Classification of Cocovers in the Double Affine Bruhat Order

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Abstract

We classify cocovers of a given element of the double affine Weyl semigroup $W_{\mathcal{T}}$ with respect to the Bruhat order, specifically when $W_{\mathcal{T}}$ is associated to a finite root system that is irreducible and simply laced. We do so by introducing a graphical representation of the length difference set defined by Muthiah and Orr and identifying the cocovering relations with the corners of those graphs. This new method allows us to prove that there are finitely many cocovers of each $x \in W_{\mathcal{T}}$. Further, we show that the Bruhat intervals in the double affine Bruhat order are finite.

Mathematics Subject Classifications: 05E99, 20F55, 06A07

1 Introduction

The double affine Bruhat order was first introduced by Braverman, Kazhdan, and Patnaik [2] in their study of Iwahori-Hecke algebras for affine Kac-Moody groups. They called it the Bruhat preorder and conjectured that it was an order (it was already known that it is an order in the finite and affine cases). In [10] it was shown that the preorder is in fact an order, and in [12] it was shown that the order coincides with the order generated by relations involving the length function on the double affine Weyl semigroup $W_{\mathcal{T}}$, which was defined in [10].

The double affine Bruhat order is an extension of the Bruhat order on the (single) affine Weyl group, but when working in the double affine case, one cannot rely on Coxeter theory. This makes it more difficult to extend key ideas. However, recent progress has been made by Muthiah and Orr. In [12] they related the cocover and cover relationships to a difference in lengths when the finite root system in question is simply laced. Additionally, Muthiah [11] has made use of the results in this paper to make further advances in the field.

This paper focuses on classifying cocovers under the double affine Bruhat order. For $x \in W_{\mathcal{T}}$, a cover of x is defined to be an element $y \in W_{\mathcal{T}}$ such that $x < y$ under the Bruhat order and there is no $z \in W_{\mathcal{T}}$ such that $x < z < y$. Similarly, y is said to be a cocover of x if $y < x$ and there is no $z \in W_{\mathcal{T}}$ such that $y < z < x$.

Our classification allows us to better understand the intervals that arise from the order. Specifically, we show that these intervals are finite. To do so, we develop a new technique for representing covering relations by introducing a graphical representation of the length difference set defined by Muthiah and Orr [12] and identifying cocovers as "corners" of these graphs.

Proposition 1. (*Proposition 21 below*) *If $y = s_\alpha x$ is a cocover of x, then α must correspond to a corner in the graph $\Gamma_{x,\mathrm{fin}(\alpha)}$.*

Using this new technique, and extending the work done by Lam and Shimozono [6] and further strengthened by Milićević [9], who classified cocovers in the affine case, we are able to fully classify cocovers under the double affine Bruhat order:

Theorem 2. (*Theorem 28 below*) *Let $x = X^{\widetilde{v}\zeta}\widetilde{w}$ and $y = s_\alpha x$ where $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi$ is a positive double affine root and $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, \ldots, n$. Then y is a cocover of x if and only if one of the following holds:*

1. $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$ *and* $j = 0$.
2. $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$ *and* $j = 1$.
3. $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1$ *and* $j = \langle \zeta, \widetilde{\alpha} \rangle$.
4. $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$ *and* $j = \langle \zeta, \widetilde{\alpha} \rangle - 1$.

Further, we show that there are finitely many corners of the length difference graphs, and thus, finitely many cocovers for each element of $W_{\mathcal{T}}$. This leads to the following theorem concerning the Bruhat intervals.

Theorem 3. (*Theorem 25 below*) *Let $x, y \in W_{\mathcal{T}}$ such that $y \leqslant x$. Then the double affine Bruhat interval $[y, x]$ is finite.*

We end by mirroring the work done by Lam and Shimozono [6] and Milićević [9] in the affine case to create a relationship between cocovers in the double affine setting and the affine quantum Bruhat graph.

2 Background

Before we begin, we must introduce our notation. We note that the finite and single affine objects and notation mostly follows that of Kac's [5] and Humphreys' [4]. For the double affine case, we follow the work of Muthiah and Orr [10], [12].

Let W_{fin} denote our finite Weyl group with associated root system Φ_{fin} irreducible and simply laced. Let W_{aff} denote the affine Weyl group created from the semidirect product of the translation group associated to $Q = \mathbb{Z}\Phi_{\text{fin}}$ with W_{fin}. Let $\{\alpha_i \mid 0 \leqslant i \leqslant n\}$ be the simple roots such that W_{aff} is generated by the affine reflections s_{α_i} for $0 \leqslant i \leqslant n$, and let $\delta = \alpha_0 + \theta$, where θ is the highest root in Φ_{fin}. Commonly, δ is referred to as the null root. Because Φ_{fin} is simply laced, we have a pairing $\langle\ ,\ \rangle$ such that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi_{\text{fin}}$. This allows us to identify Φ_{fin} with $\Phi_{\text{fin}}^{\vee}$, the set of coroots.

Let P_{fin} be the finite weight lattice. We choose the weight Λ_0 so that $\langle \Lambda_0, \alpha_i \rangle = 0$ for $0 < i \leqslant n$ and $\langle \Lambda_0, \delta \rangle = 1$. Let $X = P_{\text{fin}} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0$, the affine weight lattice. Given $\zeta = \mu + m\delta + l\Lambda_0 \in X$, we call l the *level* of ζ and denote it by $\text{lev}(\zeta) = l$. The level of the element is preserved under the action of W_{aff} on X. Let $\lambda \in Q$ and $w \in W_{\text{fin}}$. We denote a typical element of W_{aff} as $\widetilde{w} = Y^{\lambda} w$ where Y^{λ} is the translation by λ. The action of W_{aff} on X is defined by

$$Y^{\lambda} w(\mu + m\delta + l\Lambda_0) = w(\mu) + l\lambda + (m - \langle w(\mu), \lambda \rangle - l\frac{\langle \lambda, \lambda \rangle}{2})\delta + l\Lambda_0.$$

Let X_{dom} be the set of all dominant elements of X. That is, let X_{dom} contain all $\zeta \in X$ such that $\langle \zeta, \alpha_i \rangle \geqslant 0$ for $0 \leqslant i \leqslant n$. Then the *Tits cone* $\mathcal{T}$ is given by

$$\mathcal{T} = \bigcup_{w \in W_{\text{aff}}} w(X_{\text{dom}}).$$

Note that the Tits cone is the subset of X containing all elements that can be made dominant by some element of W_{aff}. Alternatively, we can view the Tits cone as a union of two sets, with one set containing elements of level zero and the other containing the elements with positive level:

$$\mathcal{T} = \{m\delta : m \in \mathbb{Z}\} \cup \{\mu + m\delta + l\Lambda_0 : \mu \in P_{\text{fin}},\ m \in \mathbb{Z},\ l \in \mathbb{Z}_{>0}\},$$

where $\mathbb{Z}_{>0}$ represents the positive integers. Under this decomposition, we see that $\mathcal{T}$ contains all the imaginary roots (roots of the form $m\delta$) and all the roots with $l > 0$, but it contains no elements with a negative level.

The *double affine Weyl semigroup* [2] [10], which we denote by $W_{\mathcal{T}}$, is the semidirect product of the the translation semigroup associated to $\mathcal{T}$ with W_{aff}:

$$\begin{aligned} W_{\mathcal{T}} &= \mathcal{T} \rtimes W_{\text{aff}} \\ &= \{X^{\zeta}\widetilde{w} : \zeta \in \mathcal{T}, \widetilde{w} \in W_{\text{aff}}\} \\ &= \{X^{\zeta} Y^{\lambda} w : \zeta \in \mathcal{T}, \lambda \in Q, w \in W_{\text{fin}}\}. \end{aligned}$$

For simplicity, we will use $\text{lev}(x)$ to denote the level of the X-weight of $x \in W_{\mathcal{T}}$ (i.e. if $x = X^{\zeta} Y^{\lambda} w \in W_{\mathcal{T}}$ then $\text{lev}(x) = \text{lev}(\zeta)$). As $W_{\mathcal{T}}$ contains no elements with $\text{lev}(x) < 0$, we see that $W_{\mathcal{T}}$ is a semigroup, but not a group.

