Ramsey-Type Results for Path Covers and Path Partitions

Shuya Chiba[*]

Applied Mathematics, Faculty of Advanced Science and Technology
Kumamoto University
2-39-1 Kurokami, Kumamoto 860-8555, Japan

schiba@kumamoto-u.ac.jp

Michitaka Furuya[†]

College of Liberal Arts and Sciences
Kitasato University
1-15-1 Kitasato, Minami-ku, Sagamihara, Kanagawa 252-0373, Japan

michitaka.furuya@gmail.com

Submitted: Aug 14, 2021; Accepted: Aug 20, 2022; Published: Oct 7, 2022

Abstract

A family $\mathcal{P}$ of subgraphs of G is called a *path cover* (resp. a *path partition*) of G if $\bigcup_{P\in\mathcal{P}} V(P) = V(G)$ (resp. $\dot{\bigcup}_{P\in\mathcal{P}} V(P) = V(G)$) and every element of $\mathcal{P}$ is a path. The minimum cardinality of a path cover (resp. a path partition) of G is denoted by $\mathrm{pc}(G)$ (resp. $\mathrm{pp}(G)$). In this paper, we characterize the forbidden subgraph conditions assuring us that $\mathrm{pc}(G)$ (or $\mathrm{pp}(G)$) is bounded by a constant. Our main results introduce a new Ramsey-type problem.

Mathematics Subject Classifications: 05C38, 05C55

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. For terms and symbols not defined in this paper, we refer the reader to [2].

Let G be a graph. Let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For a vertex $x \in V(G)$, let $N_G(x)$ denote the *neighborhood* of x in G; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. For a subset X of $V(G)$, let $N_G(X) =$

[*]Supported by JSPS KAKENHI Grant number JP20K03720.
[†]Supported by JSPS KAKENHI Grant number JP18K13449.

$(\bigcup_{x\in X} N_G(x)) \setminus X$, and let $G[X]$ (resp. $G - X$) denote the subgraph of G induced by X (resp. $V(G) \setminus X$). Let $\alpha(G)$ denote the *independence number* of G, i.e., the maximum cardinality of an independent set of G. Let K_n, P_n, C_n and $K_{1,n}$ denote the *complete graph* of order n, the *path* of order n, the *cycle* of order n and the *star* of order $n+1$, respectively. For two positive integers n_1 and n_2, the *Ramsey number* $R(n_1, n_2)$ is the minimum positive integer R such that any graph of order at least R contains a clique of cardinality n_1 or an independent set of cardinality n_2.

For two graphs G and H, G is said to be *H-free* if G contains no induced copy of H. For a family $\mathcal{H}$ of graphs, a graph G is said to be *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. In this context, the members of $\mathcal{H}$ are called *forbidden subgraphs*. For two families $\mathcal{H}_1$ and $\mathcal{H}_2$ of graphs, we write $\mathcal{H}_1 \leqslant \mathcal{H}_2$ if for every $H_2 \in \mathcal{H}_2$, there exists $H_1 \in \mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. The relation "$\leqslant$" between two families of forbidden subgraphs was introduced in [7]. Note that if $\mathcal{H}_1 \leqslant \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

Let $\mathcal{A}$ be a family of graphs. A family $\mathcal{P}$ of subgraphs of G is called an *$\mathcal{A}$-cover* of G if $\bigcup_{P\in\mathcal{P}} V(P) = V(G)$ and each element of $\mathcal{P}$ is isomorphic to a graph belonging to $\mathcal{A}$. Note that some elements of an $\mathcal{A}$-cover of G might have common vertices. An $\mathcal{A}$-cover $\mathcal{P}$ of G is called an *$\mathcal{A}$-partition* of G if the elements of $\mathcal{P}$ are pairwise vertex-disjoint. A $\{P_i : i \geqslant 1\}$-cover (resp. a $\{P_i : i \geqslant 1\}$-partition) of G is called a *path cover* (resp. a *path partition*) of G. Since $\{G[\{x\}] : x \in V(G)\}$ is a path partition of G (and so a path cover of G), the minimum cardinality of a path cover (or a path partition) of any graph is well-defined. The value $\min\{|\mathcal{P}| : \mathcal{P}$ is a path cover of $G\}$ (resp. $\min\{|\mathcal{P}| : \mathcal{P}$ is a path partition of $G\}$), denoted by $\mathrm{pc}(G)$ (resp. $\mathrm{pp}(G)$), is called the *path cover number* (resp. the *path partition number*) of G. It is trivial that $\mathrm{pc}(G) \leqslant \mathrm{pp}(G)$. Since a graph G has a Hamiltonian path if and only if $\mathrm{pp}(G) = 1$, the decision problem for the path partition number is a natural generalization of the Hamiltonian path problem. In fact, it has been widely studied in, for example, [12–14, 16, 17]. Throughout this paper, we implicitly use the following fact.

