Weighted Modulo Orientations of Graphs and Signed Graphs

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Abstract

Given a graph G and an odd prime p, for a mapping $f : E(G) \to \mathbb{Z}_p \setminus \{0\}$ and a $\mathbb{Z}_p$-boundary b of G, an orientation D is called an $(f, b; p)$-orientation if the net out f-flow is the same as $b(v)$ in $\mathbb{Z}_p$ at each vertex $v \in V(G)$ under orientation D. This concept was introduced by Esperet et al. (2018), generalizing mod p-orientations and closely related to Tutte's nowhere zero 3-flow conjecture. They proved that $(6p^2 - 14p + 8)$-edge-connected graphs have all possible $(f, b; p)$-orientations. In this paper, the framework of such orientations is extended to signed graph through additive bases. We also study the $(f, b; p)$-orientation problem for some (signed) graphs families including complete graphs, chordal graphs, series-parallel graphs and bipartite graphs, indicating that much lower edge-connectivity bound still guarantees the existence of such orientations for those graph families.

Mathematics Subject Classifications: 05C21, 05C22

1 Introduction

In this paper, our terms and notation follow [2], and graphs considered are loopless and finite with possible parallel edges. As in [2], $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the matching

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number, the connectivity and the edge-connectivity of a graph G, respectively. For $v \in V(G)$, let $N_G(v)$ be the vertices adjacent to v in G. For vertex subsets $S, T \subseteq V(G)$, define $[S,T]_G = \{st \in E(G) | s \in S, t \in T\}$, and we also use $\partial_G(S) = [S, V(G) - S]_G$ for convenience. We often omit subscript whenever no confusion occurs. As in [2], (s,t) in a digraph D is an arc directed from s to t, and we denote

$$E_D^-(s) = \{(t,s) \in A(D) : t \in V(D)\} \text{ and } E_D^+(s) = \{(s,t) \in A(D) : t \in V(D)\}.$$

Let $\mathbb{Z}_k$ denote the (additive) cyclic group of order $k > 1$ with additive identity 0, and let $\mathbb{Z}_k^* = \mathbb{Z}_k \setminus \{0\}$. A **$\mathbb{Z}_k$-boundary** of a graph G is a mapping $b : V(G) \to \mathbb{Z}_k$ satisfying $\sum_{s \in V(G)} b(s) \equiv 0 \pmod{k}$. The collection of all $\mathbb{Z}_k$-boundaries of G is denoted by $Z(G, \mathbb{Z}_k)$. For $A \subseteq \mathbb{Z}_k$, we define $F(G, A) = \{f : E(G) \to A\}$. Fix an orientation $\tau = \tau(G)$ for a graph G. For any $f \in F(G, \mathbb{Z}_k)$, define $\partial_\tau(f) : V(G) \to \mathbb{Z}_k$ as, for any vertex $s \in V(G)$,

$$\partial_\tau(f)(s) = \sum_{e \in E_\tau^+(s)} f(e) - \sum_{e \in E_\tau^-(s)} f(e).$$

For convenience, we sometimes omit the subscript τ in the notation above and write ∂f for $\partial_\tau(f)$. A mapping $f \in F(G, \mathbb{Z}_k)$ if a **$\mathbb{Z}_k$-flow** if $\partial f = 0$. It is known that ∂f is always a $\mathbb{Z}_k$-boundary for any $f \in F(G, \mathbb{Z}_k)$. Jaeger et al. [9] defined group connectivity as follows. A graph G is $\mathbb{Z}_k$-connected if for any $b \in Z(G, \mathbb{Z}_k)$, there exist a mapping $f \in F(G, \mathbb{Z}_k^*)$ and an orientation $\tau(G)$ such that $\partial_\tau f = b$ in $\mathbb{Z}_k$. The following conjecture is proposed in [9] and remains unsolved as of today.

Conjecture 1. (i) If a graph G satisfies $\kappa'(G) \geqslant 3$, then G is $\mathbb{Z}_5$-connected.
(ii) If a graph G satisfies $\kappa'(G) \geqslant 5$, then G is $\mathbb{Z}_3$-connected.

Given a $\mathbb{Z}_k$-boundary b of a graph G, an orientation $\tau = \tau(G)$ is a **b-orientation** of G if for the constant mapping $f = 1$, we have $\partial f \equiv b \pmod{k}$. In particular, when $b = 0$, any b-orientation is a **mod k-orientation** of G. The studies of group connectivity and modulo orientation of graphs are motivated by the most fascinating nowhere zero flow conjectures of Tutte, as shown in the surveys [8, 15], among others. Some of the recent breakthroughs are the following.

Theorem 2. (Lovász et al. [20]) *Every $6k$-edge-connected graph G admits a b-orientation for any $\mathbb{Z}_{2k+1}$-boundary b of G.*

Theorem 3. (Han et al. [7] and Li [16])
(i) If $k \geqslant 3$, then there exist $4k$-edge-connected graphs admitting no mod $(2k+1)$-orientation.
(i) If $k \geqslant 5$, then there exist $(4k+1)$-edge-connected graphs admitting no mod $(2k+1)$-orientation.

In particular, Theorem 3 disproved the Circular Flow Conjecture, in which Jaeger [8] conjectured that all $4k$-edge-connected graphs admit mod $(2k+1)$-orientations. Further expository of the problem can be found in the informative monograph by Zhang [21].

Aiming at extending Theorem 2, Esperet et al. in [5] defined a mod k **f-weighted b-orientation** of a graph G, for given $b \in Z(G, \mathbb{Z}_k)$ and mapping $f \in F(G, \mathbb{Z}_k)$, to be an orientation $\tau = \tau(G)$ satisfying $\partial_\tau(f) \equiv b \pmod{k}$. Throughout the rest of this paper, we shall abbreviate a mod k f-weighted b-orientation as an **$(f, b; k)$-orientation**. Esperet et al indicated in [5] that to investigate $(f, b; k)$-orientation of graphs, it is necessary to assume that k is an odd prime number, and they proved the following.

Theorem 4. (Esperet, De Verclos, Le and Thomassé, [5]) *Given an odd prime p, if G is a $(6p^2 - 14p + 8)$-edge-connected graph, then for any $b \in Z(G, \mathbb{Z}_p)$ and any mapping $f \in F(G, \mathbb{Z}_p^*)$, G admits an $(f, b; p)$-orientation.*

The current study is motivated by Theorems 2, 3 and 4. We are to investigate the relationship between the edge-connectivity of graphs in certain graph families and the $(f, b; p)$-orientability of these graphs over the finite field $\mathbb{Z}_p$. In Section 2, we prepare some of the tools for our arguments in the proofs. We then will show improved edge-connectivity bounds in certain graph families in Sections 3-4. In Section 5, we generalize the framework to the study of signed graph, in which we introduce the $(f, b; p)$-orientation of signed graphs and show that every $(12p^2 - 28p + 15)$-edge-connected signed graph admits an $(f, b; p)$-orientation. Further discussions and conjectures are presented in the last section.

2 Preliminaries

Let $\mathbb{F}$ denote a finite field and let $p \geqslant 3$ be a prime number throughout the rest of this paper. It has been noted that the concept of modulo orientation is closely related to additive bases over finite fields. Given a subset $S \subseteq \mathbb{F}$, an **S-additive basis** of $\mathbb{F}^n$ is a multiset $\{x_1, x_2, \cdots, x_m\}$ of the n-dimensional vectors such that for every $x \in \mathbb{F}^n$, there are scalars $c_i \in S$ such that $x = \sum_{i=1}^m c_i x_i$, which is called an S-linear-combination of x. An **additive basis** of $\mathbb{F}^n$ is a $\{0, 1\}$-additive basis.

Let $B_1, \ldots, B_t$ be a collection of bases of $\mathbb{F}^n$. Define $\uplus_{i=1}^t B_i$ to be the (multiset) union with possible repetitions of $B_1, \ldots, B_t$. Let $c(n, \mathbb{F})$ be the smallest positive integer m such that for any m bases $B_1, \ldots, B_m$ of $\mathbb{F}^n$, the multiset $\uplus_{i=1}^m B_i$ forms an additive basis of $\mathbb{F}^n$. Define $c(n, p) = c(n, \mathbb{Z}_p)$. Alon, Linial and Meshulam [1] obtained a theorem below, indicating the existence of $c(n, p)$, where the logarithm function is of base 2.

Theorem 5. (Alon et al. [1]) $c(n, p) \leqslant (p - 1)\log n + p - 2$.

Lemma 6. (Lemma 9 of Esperet et al.[5]) *Let $k \geqslant 1$ be an integer and $p = 2k + 1$ be a prime. Let $\tau(G) = D = (V, A)$ be a digraph obtained from the orientation τ of a graph G. A 2-list L is to assign two distinct elements of $\mathbb{Z}_{2k+1}$ to $L(e)$ for each arc $e \in A(D)$. The following are equivalent.*

(i) For any $\mathbb{Z}_{2k+1}$-boundary b and any mapping $f : E \to \mathbb{Z}_{2k+1} - \{0\}$, the undirected graph G has an $(f, b; p)$-orientation.

(ii) For any 2-list L and any $\mathbb{Z}_{2k+1}$-boundary b, D has a $\mathbb{Z}_{2k+1}$-flow g satisfying $\partial g = b$ and $g(e) \in L(e)$, for any $e \in A(D)$.

