Skew Schur Polynomials and Cyclic Sieving Phenomenon

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Abstract

Let k and m be positive integers and λ/μ a skew partition. We compute the principal specialization of the skew Schur polynomials $s_{\lambda/\mu}(x_1, \ldots, x_k)$ modulo $q^m - 1$ under suitable conditions. We interpret the results thus obtained from the viewpoint of the cyclic sieving phenomenon on semistandard Young skew tableaux of shape λ/μ. As an application, we deal with evaluations of the principal specialization of the skew Schur polynomials at roots of unity.

Mathematics Subject Classifications: 05E18, 05E05, 05E10

1 Introduction

The cyclic sieving phenomenon was introduced by Reiner, Stanton, and White in [16] as a generalization of Stembridge's $q = -1$ phenomenon [19, 20]. Let X be a finite set on which a finite cyclic group C acts and let $X(q)$ be a polynomial in q with nonnegative integer coefficients. For $d \in \mathbb{Z}_{>0}$, let ω_d be a dth primitive root of unity. We say that the triple $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon* if $\#X^c = X(\omega_{\text{o}(c)})$ for all $c \in C$, where X^c is the set of fixed points under the action of c and $\text{o}(c)$ is the order of c.

We identify a partition with the corresponding Young diagram. Given a Young diagram λ, let $\mathsf{SSYT}_m(\lambda)$ be the set of semistandard Young tableaux of shape λ with entries in $\{1, \ldots, m\}$. In [17], Rhoades proved representation-theoretically that if λ is of rectangular shape, the triple

$$\left(\mathsf{SSYT}_m(\lambda),\ \langle \mathsf{pr} \rangle,\ q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1}) \right)$$

exhibits the cyclic sieving phenomenon, where pr is *jeu-de-taquin* promotion, $\kappa(\lambda) = \sum_{i \geqslant 1} (i-1)\lambda_i$, and $s_\lambda(1, q, \ldots, q^{m-1})$ is the principal specialization of the *Schur polynomial* $s_\lambda(x_1, \ldots, x_m)$. Since then, there have been several studies on the following issues:

1. When λ is not of rectangular shape, what conditions guarantee the existence of an action of a finite cyclic group C on $\mathsf{SSYT}_m(\lambda)$ such that the triple

$$\left(\mathsf{SSYT}_m(\lambda), \ C, \ q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1}) \right)$$

   exhibits the cyclic sieving phenomenon?

2. In (1), can we describe the cyclic group action explicitly?

For instance, see [4, 6, 9, 13, 14, 15]. In particular, it was shown in [13] that for any Young diagram that satisfies the condition $\gcd(|\lambda|, m) = 1$, the triple

$$\left(\mathsf{SSYT}_m(\lambda), \ \langle \mathsf{c} \rangle, \ q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1}) \right)$$

exhibits the cyclic sieving phenomenon. Here the action c arises naturally from the $U_q(\mathfrak{sl}_m)$-*crystal* structure of $\mathsf{SSYT}_m(\lambda)$.

In this paper, we consider cyclic sieving phenomena associated with semistandard Young skew tableaux. Unlike tableaux of normal shape, little is known about cyclic sieving phenomena that occur on semistandard Young skew tableaux. It has been conjectured by Alexandersson and Amini [2, Conjecture 3.4] that for any $k, m \in \mathbb{Z}_{>0}$, there exists an action of a cyclic group C_m of order m on $\mathsf{SSYT}_k(\lambda)$ such that the triple

$$(\mathsf{SSYT}_k(m\lambda), \ C_m, \ s_{m\lambda}(1, q, \ldots, q^{k-1}))$$

exhibits the cyclic sieving phenomenon. Here $m\lambda$ denotes the stretched Young diagram of λ by m. This conjecture turns out to be true in [14], where a crystal-theoretical generalization of this phenomenon is also provided. Recently, a skew version of this conjecture has been proposed by Alexandersson-Pfannerer-Rubey-Uhlin [4, Conjecture 50], which states that there is an action of a cyclic group C_m of order m on the set $\mathsf{SSYT}_k(m\lambda/m\mu)$ such that the triple

$$(\mathsf{SSYT}_k(m\lambda/m\mu), \ C_m, \ s_{m\lambda/m\mu}(1, q, \ldots, q^{k-1}))$$

exhibits the cyclic sieving phenomenon.

The present paper concerns the conjecture given in [4, Conjecture 50]. We begin by computing the principal specialization of $s_{m\lambda/m\mu}(x_1, \ldots, x_k)$ modulo $q^m - 1$. For instance, in case where $\lambda = (3, 3, 2, 1), \mu = (2, 1), m = 9$, and $k = 4$, we observe that

$$s_{9\lambda/9\mu}(1, q, q^2, q^3) \equiv 54665112 \frac{q^9 - 1}{q - 1} - 3 \frac{q^9 - 1}{q^3 - 1} + 1 \pmod{q^9 - 1}.$$

This congruence tells us that the conjecture is not necessarily true. However, under the condition that k is divisible by m, we prove that the conjecture is true. To be precise, our first main result is the following.

Theorem 1. *Let k and m be positive integers and λ/μ a skew partition. If $\lambda_i - \mu_i$ is divisible by m for all $i \geqslant 1$, then there exists an action of a cyclic group C_m of order m such that the triple*

$$(\mathsf{SSYT}_{km}(\lambda/\mu),\ C_m,\ s_{\lambda/\mu}(1, q, \ldots, q^{km-1}))$$

exhibits the cyclic sieving phenomenon.

We also prove that the conjecture is true whenever λ/μ is a border strip, which is our second main result.

Theorem 2. *Let k and m be positive integers and λ/μ a border strip. If $\lambda_i - \mu_i$ is divisible by m for all $i \geqslant 1$, then there exists an action of a cyclic group C_m of order m such that the triple*

$$(\mathsf{SSYT}_k(\lambda/\mu),\ C_m,\ s_{\lambda/\mu}(1, q, \ldots, q^{k-1}))$$

exhibits the cyclic sieving phenomenon.

As an application, we deal with evaluations of the principal specialization of the skew Schur polynomials at roots of unity. For each $d \mid k$, Reiner, Stanton, and White [16, Theorem 4.3] provide a formula for $s_\lambda(1, q, \ldots, q^{k-1})$ when q is specialized to a dth primitive root of unity, which involves the notion of d-cores and d-quotients of λ. Using a result obtained in the proof of Theorem 1, we successfully generalize this formula to skew shapes (see Theorem 16). In particular, the case where $d = k$ is investigated in detail.

This paper is organized as follows. In Section 2, we collect the materials required to develop our arguments. In Section 3 and Section 4, the proofs of Theorem 1 and Theorem 2 are provided, respectively. In Section 5, a skew version of [16, Theorem 4.3] is provided. The final section is devoted to a few remarks for readers' understanding.

2 Preliminaries

2.1 Partitions, skew partitions, and border strip tableaux

A *partition* is any sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers in decreasing order and containing only finitely many nonzero terms. The *length* $\ell(\lambda)$ of λ is defined to be the number of positive parts of λ. Unless otherwise specified, we usually omit the zero parts of λ and simply write as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$. Given any positive integer m, we set $m\lambda$ to be the stretched partition $(m\lambda_1, m\lambda_2, \ldots, m\lambda_{\ell(\lambda)})$. A *skew partition* is a pair of partitions (λ, μ) such that the Young diagram of λ contains the Young diagram of μ; it is denoted by λ/μ. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, then the containment of diagrams means that $\lambda_i \geqslant \mu_i$ for all i.

A *border strip* is a connected skew shape with no 2×2 square. Define the *height* $\mathrm{ht}(B)$ of a border strip B to be one less than its number of rows. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a weak composition of n. Define a *border-strip tableau* of shape λ/μ (where $|\lambda/\mu| = n$) and type α to be an assignment of positive integers to the squares of λ/μ such that

- every row and column is weakly increasing,

- the integer i appears α_i times, and

- the set of squares occupied by i forms a border strip.

Define the *height* $\mathrm{ht}(\mathrm{T})$ of a border-strip tableau T to be

$$\mathrm{ht}(T) = \mathrm{ht}(\mathrm{B}_1) + \mathrm{ht}(\mathrm{B}_2) + \cdots + \mathrm{ht}(\mathrm{B}_k),$$

where $\mathrm{B}_1, \ldots, \mathrm{B}_k$ are the nonempty border strips appearing in T. For more details, see [18, Section 7.17].

2.2 The abacus model for partitions, cores and quotients

Let us review the abacus model introduced by James and Kerber [10, Ch. 2.7] to encode partitions and their quotients, some of which will be slightly modified for the simplicity of the arguments.