Fact 1. *Let G be a graph, and let $\{X_1, X_2, \ldots, X_m\}$ be a partition of $V(G)$. Then* $\mathrm{pc}(G) \leqslant \sum_{1\leqslant i\leqslant m} \mathrm{pc}(G[X_i])$ *and* $\mathrm{pp}(G) \leqslant \sum_{1\leqslant i\leqslant m} \mathrm{pp}(G[X_i])$.

In this paper, we focus on the following conditions concerning a family $\mathcal{H}$ of forbidden subgraphs:

(A1) There exists a constant $c_1 = c_1(\mathcal{H})$ such that $\mathrm{pc}(G) \leqslant c_1$ for every connected $\mathcal{H}$-free graph G.

(A2) There exists a constant $c_2 = c_2(\mathcal{H})$ such that $\mathrm{pp}(G) \leqslant c_2$ for every connected $\mathcal{H}$-free graph G.

Our main aim is to characterize the finite families $\mathcal{H}$ of connected graphs satisfying (A1) or (A2).

Let m and n be two positive integers. We define five graphs which will be used as forbidden subgraphs in our main result (see Figure 1).

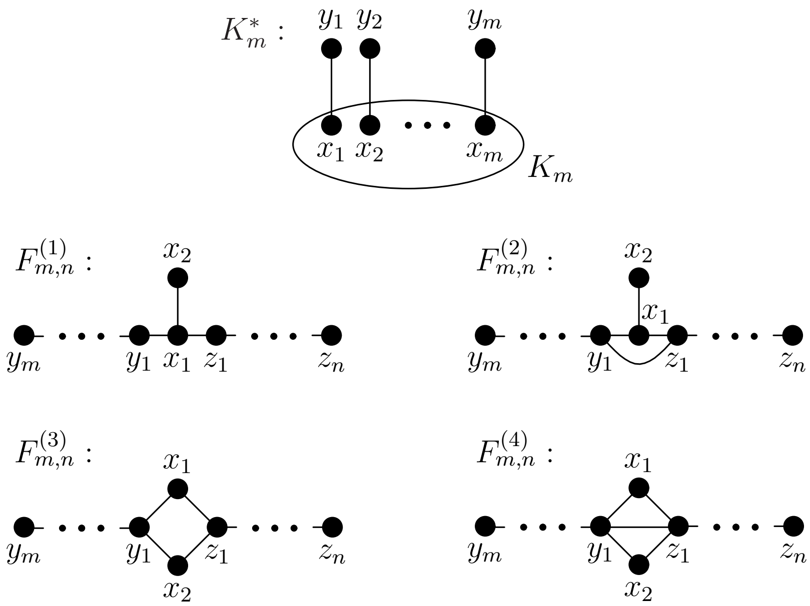

Figure 1: Graphs K_m^*, $F_{m,n}^{(1)}$, $F_{m,n}^{(2)}$, $F_{m,n}^{(3)}$ and $F_{m,n}^{(4)}$

- Let K_m^* denote the graph with $V(K_m^*) = \{x_i, y_i : 1 \leqslant i \leqslant m\}$ and $E(K_m^*) = \{x_i x_j : 1 \leqslant i < j \leqslant m\} \cup \{x_i y_i : 1 \leqslant i \leqslant m\}$.
- Let $A := \{x_1, x_2\} \cup \{y_i : 1 \leqslant i \leqslant m\} \cup \{z_i : 1 \leqslant i \leqslant n\}$. We define four graphs as follows:
  - Let $F_{m,n}^{(1)}$ denote the graph on A such that $E(F_{m,n}^{(1)}) = \{x_1x_2, x_1y_1, x_1z_1\} \cup \{y_iy_{i+1} : 1 \leqslant i \leqslant m-1\} \cup \{z_iz_{i+1} : 1 \leqslant i \leqslant n-1\}$.
  - Let $F_{m,n}^{(2)}$ be the graph obtained from $F_{m,n}^{(1)}$ by adding the edge y_1z_1.
  - Let $F_{m,n}^{(3)}$ denote the graph on A such that $E(F_{m,n}^{(3)}) = \{x_1y_1, x_1z_1, x_2y_1, x_2z_1\} \cup \{y_iy_{i+1} : 1 \leqslant i \leqslant m-1\} \cup \{z_iz_{i+1} : 1 \leqslant i \leqslant n-1\}$.
  - Let $F_{m,n}^{(4)}$ be the graph obtained from $F_{m,n}^{(3)}$ by adding the edge y_1z_1.