2.1 Roots and Reflections

Let $Q_{\text{daff}} = \mathbb{Z}\Phi_{\text{fin}} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\pi$, where π is an additional null root that arises from the double affinization (see [2] and [10]). The set of *double affine roots* is a subset of Q_{daff} given by

$$\Phi = \{\nu + r\delta + j\pi : \nu \in \Phi_{\text{fin}}, r, j \in \mathbb{Z}\}.$$

Like the affine roots, the double affine root system can be divided into *positive* and *negative* roots. Recall that we consider an affine root $\widetilde{\alpha}$ to be positive if $\widetilde{\alpha} = \nu + r\delta$ where ν is a positive root in Φ_{fin} and $r \geqslant 0$ or ν is a negative root and $r > 0$. Otherwise, we say $\widetilde{\alpha}$ is negative. For the double affine roots, this follows similarly. We say that a double affine root $\alpha = \widetilde{\alpha} + j\pi$ is positive if $\widetilde{\alpha}$ is a positive affine root and $j \geqslant 0$ or if $\widetilde{\alpha}$ is a negative affine root and $j > 0$. Otherwise, we say α is negative.

Also by analogy with the affine case, each double affine root $\alpha = \nu + r\delta + j\pi$ can be associated to a reflection s_α. Let $\widetilde{\alpha} = \nu + r\delta$. Then define:

$$\begin{aligned} s_\alpha &= s_{\widetilde{\alpha}+j\pi} \\ &= X^{-j\widetilde{\alpha}} s_{\nu+r\delta} \\ &= X^{-j\widetilde{\alpha}} Y^{-r\nu} s_\nu. \end{aligned}$$

We note that our notation for roots and reflections differs from that in [12] by a sign (see [12, (7) and (8)]). We also note that if $\alpha = \nu + r\delta + j\pi$ is a double affine root, and $j \neq 0$, then s_α is not an element of $W_{\mathcal{T}}$. Instead, s_α is an element of $X \rtimes W_{\text{aff}}$, which contains $W_{\mathcal{T}}$ as a sub-semigroup. However, when we consider $x = X^\zeta \widetilde{w} \in W_{\mathcal{T}}$ with $\text{lev}(x) > 0$, then $x s_{\nu+r\delta+j\pi} \in W_{\mathcal{T}}$.

Because the semigroup $W_{\mathcal{T}}$ is not generated by reflections, we cannot use Coxeter theory for the double affine Bruhat order, which makes it more difficult to extend from the affine to the double affine case. Consider $x = X^{\mu+m\delta+l\Lambda_0} \in W_{\mathcal{T}}$ with $\text{lev}(x) > 0$. Then x cannot be written as a product of reflections because the reflections contain no $X^{l\Lambda_0}$ part.

Let $\zeta \in X$ and $\widetilde{w} \in W_{\text{aff}}$. We define an action of $X \rtimes W_{\text{aff}}$ on Φ by

$$X^\zeta \widetilde{w}(\widetilde{\alpha} + j\pi) = \widetilde{w}(\widetilde{\alpha}) + (j - \langle \zeta, \widetilde{w}(\widetilde{\alpha})\rangle)\pi.$$

This is similar to the action defined for W_{aff} on Φ_{aff}. Letting $\zeta = \mu + m\delta + l\Lambda_0$ and $\widetilde{w} = Y^\lambda w$, we can expand this to

$$X^\zeta Y^\lambda w(\alpha + r\delta + j\pi) = \ Y^\lambda w(\alpha + r\delta) + (j - \langle \mu, w(\alpha)\rangle - l(r - \langle \lambda, w(\alpha)\rangle))\pi.$$

If α and β are double affine roots, then $s_\alpha(\beta)$ as defined by the action above is the same as

$$s_\alpha(\beta) = \beta - \langle \alpha, \beta\rangle \alpha.$$

2.2 Length Function

When considering elements of $W_{\mathcal{T}}$, it no longer makes sense to define a length function based on reduced words (as is done for W_{fin} or W_{aff}) because we lack the notion of simple reflections. Instead, we use the length function defined in [10].

Let ρ be the sum of the affine fundamental weights, which we choose now and keep consistent throughout the paper. Let $x = X^{\zeta}\widetilde{w}$ be an element of $W_{\mathcal{T}}$. We denote by $\mathrm{Inv}(\widetilde{w})$ the set of inversions of $\widetilde{w} \in W_{\text{aff}}$. That is, the set of positive affine roots $\widetilde{\alpha}$ such that $\widetilde{w}(\widetilde{\alpha})$ is a negative root. Then the *length* of x is defined by [10] to be

$$\ell(x) = \langle \zeta_+, 2\rho\rangle + |\{\widetilde{\alpha} \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle \leqslant 0\}| - |\{\widetilde{\alpha} \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle > 0\}|,$$

where ζ_+ is the dominant element associated to ζ and $\widetilde{\alpha} = \nu + r\delta$ is an affine root. Here we see why we must use $\mathcal{T}$ and not all of X when defining $W_{\mathcal{T}}$ because we need the X-weight of $x \in W_{\mathcal{T}}$ to be made dominant.

In the affine case, this definition is consistent with ℓ_{aff}, the Coxeter length function on W_{aff}. For $\widetilde{w} = Y^{\lambda}w \in W_{\mathcal{T}}$, $\ell(\widetilde{w}) = \ell(X^0\widetilde{w}) = \ell_{\text{aff}}(\widetilde{w})$. For convenience, the length function is split into big and small parts by defining the *big length* as

$$\ell_{\text{big}}(x) = \langle \zeta_+, 2\rho\rangle$$

and the *small length* as

$$\ell_{\text{small}}(x) = |\{\widetilde{\alpha} \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle \leqslant 0\}| - |\{\widetilde{\alpha} \in \mathrm{Inv}(\widetilde{w}^{-1}) : \langle \zeta, \widetilde{\alpha}\rangle > 0\}|.$$

Before ending our discussion of the length function, we need a proposition that splits the length of an element $x \in W_{\mathcal{T}}$ into the sum of two lengths, the first considering only the translation part of x and the second considering only the affine part of x. This way of re-writing the length function will be fundamental when proving our classification theorem.

We say $\zeta \in \mathcal{T}$ is *regular* if $\langle \zeta, \widetilde{\alpha}\rangle \neq 0$ for all $\widetilde{\alpha} \in \Phi_{\text{aff}}$. The following proposition is an extension of work done by Lam and Shimozono [6, Lem 3.4].

Proposition 4. *Let proposition$\zeta \in \mathcal{T}$ be regular and dominant and let $x = X^{\widetilde{v}\zeta}\widetilde{w}$ where $\widetilde{w}, \widetilde{v} \in W_{\text{aff}}$. Then*

$$\begin{aligned}\ell(x) &= \ell(X^{\zeta}) - \ell(\widetilde{v}^{-1}\widetilde{w}) + \ell(\widetilde{v}) \\ &= \langle \zeta, 2\rho\rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}).\end{aligned}$$

Before we can prove Proposition 4, we need the following lemmas.

Lemma 5. *[7, proof of (2.2.4)] Let $x, y \in W_{\text{aff}}$. Then*

$$\begin{aligned}\ell(xy) &= \ell(x) + \ell(y) - 2|\{\mathrm{Inv}(y) \cap -y^{-1}\,\mathrm{Inv}(x)\}| \\ &= \ell(x) + \ell(y) - 2|\{\alpha \in \mathrm{Inv}(y) : \alpha \notin \mathrm{Inv}(xy)\}|.\end{aligned}$$

Proof. Let $\alpha \in \mathrm{Inv}(xy)$. There are two possibilities:

1. $\alpha > 0$, $y(\alpha) < 0$, and $xy(\alpha) < 0$

2. $\alpha > 0$, $y(\alpha) > 0$, and $xy(\alpha) < 0$.