Let mG denote the graph formed by replacing every edge of G with m parallel edges. For an odd prime p, let $\mathcal{O}_p$ be the family of graphs such that a graph $G \in \mathcal{O}_p$ if and only if it admits an $(f, b; p)$-orientation for any $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b. The lemma below summarizes some basic properties of the graphs admitting $(f, b; p)$-orientations. The proofs are slight modifications of those in [12, 14] justifying the corresponding results for modulo orientations and strong group connectivity of graphs.

Lemma 7. ([18, 19]) *The following properties of $\mathcal{O}_p$ hold:*
(i) $K_1 \in \mathcal{O}_p$.
(ii) If $G \in \mathcal{O}_p$, then $G/e \in \mathcal{O}_p$ for any $e \in E(G)$.
(iii) For $H \subseteq G$, if $G/H \in \mathcal{O}_p$ and $H \in \mathcal{O}_p$, then $G \in \mathcal{O}_p$.
(iv) $G \in \mathcal{O}_p$ if and only if every block of G is in $\mathcal{O}_p$.
(v) Every graph in $\mathcal{O}_p$ contains $(p-1)$ edge-disjoint spanning trees.
(vi) $mK_2 \in \mathcal{O}_p$ if and only if $m \geqslant p-1$.

Assume that D is an $(f, b; p)$-orientation of a graph G for some given $f \in F(G, \mathbb{Z}_p^*)$ and $b \in Z(G, \mathbb{Z}_p)$. Let $e_0 = st \in E(G)$ such that $(s, t) \in A(D)$, and $f' \in F(G, \mathbb{Z}_p^*)$ be a mapping satisfying $f'(e_0) = -f(e_0)$ and $f'(e) = f(e)$ in $\mathbb{Z}_p$ whenever $e \neq e_0$. Define D' to be the orientation of G by reversing the orientation of e_0 from (s, t) to (t, s). Then by definition, D' is an $(f', b; p)$-orientation of G. This leads to the following observation.

Observation 8. If for any $b \in Z(G, \mathbb{Z}_p)$ and any $f : E(G) \to \{1, 2, \ldots, \frac{p-1}{2}\}$, G always admits an $(f, b; p)$-orientation, then $G \in \mathcal{O}_p$.

Definition 9. For $H \subseteq G$, the $\boldsymbol{\mathcal{O}_p}$**-closure** of H in G, denoted by $cl_G(H)$, is the maximal subgraph of G that contains H such that $V(cl_G(H)) - V(H)$ can be ordered as a sequence $\{v_1, v_2, \cdots, v_t\}$ such that there are at least $p-1$ edges joining v_1 and vertices in H, and for each i with $1 \leqslant i \leqslant t-1$, there are at least $p-1$ edges joining v_{i+1} and $V(H) \cup \{v_1, v_2, \cdots, v_i\}$.

As a corollary of Lemma 7(iii) and (vi), we have the following.

$$\text{If } H \in \mathcal{O}_p, \text{ then } cl_G(H) \in \mathcal{O}_p. \tag{1}$$

Lemma 10. *Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a subgraph $H_e \in \mathcal{O}_p$ with $e \in E(H_e)$, then $G \in \mathcal{O}_p$.*

Proof. We prove by induction on $|V(G)|$. Since $K_1 \in \mathcal{O}_p$, the lemma is true when $|V(G)| = 1$. Assume $|V(G)| > 1$ and pick an arbitrary edge $e_1 \in E(T)$. Then G has a subgraph $H_1 \in \mathcal{O}_p$ such that $e_1 \in E(H_1)$. Denote $G_1 = G/H_1$ and define $T_1 = T/(E(H_1) \cap E(T))$. Clearly, T_1 is a connected spanning subgraph of G_1 as it is obtained by contracting a connected graph T. Moreover, every edge e in $E(T_1)$ is also an edge in $E(T)$. From the assumption, G contains a subgraph $H_e \in \mathcal{O}_p$ with $e \in E(H_e)$. It follows by Lemma 7(ii) that $\Gamma_e = H_e/(E(H_e) \cap E(H_1)) \in \mathcal{O}_p$ and $e \in \Gamma_e \subseteq G_1$. Therefore by induction $G_1 \in \mathcal{O}_p$. As $H_1 \in \mathcal{O}_p$ and $G_1 = G/H_1 \in \mathcal{O}_p$, it follows by Lemma 7(iii) that $G \in \mathcal{O}_p$ as well. □

3 Weighted Modulo Orientations of Certain Graphs

In this section, we first investigate the edge connectivity of complete graphs in $\mathcal{O}_p$ and then apply it to study chordal graphs. We also determine, in Section 3.3, a sharp edge connectivity bound for series-parallel graphs to be in $\mathcal{O}_p$.

3.1 Complete Graphs

The main result of this subsection is the following theorem.

Theorem 11. *If $n \geqslant 2(p-1)(5+3\log(p-1))-1$, then the complete graph K_n belongs to $\mathcal{O}_p$.*

To justify Theorem 11, we start with a lemma.

Lemma 12. *Let G be a graph of order n with $c(n-1,p)$ edge-disjoint spanning trees. Then $G \in \mathcal{O}_p$.*

Proof. Let $T_1, \ldots, T_{c(n-1,p)}$ be edge-disjoint spanning trees of G, and $H = G[\cup_{i=1}^{c(n-1,p)} E(T_i)]$ be the subgraph induced by the edge subset $\cup_{i=1}^{c(n-1,p)} E(T_i)$. As T_i's are spanning trees of G, H is a spanning subgraph of G. We shall first show that $H \in \mathcal{O}_p$ using Lemma 6, that is, for any 2-list L and any $\mathbb{Z}_p$-boundary b, we shall show that H has a $\mathbb{Z}_p$-flow g satisfying $\partial g = b$ and $g(e) \in L(e)$ for each $e \in E(H)$.

For any $\mathbb{Z}_p$-boundary b, $b(v_n) = -(b(v_1) + \cdots + b(v_{n-1}))$ and so one can view b as a vector $(b(v_1), \ldots, b(v_{n-1}))$ in $\mathbb{Z}_p^{n-1}$. Choose $T \in \{T_1, T_2, \ldots, T_{c(n-1,p)}\}$ and assign to H an arbitrary orientation $D = D(H)$. Thus every subgraph of H is a subdigraph of D under this given orientation, and each $e \in E(H)$ is now an arc in $A(D)$. Since $|V(H)| = n$, we denote $A(T) = \{e_1, \ldots, e_{n-1}\}$. For each $e \in A(T)$, set $L(e) = \{a_e, b_e\}$ for two distinct elements $a_e, b_e \in \mathbb{Z}_p$.

Define a mapping $f_0 : E(H) \to \mathbb{Z}_p$ by $f_0(e) = a_e$ for any $e \in E(T)$, and $f_0(e') = 0$ if $e' \notin E(T)$. Let $b_0(v) = \partial f_0(v)$ and $b'(v) = b(v) - b_0(v)$, for any $v \in V(G)$. As b and b_0 are $\mathbb{Z}_p$-boundaries, b' is also a $\mathbb{Z}_p$-boundary of G. For any $e = (v_i, v_j) \in A(T)$, set $L'(e) = \{0, b_e - a_e\}$ and define $x_e = (x_1^e, x_2^e, \ldots, x_n^e)$ with

$$x_t^e = \begin{cases} b_e - a_e & \text{if } t = i, \\ a_e - b_e & \text{if } t = j, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of x_e, one can see that x_e is a $\mathbb{Z}_p$-boundary and so x_e can be viewed as a vector in $\mathbb{Z}_p^{n-1}$. As T is a spanning tree and $|E(T)| = n-1$, $B(T) = \{x_e : e \in A(T)\}$ is a base of $\mathbb{Z}_p^{n-1}$. For each i with $1 \leqslant i \leqslant n$, let $B_i = B(T_i)$. Then by the definition of $c(n-1,p)$, the union $B_1 \cup \cdots \cup B_{c(n-1,p)}$ forms an additive basis of $\mathbb{Z}_p^{n-1}$. Hence there exist scalars $\lambda_e \in \{0,1\}$, where $e \in E(T_1 \cup \cdots \cup T_{c(n-1,p)})$, such that $\sum \lambda_e x_e = b - b_0$. Define $g_0 : E(H) \to \mathbb{Z}_p$ by

$$g_0(e) = \begin{cases} 0 & \text{if } \lambda_e = 0, \\ b_e - a_e & \text{if } \lambda_e = 1. \end{cases}$$

Next we show that $\partial g_0 = \sum \lambda_e x_e$. For any $v_i \in V(G)$,

$$\begin{aligned}\partial g_0(v_i) &= \sum_{e\in E^+(v_i)} g_0(e) - \sum_{e\in E^-(v_i)} g_0(e)\\ &= \sum_{e\in E^+(v_i)} \lambda_e(b_e-a_e) - \sum_{e\in E^-(v_i)} \lambda_e(b_e-a_e)\\ &= \sum_{e\in E^+(v_i)} \lambda_e(b_e-a_e) + \sum_{e\in E^-(v_i)} \lambda_e(a_e-b_e).\end{aligned}$$

This shows that $\partial g_0(v_i)$ is the i-th entry of $\sum \lambda_e x_e$. By the arbitrary of v_i, one has $\partial g_0 = \sum \lambda_e x_e = b - b_0$. Define $g(e) = g_0(e) + f_0(e)$, for any $e \in E(H)$. So $\partial g = \partial g_0 + \partial f_0 = b - b_0 + b_0 = b$. Since $g(e) = g_0(e) + f_0(e) = g_0(e) + a_e \in \{a_e, b_e\}$ for each $e \in E(T_1 \cup \cdots \cup T_{c(n-1,p)})$, by Lemma 6 (ii), H has an $(f, b; p)$-orientation. As f and b are arbitrarily given, $H \in \mathcal{O}_p$. Since H is spanning in G, it follows by Lemma 7 (i) and (iii) that $G \in \mathcal{O}_p$. □

Proof of Theorem 11. When $p = 3$, a graph $G \in \mathcal{O}_p$ which is equivalent to G is $\mathbb{Z}_3$-connected. It is known that K_n is $\mathbb{Z}_3$-connected if $n \geqslant 5$ (see Proposition 3.6 of [11]), and so theorem holds for $p = 3$. In the following we assume $p \geqslant 5$.