Our first main result is the following, which is proved in Section 2.

Theorem 2. *Let $\mathcal{H}$ be a finite family of connected graphs. Then the following hold:*

(i) *The family $\mathcal{H}$ satisfies (A1) if and only if $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$ for an integer $n \geqslant 2$.*

(ii) *The family $\mathcal{H}$ satisfies (A2) if and only if $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}, F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$ for an integer $n \geqslant 2$.*

Our motivation derives from two different lines of research. The first one is forbidden subgraph conditions for the existence of a Hamiltonian path. Now we focus on the condition that

(H1) every connected $\mathcal{H}$-free graph (of sufficiently large order) has a Hamiltonian path

for a family $\mathcal{H}$ of connected graphs. Duffus et al. [3] proved $\mathcal{H} = \{K_{1,3}, K_3^*\}$ satisfies (H1), and Faudree and Gould [5] showed that if a family $\mathcal{H}$ satisfying (H1) consists of two connected graphs, then $\mathcal{H} \leqslant \{K_{1,3}, K_3^*\}$. Thereafter a series by Gould and Harris [9–11] characterized the families $\mathcal{H}$ of connected graphs with $|\mathcal{H}| = 3$ satisfying (H1). Since a graph has a Hamiltonian path if and only if its path cover number (or its path partition number) is exactly one, it is natural to study the forbidden subgraph conditions assuring us that the path cover/partition number is bounded by a constant as a next step. Our main result gives a complete solution for the problem in a sense.

Our second motivation is an analysis of gap between minimum $\mathcal{A}$-covers and minimum $\mathcal{A}$-partitions. A path cover/partition, which is the main topic in this paper, is just one of examples of $\mathcal{A}$-cover/partition problems, and there also exist many other cover/partition problems. One of representative other examples is the case where $\mathcal{A}$ is the family of all stars, where we regard K_1 as one of stars. If we define the star cover number and the star partition number in the same way as $\mathrm{pc}(G)$ and $\mathrm{pp}(G)$, we can easily verify that the values are always equivalent. (Indeed, the star cover number also equals to the *domination number*, which is one of classical invariants in graph theory. The forbidden subgraph conditions assuring us that the domination number is bounded by a constant were characterized in [8].) On the other hand, as it is evident from Theorem 2, there is a gap between the path cover number and the path partition number. By Theorem 2, we discover that $F_{n,n}^{(3)}$ and $F_{n,n}^{(4)}$ play an important role for essential structures giving such a gap.

We also obtain an analogy of Theorem 2 considering a cycle cover/partition problem. A $\{K_1, K_2, C_i : i \geqslant 3\}$-cover (resp. a $\{K_1, K_2, C_i : i \geqslant 3\}$-partition) of G is called a *cycle cover* (resp. a *cycle partition*) of G. The value $\min\{|\mathcal{P}| : \mathcal{P}$ is a cycle cover of $G\}$ (resp. $\min\{|\mathcal{P}| : \mathcal{P}$ is a cycle partition of $G\}$), denoted by $\mathrm{cc}(G)$ (resp. $\mathrm{cp}(G)$), is called the *cycle cover number* (resp. the *cycle partition number*) of G. One might feel that it is strange to admit the existence of K_1 and K_2 in cycle covers/partitions. However, if we exclude K_1 and K_2 from the definition of cycle covers/partitions, then the minimum cardinality of a $\{C_i : i \geqslant 3\}$-cover cannot be defined for some graphs, for example, trees and graphs having a vertex of degree one. Thus the cycle covers/partitions defined above are sometimes considered instead of $\{C_i : i \geqslant 3\}$-covers/partitions (see, for example, [4,6]), and we focus on it in this paper. In Section 3, as the second result, we characterize the families $\mathcal{H}$ of forbidden subgraphs satisfying one of the following:

(A'1) There exists a constant $c_1 = c_1(\mathcal{H})$ such that $\mathrm{cc}(G) \leqslant c_1$ for every connected $\mathcal{H}$-free graph G.