So $\operatorname{Inv}(xy) \subset y^{-1}\operatorname{Inv}(x) \sqcup \operatorname{Inv}(y)$ (this is a disjoint union because if $\alpha \in y^{-1}\operatorname{Inv}(x)$, then $y(\alpha) > 0$ and so $\alpha \notin \operatorname{Inv}(y)$). In general, this is a proper subset because there could be $\alpha \in \operatorname{Inv}(y)$ such that $-y(\alpha) \in \operatorname{Inv}(x)$ (so $\alpha \notin \operatorname{Inv}(xy)$), or there could be $\alpha < 0$ such that $y(\alpha) \in \operatorname{Inv}(x)$ (so $\alpha \in y^{-1}\operatorname{Inv}(x)$ but $\alpha \notin \operatorname{Inv}(xy)$).

Thus $|\operatorname{Inv}(xy)| \leqslant |y^{-1}\operatorname{Inv}(x)| + |\operatorname{Inv}(y)|$, and to find an exact representation of $|\operatorname{Inv}(xy)|$, we must subtract $|\{\alpha \in \operatorname{Inv}(y) : -y(\alpha) \in \operatorname{Inv}(x)\}| = |\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)|$ and $|\{\alpha < 0 : y(\alpha) \in \operatorname{Inv}(x)\}| = |\{\beta > 0 : -y(\beta) \in \operatorname{Inv}(x)\}| = |\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)|$.

Using $|\operatorname{Inv}(x)| = |y^{-1}\operatorname{Inv}(x)|$, we have

$$\begin{aligned}|\operatorname{Inv}(xy)| &= |\operatorname{Inv}(x)| + |\operatorname{Inv}(y)| - 2|\operatorname{Inv}(y) \cap -y^{-1}\operatorname{Inv}(x)| \\ &= |\operatorname{Inv}(x)| + |\operatorname{Inv}(y)| - 2|\{\alpha \in \operatorname{Inv}(y) : \alpha \notin \operatorname{Inv}(xy)\}|.\end{aligned}$$ □

Lemma 6. *Let x, y be elements of W_{aff}. Then*

$$\ell(xy) = \ell(x) + \ell(y) - 2|\operatorname{Inv}(x) \cap \operatorname{Inv}(y^{-1})|.$$

Proof. Using Lemma 5, this is equivalent to showing $|\operatorname{Inv}(x) \cap \operatorname{Inv}(y^{-1})| = |\{\gamma \in \operatorname{Inv}(y) : -y(\gamma) \in \operatorname{Inv}(x)\}|$. We create a bijection by mapping $\gamma \in \{\gamma \in \operatorname{Inv}(y) : -y(\gamma) \in \operatorname{Inv}(x)\}$ to $-y(\gamma) \in \operatorname{Inv}(x) \cap \operatorname{Inv}(y^{-1})$, so the sets have the same size. □

Proof of Proposition 4. We have

$$\begin{aligned}\ell(X^{\widetilde{v}\zeta}\widetilde{w}) &= \ell_{\text{big}}(X^{\widetilde{v}\zeta}\widetilde{w}) + \ell_{\text{small}}(X^{\widetilde{v}\zeta}\widetilde{w}) \\ &= \langle \zeta, 2\rho \rangle + |\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle \leqslant 0\}| - |\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle > 0\}|.\end{aligned}$$

We need to show that $\ell(\widetilde{v}) - \ell(\widetilde{w}^{-1}\widetilde{v}) = \ell_{\text{small}}(X^{\widetilde{v}\zeta}\widetilde{w})$. Note that $\langle \widetilde{v}\zeta, \gamma \rangle = \langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle$ and since ζ is dominant and regular, $\langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle > 0$ if and only if $\widetilde{v}^{-1}(\gamma) > 0$. Similarly, $\langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle < 0$ if and only if $\widetilde{v}^{-1}(\gamma) < 0$ (since ζ is regular, we know $\langle \zeta, \widetilde{v}^{-1}(\gamma) \rangle \neq 0$).

So $\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle \leqslant 0\} = \operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})$ and $\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle > 0\} = \{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \widetilde{v}^{-1}(\gamma) > 0\}$.

By Lemma 6, $\ell(\widetilde{w}^{-1}\widetilde{v}) = \ell(\widetilde{w}^{-1}) + \ell(\widetilde{v}) - 2|\operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})|$ so $2|\operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})| - \ell(\widetilde{w}^{-1}) = \ell(\widetilde{v}) - \ell(\widetilde{w}^{-1}\widetilde{v})$. Therefore,

$$\begin{aligned}\ell_{\text{small}}(X^{\widetilde{v}\zeta}\widetilde{w}) &= |\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle \leqslant 0\}| - |\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \langle \widetilde{v}\zeta, \gamma \rangle > 0\}| \\ &= |\operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})| - |\{\gamma \in \operatorname{Inv}(\widetilde{w}^{-1}) : \widetilde{v}^{-1}(\gamma) > 0\}| \\ &= |\operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})| - (|\operatorname{Inv}(\widetilde{w}^{-1})| - |\operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})|) \\ &= 2|\operatorname{Inv}(\widetilde{w}^{-1}) \cap \operatorname{Inv}(\widetilde{v}^{-1})| - \ell(\widetilde{w}^{-1}) \\ &= \ell(\widetilde{v}) - \ell(\widetilde{w}^{-1}\widetilde{v}).\end{aligned}$$ □

2.3 Bruhat Order

Given $x \in W_{\mathcal{T}}$ with $\operatorname{lev}(x) > 0$ and α a positive double affine root, [2, 5 Section B.2] defined $x \to xs_\alpha$ if $x(\alpha) > 0$. They defined the *double affine Bruhat preorder* to be the preorder generated by these relations, (that is, $x \leqslant y$ if there is some chain $x \to xs_{\alpha_1} \to \cdots \to y$), and they conjectured that it was an order. In [10] it was shown that the preorder is in fact an order, and in [12] it was shown that this order coincides with the order generated by the relations: $x \to xs_\alpha$ if $\ell(x) \leqslant \ell(xs_\alpha)$. Note $x \in W_{\mathcal{T}}$ with $\operatorname{lev}(x) = 0$ is excluded because in that case xs_α is not always in $W_{\mathcal{T}}$. When multiplying on the left, we use the relation $x \to s_\alpha x$ if $x^{-1}(\alpha) > 0$.

We are interested in classifying cocovers for a fixed $x \in W_{\mathcal{T}}$ where the associated finite root system Φ_{fin} is irreducible and simply laced. Muthiah and Orr [12] proved the following theorem that will allow us to identify cocovers by a difference in length.

Theorem 7. *[12, Thm 1.6] For α a positive double affine root associated to an irreducible and simply laced finite root system and $x \in W_{\mathcal{T}}$ with $lev(x) > 0$, xs_α is a cover of x if and only if $\ell(x) = \ell(xs_\alpha) - 1$.*

We can similarly say that xs_α is a cocover of x if and only if $\ell(x) = \ell(xs_\alpha) + 1$.

3 Classifying Cocovers

Recall that we are assuming $\operatorname{lev}(x) > 0$ whenever considering $x \in W_{\mathcal{T}}$, and we are restricting Φ_{fin} to be irreducible and simply laced.