With $d \geqslant 1$ fixed, we take an abacus with d vertical runners, numbered $0, 1, \ldots, d-1$ from left to right, and we mark positions $0, 1, 2, \ldots$ on these runners, reading from left to right along successive rows. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition. For $r \geqslant \ell$, let δ_r be the staircase partition $(r-1, r-2, \ldots, 1)$ and let $\lambda + \delta_r$ be the sequence

$$(\lambda_1 + r - 1, \ldots, \lambda_i + (r-i), \ldots, \lambda_{r-1} + 1, \lambda_r),$$

where $\lambda_i = 0$ if $i > \ell$. Once r is chosen, the *d-abacus display for λ* is obtained by circling the elements of $\lambda + \delta_r$ from the row containing the position labeled by 0 to the row containing $\lambda_1 + r - 1$. It should be mentioned that $r > \ell$ if and only if the position labeled by 0 is circled. We call a circled position a *bead* and a uncircled position a *non-bead* for simplicity. See Figure 1 for an illustration.

Let $\lambda_\emptyset$ be the *d-core* of λ, which can be obtained by removing border strips of size d from the diagram of λ as many as possible. It is well known that the abacus display for $\lambda_\emptyset$ can be obtained from the abacus display for λ by moving all the beads as far up their runners as they will go and then removing the rows containing beads only. For each $0 \leqslant i \leqslant d-1$, let $\lambda^{(i)}$ be the partition whose j-part is the number of non-beads above the jth bead from the bottom on runner i of the abacus display for λ. The sequence $(\lambda^{(0)}, \ldots, \lambda^{(d-1)})$ is called *the d-quotient of λ*.

Next, let us introduce the abacus display for skew partitions. Let λ/μ be a skew partition with $\ell(\lambda) = \ell$. Once d and $r \geqslant \ell$ are chosen, the *d-abacus display for λ/μ* can be obtained by drawing the d-abacus for μ above the d-abacus for λ. It should be mentioned that we are using the same r in obtaining the d-abaci for λ and μ. In the present paper, r will always be set to be ℓ. If the Young diagram of $\lambda^{(i)}$ contains the Young diagram of $\mu^{(i)}$ for all $0 \leqslant i \leqslant d-1$, the *$d$-quotient* of λ/μ is defined to be the sequence $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(d-1)}/\mu^{(d-1)})$. Otherwise, we say that the *d-quotient* of λ/μ does not exist. For more information on the abacus display for skew partitions, we refer the readers to [4, Section 5].

Example 3. Let $\lambda/\mu = (9^2, 6^3, 4, 1)/(2, 1^3)$. The 3-abacus for λ/μ can be found in Figure 1. And, the 3-quotient of λ/μ is $((4, 3)/(1), (2), (2, 1^2))$.
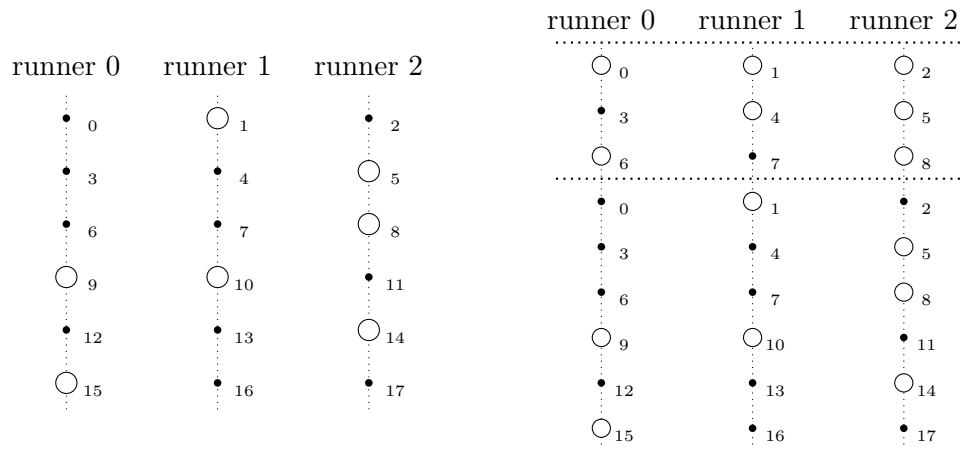

Figure 1: The 3-abaci for $(9^2, 6^3, 4, 1)$ and $(9^2, 6^3, 4, 1)/(2, 1^3)$.

2.3 Basic results and definitions necessary for proofs

Here we collect the basic results and definitions necessary to develop our arguments. The first result to introduce is due to Reiner, Stanton, and White. Let k, m, and n be positive integers. Suppose that a cyclic group C_m of order m acts *nearly freely* on $\{1, \ldots, k\}$, equivalently it is generated by an element $c \in \mathfrak{S}_k$ whose cycle type is either

1. a cycles of size m, so that $k = am$ (and C_m acts freely), or

2. a cycles of size m and one singleton cycle, so that $k = am + 1$

for some positive integer a. Let X be an n-multiset of $\{1, \ldots, k\}$, and

$$X(q) := \begin{bmatrix} n + k - 1 \\ n \end{bmatrix}_q.$$

Under this assumption, they prove in [16, Theorem 1.1] that the triple $(X, C_m, X(q))$ exhibits the cyclic sieving phenomenon. Furthermore, letting $A_l(k, n)$ denote the coefficient of q^l in

$$\begin{bmatrix} n + k - 1 \\ n \end{bmatrix}_q \pmod{q^k - 1}$$

with the convention that $A_1(1, n) = 1$, they show in [16, Corollary 1.3] that

$$A_l(k, n) = \sum_{d \mid n, k, l} A_1\left(\frac{k}{d}, \frac{n}{d}\right). \tag{1}$$

Let $h_n(x_1, \ldots, x_k)$ be the nth homogeneous symmetric polynomial in variables $x_1, \ldots, x_k$. It is easy to see that the principal specialization of $h_n(x_1, \ldots, x_k)$ equals $X(q)$, that is,

$$h_n(1, q, \ldots, q^{k-1}) = \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_q.$$

Thus the theorem [16, Theorem 1.1] tells us that if a cyclic group of order m act nearly freely on $\{1, \ldots, k\}$, then

$$h_n(1, q, \ldots, q^{k-1}) \equiv \sum_{d \mid m} a_d \frac{q^m - 1}{q^{\frac{m}{d}} - 1} \pmod{q^m - 1}$$

for some nonnegative integers a_d's.

The second result to introduce is due to Alexandersson and Amini.

Lemma 4. (Alexandersson and Amini [2, Theorem 2.7]) *Let $f(q) \in \mathbb{Z}_{\geq 0}[q]$ and suppose $f(\omega_m^j) \in \mathbb{Z}_{\geq 0}$ for each $j = 1, \ldots, m$, where ω_m denotes a primitive mth root of unity. Let X be any set of size $f(1)$. Then there exists an action of a cyclic group C of order m on X such that $(X, C, f(q))$ exhibits the cyclic sieving phenomenon if and only if for each $d \mid m$,*

$$\sum_{j \mid d} \mu(d/j) f(\omega_m^j) \geq 0.$$

Here, μ is the Möbius function.

In Lemma 4, suppose that $f(q)$ satisfies the condition that

$$f(q) \equiv \sum_{d \mid m} a_d \frac{q^m - 1}{q^{\frac{m}{d}} - 1} \pmod{q^m - 1}$$

for some integers a_d's. Then it holds that

$$\sum_{j \mid d} \mu(d/j) f(\omega_m^j) = d a_d \tag{2}$$

(for instance, see the proof of [14, Theorem 3.2]). Hence, if $a_d \geq 0$ for all $d \mid n$, then $(X, C, f(q))$ exhibits the cyclic sieving phenomenon for a unknown cyclic group action and a_d is the number of C-orbits with exactly d elements.

Motivated by this observation, we introduce the following definition.

Definition 5. Let $f(q) \in \mathbb{Z}[q]$.

(a) We say that $f(q)$ is a *pre-CSP polynomial modulo m* if

$$f(q) \equiv \sum_{d \mid m} a_d \frac{q^m - 1}{q^{\frac{m}{d}} - 1} \pmod{q^m - 1}$$

for some integers a_d's.

(b) We say that $f(q)$ is a *CSP polynomial modulo m* if it is a pre-CSP polynomial modulo m with $a_d \geqslant 0$ for all divisors d of m.

In Definition 5(a), the coefficient a_d is denoted by

$$\left[\frac{q^m - 1}{q^{\frac{m}{d}} - 1} \right] f(q)$$

for each divisor d of m because it is uniquely determined.

3 Proof of Theorem 1

In this section, we provide the proof of Theorem 1. We begin by introducing two important lemmas. Given positive integers m and n, we denote by (m, n) the greatest common divisor of m and n and $[m, n]$ the least common multiple of m and n.