Let $\phi(p) = 2 + 2\log(p-1) - \sqrt{2\log(2p-2)}$. Then as $\phi(2) = 2 - \sqrt{2} > 0$ and when $p \geqslant 5$, the derivative of ϕ at p is greater than 0, that is $\phi'(p) > 0$, it follows that $2+2\log(p-1) \geqslant \sqrt{2\log(2p-2)}$, and so algebraic manipulation leads to $5+3\log(p-1) \geqslant \log(p-1) + \sqrt{2\log(2(p-1))} + 3 = \log(2(p-1)) + \sqrt{2\log(2(p-1))} + 2$. Consequently,

$$\begin{aligned} n-1 &\geqslant 2(p-1)(5+3\log(p-1)) \\ &\geqslant 2(p-1)(\log(2(p-1)) + \sqrt{2\log(2(p-1))} + 1) + 2(p-1). \end{aligned} \tag{2}$$

Set

$$x = \frac{(n-1)-2(p-1)}{2(p-1)}, \text{ and } y = x - \log(2(p-1)).$$

By (2),

$$\begin{aligned} &x = \frac{(n-1)-2(p-1)}{2(p-1)} \geqslant \log(2(p-1)) + \sqrt{2\log(2(p-1))} + 1, \text{ and} \\ &y \geqslant \sqrt{2\log(2(p-1))} + 1. \end{aligned} \tag{3}$$

By (3), $(y-1)^2 \geqslant 2\log(2(p-1))$, and so $1 + y + \frac{1}{2}(y-1)^2 \geqslant \log(2(p-1)) + y + 1$. Let $\psi(y) = 2^y - \left(1 + y + \frac{1}{2}(y-1)^2\right)$. When $y \geqslant 3$, we have $\psi(3) = 2 > 0$ and $\psi'(y) = 2^y \ln(2) - y > 0$. It follows that as long as $y \geqslant 3$, $2^y \geqslant 1 + y + \frac{1}{2}(y-1)^2$. Since $p \geqslant 5$, it follows by (3) that $y \geqslant \sqrt{2\log(2(p-1))} + 1 \geqslant \sqrt{6} + 1 > 3$, and so we substitute $y - 1$ by $\sqrt{2\log(2(p-1))}$ in the inequality $2^y \geqslant 1 + y + \frac{1}{2}(y-1)^2$ to obtain $2^y \geqslant \log(2(p-1)) + y + 1$. Hence $y \geqslant \log(\log(2(p-1)) + y + 1)$, and so, as $x = \log(2(p-1)) + y$, $y \geqslant \log(\log(2(p-1)) + y + 1) = \log(1+x)$. This implies that $x = \log(2(p-1)) + y \geqslant \log(2(p-1)) + \log(1+x) = \log(2(p-1)(1+x)) = \log(2(p-1) + 2(p-1)x)$. Since $(n-1) - 2(p-1) = 2(p-1)x$, one has $x \geqslant \log(n-1)$. So $n-1 = 2(p-1)x + 2(p-1) \geqslant 2(p-1)\log(n-1) + 2(p-1) \geqslant 2(p-1)\log(n-1) + 2(p-2)$. By Theorem 5, $\frac{n-1}{2} \geqslant (p-1)\log(n-1) + (p-2) \geqslant c(n-1,p)$. As K_n has $\frac{n}{2}$ edge-disjoint spanning trees, by Lemma 10, we conclude that if $n-1 \geqslant 2(p-1)(5+3\log(p-1))$, then $K_n \in \mathcal{O}_p$. □

3.2 Chordal Graphs

A simple graph G is **chordal** if every cycle of length greater than 3 possesses a chord. Equivalently speaking, a simple graph G is chordal if every induced cycle of G has length 3. We need the following structure property of chordal graphs.

Lemma 13. (Lemma 2.1.2 of [10]) *A simple graph G is chordal if and only if every minimal vertex-cut induces a clique of G.*

The rest of this subsection is to show the following theorem.

Theorem 14. *Every simple chordal graph G with $\kappa(G) \geqslant 2(p-1)(5+3\log(p-1)) - 1$ is in $\mathcal{O}_p$.*

Proof. Let G be a chordal graph with $\kappa(G) \geqslant 2(p-1)(5+3\log(p-1)) - 1$. If G is a complete graph, say $G \cong K_n$, then $n \geqslant \kappa(G) + 1 \geqslant 2(p-1)(5+3\log(p-1))$ and $G \in \mathcal{O}_p$ by Theorem 11. Thus we assume G is not a clique.

Let $e = xy \in E(G)$ be an arbitrary edge. By Lemma 10, it suffices to prove that e lies in a subgraph H_e of G with $H_e \in \mathcal{O}_p$. We shall show that in any case, a subgraph $H_e \in \mathcal{O}_p$ with $e \in E(H_e)$ can always be found.

In the first case, we assume that either $N_G(x) \neq V(G) \setminus \{x\}$ or $N_G(y) \neq V(G) \setminus \{y\}$. Then by symmetry, we assume $N_G(x) \neq V(G) \setminus \{x\}$. So there exists a vertex $z \in V(G) - (N_G(x) \cup \{x\})$. Since $\kappa(G) \geqslant k \geqslant 2$ and G is not a clique, $N_G(x)$ contains a minimal vertex-cut X of G separating x and z. By Lemma 13, $G[X]$ is a clique, and so $G[X \cup \{x\}] \cong K_{m_x}$ with $m_x = |X| + 1 \geqslant \kappa(G) + 1 \geqslant 2(p-1)(5+3\log(p-1))$. By Lemma 11, $G[X \cup \{x\}] \in \mathcal{O}_p$. If $y \in X$, then as $G[X \cup \{x\}] \in \mathcal{O}_p$, we are done with $H_e = G[X \cup \{x\}]$. Hence we assume that

$$\begin{aligned}&\text{for any minimal vertex cut } X \subseteq N_G(x) \text{ separating } x \text{ from}\\ &V(G) \setminus \{N_G(x) \cup \{x\}\},\ y \notin X.\end{aligned} \tag{4}$$

If there exists $t \in N_G(y) \cap (V(G) \setminus (N_G(x) \cup \{x\}))$, then there is a minimal vertex cut of $N_G(x)$ containing y which separates x and t, contrary to (4). It follows that $N_G(y) \subseteq N_G(x) \cup \{x\}$. Since $z \in V(G) \setminus (N_G(x) \cup \{x\})$, we have $yz \notin E(G)$, and so $N_G(y)$ contains a minimal vertex cut separating y and z.

Let Y be an arbitrarily chosen minimal vertex cut in $N_G(y)$ separating y and z. By Lemma 13 and as $\kappa(G) \geqslant 2(p-1)(5+3\log(p-1)) - 1$, $G[Y \cup y] \cong K_{m_y}$ with

$m_y = |Y|+1 \geqslant \kappa(G)+1 \geqslant 2(p-1)(5+3\log(p-1))$. By Lemma 11, $G[Y \cup \{y\}] \in \mathcal{O}_p$. We may further assume that $x \notin Y$, as otherwise we are done with $H_e = G[Y \cup \{y\}] \in \mathcal{O}_p$. Thus $xy \in E(G-Y)$ and so x and y are in the same component of $G-Y$. It follows that $H_e = G[Y \cup \{x, y\}]$ is a complete graph with order $|Y| + 2 \geqslant \kappa(G) + 2 \geqslant 2(p-1)(5 + 3\log(p-1)) + 1$. By Lemma 11, $H_e \in \mathcal{O}_p$, and so this justifies the first case.