Theorem 8 (MO)**.** *Let $x = X^\zeta \widetilde{w}$ with $\zeta \in \mathcal{T}$ and $\widetilde{w} \in W_{\text{aff}}$. Let α be a positive double affine root such that $x^{-1}(\alpha) < 0$. Then $y = s_\alpha x \leqslant x$ with respect to the Bruhat order by definition, and*

$$\ell(y) = \ell(x) - |\{\beta \in \Phi^+ \ : \ x^{-1}(\beta) < 0, \quad s_\alpha(\beta) < 0, \quad x^{-1}s_\alpha(\beta) > 0\}|.$$

In particular, $L_{x,\alpha} := \{\beta \in \Phi^+ \ : \ x^{-1}(\beta) < 0, \quad s_\alpha(\beta) < 0, \quad x^{-1}s_\alpha(\beta) > 0\}$ is finite.

We call $L_{x,\alpha}$ the *length difference set* for x and $y = s_\alpha x$, and note that y is a cocover of x if and only if $L_{x,\alpha} = \{\alpha\}$. This is because y is a cocover if and only if the length difference is 1, and α is always in $L_{x,\alpha}$ if $y = s_\alpha x \leqslant x$.

Example 9. Let W_{aff} be of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$, and $\alpha = \alpha_1 - 2\delta + \pi$. With this setup,

$$L_{x,\alpha} = \{\alpha, \theta - 3\delta + \pi, -\alpha_2 + \delta\}.$$

At this point, we will not go into the detail of checking that these elements do in fact belong to the length difference set (and are the only elements that do belong), but we can do a quick check of the cardinality. Using Sage [13], we find $\ell(x) = 12$ and $\ell(s_\alpha x) = 9$, so the length difference set must indeed contain 3 elements.

Note that the two elements of the length difference set that are not α are $\theta - 3\delta + \pi$ and $-\alpha_2 + \delta = -s_\alpha(\theta - 3\delta + \pi)$. In general, the elements of the length difference set that are not equal to α will come in such pairs. If $\beta \in L_{x,\alpha}$ and $\beta \neq \alpha$ then $-s_\alpha(\beta) \in L_{x,\alpha}$.

3.1 Graphs

We begin by graphing the positive double affine roots α such that $y = s_\alpha x$ is less than x with respect to the Bruhat order.

Definition 10. Let $\nu \in \Phi_{\text{fin}}$ and let $\Gamma_{x,\nu}$ denote the points $(r, j) \in \mathbb{Z}^2$ such that $\alpha = \nu + r\delta + j\pi > 0$ and $x^{-1}(\alpha) < 0$. We call this the *lower graph of x corresponding to ν* and say α corresponds to a point in $\Gamma_{x,\nu}$ if $\alpha = \nu + r\delta + j\pi$ such that $(r, j) \in \Gamma_{x,\nu}$.

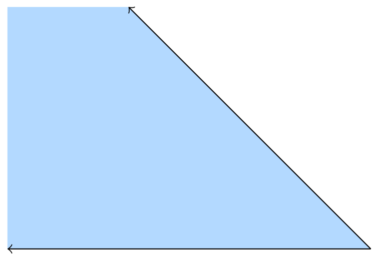


Figure 1: A general $\Gamma_{x,\nu}$.

It is important to note that the two outer edges appearing in the graph above may or may not be included (and it is very possible that only part of an edge will be included) depending on the choice of x and ν. Because the graph shows all α such that $x \geqslant s_\alpha x$, it is clear that the cardinality of $L_{x,\alpha}$ will be greater than or equal to one if α corresponds to a point in $\Gamma_{x,\nu}$.

Definition 11. Let $\alpha = \nu + r\delta + j\pi$ be a double affine root. Then we say ν is the *finite part of* α because $\nu \in \Phi_{\text{fin}}$. We denote this by $\text{fin}(\alpha) = \nu$.

Proposition 12. *Let $\alpha = \nu + r\delta + j\pi$ and $\beta = \gamma + p\delta + q\pi$. The double affine root β is in $L_{x,\alpha}$ if and only if $\beta \in \Gamma_{x,\text{fin}(\beta)}$ and $-s_\alpha\beta \in \Gamma_{x,\text{fin}(-s_\alpha(\beta))}$.*

Proof. Note that $\text{fin}(\alpha) = \nu$, $\text{fin}(\beta) = \gamma$, and $\text{fin}(-s_\alpha(\beta)) = -s_\nu(\gamma)$.

Let $\beta \in L_{x,\alpha}$. Then $\beta > 0$ and $x^{-1}(\beta) < 0$, so $\beta \in \Gamma_{x,\gamma}$. Additionally, $s_\alpha(\beta) < 0$ and $x^{-1}(s_\alpha\beta) > 0$, so $-s_\alpha(\beta) \in \Gamma_{x,-s_\nu(\gamma)}$.

Let $\beta \in \Gamma_{x,\gamma}$ and $-s_\alpha(\beta) \in \Gamma_{x,-s_\nu(\gamma)}$. Then $\beta > 0$, $x^{-1}(\beta) < 0$, $-s_\alpha(\beta) > 0$, and $-x^{-1}(s_\alpha(\beta)) < 0$, so $\beta \in L_{x,\alpha}$. $\square$

Example 13. Consider W_{aff} of type $\widetilde{A}_2$ and $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{\alpha_2}$ (the same choices from Example 9). The lower graph of x corresponding to α_1 is shown in Figure 2.

To see why this is the graph for Γ_{x,α_1}, we need to examine when $\alpha > 0$ and $x^{-1}(\alpha) < 0$. The double affine root $\alpha = \alpha_1 + r\delta + j\pi$ is positive if and only if one of the following holds:

1. $j > 0$

2. $j = 0$ and $r \geqslant 0$.

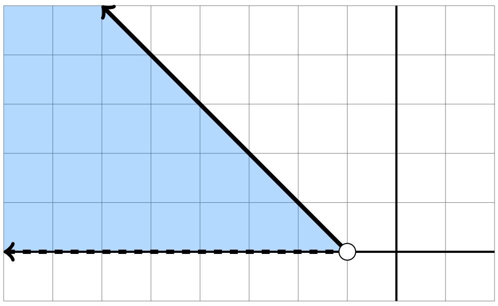


Figure 2: Γ_{x,α_1}.

To determine when $x^{-1}(\alpha) < 0$, it will help to expand $x^{-1}(\alpha)$:

$$\begin{aligned} x^{-1}(\alpha) &= Y^{-\alpha_2} X^{-\alpha_1-\alpha_2-\delta-\Lambda_0}(\alpha_1 + r\delta + j\pi) \\ &= \alpha_1 + (r + \langle \alpha_1, \alpha_2 \rangle)\delta + (j + \langle \alpha_1 + r\delta, -\alpha_1 - \alpha_2 - \delta - \Lambda_0 \rangle)\pi \\ &= \alpha_1 + (r-1)\delta + (j + r + 1)\pi. \end{aligned}$$

Now we can see that $x^{-1}(\alpha) < 0$ if and only if one of the following holds:

1. $j < -r - 1$
2. $j = -r - 1$ and $r < 1$.

Combining these restrictions results in the graph shown above.

Proposition 14. *Fix $x = X^{\zeta}\widetilde{w} \in W_{\mathcal{T}}$ with $\widetilde{w} = Y^{\lambda}w \in W_{\mathrm{aff}}$ and fix $\nu \in \Phi_{\mathrm{fin}}$. The point $(r, j) \in \Gamma_{x,\nu}$ if and only if one of the following holds:*

1. $0 < j < \langle -\zeta, \widetilde{\alpha} \rangle = -\langle \zeta, \nu + r\delta \rangle$
2. $(r, j) = (r, 0)$ *with* $0 \leqslant r \leqslant \frac{\langle \nu, \mu \rangle}{-l}$*, and if* $r = 0$*, then* $\nu > 0$
3. $(r, j) = (r, \langle -\zeta, \widetilde{\alpha} \rangle) = (r, -\langle \nu, \mu \rangle - lr)$ *with* $r \leqslant min\{\frac{\langle \nu, \mu \rangle}{-l}, -\langle \lambda, \nu \rangle\}$*, and if* $r = -\langle \lambda, \nu \rangle$*, then* $w^{-1}(\nu) < 0$.