Lemma 6. *For any positive integers k, m and n, $h_n(1, q, \ldots, q^{km-1})$ is a CSP polynomial modulo m. Furthermore, for each divisor d of m,*

$$\left[\frac{q^m - 1}{q^{\frac{m}{d}} - 1} \right] h_n(1, q, \ldots, q^{km-1}) = \begin{cases} \left[\frac{q^d - 1}{q - 1} \right] h_{nd/m}(1, q, \ldots, q^{kd-1}) & \text{if } \frac{m}{d} \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the theorem [16, Theorem 1.1], we can see that $h_n(1, q, \ldots, q^{km-1})$ is a CSP polynomial modulo m as well as a CSP polynomial modulo km. Let e be any divisor of km and set

$$b_e := \left[\frac{q^{km} - 1}{q^{\frac{km}{e}} - 1} \right] h_n(1, q, \ldots, q^{km-1}).$$

In view of (1), we have that

$$b_e = \begin{cases} A_1\left(e, \frac{ne}{km}\right) & \text{if } \frac{km}{e} \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Fix a divisor d of m and let $\mathcal{A}_d$ be the set of divisors e of km satisfying that $\left(\frac{km}{e}, m\right) = \frac{m}{d}$. Setting

$$a_d := \left[\frac{q^m - 1}{q^{\frac{m}{d}} - 1} \right] h_n(1, q, \ldots, q^{km-1})$$

gives rise to the following relation between a_d and b_e:

$$a_d = \sum_{e \in \mathcal{A}_d} \frac{e}{d} b_e. \tag{4}$$

In case where $\frac{m}{d}$ divides n, applying (3) to this equality yields that

$$a_d = \sum_{\substack{e \in \mathcal{A}_d \\ \frac{km}{e} \mid n}} \frac{e}{d} b_e + \sum_{\substack{e \in \mathcal{A}_d \\ \frac{km}{e} \nmid n}} \frac{e}{d} b_e = \sum_{\substack{e \in \mathcal{A}_d \\ \frac{km}{e} \mid n}} \frac{e}{d} b_e = \sum_{\substack{e \in \mathcal{A}_d \\ \frac{km}{e} \mid n}} \frac{e}{d} A_1\left(e, \frac{ne}{km}\right).$$

Otherwise, there is no element $e \in \mathcal{A}_d$ with $\frac{km}{e}|n$, and thus $a_d = 0$ by (3) together with (4).

For the second assertion, observe that $\mathcal{A}_d$ is identical to the set of divisors e of kd satisfying that $(\frac{kd}{e}, d) = 1$. Set

$$a'_d := \left[\frac{q^d - 1}{q - 1} \right] h_{nd/m}(1, q, \dots, q^{kd-1}).$$

Then we have

$$a'_d = \sum_{\substack{e \in \mathcal{A}_d \\ \frac{kd}{e} | \frac{nd}{m}}} \frac{e}{d} A_1 \left(e, \frac{ne}{km} \right).$$

Now our assertion follows from the fact that, for each $e \in \mathcal{A}_d$, $\frac{km}{e}$ divides n if and only if $\frac{kd}{e}$ divides $\frac{nd}{m}$. $\qquad \square$

Lemma 7. *Let m be a positive integer and let a, b be divisors of m. Then,*

$$\frac{q^m - 1}{q^a - 1} \cdot \frac{q^m - 1}{q^b - 1} \equiv \frac{m}{[a, b]} \cdot \frac{q^m - 1}{q^{(a,b)} - 1} \pmod{q^m - 1}. \tag{5}$$

Proof. For simplicity, let $d := (a, b)$ and $l := [a, b]$. Then we can write $a = da'$ and $b = db'$ with $(a', b') = 1$. The left hand side of (5) equals

$$\sum_{\substack{0 \leqslant i < m/a \\ 0 \leqslant j < m/b}} q^{ai+bj}$$

and the right hand side of (5) equals

$$\frac{m}{l} \sum_{0 \leqslant k < m/d} q^{dk}.$$

Hence the assertion can be proven by showing that, for each $0 \leqslant k < \frac{m}{d}$, the number of solutions to the linear congruence

$$ai + bj \equiv dk \pmod{m} \quad \text{with } 0 \leqslant i < \frac{m}{a} \text{ and } 0 \leqslant j < \frac{m}{b} \tag{6}$$

is given by $\frac{m}{l}$. To do this, we first note that the linear congruence $ai + bj \equiv dk \pmod{m}$ is equivalent to $a'i + b'j \equiv k \pmod{\frac{m}{d}}$. Then we consider the set

$$\left\{ a'i + b'j \ : \ 0 \leqslant i < \frac{m}{a}, \ 0 \leqslant j < \frac{m}{b} \right\}. \tag{7}$$

For each $0 \leqslant c < \frac{m}{ab'}$, we claim that the subset

$$\left\{ a'(i + cb') + b'j \ : \ 0 \leqslant i < b', \ 0 \leqslant j < \frac{m}{b} \right\}$$

is a complete system of residues modulo $\frac{m}{d}$. Note that the number of elements in this set is $\frac{m}{d}$. Suppose that

$$a'(i_1 + cb') + b'j_1 \equiv a'(i_2 + cb') + b'j_2 \pmod{\frac{m}{d}},$$

where $0 \leqslant i_1, i_2 < b'$ and $0 \leqslant j_1, j_2 < \frac{m}{b}$. Then $a'(i_1 - i_2) + b'(j_1 - j_2) = \frac{m}{d}k$ for some k. Applying $b' \mid \frac{m}{d}$ and $(a', b') = 1$ to this equality yields that $b' \mid (i_1 - i_2)$. Since $-b' < i_1 - i_2 < b'$, this division relation implies that $i_1 = i_2$, thus $j_1 - j_2 = \frac{m}{b}k$. It says that $j_1 = j_2$ since $-\frac{m}{b} < j_1 - j_2 < \frac{m}{b}$, which completes the verification of the claim.

Applying the above claim to (7) shows that the number of solutions to (6) is given by

$$\frac{\frac{m}{a} \cdot \frac{m}{b}}{\frac{m}{d}} = \frac{m}{l},$$

as required. $\qquad\square$

Remark 8. Let m and k be positive integers. For every partition λ with at most km rows, by applying Lemma 6 and Lemma 7 repeatedly, we deduce that $h_\lambda(1, q, \ldots, q^{km-1})$ is a CSP polynomial modulo m. Going further, we deduce that the principal specialization of every symmetric polynomial in $\mathbb{Z}[x_1, \ldots, x_{km}]$ is a pre-CSP polynomial modulo m since

$$\{h_\lambda(x_1, \ldots, x_{km}) : \lambda \text{ a partition with at most } km \text{ rows}\}$$

is a basis over $\mathbb{Z}$ of the symmetric polynomials in $x_1, \ldots, x_{km}$.

To each skew partition λ/μ with $\ell(\lambda) = l$, we associate the $l \times l$ matrix $\mathsf{M}(\lambda/\mu)$ defined by

$$\mathsf{M}(\lambda/\mu)_{i,j} = \lambda_i - \mu_j + j - i. \tag{8}$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let k be an arbitrary positive integer and be fixed throughout the proof. By Remark 8, we have

$$s_{\lambda/\mu}(1, q, \ldots, q^{km-1}) \equiv \sum_{d|m} c_d \frac{q^m - 1}{q^{\frac{m}{d}} - 1} \pmod{q^m - 1},$$

where c_d's are integers. Due to Lemma 4 together with (2), for the assertion, we have only to show that $s_{\lambda/\mu}(1, q, \ldots, q^{km-1})$ is a CSP polynomial modulo m. This will be accomplished by applying mathematical induction on m. When $m = 1$, there is nothing to prove. Suppose that $m \geqslant 2$ and our assertion holds for all positive integers less than m. For any $n \in \mathbb{Z}$ and any divisor d of m, let

$$a_d(n, m) := \left[\frac{q^m - 1}{q^{\frac{m}{d}} - 1}\right] h_n(1, q, \ldots, q^{km-1}). \tag{9}$$

If n is a positive integer, by Lemma 6, we have

$$a_d(n, m) = \begin{cases} a_d(\frac{nd}{m}, d) > 0 & \text{if } \frac{m}{d} \mid (m, n), \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

One can also see that $a_d(0, m) = \delta_{d,1}$ and that when n is a negative integer, $a_d(n, m) = 0$ for all $d \mid m$. Here $\delta_{d,1}$ denotes the Kronecker delta.