Otherwise, we may assume that both $N_G(x) = V(G) \setminus \{x\}$ and $N_G(y) = V(G) \setminus \{y\}$. Since G itself is not a complete graph, G contains vertices $v, v' \in V(G) - \{x, y\}$ such that $vv' \notin E(G)$. Therefore, $N(v)$ contains a minimal vertex cut X' separating v and v' in G. By Lemma 13 and as $\kappa(G) \geqslant 2(p-1)(5+3\log(p-1)) - 1$, $G[X' \cup \{v\}]$ is a complete graph of order at least $2(p-1)(5+3\log(p-1))$, and so by Lemma 11, it is in $\mathcal{O}_p$. Let $H_e = G[X' \cup \{v\}]$. Since $N_G(x) = V(G) \setminus \{x\}$ and $N_G(y) = V(G) \setminus \{y\}$, both x and y must be in X', and so $e = xy \in E(H_e)$. This completes the proof of the lemma. □

3.3 Series-parallel graphs

For a graph G, if K_4 can not be obtained from G by contraction, then G is called K_4-minor free. In this section, we will present a sharp lower bound of edge-connectivity for a K_4-minor free graph to be in $\mathcal{O}_p$. The following is a theorem of Dirac [4].

Theorem 15. (Dirac [4]) *If G is a simple K_4-minor free graph, then $\delta(G) \leqslant 2$.*

Corollary 16. *Let G be a K_4-minor free graph. If $\kappa'(G) \geqslant 2p-3$, then $G \in \mathcal{O}_p$.*

Proof. Let G be a $(2p-3)$-edge-connected K_4-minor free graph, and let G_0 be the underlying simple graph of G (see p. 47 of [2]). By Lemma 7(i), $K_1 \in \mathcal{O}_p$. Hence we assume that $|V(G)| > 1$ and let G be a minimal counterexample with $|V(G)|$ minimized.

Since G is K_4-minor free, we have G_0 is also K_4-minor free. By Theorem 15, there is a vertex $w \in V(G_0)$ with degree 1 or 2. If $d_{G_0}(w) = 1$, since $\kappa'(G) \geqslant 2p-3$, we have a subgraph $H \subseteq G$ such that $H \cong (2p-3)K_2$. If $d_{G_0}(w) = 2$, let e_1 and e_2 be two edges incident with w in G_0. By $\kappa'(G) \geqslant 2p-3$, at least one of e_1 and e_2 must be contained in a subgraph $H \subseteq G$ with $H \cong (p-1)K_2$. In either case, by Lemma 7(vi), $H \in \mathcal{O}_p$. Since G is K_4-minor free, we have G/H is also K_4-minor free. By the property of contractions, we have $\kappa'(G/H) \geqslant \kappa'(G)$. By the minimality of G, we obtain $G/H \in \mathcal{O}_p$. Since $H \in \mathcal{O}_p$ and by Lemma 7(iii), $G \in \mathcal{O}_p$, and so the corollary is complete. □

4 Complete Bipartite Graphs and Graphs with Small Matching Number

In this section we will determine sufficient conditions for a complete bipartite graph to be in $\mathcal{O}_p$. From definition, a graph G is $\mathbb{Z}_3$-connected if and only if it is in $\mathcal{O}_3$. As Theorem 4.6 of [3] characterizes all complete bipartite graphs in $\mathcal{O}_3$, we shall, throughout this section, assume that $p \geqslant 5$ is an odd prime. Using the arguments similar to those justifying Theorem 3.2 of [13], the following lifting lemma can be routinely verified from the definition of graphs in $\mathcal{O}_p$.

Lemma 17 (Lifting). *Let G be a graph and $p > 0$ be an odd prime. For every function $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b of G, let v_1v_2, v_1v_3 be two edges of G with $f(v_1v_2) = f(v_1v_3)$. Let $G_{[v_1,v_2v_3]}$ be the graph obtained from G by deleting v_1v_2, v_1v_3 and adding a new edge $e = v_2v_3$, and $f' \in F(G_{[v_1,v_2v_3]}, \mathbb{Z}_p^*)$ be formed from the restriction of f to $E - \{v_1v_2, v_1v_3\}$ by defining $f'(v_2v_3) = f(v_1v_2)$. If $G_{[v_1,v_2v_3]}$ has an $(f', b; p)$-orientation, then G has an $(f, b; p)$-orientation.*

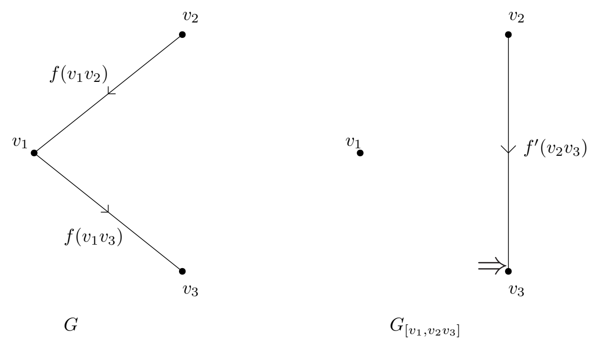

Figure 1: $G_{[v_1,v_2v_3]}$ is the graph by lifting two edges v_1v_2, v_1v_3.

Proof. Let (v_2, v_3) be an arc in D and let (v_2, v_1) and (v_1, v_3) be two arcs in D. By assumption, $G_{[v_1,v_2v_3]}$ has an $(f', b; p)$-orientation, say D'. Without loss of any generality, assume that the direction of v_2v_3 is (v_2, v_3) as in Figure 1. Then define an orientation D of the graph G as follows: D is the same as D' restricted on $E(G) - \{v_1v_2, v_1v_3\}$ and the directions of $\{v_1v_2, v_1v_3\}$ are (v_2, v_1) and (v_1, v_3), see Figure 1. Since $f'(v_2v_3) = f(v_1v_2) = f(v_1v_3)$, one can verify that D is an $(f, b; p)$-orientation of G. □

Definition 18. Let G be a graph, $f \in F(G, \mathbb{Z}_p^*)$ and b be any given $\mathbb{Z}_p$-boundary of G. Fix two vertices $u_1, u_2 \in V(G)$ such that $N_G(u_1) \cap N_G(u_2)$ contains a subset $W = \{v_1, \ldots, v_{p-1}\} \subseteq N_G(u_1) \cap N_G(u_2)$ satisfying that $f(u_1v_i) = f(u_2v_i)$ for each $i \in \{1, \ldots, p-1\}$. We obtain a new graph $G^L_{u_1,u_2,W}$ from G by lifting each edge pair in $\{u_1v_1, u_2v_1\}, \ldots, \{u_1v_{p-1}, u_2v_{p-1}\}$. For notational convenience, when u_1, u_2 and W are understood from the context, we simply use G^L for $G^L_{u_1,u_2,W}$, and we say that G^L is obtained by performing the **L-operation** on G at $\{u_1, u_2\}$. By definition, G^L contains a subgraph L_{u_1,u_2} with vertex set $\{u_1, u_2\}$ and with at least $(p-1)$ multiple edges between u_1, u_2.

By Lemmas 7(vi), $L_{u_1,u_2} \in \mathcal{O}_p$ and so by Lemma 17,

$$\text{if } G^L/L_{u_1,u_2} \in \mathcal{O}_p, \text{ then } G \in \mathcal{O}_p. \tag{5}$$

If $p = 3$, then an $(f, b; p)$-orientation is equivalent to $\mathbb{Z}_3$-connectivity and $K_{m,n} \in \mathcal{O}_p$ if and only if $m \geqslant n \geqslant 4$ from [3]. In the rest of this section, let $p \geqslant 5$ be a prime and we define

$$n_1 = \frac{1}{2}(p-1)(p-2) + 1, \tag{6}$$

$$n_2 \;=\; \frac{1}{2}n_1(n_1-1)(p-1).$$

Lemma 19. *Let $p > 0$ be an odd prime, $G = K_{n_1,n}$ be a complete bipartite graph with vertex bipartition (U, V), where*

$$U = \{u_1, \ldots, u_{n_1}\} \text{ and } V = \{v_1, \ldots, v_n\}. \tag{7}$$

Let $b \in Z(G, \mathbb{Z}_p)$ and $f \in F(G, \mathbb{Z}_p^*)$ be given such that (by Observation 8),

$$\text{for any } e \in E(G),\ f(e) \in \{1, \ldots, \tfrac{p-1}{2}\}. \tag{8}$$

Let K_{n_1} be the complete graph with $V(K_{n_1}) = U$ and $E(K_{n_1}) = \{e_1, \ldots, e_m\}$, where $m := m(|U|) = \frac{|U|(|U|-1)}{2}$. Define a new bipartite graph $B = B(G)$ with a vertex partition (W_1, W_2), where $W_1 = V$ and $W_2 = E(K_{n_1})$, such that v_j is adjacent to $e_i = u_{i_1}u_{i_2}$ if and only if $f(v_j u_{i_1}) = f(v_j u_{i_2})$. (Thus an element $e_i \in W_2$ represents both an edge in the complete graph K_{n_1} as well as a vertex in $V \subset V(B)$.) If $|U| = n_1 > \frac{p-1}{2}$ and $|V| = n \geqslant m(p-2)+2$, then each of the following holds.
(i) For any $v_j \in V$, $d_B(v_j) \geqslant 1$.
(ii) There exists an $e_i \in W_2$ with $d_B(e_i) \geqslant p-1$.