Proof. For $\alpha = \nu + r\delta + j\pi \in \Phi$ to correspond to a point in $\Gamma_{x,\nu}$, we need both $\alpha > 0$ and $x^{-1}(\alpha) < 0$.

For $\alpha = \nu + r\delta + j\pi > 0$, we need one of the following:

1. $j > 0$
2. $j = 0,\ r > 0$
3. $j = 0,\ r = 0, \nu > 0$.

For $x^{-1}(\alpha) < 0$, we need

$$\begin{aligned} x^{-1}(\alpha) &= \widetilde{w}^{-1}(\widetilde{\alpha}) + (j - \langle -\zeta, \widetilde{\alpha}\rangle)\pi \\ &= w^{-1}(\nu) + (r + \langle \lambda, \nu\rangle)\delta + (j + \langle \mu, \nu\rangle + lr)\pi < 0, \end{aligned}$$

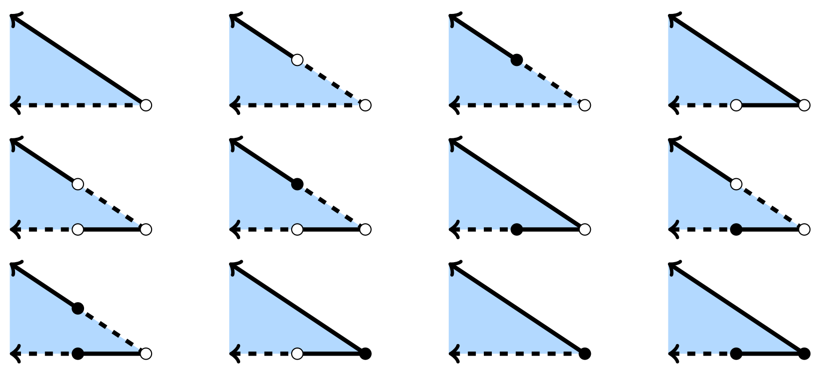
so we need one of the following:

1. $j < \langle -\zeta, \widetilde{\alpha}\rangle = -\langle \mu, \nu\rangle - lr$
2. $j = \langle -\zeta, \widetilde{\alpha}\rangle,\ r < -\langle \lambda, \nu\rangle$
3. $j = \langle -\zeta, \widetilde{\alpha}\rangle,\ r = -\langle \lambda, \nu\rangle, w^{-1}(\nu) < 0.$

Combining these results, we see that if (r, j) is in the graph, then $0 \leqslant j \leqslant \langle -\zeta, \widetilde{\alpha}\rangle$. This tells us that $-\langle \zeta, \widetilde{\alpha}\rangle \geqslant 0$ and since $-\langle \zeta, \widetilde{\alpha}\rangle = \langle -\mu - m\delta - l\Lambda_0, \nu + r\delta\rangle = -\langle \mu, \nu\rangle - lr$, we can solve for r and get $r \leqslant \frac{\langle \nu,\mu\rangle}{-l}$. The point $(r, j) = (\frac{\langle \nu,\mu\rangle}{-l}, 0)$ is the intersection point of $j = 0$ and $j = \langle -\zeta, \widetilde{\alpha}\rangle$. When $j = 0$, we can restrict r to $0 \leqslant r \leqslant \frac{\langle \nu,\mu\rangle}{-l}$, and when $j = \langle -\zeta, \widetilde{\alpha}\rangle$, we can restrict r to $r \leqslant \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda, \nu\rangle\}$. □

Definition 15. For simplicity, we will refer to the line segment of $j = 0$ that is included in the graph and the ray of $j = -\langle \zeta, \widetilde{\alpha}\rangle = -\langle \zeta, \nu + r\delta\rangle$ that is included in the graph as the *lower and upper outer edges* respectively. We will refer to the ray of $j = 1$ that is included in the graph and the ray of $j = -\langle \zeta, \nu + r\delta\rangle - 1$ that is included in the graph as the *lower and upper inner edges* respectively.

Proposition 16. *For a fixed $x \in W_{\mathcal{T}}$ and $\nu \in \Phi_{\text{fin}}$, there are 12 possible forms for $\Gamma_{x,\nu}$, and they are represented by the graphs below.*



Proof. Because the lower outer edge will be the line segment $j = 0$ with $0 \leqslant r \leqslant \frac{\langle \nu,\mu\rangle}{-l}$ and endpoints possibly not included, there are four possibilities:

L1 L2 L3 L4

L1* L4*

Note that the first two types do not include the intersection point of $j = 0$ and $j = -\langle \zeta, \widetilde{\alpha} \rangle$ (represented by the right endpoint), but the last two types do. Also note that type L1 and type L4 may or may not include the line segment between the endpoints. If the line segment is not included, we will refer to these as type L1* or L4* respectively:

Now we will look at the possibilities for the upper outer edge given by $j = -\langle \zeta, \widetilde{\alpha} \rangle$ with $r \leqslant \min\{\frac{\langle \nu,\mu \rangle}{-l}, -\langle \lambda, \nu \rangle\}$ and endpoint possibly not included:

U1 U2 U3 U4

Note that the first three types do not include the intersection point of $j = 0$ and $j = -\langle \zeta, \widetilde{\alpha} \rangle$ (represented by the right point). These upper outer edges will match with the lower outer edges of type L1, L1*, and L2. The only upper outer edge containing the intersection point is of type U4, so this will match with the lower outer edges of type L3, L4, and L4*. In total, this gives 12 possibilities for the graph. $\square$

3.2 Corners

Recall that we are interested in determining which α of $\Gamma_{x,\mathrm{fin}(\alpha)}$ correspond to cocovers (meaning $y = s_\alpha x$ is a cocover of x). To do this, we must examine specific $(r, j) \in \Gamma_{x,\mathrm{fin}(\alpha)}$.

Definition 17. For double affine roots $\alpha = \nu + r\delta + j\pi$ and $\beta = \nu + p\delta + q\pi$, define β_α^- to be the root found by rotating (p, q) 180 degrees about (r, j).

Proposition 18. *If β and α are double affine roots such that $\mathrm{fin}(\alpha) = \mathrm{fin}(\beta)$, then $\beta_\alpha^- = -s_\alpha\beta$.*

Proof. Let $\alpha = \nu + r\delta + j\pi$ and $\beta = \nu + p\delta + j\pi$. Then

$$\begin{aligned} -s_\alpha(\beta) &= -\beta + \langle \beta, \alpha \rangle \alpha \\ &= -\beta + \langle \nu, \nu \rangle \alpha \\ &= -\beta + 2\alpha \\ &= \nu + (2r - p)\delta + (2j - q)\pi. \end{aligned}$$

The root β_α^- is equal to $\nu + p'\delta + q'\pi$ where (p', q') is the result of rotating (p, q) 180 degrees about (r, j). To determine (p', q'), first shift so that we are rotating about the center: $(p, q) \to (p - r, q - j)$ and $(r, j) \to (0, 0)$, then reflect over the x and y axes: $(p-r, q-j) \to (-p+r, -q+j)$, and now shift back to original orientation: $(0, 0) \to (r, j)$ and $(-p + r, -q + j) \to (-p + 2r, -q + 2j) = (2r - p, 2j - q)$.

So $(p', q') = (2r - p, 2j - q)$ and $\beta_\alpha^- = \nu + (2r - p)\delta + (2j - q)\pi = -s_\alpha(\beta)$. $\square$

Definition 19. We say that α is a *corner of the graph* $\Gamma_{x,\mathrm{fin}(\alpha)}$, or a *corner relative to* x, if α corresponds to a point in $\Gamma_{x,\mathrm{fin}(\alpha)}$, and if for any $\beta = \mathrm{fin}(\alpha) + p\delta + q\pi$ corresponding to a point in $\Gamma_{x,\mathrm{fin}(\alpha)}$, β_α^- is not in the graph.