Since every coefficient of $s_{\lambda/\mu}(1, q, \ldots, q^{km-1})$ is nonnegative, one can see that $c_m(m) \geqslant 0$. Let d be a divisor of m smaller than m, which will be fixed until the end of the proof. In what follows, we will show that c_d is nonnegative.

Let $\mathsf{M} := \mathsf{M}(\lambda/\mu)$ and $l := \ell(\lambda)$. Consider the $\frac{m}{d}$-abacus display for λ/μ and let $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(\frac{m}{d}-1)}/\mu^{(\frac{m}{d}-1)})$ be the $\frac{m}{d}$-quotient of λ/μ. For $0 \leqslant t \leqslant \frac{m}{d} - 1$, let

$$\mathcal{C}_t := \{1 \leqslant i \leqslant l : \text{the bead for } \lambda_i + l - i \text{ is on runner } t \text{ in the abacus display}\}.$$

Note that for $1 \leqslant i \leqslant l$, the bead for $\mu_i + l - i$ is on runner t if and only if $i \in \mathcal{C}_t$ since $\lambda_i - \mu_i$ is divisible by m. For simplicity, let $l(t) := |\mathcal{C}_t|$ and write

$$\mathcal{C}_t = \{c_{t,1} < \cdots < c_{t,l(t)}\}.$$

Pick any integer $0 \leqslant t \leqslant \frac{m}{d} - 1$, which will be fixed until the end of the proof. Let us introduce three $l(t) \times l(t)$ matrices $\mathsf{M}(t)$, $\mathsf{P}(t)$, and $\widehat{\mathsf{P}}(t)$ whose (i, j) entry is given by

$$\mathsf{M}(t)_{i,j} = \mathsf{M}_{c_{t,i}, c_{t,j}},$$
$$\mathsf{P}(t)_{i,j} = h_{\mathsf{M}(t)_{i,j}}(1, q, \ldots, q^{km-1}), \quad \text{and}$$
$$\widehat{\mathsf{P}}(t)_{i,j} = \sum_{e \mid d} a_e\left(\mathsf{M}(t)_{i,j}, m\right) \frac{q^m - 1}{q^{\frac{m}{e}} - 1}.$$

For $\sigma \in \mathfrak{S}_l$, let

$$f_\sigma(q) := \operatorname{sgn}(\sigma) \prod_{1 \leqslant i \leqslant l} h_{\lambda_i - \mu_{\sigma(i)} + \sigma(i) - i}(1, q, \ldots, q^{km-1}).$$

Let $\mathfrak{S} := \mathfrak{S}_{\mathcal{C}_1} \times \mathfrak{S}_{\mathcal{C}_2} \times \ldots \times \mathfrak{S}_{\mathcal{C}_n}$ and $\mathfrak{S}_{\mathcal{C}_t}$ is the symmetric group on $\mathcal{C}_t$. In case where $\sigma \in \mathfrak{S}_l \setminus \mathfrak{S}$, by applying Lemma 6 and Lemma 7 to $f_\sigma(q)$, we derive that

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] f_\sigma(q) = 0 \quad \text{for all } e \mid d.$$

Combining this with Jacobi-Trudi formula, we also derive that

$$c_e = \sum_{\sigma \in \mathfrak{S}_l} \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] f_\sigma(q) = \sum_{\sigma \in \mathfrak{S}} \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] f_\sigma(q) = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] \prod_{0 \leqslant t \leqslant \frac{m}{d} - 1} \det(\mathsf{P}(t)) \tag{11}$$

for all divisors e of d, particularly for $e = d$. On the other hand, applying Lemma 7 to

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] \mathsf{P}(t)_{i,j} = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] \widehat{\mathsf{P}}(t)_{i,j} \quad \text{for all } 1 \leqslant i, j \leqslant l \text{ and } e \mid d$$

gives rise to the identity

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] \det(\mathsf{P}(t)) = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] \det(\widehat{\mathsf{P}}(t)) \quad \text{for all } e \mid d. \tag{12}$$

So our assertion reduces to showing that the right hand side of (12) is always nonnegative.

We start this by comparing $\mathsf{M}(\lambda^{(t)}/\mu^{(t)})$ and $\mathsf{M}(t)$. Note that for $1 \leqslant i \leqslant l(t)$,

$$\lambda_i^{(t)} = \frac{d(\lambda_{c_{t,i}} + l - c_{t,i} - t)}{m} - l(t) + i \quad \text{and} \quad \mu_i^{(t)} = \frac{d(\mu_{c_{t,i}} + l - c_{t,i} - t)}{m} - l(t) + i.$$

Therefore, we have $\lambda_i^{(t)} - \mu_j^{(t)} + j - i = \frac{d\mathsf{M}(t)_{i,j}}{m}$ for $1 \leqslant i, j \leqslant l(t)$.

Since $\lambda_i^{(t)} - \mu_i^{(t)}$ is divisible by d for all $1 \leqslant i \leqslant l(t)$ and $d < m$, by induction hypothesis, we have

$$s_{\lambda^{(t)}/\mu^{(t)}}(1, q, \ldots, q^{kd-1}) \equiv \sum_{e \mid d} b_e \frac{q^d - 1}{q^{\frac{d}{e}} - 1} \pmod{q^d - 1},$$

where b_e's are nonnegative integers. By replacing q by $q^{\frac{m}{d}}$, we have

$$s_{\lambda^{(t)}/\mu^{(t)}}(1, q^{\frac{m}{d}}, \ldots, (q^{\frac{m}{d}})^{kd-1}) \equiv \sum_{e \mid d} b_e \frac{q^m - 1}{q^{\frac{m}{e}} - 1} \pmod{q^m - 1}.$$

On the other hand, since $\frac{d\mathsf{M}(t)_{i,j}}{m} = \lambda_i^{(t)} - \mu_j^{(t)} + j - i$, (9) yields the following congruence:

$$s_{\lambda^{(t)}/\mu^{(t)}}(1, q, \ldots, q^{kd-1}) \equiv \det\left(\sum_{e \mid d} a_e \left(\frac{d\mathsf{M}(t)_{i,j}}{m}, d\right) \frac{q^d - 1}{q^{\frac{d}{e}} - 1}\right)_{1 \leqslant i, j \leqslant l(t)} \pmod{q^d - 1}.$$

This tells us that

$$
\begin{aligned}
s_{\lambda^{(t)}/\mu^{(t)}}(1, q^{\frac{m}{d}}, \ldots, (q^{\frac{m}{d}})^{kd-1}) &\equiv \det\left(\sum_{e \mid d} a_e \left(\frac{d\mathsf{M}(t)_{i,j}}{m}, d\right) \frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right)_{1 \leqslant i, j \leqslant l(t)} \\
&\equiv \det\left(\sum_{e \mid d} a_e(\mathsf{M}(t)_{i,j}, m) \frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right)_{1 \leqslant i, j \leqslant l(t)} \\
&\equiv \det\left(\widehat{\mathsf{P}}(t)_{i,j}\right)_{1 \leqslant i, j \leqslant l(t)} \pmod{q^m - 1}.
\end{aligned}
\tag{13}
$$

Here the second congruence follows from the formula (10). As a consequence,

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1}\right] \det(\widehat{\mathsf{P}}(t)) = b_e \geqslant 0 \text{ for all } e \mid d.$$

This completes the proof. $\qquad \square$

Remark 9. Let d be a divisor of m and e a divisor of d. In the proof of Theorem 1, combining (11), (12), and (13), one can derive the equality

$$c_e = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] \prod_{0 \leqslant t \leqslant \frac{m}{d} - 1} s_{\lambda^{(t)}/\mu^{(t)}}(1, q^{\frac{m}{d}}, \ldots, (q^{\frac{m}{d}})^{kd-1}). \tag{14}$$

The idea of obtaining this equality plays an important role in proving Proposition 15.

