suppose $v \in \Gamma_1(u) \cap \Gamma_2(x)$ lies on the unique line through u meeting $\Gamma_1(x)$. This case does not contribute to the total number of quadruples as any point $z \in \Gamma_1(v)$ not on uv lies at distance 3 from x.

Suppose $v \in \Gamma_1(u) \cap \Gamma_2(y)$ does not lie on the unique line through u meeting $\Gamma_1(x)$. This implies that the unique neighbour $w \in \Gamma_2(y)$ of x and z cannot be contained on the unique line K through x meeting $\Gamma_1(u)$. So, there are t choices for $w \in \Gamma_2(y)$, one for each of the t lines through x distinct from K. For given w, the point z must lie on one of the $t-1$ lines through w distinct from xw and the unique line through w meeting $\Gamma_1(y)$. As $\mathrm{d}(w,u) = 3$, each of these $t-1$ lines contains a unique point z at distance 2 from u and this point z belongs to $\Gamma_2(x) \cap \Gamma_2(u) \cap \Gamma_3(y)$, i.e. belongs to Z_1'. The point v is then the unique neighbour of u and z. This case thus distributes $(t+1)t(t-1)$ to the total number of quadruples.

So, the total number of quadruples is also equal to $(t+1)s(t-1) + (t+1)t(t-1) = (t+1)(t-1)(s+t)$. ☐

Lemma 16. *We have*

$$\sum_{z \in Z_1''} |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| = (t+1)t(s-1)(t-1).$$

Proof. We know that $\sum_{z \in Z_1} |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| = \sum_{z \in Z_1'} |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| + \sum_{z \in Z_1''} |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$. The claim then follows from Lemmas 9 and 15. ☐

Put

$$\gamma_1 := \frac{t+s^2+s}{s}, \qquad \gamma_2 := \frac{t-1}{s-1}.$$

By Lemmas 13, 14 and 15, γ_1 is the average value of the numbers

$$s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z')| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z')|, \ z' \in Z_1',$$

and γ_2 is the average value of the numbers

$$s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z'')| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z'')|, \ z'' \in Z_1''.$$

Putting

$$\begin{aligned}
N'(x,y,z') &:= s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z')| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z')| - \gamma_1 \\
&= |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z')| - \frac{t+s}{s}, \\
N'(x,y,z'') &:= s \cdot |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_1(z'')| + |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z'')| - \gamma_2 \\
&= |\Gamma_2(x) \cap \Gamma_1(y) \cap \Gamma_2(z'')| - \frac{t-1}{s-1}
\end{aligned}$$

for all $z' \in Z_1'$ and $z'' \in Z_1''$, we see that $\sum_{z \in Z_1'} N'(x,y,z') = 0$, $\sum_{z \in Z_1''} N'(x,y,z'') = 0$ and

$$\begin{aligned} N(x,y,z') &= N'(x,y,z') + \gamma_1 - (s+1)s - 1 = N'(x,y,z') - \frac{s^3-t}{s}, \\ N(x,y,z'') &= N'(x,y,z'') + \gamma_2 - (s+1)s - 1 = N'(x,y,z'') - \frac{s^3-t}{s-1} \end{aligned}$$

for all $z' \in Z_1$ and $z'' \in Z_1''$. The latter equations in combination with Lemma 13 imply that

$$\begin{aligned} \sum_{z \in Z_1'} N(x,y,z)^2 &= \sum_{z \in Z_1'} N'(x,y,z)^2 + \frac{(s^3-t)^2(t+1)(t-1)}{s}, \\ \sum_{z \in Z_1''} N(x,y,z)^2 &= \sum_{z \in Z_1''} N'(x,y,z)^2 + (s^3-t)^2(t+1)t. \end{aligned}$$