Example 20. Consider W_{aff} of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{\alpha_2}$ (the same choices from Example 9).

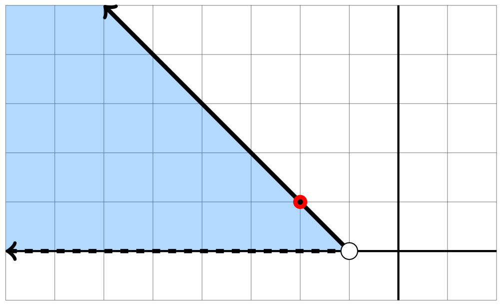


Figure 3: Γ_{x,α_1} with highlighted corner.

With this setup, $\alpha = \alpha_1 - 2\delta + \pi$ corresponds to a corner of Γ_{x,α_1}.

Proposition 21. *If $y = s_\alpha x$ is a cocover of x, then α must correspond to a corner in the graph $\Gamma_{x,\mathrm{fin}(\alpha)}$.*

Proof. Suppose α is not a corner of $\Gamma_{x,\mathrm{fin}(\alpha)}$. Then there is some β with $\mathrm{fin}(\beta) = \mathrm{fin}(\alpha)$ such that $\beta \neq \alpha$, $\beta \in \Gamma_{x,\mathrm{fin}(\alpha)}$, and $\beta_\alpha^- \in \Gamma_{x,\mathrm{fin}(\alpha)}$. But $\beta_\alpha^- = -s_\alpha(\beta)$, so by Proposition 12, $\beta \in L_{x,\alpha}$. So $|L_{x,\alpha}| > 1$, and y is not a cocover of x. □

Remark 22. In general, the set of corners will be larger than the set of roots corresponding to cocovers of a fixed $x \in W_{\mathcal{T}}$. Consider our continuing example where W_{aff} is of type $\widetilde{A}_2$, $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{\alpha_2}$, $\alpha = \alpha_1 - 2\delta + \pi$. Then α corresponds to a corner of Γ_{x,α_1}, as shown in Example 20, but $s_\alpha x$ is not a cocover of x (we saw in Example 9 that the length difference set contains 3 elements).

We would like to show that there are finitely many α that are corners relative to a fixed x, but before we do, we need to make some observations about the graphs.

Fix $\nu \in \Phi_{\mathrm{fin}}$ and $x \in W_{\mathcal{T}}$. Then:

- If (r, j) is a point of $\Gamma_{x,\nu}$ and $j \neq 0$, then (p, j) is in $\Gamma_{x,\nu}$ for all $p < r$. This is true because if (r, j) is a point in $\Gamma_{x,\nu}$ with $j \neq 0$, then the only possible bound on r is the upper bound $r \leqslant \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda, \nu\rangle\}$. So if $p < r$, then $p < \min\{\frac{\langle \nu,\mu\rangle}{-l}, -\langle \lambda, \nu\rangle\}$ and (p, j) is in $\Gamma_{x,\nu}$.

- The upper outer edge falls on the line $y = -\langle \zeta, \nu + x\delta\rangle = -\langle \mu, \nu\rangle - xl$. The slope of this line is $-l$, which is an integer (specifically, $l =$lev(ζ)), and for any $r \in \mathbb{Z}$, $j = -\langle \zeta, \nu + r\delta\rangle$ is also an integer because $-\langle \zeta, \nu + r\delta\rangle = -\langle \mu, \nu\rangle - rl$ where $r, l, \langle \mu, \nu\rangle \in \mathbb{Z}$.

- Let L_0 represent the line given by $y = -\langle \zeta, \nu + x\delta \rangle$. Let (r, j) be a point of L_0 that falls in $\Gamma_{x,\nu}$. Then $r \leqslant \min\{\frac{\langle \nu, \mu \rangle}{-l}, -\langle \lambda, \nu \rangle\}$. Additionally, for any (p, q) of L_0 with $p < r$, (p, q) is in $\Gamma_{x,\nu}$. This is true because (p, q) satisfies $q = -\langle \zeta, \nu + p\delta \rangle$ and $p < r \leqslant \min\{\frac{\langle \nu, \mu \rangle}{-l}, -\langle \lambda, \nu \rangle\}$.

- Let L_k represent the line given by $y = -\langle \zeta, \nu + x\delta \rangle - k$, where k is a positive integer. Let (r, j) be a point on L_k that falls in the graph. Then $j \geqslant 0$, and for any (p, q) of L_k with $p < r$, (p, q) is in $\Gamma_{x,\nu}$. This is true because $p < r$ means $q > j$ (since the slope of L_k is negative), so $0 \leqslant j < q < -\langle \zeta, \nu + p\delta \rangle$, and (p, q) is a point of $\Gamma_{x,\nu}$.

Proposition 23. *Fix $x \in W_{\mathcal{T}}$ and $\nu \in \Phi_{\text{fin}}$. The number of corners of $\Gamma_{x,\nu}$ is finite.*

Idea: We show that if $\alpha = \nu + r\delta + j\pi$ is a corner relative to x, then α must fall on one of the two outer edges or one of the two inner edges of $\Gamma_{x,\nu}$. But on these edges, only the $(r, j) \in \mathbb{Z}^2$ closest to endpoints can be corners.

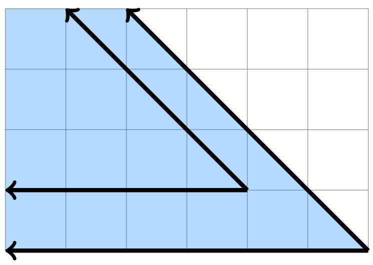


Figure 4: Inner and Outer Edges of a General $\Gamma_{x,\nu}$.

Proof. We break the proof into several cases. Let (r', j') represent a corner.

Case 1: Assume $j' = 0$. Then (r', j') falls along the lower outer edge, which is either a line segment or a single point. In either case, there are finitely many possibilities for (r', j').

Case 2: Assume (r', j') falls along the upper outer edge (the diagonal $j = -\langle \zeta, \nu + r\delta \rangle$). Then any other $(r, j) \in \Gamma_{x,\nu}$ on the upper outer edge must have $r < r'$. If there exists some (r, j) on the graph's upper outer edge such that $r > r'$, then it can be rotated 180 degrees about (r', j') and end up in the graph (because it will land on the diagonal and be higher up than (r', j')), which contradicts the fact that (r', j') is a corner. So there is only one possibility for (r', j').

Case 3: Assume (r', j') falls along the upper inner edge (the diagonal given by $j = -\langle \zeta, \nu + r\delta \rangle - 1$). Then using the same logic from above, (r', j') must have largest possible r', so there is only one possibility for (r', j').

Case 4: Assume $j' = 1$. Again, r' must be maximal. Suppose $(r, 1)$ is another point of the graph such that $r > r'$. Then $(r, 1)$ rotated 180 degrees about $(r', 1)$ results in some $(p, 1)$ with $p < r'$. This would mean that $(p, 1)$ is in the graph, but that contradicts the fact that (r', j') is a corner.

Case 5: Assume (r', j') does not lie on any of the outer or inner edges. Then $1 < j' < \langle -\mu, \nu \rangle - r'l - 1$. So $1 \leqslant j' - 1 < j' < \langle -\mu, \nu \rangle - r'l - 1$, and $(r', j' - 1)$ is a point of the graph as it falls strictly between the outer edges. Similarly, $1 < j' < j' + 1 \leqslant \langle -\mu, \nu \rangle - r'l - 1$, so $(r', j' + 1)$ is also a point on the graph as it falls strictly between the outer edges. Thus (r', j') cannot be a corner.