Example 10. Let $m = 6$, $k = 1$, and $d = 2$. We here simply write $h_n(q)$ for $h_n(1, q, \ldots, q^5)$ for all $n \in \mathbb{Z}$. If $\lambda = (13, 10^3, 6)$ and $\mu = (7, 4^3)$, then 3-abacus display for λ/μ is
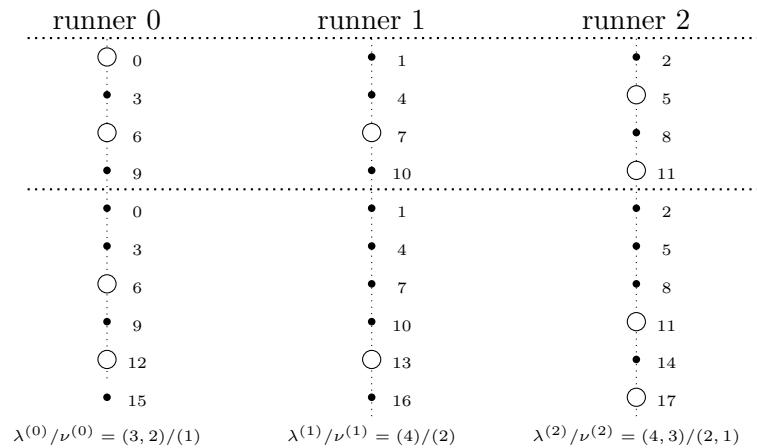
and $s_{\lambda/\mu}(1, q, \ldots, q^5) = \det(\mathsf{P})$ with

$$\mathsf{P} := \begin{bmatrix} h_6(q) & h_{10}(q) & h_{11}(q) & h_{12}(q) & h_{17}(q) \\ h_2(q) & h_6(q) & h_7(q) & h_8(q) & h_{13}(q) \\ h_1(q) & h_5(q) & h_6(q) & h_7(q) & h_{12}(q) \\ h_0(q) & h_4(q) & h_5(q) & h_6(q) & h_{11}(q) \\ h_{-5}(q) & h_{-1}(q) & h_0(q) & h_1(q) & h_6(q) \end{bmatrix}.$$

Since $\mathcal{C}_0 = \{3, 5\}$, $\mathcal{C}_1 = \{2\}$, $\mathcal{C}_2 = \{1, 4\}$, and $\mathsf{P}(t) = \mathsf{P}|_{\mathcal{C}_t \times \mathcal{C}_t}$, it follows that

$$\mathsf{P}(0) = \begin{bmatrix} h_6(q) & h_{12}(q) \\ h_0(q) & h_6(q) \end{bmatrix}, \quad \mathsf{P}(1) = \begin{bmatrix} h_6(q) \end{bmatrix}, \quad \text{and} \quad \mathsf{P}(2) = \begin{bmatrix} h_6(q) & h_{12}(q) \\ h_0(q) & h_6(q) \end{bmatrix}.$$

4 Proof of Theorem 2

In this section, we provide the proof of Theorem 2.

Lemma 11. *Let k, m, and n be positive integers. If n is a multiple of m, then*

$$h_n(1, q, \ldots, q^{k-1}) \equiv \sum_{0 \leqslant j \leqslant k-1} h_j(1, q, \ldots, q^{n-1}) \pmod{q^m - 1}.$$

Proof. Note that
$$h_n(1, q, \ldots, q^{k-1}) = \sum_{T \in \mathsf{SSYT}_k((n))} q^{\omega(T)},$$
where $\omega(T) = \sum_{1 \leqslant i \leqslant k}(i-1)m_i$ and m_i is the number of occurrences of i in T. The decomposition
$$\mathsf{SSYT}_k((n)) = \bigcup_{1 \leqslant i \leqslant k} \mathcal{A}_i$$
with
$$\mathcal{A}_i = \{T \in \mathsf{SSYT}_k((n)) : \text{the leftmost entry in } T \text{ is } i\}$$
gives rise to the equality
$$h_n(1, q, \ldots, q^{k-1}) = \sum_{1 \leqslant i \leqslant k} q^{i-1} h_{n-1}(q^{i-1}, \ldots, q^{k-1}).$$

Applying $h_{n-1}(q^{i-1}, \ldots, q^{k-1}) = q^{(n-1)(i-1)} h_{n-1}(1, q, \ldots, q^{k-i})$ to the above equality, we derive that
$$h_n(1, q, \ldots, q^{k-1}) \equiv \sum_{1 \leqslant i \leqslant k} h_{n-1}(1, q, \ldots, q^{k-i})$$
$$\equiv \sum_{0 \leqslant j \leqslant k-1} h_{n-1}(1, q, \ldots, q^j) \quad (\text{by letting } j := k - i)$$
$$\equiv \sum_{0 \leqslant j \leqslant k-1} h_j(1, q, \ldots, q^{n-1}) \pmod{q^m - 1}.$$

The third congruence follows from the equality $h_{n-1}(1, q, \ldots, q^j) = h_j(1, q, \ldots, q^{n-1})$. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $l := \ell(\lambda)$. Since λ/μ is a border strip, we may assume that $\ell(\mu) = l - 1$. Under this assumption, λ/μ can be naturally identified with the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ of $|\lambda| - |\mu|$, where $\alpha_i = \lambda_i - \mu_i$ for $1 \leqslant i \leqslant l$. From now on, we simply write $r_\alpha(1, q, \ldots, q^{k-1})$ and $\mathsf{M}(\alpha)$ for $s_{\lambda/\mu}(1, q, \ldots, q^{k-1})$ and $\mathsf{M}(\lambda/\mu)$, respectively. By (8), it holds that
$$\mathsf{M}(\alpha)_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_j & \text{if } 1 \leqslant i \leqslant j \leqslant l, \\ 0 & \text{if } 2 \leqslant i \leqslant l \text{ and } i - j = 1, \\ < 0 & \text{otherwise.} \end{cases}$$

Let $\mathsf{P}(\alpha; k)$ be the $l \times l$ matrix defined by
$$\mathsf{P}(\alpha; k)_{i,j} = h_{\mathsf{M}(\alpha)_{i,j}}(1, q, \ldots, q^{k-1}).$$

Combining Jacobi-Trudi formula with Lemma 6 and Lemma 7, we can easily derive that
$$r_\alpha(1, q, \ldots, q^{k-1}) = \det(\mathsf{P}(\alpha; k)) \equiv \sum_{d | m} c_d(\alpha, k, m) \frac{q^m - 1}{q^{\frac{m}{d}} - 1} \pmod{q^m - 1}, \qquad (15)$$

where $c_d(\alpha, k, m)$'s are integers. For our purpose, it should be verified that $c_d(\alpha, k, m) \geqslant 0$ for every divisor d of m. This will be accomplished by applying induction on m. When $m = 1$, there is nothing to prove. Suppose that $m \geqslant 2$ and our assertion holds for all positive integers less than m. Since every coefficient of $r_\alpha(1, q, \ldots, q^{k-1})$ is nonnegative, it follows that $c_m(\alpha, k, m) \geqslant 0$. We now choose an arbitrary divisor d of m smaller than m and set $k' := \lfloor \frac{d(k-1)}{m} \rfloor + 1$, where $\lfloor \frac{d(k-1)}{m} \rfloor$ denotes the integer part of $\frac{d(k-1)}{m}$.

Let $\widehat{\mathsf{P}}(d)$ be the $l \times l$ matrix defined by

$$\widehat{\mathsf{P}}(d)_{i,j} = \begin{cases} \sum_{0 \leqslant t \leqslant k'-1} h_{mt/d}(1, q, \ldots, q^{\mathsf{M}(\alpha)_{i,j}-1}) & \text{if } 1 \leqslant i \leqslant j \leqslant l, \\ 1 & \text{if } 2 \leqslant i \leqslant l, \, j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathsf{P}(\alpha; k)_{i,j} = \widehat{\mathsf{P}}(d)_{i,j}$ for $1 \leqslant j < i \leqslant l$. On the other hand, in case where $1 \leqslant i \leqslant j \leqslant l$, Lemma 6 and Lemma 11 imply that $\mathsf{P}(\alpha; k)_{i,j}$ is a CSP polynomial modulo m satisfying that

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] \mathsf{P}(\alpha; k)_{i,j} = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] \widehat{\mathsf{P}}(d)_{i,j} \quad \text{for all } e \,|\, d.$$

So, from Lemma 7 it follows that $c_d(\alpha, k, m)$ is equal to $c'_d := \left[\frac{q^m - 1}{q^{\frac{m}{d}} - 1} \right] \widehat{\mathsf{P}}(d)$.

In the following, we will see that $c'_d \geqslant 0$. Let $\beta := \left(\frac{d\alpha_1}{m}, \frac{d\alpha_2}{m}, \ldots, \frac{d\alpha_l}{m} \right)$. For $1 \leqslant j < i \leqslant l$, it holds that $\widehat{\mathsf{P}}(d)_{i,j} = \mathsf{P}(\beta; k')_{i,j}$. We next consider the case where $1 \leqslant i \leqslant j \leqslant l$. Recall that

$$\mathsf{M}(\beta)_{i,j} = \frac{d\mathsf{M}(\alpha)_{i,j}}{m}.$$

For each $0 \leqslant t \leqslant k' - 1$, Lemma 6 shows that $h_t(1, q^{\frac{m}{d}}, \ldots, (q^{\frac{m}{d}})^{\mathsf{M}(\beta)_{i,j}-1})$ is a CSP polynomial modulo m and

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] h_t(1, q^{\frac{m}{d}}, \ldots, (q^{\frac{m}{d}})^{\mathsf{M}(\beta)_{i,j}-1}) = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] h_{mt/d}(1, q, \ldots, q^{\mathsf{M}(\alpha)_{i,j}-1}) \quad \text{for all } e \,|\, d.$$

Hence $\widehat{\mathsf{P}}(d)_{i,j}$ and $\mathsf{P}(\beta, k')_{i,j}$ are CSP polynomials modulo m satisfying that

$$\left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] \mathsf{P}(\beta; k')_{i,j} = \left[\frac{q^m - 1}{q^{\frac{m}{e}} - 1} \right] \widehat{\mathsf{P}}(d)_{i,j} \quad \text{for all } e \,|\, d.$$

Now, by applying Lemma 7 to this identity, we derive that $c'_d = c_d(\beta, k', d)$. Since β_i is divisible by d for all $1 \leqslant i \leqslant l$ and $d < m$, our induction hypothesis implies that $c_d(\beta, k', d) \geqslant 0$, as required. $\qquad\square$

Remark 12. In the proof of Theorem 2, we can additionally observe the following.