Proof. For any $v_j \in V$, by (8) and as $|U| = n_1 > \frac{p-1}{2}$, there exist distinct $u_{i_1}, u_{i_2} \in U$ such that $f(v_j u_{i_1}) = f(v_j u_{i_2})$. Hence every vertex v_j is incident with at least one edge $e \in E(G)$, and so $d_B(v_j) \geqslant 1$. Counting the number of edges in B, we have

$$\sum_{v \in W_1} d_B(v) = |E(B)| = \sum_{e \in W_2} d_B(e). \tag{9}$$

As $n > m(p-2)+1$ and by (9), we conclude that there must be an $e_i \in W_2$ with $d_B(e_i) \geqslant p-1$. This justifies Lemma 19. $\square$

The bipartite graph $B = B(G)$ defined in Lemma 19 will be referred as to the associate bipartite graph of G.

Theorem 20. *Suppose n_1, n_2 are integers satisfying (6). Let $G = K_{n_1,n_2}$ and $p \geqslant 5$ be a prime integer. For every function $f \in F(G, \mathbb{Z}_p^*)$ and every $\mathbb{Z}_p$-boundary b of G, G has an $(f, b; p)$-orientation. Consequently, $K_{n_1,n} \in \mathcal{O}_p$ for every $n \geqslant n_2$.*

Proof. Let (U, V) denote the bipartition of G using the notation in (7), and let $b \in Z(G, \mathbb{Z}_p)$ and $f \in F(G, \mathbb{Z}_p^*)$ be given. We shall show that K_{n_1,n_2} has an $(f, b; p)$-orientation. By Observation 8, we may assume that (8) holds. In the arguments below, we let K_{n_1} be the complete graph with $V(K_{n_1}) = U$ and $E(K_{n_1}) = \{e_1, \ldots, e_m\}$, where $m = \frac{n_1(n_1-1)}{2}$, and let B be the associate bipartite graph of G as defined in Lemma 19.

By (6), $|U| = n_1 > \frac{p-1}{2}$, $|V| = n_2 \geqslant m(p-2)+2$, and so Lemma 19 is applicable.

Assume that $e_i = u_{i_1}u_{i_2}$ is the edge assured in Lemma 19(ii), and $N_B(e_i)$ contains $Q_1 = \{v_{j_1}, \ldots, v_{j_{p-1}}\} \subseteq W_1$. By the definition of B,

$$\text{for any } \ell \in \{1, \ldots, p-1\},\ f(u_{i_1}v_{j_\ell}) = f(u_{i_2}v_{j_\ell}). \tag{10}$$

Let $G^L = G^L_{u_{i_1},u_{i_2},Q_1}$ and $L_{u_{i_1},u_{i_2}}$ be the graphs arising in the process of performing L-operations to G, as defined in Definition 18. Define $G_1 = G^L/L_{u_{i_1},u_{i_2}}$ and v_{L_1} be the vertex in G_1 onto which $L_{u_{i_1},u_{i_2}}$ is contracted, and $G'_1 = G_1 - Q_1$. Then G'_1 is again a complete bipartite graph with bipartition (U_1, V_1) where $U_1 = (U - \{u_{i_1}, u_{i_2}\}) \cup \{v_{L_1}\}$ and $V_1 = V - Q_1$. Thus we have

$$|U_1| = n_1 - 1 \text{ and } |V_1| = (m-1)(p-1) \geqslant m_1(p-1) \geqslant m_1(p-2) + 2,$$

where $m_1 := \frac{|U_1|(|U_1|-1)}{2}$.

Assume that for some j with $1 \leqslant j \leqslant \frac{1}{2}(p-1)(p-3)$, the complete bipartite graph $G'_j = (U_j, V_j)$ is defined such that

$$|U_j| = n_1 - j \text{ and } |V_j| = (m-j)(p-1) \geqslant m_j(p-1) \geqslant m_j(p-2) + 2, \tag{11}$$

where $m_j := \frac{|U_j|(|U_j|-1)}{2}$. Define the associate bipartite graph $B(G'_j)$ as defined in Lemma 19. By (6) and $j \leqslant \frac{1}{2}(p-1)(p-3)$, we have $|U_j| = n_1 - j = \frac{1}{2}(p-1)(p-2)+1-j > \frac{1}{2}(p-1)$. Hence by replacing G with G'_j, there exists a vertex $e_j = u_{j_1}u_{j_2} \in E(K_{|V_j|})$ of degree at least $p-1$ in $B(G'_j)$, then a subset $Q_{j+1} \subseteq N_{B(G'_j)}(e_j) \subseteq V_j$ is identified with $|Q_{j+1}| = p-1$. Let $G^L_j = (G'_j)^L{}_{u_{j_1},u_{j_2},Q_{j+1}}$ with $L_{j+1} = L_{u_{j_1},u_{j_2}}$ be the graphs arising in the process of performing L-operations to G'_j. Let $G_{j+1} = (G'_j)^L/L_{j+1}$, and $G'_{j+1} = G_{j+1} - Q_{j+1}$. With the same arguments, G'_{j+1} is also a complete bipartite graph with the bipartition (U_{j+1}, V_{j+1}). As G is finite, this process must end at $j = \ell$ for some integer $\ell > 0$, and so no further L-operations can be performed in the way above on the bipartite graph G'_ℓ. Let (U_ℓ, V_ℓ) be the bipartition of G'_ℓ. It follows $|U_\ell| \leqslant \frac{p-1}{2}$.

By Definition 18, there exists a sequence of ordered pairs

$$(L_1, Q_1), (L_2, Q_2), \ldots, (L_\ell, Q_\ell)$$

arising in the process of the L-operations to obtain G_ℓ, and satisfying both (S1) and (S2) below.

(S1) Let $U_0 = U$. For $i = 1, 2, \ldots, \ell$, each L_i is spanned by a $(p-1)K_2$, with $V(L_i)$ consisting of two vertices in U_{i-1}, formed by, for $i > 1$, identifying the two vertices in $V(L_{i-1})$ in U_{i-2}.

(S2) Let $Q_0 = \emptyset$. For $i = 1, 2, \ldots, \ell$, $|Q_i| = p-1$, $Q_i \subseteq V - (\cup_{j<i} Q_j)$, and no edges joining vertices in Q_i to the contraction image of L_i.

Let G' be the graph obtained from G by recursively applying the L-operation at the two vertices of each L_i, and then contract the edges in $E(L_i)$, recursively for each $i = 1, 2, \ldots, \ell$. As all the contractions are taken with vertices in U, G' is a graph whose vertex set is a disjoint union of V and U_ℓ. Since $|U| = \frac{1}{2}(p^2 - 3p + 4) = \frac{1}{2}(p-1)(p-2) + 1$, by (S1) and $|U_\ell| \leqslant \frac{p-1}{2}$, there must be a vertex $u' \in U_\ell$ which is obtained by identifying at least $p-1$ vertices in U.

Let $J = cl_{G'}(\{u'\})$, the $\mathcal{O}_p$-closure of the single vertex u' in G' and let $V' = V - (\cup_{j=1}^{\ell} Q_j)$. By (S2) and (6), and as $\ell \leqslant p-2 < m$, we have $n' = |V'| \geqslant n_2 - \ell(p-1) \geqslant p-1$. It follows that for every $v' \in V'$, there are at least $(p-1)$ parallel edges joining u' and v'

in G'. Hence we may write $V' = \{v'_1, v'_2, \ldots, v'_n\}$ such that for any i with $1 \leqslant i \leqslant n' - 1$, there are at least $p-1$ edges in G' joining v_{i+1} to $\{u', v'_1, \ldots, v'_i\}$. It follows by Definition 9, $V' \subseteq V(J)$. By (S2), any vertex in V' is adjacent to every vertex in U_ℓ. Since $|V'| \geqslant p-1$, it follows by Definition 9, that $U_\ell \subseteq V(J)$. By (S2) again, every $v \in V$ is in at most one Q_j's, and so by $p \geqslant 5$, $d_{G'}(v') \geqslant d_G(v) - 2 = n_1 - 2 \geqslant p - 1$. Therefore we must have $G' = J$ and so by (1), $G' \in \mathcal{O}_p$.

Let G'' be the graph obtained from G by recursively performing the L-operation at the two vertices of each L_i, recursively for each $i = 1, 2, \ldots, \ell$. Then by Definition 18, G'' is a bipartite graph with bipartition (U, V) as G with $E(G'') - \cup_{j=1}^{\ell} E(L_j) \subset E(G)$. Fix j with $1 \leqslant j \leqslant \ell$, for each edge $e_j \in E(L_j)$, by (10), there exists a pair of edges $e'_j, e''_j \in E_G(v)$ for some $v \in V$ with $f(e'_j) = f(e''_j)$ such that in the lifting process, e'_j and e''_j become e_j in G''. Define

$$f''(e_j) = f(e'_j), \text{ for each edge } e_j \in E(L_j), \text{ where } 1 \leqslant j \leqslant \ell. \tag{12}$$

Recall that $b \in Z(G, \mathbb{Z}_p)$ and $f \in F(G, \mathbb{Z}_p^*)$ are given with f satisfying (8). Define $b' = b \in Z(G, \mathbb{Z}_p)$, and $f' : E(G'') \to \mathbb{Z}_p^*$ by utilizing (12) as follows:

$$f'(e) = \begin{cases} f(e) & \text{if } e \in E(G) - \cup_{j=1}^{\ell} E(L_j), \\ f''(e) & \text{if } e \in \cup_{j=1}^{\ell} E(L_j). \end{cases}$$

By Lemma 7(iii) and (vi), and since $G' \in \mathcal{O}_p$, we conclude that $G'' \in \mathcal{O}_p$. Hence G'' has an $(f', b; p)$-orientation D'. By repeated application of Lemma 17, we conclude that G has an $(f, b; p)$-orientation, as desired.