To be a corner, (r', j') must fall along one of the two outer edges or one of the two inner edges. On those edges there are finitely many possibilities for corners. Thus for any given x and ν, the corresponding graph $\Gamma_{x,\nu}$ contains finitely many corners. □

Corollary 24. *The number of cocovers of x is finite.*

Proof. Fix x. For any $\nu \in \Phi_{\text{fin}}$, the graph $\Gamma_{x,\nu}$ has finitely many corners. So there are finitely many $\alpha = \nu + r\delta + j\pi$ such that $y = s_\alpha x$ is a cocover of x. Since ν is a finite root, there are finitely many possibilities for ν. So there are finitely many cocovers for a given x. □

Theorem 25. *Let $x, y \in W_{\mathcal{T}}$ such that $y \leqslant x$. Then the double affine Bruhat interval $[y, x]$ will be finite.*

Proof. If we construct a saturated chain from y to x by starting at x and selecting cocovers, then there are finitely many options for each element in the chain as there are finitely many cocovers for any element of $W_{\mathcal{T}}$. Additionally, there are finitely many elements in the chain as any path from y to x will have at most $\ell(x) - \ell(y)$ steps. Thus there are finitely many options for a saturated chain, and since every z such that $y < z < x$ is part of a saturated chain, we see there are finitely many options for $z \in [y, x]$ and $[y, x]$ must be finite. □

Example 26. Consider W_{aff} of type $\widetilde{A}_2$, and let $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$ and $\alpha = \alpha_1 - 2\delta + \pi$ (the same choices from Example 9). Then $[s_\alpha x, x]$, which is graphed in figure 5, contains 8 elements:

1. $x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{\alpha_2}$
2. $s_{\theta-3\delta+\pi} x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{2\alpha_1+3\alpha_2} s_1 s_2 s_1$, a cocover of x
3. $s_{-\alpha_2+\delta} x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} s_2$, a cocover of x
4. $s_{\alpha_1+\alpha_2-4\delta+2\pi} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{3\alpha_1+\alpha_2} s_1 s_2$, a cover of $s_\alpha x$
5. $s_{\theta-3\delta+\pi} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{2\alpha_1} s_1 s_2$, a cover of $s_\alpha x$
6. $s_{-\alpha_2+\delta} s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0} Y^{3\alpha_1+3\alpha_2} s_2 s_1$, a cover of $s_\alpha x$

7. $s_{-\alpha_2+\pi}s_\alpha x = X^{\alpha_1+\alpha_2+\delta+\Lambda_0}Y^{3\alpha_1+2\alpha_2}s_2s_1$, a cover of $s_\alpha x$

8. $s_\alpha x = s_{\alpha_1-2\delta+\pi}x$.

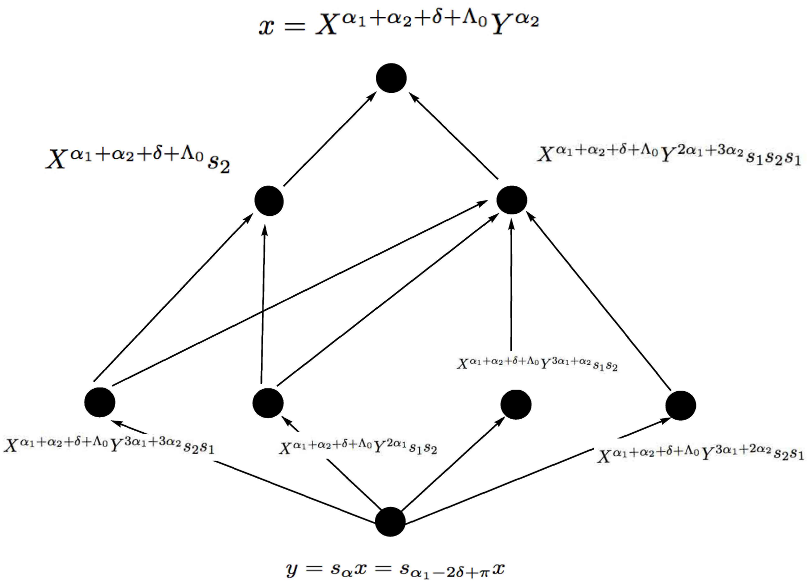


Figure 5: Graph of the Bruhat Interval.

Corollary 27. *Let* $x = X^\zeta \widetilde{w}$ *with* $\zeta \in \mathcal{T}$ *and* $\widetilde{w} \in W_{\text{aff}}$. *If* $\alpha = \nu + r\delta + j\pi$ *corresponds to a corner of the graph* $\Gamma_{x,\text{fin}(\alpha)}$, *then one of the following must hold:*

1. $j = 0$
2. $j = 1$
3. $j = -\langle \zeta, \widetilde{\alpha} \rangle$
4. $j = -\langle \zeta, \widetilde{\alpha} \rangle - 1$.

3.3 Classification

The following classification is extended from the work done in the affine case by Lam and Shimozono [6] and further strengthened by Milićević [9]. We note that the restriction in [9] is stricter than the restriction needed here. This is not due to the change from the affine case to the double affine case, but instead it is due to the method of proof. In both [6] and [9] the proof involves introducing a convex function and using it to approximate the length function in order to classify the cocovers. By using our relationship between cocovers and the corners of graphs, we are able to avoid introducing this function and in doing so we avoid using as strict of a bound.

Theorem 28. *Let $x = X^{\widetilde{v}\zeta}\widetilde{w}$ and $y = s_\alpha x$ where $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi$ is a positive double affine root and $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, \ldots, n$. Then y is a cocover of x if and only if one of the following holds:*

1. $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$ *and* $j = 0$.
2. $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$ *and* $j = 1$.
3. $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1$ *and* $j = \langle \zeta, \widetilde{\alpha} \rangle$.
4. $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) = \ell(\widetilde{w}^{-1}\widetilde{v}) + 1 - \langle \widetilde{\alpha}, 2\rho \rangle$ *and* $j = \langle \zeta, \widetilde{\alpha} \rangle - 1$.

Proof. Following Milićević [9, Proof of Prop 4.2], we re-write y as

$$\begin{aligned} y = s_{-\widetilde{v}\widetilde{\alpha}+j\pi}x &= X^{j\widetilde{v}\widetilde{\alpha}} s_{\widetilde{v}\widetilde{\alpha}} X^{\widetilde{v}\zeta}\widetilde{w} \\ &= X^{j\widetilde{v}\widetilde{\alpha} + s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}\zeta} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}(s_{\widetilde{\alpha}}\zeta + j\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta - j\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w} \\ &= X^{\widetilde{v}(\zeta - (\langle \zeta, \widetilde{\alpha} \rangle - j)\widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}. \end{aligned}$$

Using Proposition 4 and the fact that ζ is dominant and regular, we have

$$\ell(x) = \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}).$$

If y is a cocover of x, then α is a corner relative to x, and by Corollary 27, there are four possibilities for j:

1. $j = 0$ and $y = X^{\widetilde{v}s_{\widetilde{\alpha}}\zeta} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$
2. $j = 1$ and $y = X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta - \widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$
3. $j = -\langle \widetilde{v}\zeta, -\widetilde{v}\widetilde{\alpha} \rangle = \langle \zeta, \widetilde{\alpha} \rangle$ and $y = X^{\widetilde{v}\zeta} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$
4. $j = -\langle \widetilde{v}\zeta, -\widetilde{v}\widetilde{\alpha} \rangle - 1 = \langle \zeta, \widetilde{\alpha} \rangle - 1$ and $y = X^{\widetilde{v}(\zeta - \widetilde{\alpha})} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$.