(a) Let $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(\frac{m}{d}-1)}/\mu^{(\frac{m}{d}-1)})$ be the $\frac{m}{d}$-quotient of λ/μ. Then $\lambda^{(0)}/\mu^{(0)}$ is the border strip such that $\lambda_i^{(0)} - \mu_i^{(0)} = \beta_i$ for all $1 \leqslant i \leqslant l$ and $\lambda^{(t)}/\mu^{(t)} = \emptyset$ for all $1 \leqslant t \leqslant \frac{m}{d} - 1$.

(b) It holds that $c_d(\alpha, k, m) = c_d(\beta, k', d)$, which can be viewed as an analogue of Lemma 6 (for the definition of either side, see (15)).

5 Principal specializations of skew Schur polynomials at roots of unity

Let λ/μ be a skew partition of size n. The skew characters $\chi^{\lambda/\mu}(\nu)$ of the symmetric group $\mathfrak{S}_n$ are then defined implicitly via

$$s_{\lambda/\mu}(X) = \sum_\nu \chi^{\lambda/\mu}(\nu) \frac{p_\nu(X)}{z_\nu}, \tag{16}$$

where the sum is over all partitions ν of n, $z_\nu := \prod_j m_j! j^{m_j}$, m_j is the number of parts in ν equal to j, and $p_\nu(X)$ is the power-sum symmetric function indexed by ν. It can be found in [18, Theorem 7.17.3] that

$$\chi^{\lambda/\mu}(\nu) = \sum_T (-1)^{\text{ht}(\text{T})},$$

summed over all border-strip tableaux of shape λ/μ and type ν. In case where $\nu = (d^m)$ with $n = dm$, it was proved in [4, Corollary 30] that

$$\chi^{\lambda/\mu}((d^m)) = \epsilon |\text{BST}(\lambda/\mu, d)|,$$

where $\text{BST}(\lambda/\mu, d)$ is the set of all border-strip tableaux of shape λ/μ and type (d^m) and $\epsilon = (-1)^{\text{ht}(\text{T})}$ for any $T \in \text{BST}(\lambda/\mu, d)$. The following lemma was also presented in the same paper.

Lemma 13. ([4, Lemma 18]) *Let λ/μ be a skew shape. Then the d-quotient of λ/μ exists if and only if $\text{BST}(\lambda/\mu, d)$ is nonempty.*

In the following, we will show that ϵ is given by the signature of a particular permutation in $\mathfrak{S}_n$, assuming that the d-quotient of λ/μ exists. As above, let $|\lambda/\mu| = dm$ and let $l = \ell(\lambda)$. For $0 \leqslant r \leqslant d-1$, let

$$\lambda[r] := \{1 \leqslant j \leqslant l : \lambda_j + (l-j) \equiv r \pmod{d}\} \quad \text{and}$$
$$\mu[r] := \{1 \leqslant j \leqslant l : \mu_j + (l-j) \equiv r \pmod{d}\}.$$

Since the d-cores of λ and μ coincide, one sees that $\lambda[r]$ and $\mu[r]$ have the same number of elements. Let us enumerate the elements $\lambda[r]$ and $\mu[r]$ in the increasing orders:

$$\lambda[r] = \{a_{r,1} < a_{r,2} < \cdots < a_{r,i_r}\},$$
$$\mu[r] = \{b_{r,1} < b_{r,2} < \cdots < b_{r,i_r}\}.$$

Mapping $a_{r,i}$ to $b_{r,i}$ for $1 \leqslant i \leqslant i_r$ and for $r = 0, 1, \ldots, d-1$, one can obtain a permutation on $\{1, \ldots, l\}$. The resulting permutation is denoted by $\mathbf{perm}(\lambda/\mu)$. For instance, if $\lambda/\mu = (9^2, 6^3, 4, 1)/(2, 1^3)$, then

$$\lambda + \delta_l = (15, 14, 10, 9, 8, 5, 1) \equiv (0, 2, 1, 0, 2, 2, 1) \pmod{3},$$
$$\mu + \delta_l = (8, 6, 5, 4, 2, 1, 0) \equiv (2, 0, 2, 1, 2, 1, 0) \pmod{3},$$

and thus $\mathbf{perm}(\lambda/\mu) = 2147356$.

Since the d-quotient of λ/μ exists, one can obtain μ from λ by removing border strips of size d. Suppose that one removes a border strip of size d and height k whose highest and rightmost cell is placed in the ith row of λ. Let us denote the resulting partition by $\tilde{\lambda}$. It is easily seen that

$$\tilde{\lambda}_j = \begin{cases} \lambda_j & \text{if } 1 \leqslant j \leqslant i-1 \text{ or } i+k+1 \leqslant j \leqslant l, \\ \lambda_{j+1} - 1 & \text{if } i \leqslant j \leqslant i+k-1, \\ \lambda_i - d + k & \text{if } j = i+k, \end{cases}$$

and therefore

$$\tilde{\lambda} + \delta_l = (\dots, \underbrace{\lambda_{i+1} + l - (i+1)}_{i^{\text{th}}}, \lambda_{i+2} + l - (i+2), \dots, \lambda_{i+k} + l - (i+k), \underbrace{\lambda_i + (l-i) - d}_{(i+k)^{\text{th}}}, \dots).$$

Note that the symmetric group $\mathfrak{S}_l$ acts on the set of weak compositions of length l by place permutation. Using this action, one can see that

$$\tilde{\lambda} + \delta_l = \begin{cases} s_{i+k-1} \cdots s_{i+1} s_i \cdot (\lambda + \delta_l - de_i) & \text{if } k > 0, \\ \mathrm{id}_i \cdot (\lambda + \delta_l - de_i) & \text{if } k = 0, \end{cases}$$

where $e_i := (0, \dots, \overset{i\text{th}}{1}, 0, \dots, 0)$ and $s_j := (j\ j+1)$ are simple transpositions of $\mathfrak{S}_l$. In case where $k = 0$, the subscript in id_i is used to indicate the row from which the border strip of zero height is removed. One can see that k is equal to the length of $s_{i+k-1} \cdots s_{i+1} s_i$. Continue this process until we get $\mu + \delta_l$ from $\lambda + \delta_l$. Composing permutations appearing at each step, one can recover $\mathbf{perm}(\lambda/\mu)$. The expressions of $\mathbf{perm}(\lambda/\mu)$ obtained in this way are in bijection with the methods of obtaining μ from λ by removing the border strips of size d. Since each method can be uniquely described as a border-strip tableaux of shape λ/μ and type (d^m) by numbering the strips in descending order, we derive that the set of the expressions of $\mathbf{perm}(\lambda/\mu)$ obtained in this way is one-to-one correspondence with $\mathrm{BST}(\lambda/\mu, d)$.