(A'2) There exists a constant $c_2 = c_2(\mathcal{H})$ such that $\mathrm{cp}(G) \leqslant c_2$ for every connected $\mathcal{H}$-free graph G.

Theorem 3. *Let $\mathcal{H}$ be a family of connected graphs. Then the following are equivalent.*

(i) *The family $\mathcal{H}$ satisfies (A'1).*

(ii) *The family* $\mathcal{H}$ *satisfies (A'2).*

(iii) *For an integer* $n \geqslant 2$, $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, P_n\}$.

We conclude this section by defining a new Ramsey-type concept concerning the path cover/partition number. Let $\mathcal{H}$ be a family of graphs. The *path cover Ramsey number* $R^{\mathrm{pc}}(\mathcal{H})$ (resp. the *path partition Ramsey number* $R^{\mathrm{pp}}(\mathcal{H})$) is the minimum positive integer R such that any connected graph G with $\mathrm{pc}(G) \geqslant R$ (resp. $\mathrm{pp}(G) \geqslant R$) contains an induced copy of an element of $\mathcal{H}$, where $R^{\mathrm{pc}}(\mathcal{H}) = \infty$ (resp. $R^{\mathrm{pp}}(\mathcal{H}) = \infty$) if such an integer does not exist. Then it follows from Theorem 2 that the following hold:

(P1) For a finite family $\mathcal{H}$ of connected graphs, $R^{\mathrm{pc}}(\mathcal{H})$ is a finite number if and only if $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$ for an integer $n \geqslant 2$.

(P2) For a finite family $\mathcal{H}$ of connected graphs, $R^{\mathrm{pp}}(\mathcal{H})$ is a finite number if and only if $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}, F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$ for an integer $n \geqslant 2$.

Note that $R^{\mathrm{pc}}(\mathcal{H}) = 2$ if and only if $R^{\mathrm{pp}}(\mathcal{H}) = 2$. As we mentioned above, it is known that $R^{\mathrm{pc}}(\{K_{1,3}, K_3^*\}) = 2$ and the study of triples $\{H_1, H_2, H_3\}$ of connected graphs with $R^{\mathrm{pc}}(\{H_1, H_2, H_3\}) = 2$ is completed. Since the $K_{1,3}$-freeness tends to give an important structure to many Hamiltonian properties, one might be interested in a relationship between such new Ramsey-type values and $K_{1,3}$-freeness. Here we focus on the values $R^{\mathrm{pc}}(\mathcal{H})$ and $R^{\mathrm{pp}}(\mathcal{H})$ for the case where $\mathcal{H}$ contains $K_{1,3}$. Note that for positive integers m and n with $m + n \geqslant 3$, all of $F_{m,n}^{(1)}$, $F_{m,n}^{(3)}$ and $F_{m,n}^{(4)}$ contain $K_{1,3}$ as an induced copy. Thus if $K_{1,3} \in \mathcal{H}$, then

$$R^{\mathrm{pc}}(\mathcal{H}) = R^{\mathrm{pc}}(\mathcal{H} \setminus \{F_{m,n}^{(1)}, F_{m,n}^{(3)}, F_{m,n}^{(4)} : m \geqslant 1,\ n \geqslant 1,\ m + n \geqslant 3\})$$

and

$$R^{\mathrm{pp}}(\mathcal{H}) = R^{\mathrm{pp}}(\mathcal{H} \setminus \{F_{m,n}^{(1)}, F_{m,n}^{(3)}, F_{m,n}^{(4)} : m \geqslant 1,\ n \geqslant 1,\ m + n \geqslant 3\}).$$

Considering (P1) and (P2), we leave the following open problem which will be a next interesting target on this concept for readers.

Problem 4. For positive integers p, q and r with $p \geqslant 3$ and $q + r \geqslant 4$ and for a family $\mathcal{H}$ of graphs with $\mathcal{H} \leqslant \{K_{1,3}, K_p^*, F_{q,r}^{(2)}\}$, determine the value $R^{\mathrm{pc}}(\mathcal{H})$ and $R^{\mathrm{pp}}(\mathcal{H})$.

2 Proof of Theorem 2

2.1 The "if" parts of Theorem 2

In this subsection, we prove the following theorem, which implies that the "if" parts of Theorem 2 hold.