By applying contraction of K_{n_1,n_2} from $K_{n_1,n}$ with $n \geqslant n_2$ and Lemma 7 (i), one concludes that $K_{n_1,n} \in \mathcal{O}_p$. □

For positive integers m and n, let $K_{m,n}$ be the complete bipartite graph with bipartition $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$. For any subset $\{t_1, t_2, \ldots, t_\ell\}$ of $\mathbb{Z}_m$, where $t_1 \leqslant t_2 \ldots \leqslant t_\ell$, let $K(t_1, t_2, \ldots, t_\ell)$ be the graph obtained from $K_{m,n}$ by identifying $u_1, \ldots, u_{t_1}$, identifying $u_{t_i+1}, \ldots, u_{t_{i+1}}$ for each $1 \leqslant i \leqslant \ell - 1$ and identifying $u_{t_\ell+1}, \ldots, u_m$, respectively. Define

$$\mathcal{K}^*(m, n) = \{K(t_1, t_2, \ldots, t_\ell) : \{t_1, t_2, \ldots, t_\ell\} \subseteq \mathbb{Z}_m\}.$$

Since identifying two nonadjacent vertices u, v in a graph G amounts to the operation $(G+uv)/uv$. By Lemma 7(iii) and (ii), $G \in \mathcal{O}_p$ implies that $(G+uv)/uv \in \mathcal{O}_p$. Combining Theorem 20, leads to the following seemingly more general corollary.

Corollary 21. *Let $G \in \mathcal{K}^*(n_1, n_2)$ be a graph and $p > 0$ be an odd prime. Then $G \in \mathcal{O}_p$.*

As an application of corollary above, we present that if a family of graphs has a bounded matching number, then after certain reduction operations, there are only finitely many $\frac{1}{2}(p^2 - 3p + 4)$-edge-connected graphs not in $\mathcal{O}_p$. To state our theorem formally, we shall first introduce the concept of $\mathcal{O}_p$-reduction below.

As $K_1 \in \mathcal{O}_p$ by definition, for every graph G, any vertex is contained in a maximal subgraph in $\mathcal{O}_p$. Let $H_1, H_2, \cdots, H_c$ be the family of all maximal subgraphs of G which all in $\mathcal{O}_p$. Define $G' = G/(\cup_{i=1}^c E(H_i))$ to be the **$\mathcal{O}_p$-reduction** of G, or G is **$\mathcal{O}_p$-reduced** to G'. A graph G is called **trivially $\mathcal{O}_p$-reduced** if G has no non-trivial subgraph in $\mathcal{O}_p$. Our main result can be stated below.

Theorem 22. *Let G be a graph, $p > 0$ be an odd prime and $s > 0$ be an integer. Then for every function $f \in F(G, \mathbb{Z}_p^*)$ and every $\mathbb{Z}_p$-boundary b of G, there is a finite graph family $\mathcal{G}(p, s)$ such that every graph G with $\kappa'(G) \geqslant \frac{1}{2}(p^2 - 3p + 4)$ and $\alpha'(G) \leqslant s$ has an $(f, b; p)$-orientation if and only if the $\mathcal{O}_p$-reduction of G is not in $\mathcal{G}(p, s)$.*

To obtain this theorem, we also need the following elementary counting lemma, see [6, II.5*].

Lemma 23. ([6]) *Let $\ell, n > 0$ be integers. Then there are $\binom{n+\ell-1}{\ell-1}$ non-negative integral solutions $(x_1, x_2, \ldots, x_\ell)$ for the equation $x_1 + x_2 + \cdots + x_\ell = n$.*

Denote $N(p, s) = n_2 \cdot \binom{2s+n_1-1}{2s-1} + 2s$, where $n_1 = \frac{1}{2}(p^2 - 3p + 4), n_2 = \frac{1}{2}n_1(n_1 - 1)(p - 1)$. Let $\mathcal{F}(p, s)$ be the family of all n_1-edge-connected $\mathcal{O}_p$-reduced graphs of order between 2 and $N(p, s)$ with matching number at most s. Then each graph in $\mathcal{F}(p, s)$ has edge multiplicity at most $p - 2$ by Lemma 7(vi). So there are finitely many graphs in $\mathcal{F}(p, s)$. We will show the following stronger theorem, which implies Theorem 22 by Lemma 7(i), (iii) and Corollary 21.

Theorem 24. *Let G be a $\frac{1}{2}(p^2 - 3p + 4)$-edge-connected graph with $\alpha'(G) \leqslant s$. Then $G \in \mathcal{O}_p$ if and only if G cannot be $\mathcal{O}_p$-reduced to a member in $\mathcal{F}(p, s)$.*

Proof. If $G \in \mathcal{O}_p$, then G is $\mathcal{O}_p$-reduced to $K_1 \notin \mathcal{F}(p, s)$ by Lemma 7(vi). We shall show the converse that if G cannot be $\mathcal{O}_p$-reduced to a member in $\mathcal{F}(p, s)$, then $G \in \mathcal{O}_p$.

Let G be a counterexample and let G' be the $\mathcal{O}_p$-reduction of G. Then $G' \notin \mathcal{F}(p, s)$ and it leads to

$$|V(G')| > N(p, s) = n_2 \cdot \binom{2s + n_1 - 1}{2s - 1} + 2s. \tag{13}$$

By the definition of G', we achieve $\alpha'(G') \leqslant \alpha'(G) \leqslant s$. Let $M = \{w_1w_2, w_3w_4, \ldots, w_{2d-1}w_{2d}\}$ be a maximum matching of G', where $d \leqslant s$. Denote $W = \{w_1, \ldots, w_{2d}\}$. Then $Z = V(G') - W$ is an independent vertex set of G'. Since G' is n_1-edge-connected, we have $|[z, W]_{G'}| \geqslant n_1$ for any $z \in Z$. Pick arbitrary n_1 edges from $[z, W]_{G'}$, denoted by $H(z)$, for each $z \in Z$. Let $G'_1 = \cup_{z \in Z} H(z)$ be the graph induced by the edge set $\cup_{z \in Z} H(z)$ in G'.

We claim that there exists a member of $\mathcal{K}^*(n_1, n_2)$ in G'_1, therefore in G'. This will lead to a contradiction to the fact that G' is a $\mathcal{O}_p$-reduced graph by Theorem 21.

For any $w \in W$ and $z \in Z$, denote $x(w, z) = |[w, z]_{G'_1}|$ to be the number of edges between w and z in $H(z)$. Note that $x(w, z) = 0$ if w is not in the graph $H(z)$. Since $H(z)$ consists of n_1 edges, we have, for each $z \in Z$,

$$x(w_1, z) + x(w_2, z) + \cdots + x(w_{2d}, z) = n_1.$$

By (13) and $d \leqslant s$, $|Z| = |V(G')| - 2d > N(p,s) - 2s \geqslant n_2\binom{2s+n_1-1}{2s-1}$. By Lemma 23 and the Pigeon-Hole Principle, there exists a subset $Z_1 \subset Z$ of size n_2 such that, for any $z, z' \in Z_1$,

$$(x(w_1,z), x(w_2,z), \ldots, x(w_{2d},z)) = (x(w_1,z'), x(w_2,z'), \ldots, x(w_{2d},z')).$$

Denote $x_1, \ldots, x_{\ell+1}$ to be all the nonzero coordinates in $(x(w_1,z), x(w_2,z), \ldots, x(w_{2d},z))$. Then the graph $[Z_1, Y]_{G_1'} \cong K(t_1, t_2, \ldots, t_\ell)$ is a member of $\mathcal{K}^*(n_1, n_2)$, where $t_1 = x_1$, $t_{\ell+1} = (n_1) - t_\ell$ and $t_i - t_{i-1} = x_i$ for $2 \leqslant i \leqslant \ell$. This proves the claim as well as the theorem. □

5 Signed graphs

A **signed graph** is an ordered pair (G, σ) consisting of a graph G with a mapping $\sigma : E(G) \to \{1, -1\}$. An edge $e \in E(G)$ is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. The mapping σ, called the **signature** of G, is sometimes implicit in the notation of a signed graph and will be specified when needed. Both negative and positive loops are allowed in signed graphs. Define $E^+_\sigma(G) = \sigma^{-1}(1)$ and $E^-_\sigma(G) = \sigma^{-1}(-1)$. If no confusion occurs, we simply use E^+ for $E^+_\sigma(G)$ and E^- for $E^-_\sigma(G)$. An **orientation** τ assigns each edge of (G, σ) as follows: if $e = xy \in E^+(G)$, then e is either oriented from x and to y or bi-direction; if $e = xy \in E^-(G)$, then e is oriented either away from both x and y or towards both x and y. We call $e = xy$ a **sink edge** (a **source edge**, respectively) if it is oriented away from (towards, respectively) both x and y.