Example 14. Let $\lambda/\mu = (9^2, 6^3, 4, 1)/(2, 1^3)$, $l = 7$, and $d = 3$. Then $\lambda + \delta_l = (15, 14, 10, 9, 8, 5, 1)$ and repeat the above process until we get $\mu + \delta_l$ as follows:
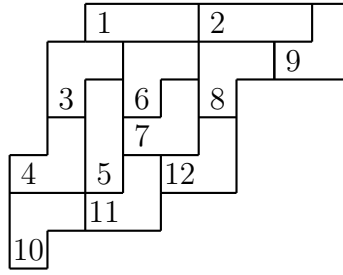
$$(15, 14, 10, 9, 8, 5, 1) \overset{s_4}{\rightsquigarrow} (15, 14, 10, 8, 6, 5, 1) \overset{s_5}{\rightsquigarrow} (15, 14, 10, 8, 5, 3, 1) \overset{s_6}{\rightsquigarrow}$$

$$(15, 14, 10, 8, 5, 1, 0) \overset{s_1}{\rightsquigarrow} (14, 12, 10, 8, 5, 1, 0) \overset{s_2}{\rightsquigarrow} (14, 10, 9, 8, 5, 1, 0) \overset{s_3}{\rightsquigarrow}$$

$$(14, 10, 8, 6, 5, 1, 0) \overset{s_2}{\rightsquigarrow} (14, 8, 7, 6, 5, 1, 0) \overset{s_4 s_3}{\rightsquigarrow} (14, 8, 6, 5, 4, 1, 0) \overset{s_4}{\rightsquigarrow}$$

$$(14, 8, 6, 4, 2, 1, 0) \overset{s_2}{\rightsquigarrow} (14, 6, 5, 4, 2, 1, 0) \overset{\mathrm{id}_1}{\rightsquigarrow} (11, 6, 5, 4, 2, 1, 0) \overset{\mathrm{id}_1}{\rightsquigarrow}$$

$$(8, 6, 5, 4, 2, 1, 0) = \mu + \delta_l$$

Thus we obtain an expression

$$\mathbf{perm}(\lambda/\mu) = \mathrm{id}_1 \cdot \mathrm{id}_1 \cdot s_2 \cdot s_4 \cdot s_4 s_3 \cdot s_2 \cdot s_3 \cdot s_2 \cdot s_1 \cdot s_6 \cdot s_5 \cdot s_4$$

and the corresponding border-strip tableau is

Proposition 15. *Let d be a positive integer. If $|\lambda/\mu| = dm$ and the d-quotient of λ/μ exists, then*

$$\mathrm{sgn}(\mathbf{perm}(\lambda/\mu)) = \mathrm{sgn}(\chi^{\lambda/\mu}((d^m))).$$

Proof. Let us obtain an expression of $\mathbf{perm}(\lambda/\mu)$ in the above way. The number of simple transpositions appearing in this expression equals the height of the corresponding border-strip tableau of shape λ/μ and type (d^m). Now the desired result follows from [4, Corollary 30]. $\qquad\square$

The following theorem is a skew version of [16, Theorem 4.3].

Theorem 16. *Let $d \mid N$ and let ω_d be a primitive dth root of unity. Then $s_{\lambda/\mu}(1, \omega_d, \ldots, \omega_d^{N-1})$ is zero unless the d-quotient of λ/μ exists, in which case*

$$s_{\lambda/\mu}(1, \omega_d, \ldots, \omega_d^{N-1}) = \mathrm{sgn}(\chi^{\lambda/\mu}((d^m))) \prod_{i=0}^{d-1} s_{\lambda^{(i)}/\mu^{(i)}}(\underbrace{1, 1, \ldots, 1}_{\frac{N}{d}}),$$

where $|\lambda/\mu| = dm$ and the d-quotient of λ/μ is $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(d-1)}/\mu^{(d-1)})$.

Proof. In the case where the d-quotient of λ/μ does not exist, the proof can be done in the same way as in [16, Theorem 4.3]. Choosing $X = (1, \omega_d, \ldots, \omega_d^{N-1}, 0, 0, \ldots)$ gives $p_j(X) = 0$ unless $d \mid j$. Hence, in view of (16), one has that

$$s_{\lambda/\mu}(1, \omega_d, \ldots, \omega_d^{N-1}) = \sum_{\nu: d\text{-stretched}} \chi^{\lambda/\mu}(\nu) \frac{p_\nu(1, \omega_d, \ldots, \omega_d^{N-1})}{z_\nu}.$$

By Lemma 13, from the condition that the d-quotient of λ/μ does not exist it follows that $\mathrm{BST}(\lambda/\mu, d)$ is empty. Thus $\mathrm{BST}(\lambda/\mu, \nu)$ is also empty. Combining this with the formula

$$\chi^{\lambda/\mu}(\nu) = \sum_{B \in \mathrm{BST}(\lambda/\mu, \nu)} (-1)^{\mathrm{ht(B)}},$$

we have $\chi^{\lambda/\mu}(\nu) = 0$ and our assertion follows.

Now suppose that the d-quotient of λ/μ exists. Let $l := \ell(\lambda)$, $\sigma := \mathbf{perm}(\lambda/\mu)$, and $\mathsf{M} := \sigma \cdot \mathsf{M}(\lambda/\mu)$. Here σ acts on $\mathsf{M}(\lambda/\mu)$ by permuting the rows of $\mathsf{M}(\lambda/\mu)$. Then

$$s_{\lambda/\mu}(1, q, \ldots, q^{N-1}) = \det(\mathsf{M}(\lambda/\mu)) = \mathrm{sgn}(\sigma) \det(\mathsf{M}) = \mathrm{sgn}(\chi^{\lambda/\mu}((d^m))) \det(\mathsf{M}).$$

The last equality follows from Proposition 15. Note that $\det(\mathsf{M})$ is a pre-CSP polynomial modulo N. Write

$$\det(\mathsf{M}) \equiv \sum_{e \mid N} c_e \frac{q^N - 1}{q^{\frac{N}{e}} - 1} \quad (\text{mod } q^N - 1),$$

where c_e's are integers.

Consider the d-abacus display for λ/μ and let $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(d-1)}/\mu^{(d-1)})$ be the d-quotient of λ/μ. For $0 \leqslant t \leqslant d-1$, let

$$\mathcal{C}_t := \{1 \leqslant i \leqslant l : \text{the bead for } \mu_i + l - i \text{ is on runner } t \text{ of the abacus display}\}.$$

By the definition of σ, one can easily see that the bead for $\lambda_i + l - i$ is on runner t if and only if $\sigma(i) \in \mathcal{C}_t$. Therefore, it follows that $\mathsf{M}_{i,j}$ is divisible by d if and only if $i, j \in \mathcal{C}_t$ for some $0 \leqslant t \leqslant d-1$. Using the same argument as in the proof of Theorem 1, for each divisor $e \mid \frac{N}{d}$, one can derive that

$$c_e = \left[\frac{q^N - 1}{q^{\frac{N}{e}} - 1}\right] \prod_{0 \leqslant t \leqslant d-1} s_{\lambda^{(t)}/\mu^{(t)}}(1, q^d, \ldots, q^{d(\frac{N}{d}-1)}). \tag{17}$$

Using (2), one can derive from (17) that

$$\det(\mathsf{M})|_{q=\omega_d} = \sum_{e \mid \frac{N}{d}} e c_e = \prod_{0 \leqslant t \leqslant d-1} s_{\lambda^{(t)}/\mu^{(t)}}(1, q^d, \ldots, q^{d(\frac{N}{d}-1)})\Big|_{q=\omega_d}$$

$$= \prod_{0 \leqslant t \leqslant d-1} s_{\lambda^{(t)}/\mu^{(t)}}(\underbrace{1, 1, \ldots, 1}_{\frac{N}{d}}).$$

$\square$

Remark 17. (a) Reiner, Stanton, and White proved [16, Theorem 4.3] by using the hook-content formula for $s_\lambda(1, q, \ldots, q^{N-1})$, so one is tempting to prove Theorem 16 in a similar manner. However, to the best of the authors' knowledge, this formula has not yet extended to general skew shapes (see [12, Section 8]).

(b) A similar result has been provided in [11, Examples I.5.24] in a slightly different form.

The rest of this section is devoted to the case where $d = N$. Let $\ell(\lambda) \leqslant N$. From [16, Theorem 4.3] it follows that $s_\lambda(1, \omega_N, \ldots, \omega_N^{N-1})$ is zero unless the N-core of λ is empty, in which case

$$s_\lambda(1, \omega_N, \ldots, \omega_N^{N-1}) = \text{sgn}(\chi^\lambda((N^m))) \quad \text{with } |\lambda| = Nm. \tag{18}$$

This evaluation can also be found in [11, Examples I.3.17]. Recall that the *Kostka-Foulkes polynomial* $K_{\lambda,\nu}(q)$ is defined via the expansion

$$s_\lambda(x_1, x_2, \ldots, x_N) = \sum_\nu K_{\lambda,\nu}(q) P_\nu(x_1, x_2, \ldots, x_N; q),$$

where $P_\nu(x_1, x_2, \ldots, x_N; q)$ is the Hall-Littlewood polynomial indexed by ν. And the *skew Kostka-Foulkes polynomial* $K_{\lambda/\mu,\nu}(q)$ is defined by

$$K_{\lambda/\mu,\nu}(q) := \langle s_{\lambda/\mu}(X),\ Q'_\nu(X;q)\rangle, \tag{19}$$

where $Q'_\nu(X;q) = \sum_\gamma K_{\gamma,\nu}(q)s_\gamma(X)$ (see [7]).