No matter which direction we are proving, we may reduce to these four cases. Using $\langle \widetilde{\alpha}, \widetilde{\beta} \rangle \leqslant 2$ for all $\widetilde{\alpha}, \widetilde{\beta} \in \Phi_{\text{aff}}$ [1, VI 1.3] and the assumption that $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1, \ldots, n$, we have that $\zeta - \widetilde{\alpha}$ is dominant and regular.

Case 1: Let $j = 0$. Then $y = X^{\widetilde{v}s_{\widetilde{\alpha}}\zeta} s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$, and by using Proposition 4 we have

$$\begin{aligned} \ell(y) &= \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}s_{\widetilde{\alpha}}) \\ &= \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}s_{\widetilde{\alpha}}) \\ &= \langle \zeta, 2\rho \rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}s_{\widetilde{\alpha}}). \end{aligned}$$

So $\ell(x) - \ell(y) = \ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}})$, and y is a cocover of x if and only if $\ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) = 1$.

Case 2: Let $j = 1$. Then $y = X^{\widetilde{v}s_{\widetilde{\alpha}}(\zeta-\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$, and by using Proposition 4 we have

$$\begin{aligned}\ell(y) =& \langle \zeta - \widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}s_{\widetilde{\alpha}})\\ =& \langle \zeta, 2\rho\rangle - \langle \widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}\widetilde{v}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}s_{\widetilde{\alpha}})\\ =& \langle \zeta, 2\rho\rangle - \langle \widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{w}^{-1}\widetilde{v}) + \ell(\widetilde{v}s_{\widetilde{\alpha}}).\end{aligned}$$

So $\ell(x) - \ell(y) = \ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) + \langle \widetilde{\alpha}, 2\rho\rangle$, and y is a cocover of x if and only if $\ell(\widetilde{v}) - \ell(\widetilde{v}s_{\widetilde{\alpha}}) + \langle \widetilde{\alpha}, 2\rho\rangle = 1$.

Case 3: Let $j = \langle \zeta, \widetilde{\alpha}\rangle$. Then $y = X^{\widetilde{v}\zeta}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$, and by using Proposition 4 we have

$$\begin{aligned}\ell(y) &= \langle \zeta, 2\rho\rangle - \ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}) + \ell(\widetilde{v})\\ &= \langle \zeta, 2\rho\rangle - \ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}).\end{aligned}$$

So $\ell(x)-\ell(y) = \ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})-\ell(\widetilde{w}^{-1}\widetilde{v})$, and y is a cocover of x if and only if $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}})-\ell(\widetilde{w}^{-1}\widetilde{v}) = 1$.

Case 4: Let $j = \langle \zeta, \widetilde{\alpha}\rangle - 1$. Then $y = X^{\widetilde{v}(\zeta-\widetilde{\alpha})}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{w}$, and by using Proposition 4 we have

$$\begin{aligned}\ell(y) &= \langle \zeta, 2\rho\rangle - \langle \widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{w}^{-1}s_{\widetilde{v}\widetilde{\alpha}}\widetilde{v}) + \ell(\widetilde{v})\\ &= \langle \zeta, 2\rho\rangle - \langle \widetilde{\alpha}, 2\rho\rangle - \ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) + \ell(\widetilde{v}).\end{aligned}$$

So $\ell(x) - \ell(y) = \ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) - \ell(\widetilde{w}^{-1}\widetilde{v}) + \langle \widetilde{\alpha}, 2\rho\rangle$, and y is a cocover of x if and only if $\ell(\widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}) - \ell(\widetilde{w}^{-1}\widetilde{v}) + \langle \widetilde{\alpha}, 2\rho\rangle = 1$. □

3.4 Quantum Bruhat Graphs

In both [6] and [9] the authors related the cocovering relation in the affine setting to the finite quantum Bruhat graph defined by Brenti, Fomin, and Postnikov [3]. Following their work, we relate the cocovering relation in the double affine Bruhat order to the affine quantum Bruhat graph, giving us another way of visualizing these relations. For other applications of the quantum Bruhat graph of the affine Weyl group, see [8] where Mihalcea and Mare use the graph in their work on Chevalley operators.

Definition 29. We define the *quantum Bruhat graph* (QBG) of W_{aff} to be the graph whose set of vertices consists of the elements of W_{aff} and whose edge set is created by making a directed edge from $\widetilde{v}s_{\widetilde{\alpha}}$ to $\widetilde{v}$ for $\widetilde{\alpha}$ a positive affine root if one of the following holds:

1. $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) + 1$
2. $\ell(\widetilde{v}) = \ell(\widetilde{v}s_{\widetilde{\alpha}}) - \langle \widetilde{\alpha}, 2\rho\rangle + 1$.

The edges are labeled by $\widetilde{\alpha}$.

Example 30. Let W_{aff} be of type $\widetilde{A}_1$. Then the QBG of W_{aff} is given below.

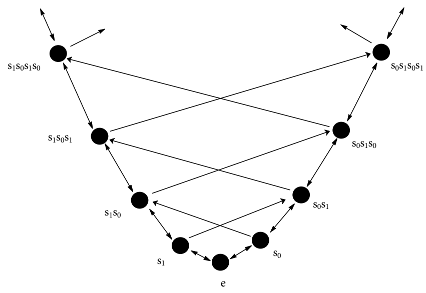


Figure 6: QBG of W_{aff}.

Remark 31. There is a correspondence from the length conditions required in Theorem 28 to the edges in the quantum Bruhat graph of W_{aff}.

- Length condition (1) corresponds to an upward edge in the QBG of the form $\widetilde{v}s_{\widetilde{\alpha}} \to \widetilde{v}$ with length change $+1$.
- Length condition (2) corresponds to a downward edge in the QBG of the form $\widetilde{v}s_{\widetilde{\alpha}} \to \widetilde{v}$ with length change $-(\langle \widetilde{\alpha}, 2\rho \rangle - 1)$.
- Length condition (3) corresponds to an upward edge in the QBG of the form $\widetilde{w}^{-1}\widetilde{v} \to \widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}$ with length change $+1$.
- Length condition (4) corresponds to a downward edge in the QBG of the form $\widetilde{w}^{-1}\widetilde{v} \to \widetilde{w}^{-1}\widetilde{v}s_{\widetilde{\alpha}}$ with length change $-(\langle \widetilde{\alpha}, 2\rho \rangle - 1)$.

Because of the correspondence in Theorem 28 between the length conditions and the QBG, we can find cocovers in $W_{\mathcal{T}}$ by examining edges in the QBG of W_{aff}.

Example 32. The QBG with W_{aff} of type $\widetilde{A}_1$ has the upward edge $s_1s_0 \to s_0s_1s_0$. If we pick $\widetilde{v} = s_0s_1s_0$ and $\widetilde{v}s_{\widetilde{\alpha}} = s_1s_0$, then this edge corresponds to the first cocover type in Theorem 28 and $j = 0$. The reflection we are extending by is $s_{\widetilde{\alpha}} = s_0s_1s_0s_1s_0$, so $\widetilde{\alpha} = s_0s_1(\alpha_0) = 3\alpha_0 + 2\alpha_1 = -\alpha_1 + 3\delta$, and $\alpha = -\widetilde{v}\widetilde{\alpha} + j\pi = -\alpha_1 + \delta = \alpha_0$.

We pick $\zeta = 2\alpha_1 + \delta + 8\Lambda_0$ and check $\langle \zeta, \alpha_i \rangle > 2$ for $i = 0, 1$. With these choices,

$$x = X^{s_0s_1s_0(\zeta)} = X^{14\alpha_1 - 23\delta + 8\Lambda_0}, \quad y = X^{s_1s_0(\zeta)}Y^{\alpha_1}s_1 = X^{-6\alpha_1 - 3\delta + 8\Lambda_0}Y^{\alpha_1}s_1,$$

and y is a cocover of x. Further, we can confirm this by using Sage [13] to check the lengths. Indeed, $\ell(x) - \ell(y) = 8 - 7 = 1$.

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