Theorem 5. *Let* $n \geqslant 2$ *be an integer. Then the following hold:*

(i) *There exists a constant* $c_1 = c_1(n)$ *depending on* n *only such that* $\mathrm{pc}(G) \leqslant c_1$ *for every connected* $\{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$*-free graph* G.

(ii) *There exists a constant* $c_2 = c_2(n)$ *depending on* n *only such that* $\mathrm{pp}(G) \leqslant c_2$ *for every connected* $\{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}, F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$*-free graph* G.

The following lemma was proved in [15].

Lemma 6 (Pósa [15])**.** *For a graph* G, $\mathrm{cp}(G) \leqslant \alpha(G)$.

Let $\mathcal{C}$ be a cycle partition of a graph G. For each cycle C belonging to $\mathcal{C}$, fix an edge $e_C \in E(C)$. Then $\mathcal{P} = \{C - e_C : C \in \mathcal{C},\ |V(C)| \geqslant 3\} \cup \{C : C \in \mathcal{C},\ |V(C)| \leqslant 2\}$ is a path partition of G. This implies that $\mathrm{pp}(G) \leqslant \mathrm{cp}(G)$ for all graphs G. Hence the following lemma is obtained by Lemma 6.

Lemma 7. *For a graph* G, $\mathrm{pc}(G) \leqslant \mathrm{pp}(G) \leqslant \alpha(G)$.

Lemma 8. *Let* $n \geqslant 2$ *and* $\alpha \geqslant 1$ *be integers. Let* G *be a* $\{K_{1,n}, K_n^*\}$*-free graph, and let* X *be a subset of* $V(G)$ *with* $\alpha(G[X]) \leqslant \alpha$. *Then* $\alpha(G[N_G(X)]) \leqslant (n-1)R(n, \alpha+1) - 1$.

Proof. By way of contradiction, we suppose that there exists a subset Y of $N_G(X)$ such that Y is an independent set of G and $|Y| = (n-1)R(n, \alpha+1)$. Take a subset X_0 of X with $Y \subseteq N_G(X_0)$ so that $|X_0|$ is as small as possible. If $|X_0| \leqslant R(n, \alpha+1) - 1$, then $\frac{|Y|}{|X_0|} \geqslant \frac{(n-1)R(n,\alpha+1)}{R(n,\alpha+1)-1} > n-1$, and hence there exists a vertex $x_0 \in X_0$ with $|N_G(x_0) \cap Y| \geqslant n$, which contradicts the $K_{1,n}$-freeness of G. Thus $|X_0| \geqslant R(n, \alpha+1)$. Since $\alpha(G[X_0]) \leqslant \alpha(G[X]) \leqslant \alpha$, this implies that there exists a subset X_1 of X_0 such that X_1 is a clique of G and $|X_1| = n$. By the minimality of X_0, $(N_G(x) \cap Y) \setminus N_G(X_0 \setminus \{x\}) \neq \varnothing$ for every $x \in X_0$. For each $x \in X_0$, let $y_x \in (N_G(x) \cap Y) \setminus N_G(X_0 \setminus \{x\})$. Then $X_1 \cup \{y_x : x \in X_1\}$ induces a copy of K_n^* in G, which contradicts the K_n^*-freeness of G. □

In the remainder of this subsection, we fix an integer $n \geqslant 2$ and a connected $\{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$-free graph G. Set $n_0 = \max\{\lceil \frac{n^2-n-2}{2} \rceil, n\}$. Take a longest induced path P of G, and write $P = u_1u_2 \cdots u_m$. Let $X_0 = \{u_i : 1 \leqslant i \leqslant n_0 \text{ or } m - n_0 + 1 \leqslant i \leqslant m\}$ and $Y = N_G(V(P) \setminus X_0) \setminus (X_0 \cup N_G(X_0))$. Note that if $|V(P)| \leqslant 2n_0$, then $X_0 = V(P)$ and $Y = \varnothing$. We further remark that $N_G(y) \cap V(P) \subseteq \{u_i : n_0+1 \leqslant i \leqslant m-n_0\}$ for every $y \in Y$ (and in the remainder of this subsection, we frequently use the fact without mentioning). For each i with $n_0+1 \leqslant i \leqslant m-n_0$, let $Y_i = \{y \in Y : \min\{j : n_0+1 \leqslant j \leqslant m-n_0,\ yu_j \in E(G)\} = i\}$. In other words, Y_i is the set of vertices $y \in Y$ whose neighbor in P with the smallest index is u_i. Now we recursively define the sets X_i ($i \geqslant 1$) as follows: Let $X_1 = N_G(X_0) \setminus V(P)$, and for i with $i \geqslant 2$, let $X_i = N_G(X_{i-1}) \setminus (V(P) \cup Y \cup (\bigcup_{1 \leqslant j \leqslant i-1} X_j))$ (see Figure 2). Then $X_1 \cap Y = \varnothing$ and $X_1 \cup Y = N_G(V(P))$.