Let τ be an orientation of (G, σ). For each vertex $v \in V(G)$, let $H_G(v)$ be the set of half edges incident with v. Define $\tau(h) = 1$ if the half edge $h \in H_G(v)$ is oriented away from v, and $\tau(h) = -1$ if the half edge $h \in H_G(v)$ is oriented towards v. Denote $d^+_\tau(v) = |H^+_{G,\tau}(v)|$ ($d^-_\tau(v) = |H^-_{G,\tau}(v)|$, respectively) to be the outdegree (indegree, respectively) of (G, σ) under orientation τ, where $E^+_\tau(v)$ ($E^-_\tau(v)$, respectively) denotes the set of outgoing (ingoing, respectively) half edges incident with v.

An edge cut of (G, σ) is just an edge cut of G. The **switch operation** $\zeta = \zeta_S$ on an edge-cut S is a mapping $\zeta : E(G) \to \{-1, 1\}$ such that $\zeta(e) = -1$ if $e \in S$ and $\zeta(e) = 1$ otherwise. Two signatures σ and σ' are **equivalent** if there exists an edge-cut S such that $\sigma(e) = \sigma'(e)\zeta(e)$ for every edge $e \in E(G)$, where ζ is the switch operation on some edge-cut S of G. For a signed graph (G, σ), let χ denote the collection of all signatures equivalent to σ. The **negativeness** of (G, σ) is denoted by $\epsilon_N(G, \sigma) = \min\{|E^-_{\sigma'}(G)| : \forall \sigma' \in \chi\}$. We use ϵ_N for short if the signed graph (G, σ) is understood from the context. A signed graph is called ***k*-unbalanced** if $\epsilon_N \geqslant k$, and a 1-unbalanced signed graph is also known as an unbalanced signed graph.

We follow [17], to define signed graph **contractions**. For an edge $e \in E(G)$, the contraction G/e is the signed graph obtained from G by identifying the two ends of e, and then deleting the resulting positive loop if $e \in E^+$, but keeping the resulting negative loop if $e \in E^-$. For $X \subseteq E(G)$, the contraction G/X is the signed graph obtained from G by contracting each edge in X. If H is a subgraph of G, then we use G/H for $G/E(H)$. By definition, for any edge subset X of G, $\epsilon_N(G/X) \leqslant \epsilon_N(G)$.

Let A be an abelian (additive) group. Define $2A = \{2\alpha : \forall \alpha \in A\}$, and $A^* = A - \{0\}$. For a signed graph (G, σ), we still denote $F(G, A) = \{f | f : E(G) \to A\}$. Let τ be an orientation of (G, σ). For each $f \in F(G, A^*)$, the **boundary** of f is the function $\partial f : V(G) \to A$ defined by

$$\partial f = \sum_{h \in H_G(v)} \tau(h) f(e_h),$$

where e_h is the edge of G containing h and the summation is taken in A. If $\partial f = 0$, then (τ, f) is an A-flow of G. In addition, (τ, f) is a nowhere-zero A-flow if both $f \in F(G, A^*)$ and $\partial f = 0$. For any $f \in F(G, A^*)$, each positive edge contributes 0, each sink edge e contributes $2f(e)$, and each source edge e contributes $-2f(e)$ to $\sum_{v \in V(G)} \partial f(v)$. Thus one has

$$\sum_{v \in V(G)} \partial f(v) = \sum_{e \text{ is a sink edge}} 2f(e) - \sum_{e \text{ is a source edge}} 2f(e) \in 2A.$$

In [17], the authors introduced the definition of group connectivity of signed graphs. We extend this notation to **a mod k f-weighted b-orientation (an $(f, b; k)$-orientation)** of signed graphs.

Let (G, σ) be a 2-unbalanced signed graph. A mapping $b : V(G) \to \mathbb{Z}_k$ is called an $\mathbb{Z}_k$-boundary of (G, σ) if

$$\sum_{v \in V(G)} b(v) = 2\alpha \text{ for some } \alpha \in \mathbb{Z}_k.$$

Let $Z(G, \mathbb{Z}_k)$ be the collection of all $\mathbb{Z}_k$-boundaries. Given a signed graph (G, σ), for every $b \in Z(G, \mathbb{Z}_k)$ and every $f \in F(G, \mathbb{Z}_k^*)$, an orientation τ of (G, σ) is an **$(f, b; k)$-orientation** if for every vertex $v \in V(G)$,

$$\partial f(v) = \sum_{h \in H_G(v)} \tau(v) f(e_h) = b(v).$$

As graphs are signed graphs with negativeness zero, it is again necessary to assume k to be a prime when studying $(f, b; k)$-orientations of signed graphs. Let $p > 1$ be a prime. For notational simplification, we continue using $\mathcal{O}_p$ to denote the signed graph family $\mathcal{O}_p$ such that $(G, \sigma) \in \mathcal{O}_p$ if and only if (G, σ) admits an $(f, b; p)$-orientation for any $f \in F(G, \mathbb{Z}_p^*)$ and any $b \in Z(G, \mathbb{Z}_p)$. To avoid triviality, throughout the rest of this section, we always assume signed graphs under discussion with negativeness at least one.

Lemma 25. *Weighted modulo orientability is invariant under the switch operation.*

Proof. Let (G, σ) be a 2-unbalanced signed graph such that $(G, \sigma) \in \mathcal{O}_p$. As every switching operation can be composed from the switching operations on trivial edge-cut, it suffices to verify this lemma for the switch operation ζ_u on the trivial edge-cut $S = E_G(u)$ for any given vertex u. We fix a vertex u and let $\zeta = \zeta_u$ in the discussion below. Then

$\sigma' = \sigma\zeta$ is an signature equivalent to σ. We are to show that for any $f' \in F(G, \mathbb{Z}_p^*)$ and any $b' \in Z(G, \mathbb{Z}_p)$, the signed graph (G, σ') also admits an $(f', b'; p)$-orientation.

Let $f = f'$ and define $b : V(G) \to \mathbb{Z}_p$ by setting $b(u) = -b'(u)$ and $b(v) = b'(v)$ for any $v \in V(G)\backslash\{u\}$. As $b' \in Z(G, \mathbb{Z}_p)$, we also have

$$\sum_{v\in V(G)} b(v) = -b'(u) + \sum_{v\in V(G)\backslash\{u\}} b'(v) = \sum_{v\in V(G)} b'(v) - 2b'(u) \in 2\mathbb{Z}_p.$$

Thus $b \in Z(G, \mathbb{Z}_p)$ is also an $\mathbb{Z}_p$-boundary of (G, σ). Since (G, σ) admits an $(f, b; p)$-orientation, there exists an orientation τ such that, for every vertex $v \in V(G)$,

$$\partial f(v) = \sum_{h\in H_G(v)} \tau(h) f(e_h) = b(v).$$

Let τ' be the orientation of (G, σ') such that $\tau'(h) = -\tau(h)$ if $h \in H_G(u)$ and $\tau'(h) = \tau(h)$ otherwise. Hence, we have $\partial f'(v) = \partial f(v) = \sum_{h\in H_G(v)} \tau'(h) f(e_h) = b(v) = b'(v)$ for any vertex $v \in V(G)\backslash\{u\}$. In addition,

$$\partial f'(u) = -\partial f(u) = \sum_{h\in H_G(u)} \tau'(h) f(e_h) = \sum_{h\in H_G(u)} -\tau(h) f(e_h) = -b(u) = b'(u).$$

Therefore, $\partial f' = b'$ in the signed graph (G, σ') with orientation τ'. □

Lemma 26. *Let K_1^{-t} be the graph obtained from K_1 by attaching t negative loops to it. Then $K_1^{-t} \in \mathcal{O}_p$ if and only if $t \geqslant p - 1$.*

Proof. Let $V(K_1^{-t}) = \{v\}$, $H = tK_2$ be the signed graph with $V(H) = \{v, v'\}$ such that there are t positive edges joining v and v'. Note that $E(H) = E(K_1^{-t})$.

Assume first that $t \geqslant p - 1$. Let $f \in F(K_1^{-t}, \mathbb{Z}_p^*)$ be an arbitrary mapping and $b(v) \in 2\mathbb{Z}_p$ by an arbitrary $\mathbb{Z}_p$-boundary of K_1^{-t}. Since $b(v) \in 2\mathbb{Z}_p$, there exists an element $\beta \in \mathbb{Z}_p$ such that $b(v) = 2\beta$. Define $b_H \in Z(H, \mathbb{Z}_p)$ by setting $b_H(v) = \beta$ and $b_H(v') = -\beta$. As $t \geqslant p - 1$, by Lemma 7(vi), there exists an orientation τ of H such that $\sum_{h\in H_G(v)} \tau(h) f(e_h) = \beta$ and $\sum_{h\in H_G(v')} \tau(h) f(e_h) = -\beta$. Since K_1^{-t} can be obtained from H by identifying v and v', the orientation of K_1^{-t} is obtained from τ of H by taking the oppositive direction of every half edge in $H_G(v')$. Thus $K_1^{-t} \in \mathcal{O}_p$.