Combining (18) with [7, Theorem 9.14], for any partition λ with $|\lambda| = Nm$ and $\ell(\lambda) \leqslant N$, we have

$$K_{\lambda,m^N}(\omega_N) = \begin{cases} (-1)^{(N-1)m}\operatorname{sgn}(\chi^\lambda((N^m))) & \text{if the } N\text{-core of } \lambda \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

We will derive a skew version of this formula. To begin with, we observe that

$$Q'_{(m^N)}(X;\omega_N) = (-1)^{(N-1)m}p_N \circ h_m(X) = (-1)^{(N-1)m}\sum_{\lambda \vdash Nm} s_\lambda(1, \omega_N, \ldots, \omega_N^{N-1})s_\lambda(X),$$

where $p_N \circ h_m$ denotes the plethysm of h_m by p_N. The first equality follows from [7, Theorem 9.2] and the second equality follows from [11, Examples I.8.7]. Plugging this identity to (19) yields that, for any λ/μ with $|\lambda/\mu| = Nm$,

$$K_{\lambda/\mu,m^N}(\omega_N) = (-1)^{(N-1)m}s_{\lambda/\mu}(1, \omega_N, \ldots, \omega_N^{N-1}). \tag{21}$$

Define a *horizontal strip* to be a skew shape with no two squares in the same column. The following corollary is the desired skew version of (20).

Corollary 18. (cf. [11, Examples I.5.24]) *Let ω_N be a primitive Nth root of unity.*

(a) $s_{\lambda/\mu}(1, \omega_N, \ldots, \omega_N^{N-1})$ *is zero unless $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip for all $0 \leqslant i \leqslant N-1$, in which case*

$$s_{\lambda/\mu}(1, \omega_N, \ldots, \omega_N^{N-1}) = \operatorname{sgn}(\chi^{\lambda/\mu}((N^m))) \quad \text{with } |\lambda/\mu| = Nm.$$

(b) *For any λ/μ with $|\lambda/\mu| = Nm$, $K_{\lambda/\mu,m^N}(\omega_N)$ is zero unless $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip for all $0 \leqslant i \leqslant N-1$, in which case*

$$K_{\lambda/\mu,m^N}(\omega_N) = (-1)^{(N-1)m}\operatorname{sgn}(\chi^{\lambda/\mu}((N^m))).$$

Proof. (a) follows from Theorem 16 when $d = N$.

(b) can be obtained by combining (a) with (21). $\qquad\square$

6 Remarks

We provide a few remarks for readers' understanding.

1. As stated in Introduction, the present paper is motivated by the conjecture in [4, Conjecture 50], which is equivalent to saying that $s_{m\lambda/m\mu}(1, q, \ldots, q^{k-1})$ is a CSP polynomial modulo m. It was shown in [1] that this is true in case where $(k, m(|\lambda| - |\mu|)) = 1$. However, one can find a counterexample in case where k is not divisible by m and $(k, m(|\lambda| - |\mu|)) \neq 1$. For instance, if $\lambda = (3, 3, 2, 1), \mu = (2, 1), m = 9$, and $k = 4$, using a computer, one can easily see that

$$s_{9\lambda/9\mu}(1, q, q^2, q^3) \equiv 54665112 \frac{q^9 - 1}{q - 1} - 3 \frac{q^9 - 1}{q^3 - 1} + 1 \pmod{q^9 - 1}.$$

On the other hand, in case where $\lambda = (3, 3, 1)$, $\mu = (2, 1)$, $m = 4$, and $k = 6$, k is not divisible by m but $(k, |m\lambda/m\mu|) = (6, 16) = 2 \neq 1$. In this case, one can see that

$$s_{4\lambda/4\mu}(1, q, \ldots, q^5) \equiv 1576440 \frac{q^4 - 1}{q - 1} + 264 \frac{q^4 - 1}{q^2 - 1} + 12 \pmod{q^4 - 1}.$$

It would be very nice to find a sharp condition for the triples $(m\lambda/m\mu, m, k)$ such that $s_{m\lambda/m\mu}(1, q, \ldots, q^{k-1})$ is a CSP polynomial modulo m.

2. Based on [3, Theorem 37], one may expect that any stretched skew Schur polynomial $s_{m\lambda/m\mu}(1, q, \ldots, q^{k-1})$ can be a CSP polynomial modulo m when multiplied by an appropriate power of q. However, this immediately turns out to be false. For instance, as shown in (1), $s_{9\lambda/9\mu}(1, q, q^2, q^3)$ is not a CSP polynomial modulo 9 when $\lambda = (3, 3, 2, 1)$, $\mu = (2, 1)$, $m = 9$, and $k = 4$. In this case, $q^i s_{9\lambda/9\mu}(1, q, q^2, q^3)$ are not CSP polynomials modulo 9 for all $1 \leqslant i \leqslant 8$, and which can be easily checked with the help of a computer. Indeed, they are not even pre-CSP polynomials modulo 9.

3. It is well known that there is a one-to-one correspondence

$$\mathsf{SSYT}_k(\lambda/\mu) \overset{1-1}{\longleftrightarrow} \bigcup_{|\lambda| = |\mu| + |\nu|} \mathsf{SSYT}_k(\nu)^{c_{\mu\nu}^{\lambda}},$$

where $c_{\mu\nu}^{\lambda}$ is the Littlewood-Richardson number indexed by λ, μ and ν (see [8, Section 5] or [5, Example 3.3]). However, some ν's appearing on the right hand side may not be m-stretched, thus [14, Corollary 3.4] cannot be applicable in attacking the conjecture in [4, Conjecture 50]. For example, in case where $\lambda = (3, 2), \mu = (1)$, $m = 4$, and $k = 4$, there is a one-to-one correspondence between $\mathsf{SSYT}_4(4\lambda/4\mu)$ and

$$\mathsf{SSYT}_4(11, 5) \cup \mathsf{SSYT}_4(12, 4) \cup \mathsf{SSYT}_4(8, 8) \cup \mathsf{SSYT}_4(10, 6) \cup \mathsf{SSYT}_4(9, 7).$$

Acknowledgments

References

[1] P. Alexandersson. Free action and cyclic sieving on skew semistandard Young tableaux. arXiv:1906.08216, 2019.

[2] P. Alexandersson and N. Amini. The cone of cyclic sieving phenomena. *Discrete Math.*, 342(6):1581–1601, 2019.

[3] P. Alexandersson, E. K. Oğuz, and S. Linusson. Promotion and cyclic sieving on families of SSYT. *Ark. Mat.*, 59(2):247–274, 2021.

[4] P. Alexandersson, S. Pfannerer, M. Rubey, and J. Uhlin. Skew characters and cyclic sieving. *Forum Math. Sigma*, 9:Paper No. e41, 31, 2021.

[5] G. Benkart, F. Sottile, and J. Stroomer. Tableau switching: algorithms and applications. *J. Combin. Theory Ser. A*, 76(1):11–43, 1996.

[6] M. Bennett, B. Madill, and A. Stokke. Jeu-de-taquin promotion and a cyclic sieving phenomenon for semistandard hook tableaux. *Discrete Math.*, 319:62–67, 2014.

[7] J. Désarménien, B. Leclerc, and J.-Y. Thibon. Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory. *Sém. Lothar. Combin.*, 32:Art. B32c, approx. 38, 1994.

[8] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[9] S. Hopkins. Cyclic sieving for plane partitions and symmetry. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 130, 40 pp., 2020.

[10] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[11] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[12] A. H. Morales, I. Pak, and G. Panova. Hook formulas for skew shapes I. q-analogues and bijections. *J. Combin. Theory Ser. A*, 154:350–405, 2018.

[13] Y.-T. Oh and E. Park. Crystals, semistandard tableaux and cyclic sieving phenomenon. *Electron. J. Combin.*, 26(4): #P4.39, 2019.

[14] Y.-T. Oh and E. Park. q-dimensions of highest weight crystals and cyclic sieving phenomenon. *European J. Combin.*, 97:Paper No. 103372, 14, 2021.

[15] S. Pon and Q. Wang. Promotion and evacuation on standard young tableaux of rectangle and staircase shape. *Electron. J. Combin.*, 18(1): #P18, 2011.

[16] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108(1):17–50, 2004.

[17] B. Rhoades. Cyclic sieving, promotion, and representation theory. *J. Combin. Theory Ser. A*, 117(1):38–76, 2010.

[18] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[19] J. R. Stembridge. On minuscule representations, plane partitions and involutions in complex Lie groups. *Duke Math. J.*, 73(2):469–490, 1994.

[20] J. R. Stembridge. Canonical bases and self-evacuating tableaux. *Duke Math. J.*, 82(3):585–606, 1996.