In a similar way, one proves that

$$\begin{aligned} \sum_{z' \in Z_2'} N(x,y,z')^2 &= \sum_{z \in Z_2'} N'(x,y,z')^2 + \frac{(s^3-t)^2(t+1)(t-1)}{s}, \\ \sum_{z'' \in Z_2''} N(x,y,z'')^2 &= \sum_{z \in Z_2''} N'(x,y,z'')^2 + (s^3-t)^2(t+1)t \end{aligned}$$

if we define

$$\begin{aligned} N'(x,y,z') &:= s \cdot |\Gamma_1(y) \cap \Gamma_2(x) \cap \Gamma_1(z')| + |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z')| - \gamma_1, \\ N'(x,y,z'') &:= s \cdot |\Gamma_1(y) \cap \Gamma_2(x) \cap \Gamma_1(z'')| + |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z'')| - \gamma_2 \end{aligned}$$

for all $z' \in Z_2'$ and $z'' \in Z_2''$. Note that γ_1 and γ_2 are again the average values of respectively $s \cdot |\Gamma_1(y) \cap \Gamma_2(x) \cap \Gamma_1(z')| + |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z')|$, $z' \in Z_1'$, and $s \cdot |\Gamma_1(y) \cap \Gamma_2(x) \cap \Gamma_1(z'')| + |\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z'')|$, $z'' \in Z_1''$. If we also define $N'(x,y,z) := N(x,y,z)$ for every $z \in \Gamma_3(x) \cap \Gamma_3(y)$, then after invoking Proposition 7, we find that $\sum_{z \in Z} N'(x,y,z)^2$ is equal to

$$\sum_{x \in Z} N(x,y,z)^2 - \frac{2(s^3-t)^2(t+1)(st+t-1)}{s} = 2(s^3-t)\Big(\Omega - \frac{(s^3-t)(t+1)(st+t-1)}{s}\Big).$$

As $\Omega = (s^3-t)(t^2-s) + (s^2-1)(s^2+st+t^2)$, we find

$$\sum_{z \in Z} N'(x,y,z)^2 = \frac{2(s^3-t)(t^2-1)t}{s}. \tag{5}$$

So, if $s > 1$, then $t \leqslant s^3$. This is precisely the Haemers-Roos inequality for generalized hexagons of order (s,t). Besides the property $(P_3)_{xy}$ defined in Section 3, we now consider two additional properties for two opposite points x and y in a generalized hexagon:

$(P_1)_{xy}$ All triads of type I containing $\{x,y\}$ have a constant number of centers.

$(P_2)_{xy}$ All triads of type II containing $\{x,y\}$ have a constant number of centers.

Proposition 17. *Suppose x and y are two opposite points in a generalized hexagon of order (s,t). Then properties $(P_1)_{xy}$, $(P_2)_{xy}$ and $(P_3)_{xy}$ hold in $\mathcal{S}$ if and only if $s=1$, $t=1$ or $t=s^3$.*

Proof. For $s,t>1$, this follows from equation (5).

If $s=1$, then we know from the discussion following Proposition 10 that all triads have precisely $t+1$ centers and that $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)| = |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)| = t+1$ for any three mutually opposite points x, y and z. Note also that in this case, there are no triads of type II.

If $t=1$, then we know from Lemmas 11 and 12 that no triad can have a center, and that $|\Gamma_1(x)\cap\Gamma_2(y)\cap\Gamma_2(z)| = |\Gamma_2(x)\cap\Gamma_1(y)\cap\Gamma_2(z)| \in \{0,1,2\}$ equals the number of lines at distance 1 from x, y and z. Note also that in this case, there are no triads of type I. $\square$

Theorem 3 is a consequence of Proposition 17. For $t=s^3>1$, we know from the discussion above that each triad of type I has $\gamma_1 - s = \frac{t+s}{s} = s^2+1$ centers and that each triad of type II has $\gamma_2 = \frac{t-1}{s-1} = s^2+s+1$ centers, hereby confirming the already mentioned results of [7, 8].

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