Conversely, we argue by contradiction and assume $K_1^{-t} \in \mathcal{O}_p$ but $t < p - 1$. By Lemma 7(iv), there exists an element $\beta \in \mathbb{Z}_p$, a mapping $b' \in Z(H, \mathbb{Z}_p)$ with $b'(v) = \beta$ and $b'(v') = -\beta$, and a mapping $f \in F(H, \mathbb{Z}_p^*)$ such that H admits no $(f, b'; p)$-orientations. Let $b \in Z(K_1^{-t}, \mathbb{Z}_p)$ be the mapping with $b(v) = 2\beta$. As $f \in F(K_1^{-t}, \mathbb{Z}_p^*)$ also, if K_1^{-t} has an $(f, b; p)$-orientation τ', then τ' also gives rise to an $(f, b'; p)$-orientation of H, contrary to the fact that H admits no $(f, b'; p)$-orientations. This contradiction indicates that we must have $t \geqslant p - 1$. □

Thus we have the following observation immediately.

Observation 27. If $(G, \sigma) \in \mathcal{O}_k$ is an unbalanced signed graph, then $\epsilon_N \geqslant k - 1$.

Lemma 28. *Let k be a positive integer and let (H,σ) be a signed graph. Assume that either $E^-_\sigma(H)=\emptyset$ and $H\in\mathcal{O}_k$ is as an ordinary graph or $(H,\sigma)\in\mathcal{O}_k$ is as a $(k-1)$-unbalanced signed graph. If (G,σ') is a $(k-1)$-unbalanced signed graph containing (H,σ) as a subgraph, then $(G,\sigma')\in\mathcal{O}_k$ if and only if $(G/H,\sigma'')\in\mathcal{O}_k$.*

Proof. For the unsigned graphs, the necessity can be proved following Lemma 7 (ii). One can prove the necessity of signed graphs analogously. It remains to prove the sufficiency.

In the sequel, for simplicity, we will use G/H to denote the signed graph $(G/H,\sigma'')$. Let $f\in F(G,\mathbb{Z}^*_k)$ and $b\in Z(G,\mathbb{Z}_k)$ be given, and let v_H be the vertex in G/H onto which H is contracted. For notational convenience, let $E^-_\sigma(H)$ denote the set of all negative edges of (H,σ), as well as the set of negative loops incident with v_H in G/H obtained by contracting H. Let $f_1\in F(G/H,\mathbb{Z}^*_k)$ be the restriction of f on $E(G/H)$, and define $b_1(v_H)=\sum_{v\in V(H)}b(v)$ and $b_1(v)=b(v)$ if $v\in V(G/H)-\{v_H\}$. Direct verification shows that $b_1\in Z(G/H,\mathbb{Z}_k)$. Since $G/H\in\mathcal{O}_k$, there exists an $(f_1,b_1;p)$-orientation τ_1 of G/H, and so $\partial f_1=b_1$.

For each vertex $v\in V(H)$, let $X_1(v)$ be the set of half edges incident with v in $E(G)-E(H)$, and $X_2(v)$ be the set of half edges incident with v in $E^-_\sigma(H)$. Define $b_2: V(H)\to\mathbb{Z}_k$ by

$$b_2(v)=b(v)-\sum_{h\in X_1(v)}\tau(h)f_1(e_h). \tag{14}$$

Since $\partial f_1=b_1$ in G/H, we have

$$\sum_{v\in V(H)}\sum_{h\in X_1(v)\cup X_2(v)}\tau(h)f_1(e_h)=\partial f_1(v_H)=b_1(v_H)=\sum_{v\in V(H)}b(v).$$

By (14),

$$\begin{aligned}\sum_{v\in V(H)}b_2(v)&=\sum_{v\in V(H)}b(v)-\sum_{v\in V(H)}\sum_{h\in X_1(v)}\tau(h)f_1(e_h)\\&=\sum_{v\in V(H)}\sum_{h\in X_2(v)}\tau(h)f_1(e_h)=\sum_{e\in E^-_\sigma(H)}\pm2f_1(e)\in2\mathbb{Z}_k.\end{aligned}$$

In the case when $E^-_\sigma(H)=\emptyset$, b_2 is a zero sum function, and so we always have $b_2\in Z(H,\mathbb{Z}_k)$. Let $f_2\in F(H,\mathbb{Z}^*_k)$ be the restriction of f in $E(H)$. Since $H\in\mathcal{O}_k$, there exists an orientation τ_2 of H such that $\partial f_2=b_2$. Let $\tau=\tau_1\cup\tau_2$ be the orientation of G formed by combing the orientation τ_2 of H and the orientation τ_1 of G/H. Then, for each vertex $v\in V(H)$, it follows from (14) that

$$\begin{aligned}\partial f(v)&=\partial f_1(v)+\partial f_2(v)\\&=\sum_{h\in X_1(v)}\tau(h)f_1(e_h)+b_2(v)\\&=\sum_{h\in X_1(v)}\tau(h)f_1(e_h)+[b(v)-\sum_{h\in X_1(v)}\tau(h)f_1(e_h)]=b(v).\end{aligned}$$

Therefore, τ is an $(f, b; k)$-orientation of (G, σ'). By definition, $(G, \sigma') \in \mathcal{O}_k$. □

Lemma 28 leads to a reduction method for verifying weighted modulo orientability of unbalanced signed graphs, which is an extension of Lemma 7(iii) for unsigned graphs. The following lemma follows Lemma 26 and Lemma 28.

Lemma 29. *An unbalanced signed graph $(G, \sigma) \in \mathcal{O}_p$ if and only if it can be contracted to K_1^{-t} for some integer $t \geqslant p-1$ by contracting its subgraphs in $\mathcal{O}_p$ recursively.*

Lemma 30 below is a consequence by combining Lemma 28 and Lemma 29.

Lemma 30. *Let (G, σ) be a $(p-1)$-unbalanced signed graph. If $G[E^+]$ is spanning and $G[E^+] \in \mathcal{O}_p$ is as an ordinary graph, then $(G, \sigma) \in \mathcal{O}_p$.*

The following theorems are our main results of this section.

Theorem 31. *Let p be an odd prime and let (G, σ) be a $(p-1)$-unbalanced signed graph with $\kappa'(G) \geqslant 12p^2 - 28p + 15$. Then $(G, \sigma) \in \mathcal{O}_p$.*

Proof. Pick any $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b. Since p is prime, we have $2\mathbb{Z}_p = \mathbb{Z}_p$ and $\sum_{v \in V(G)} b(v)$ can be any element in $\mathbb{Z}_p$. By Lemma 25, we may assume that $|E_\sigma^-(G)| = \epsilon_N$. Since (G, σ) is a $(12p^2 - 28p + 15)$-edge-connected signed graph with minimal number of negative edges in the switch equivalent class, $|S \cap E_\sigma^-(G)| \leqslant \frac{1}{2}|S|$ for each edge-cut S. Therefore $G[E_\sigma^+(G)]$ is $(6p^2 - 14p + 8)$-edge-connected and hence $G[E^+] \in \mathcal{O}_p$ by Theorem 4. By Lemma 30, one has $(G, \sigma) \in \mathcal{O}_p$. □

Theorem 32. *Let p be an odd prime and let (G, σ) be a $(p-1)$-unbalanced signed series-parallel graph with $\kappa'(G) \geqslant 4p - 7$. Then $(G, \sigma) \in \mathcal{O}_p$.*

Proof. We prove by induction on $|V(G)|$. The statement clearly holds for $|V(G)| = 1$ by Lemma 26. Assume $|V(G)| \geqslant 2$. The underlying simple graph H of G is K_4-minor-free, and so contains a vertex v of degree at most 2. Denote $N_H(v) = \{x, y\}$ if v has two neighbors and $N_H(v) = \{x\}$ if v has a unique neighbor. In the signed graph G, by the edge connectivity $\kappa'(G) \geqslant 4p - 7$, we have $|[v, x]_G| + |[v, y]_G| \geqslant 4p - 7$. Hence $\max\{|[v, x]_G, |[v, y]_G|\} \geqslant 2p - 3$. We may, with out loss of generality, assume $|[v, x]_G| \geqslant 2p - 3$. (In the case $N_H(v) = \{x\}$, we have $|[v, x]_G| \geqslant 4p - 7 \geqslant 2p - 3$ as well.) By Lemma 25, by possible some switching operation at least half of edges in $[v, x]_G$ are positive, and so there are at least $p - 1$ parallel positive edges, denoted by M, in $[v, x]_G$. Thus by Lemma 7(iv), those parallel positive edges M in $[v, x]_G$ is in $\mathcal{O}_p$. Moreover, $G/M \in \mathcal{O}_p$ by induction, and so $(G, \sigma) \in \mathcal{O}_p$ by Lemma 28. □

6 Conclusion

In this paper, we reduce the edge-connectivity $(6p^2 - 14p + 8)$ in Theorem 4 for some graph families, and we extend the $(f, b; p)$-orientation framework to signed graph. Viewing the results in this paper and in literatures, we believe that it is possible that a linear function of p would suffice for the existence of such $(f, b; p)$-orientations. We conclude this paper with the following conjectures.

Conjecture 33. There exists a constant c independent of p such that every cp-edge-connected graph is in $\mathcal{O}_p$.

Conjecture 34. There exists a constant c independent of p such that every cp-edge-connected $(p-1)$-unbalanced signed graph is in $\mathcal{O}_p$.

In fact, by Lemma 30 those two conjecture are equivalent (regardless of the constant c).

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