Lemma 9. *We have* $X_{2n_0} = \varnothing$.

Proof. Suppose that $X_{2n_0} \neq \varnothing$. Let $x_{2n_0} \in X_{2n_0}$. Then we can recursively take a vertex $x_{2n_0-i} \in N_G(x_{2n_0-i+1}) \cap X_{2n_0-i}$ for i with $1 \leqslant i \leqslant 2n_0$. Note that $x_0 = u_k$ for some k with $1 \leqslant k \leqslant n_0$ or $m - n_0 + 1 \leqslant k \leqslant m$. By symmetry, we may assume that $1 \leqslant k \leqslant n_0$. Under this condition, we choose k so that k is as large as possible. Since $x_0x_1 \cdots x_{2n_0}$ is

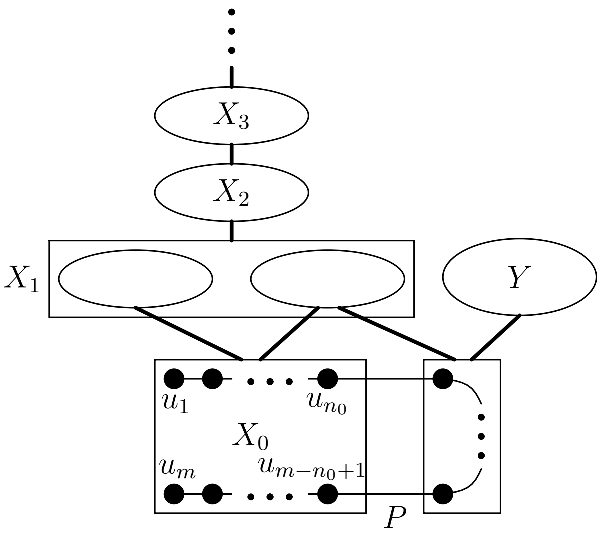

Figure 2: Path P and sets X_i and Y.

an induced path of G having $2n_0+1$ vertices, it follows from the maximality of P that $|V(P)| \geqslant 2n_0+1$. In particular, $V(P) \setminus X_0 \neq \varnothing$.

If $N_G(x_1) \cap (V(P) \setminus X_0) = \varnothing$, then $x_{2n_0}x_{2n_0-1} \cdots x_1 u_k u_{k+1} \cdots u_{m-n_0}$ is an induced path of G having $2n_0+m-n_0-k+1$ ($\geqslant m+1$) vertices, which contradicts the maximality of P. Thus $N_G(x_1) \cap (V(P) \setminus X_0) \neq \varnothing$.

Now we consider an operation recursively defining integers $j_1, j_2, \ldots$ with $1 \leqslant j_p \leqslant m$ ($p \geqslant 1$) and $j_1 < j_2 < \cdots$ as follows (see Figure 3): Let $j_1 = \min\{j : 1 \leqslant j \leqslant m,\ x_1u_j \in E(G)\}$. For $p \geqslant 2$, we assume that the integer j_{p-1} has defined. If $\{j : j_{p-1}+2 \leqslant j \leqslant m,\ x_1u_j \in E(G)\} \neq \varnothing$, we let $j_p = \min\{j : j_{p-1}+2 \leqslant j \leqslant m,\ x_1u_j \in E(G)\}$; otherwise, we finish the operation. Let $S = \{u_{j_p} : p \geqslant 1\}$, and set $s = |S|$. Let $j^* = \max\{j : 1 \leqslant j \leqslant m,\ x_1u_j \in E(G)\}$. Note that $j^* \in \{j_s, j_s+1\}$. Since $j_p \geqslant j_{p-1}+2$, S is an independent set of G. Since G is $K_{1,n}$-free and $\{x_1, x_2\} \cup S$ induces a copy of $K_{1,s+1}$ in G, we have $s+1 \leqslant n-1$.

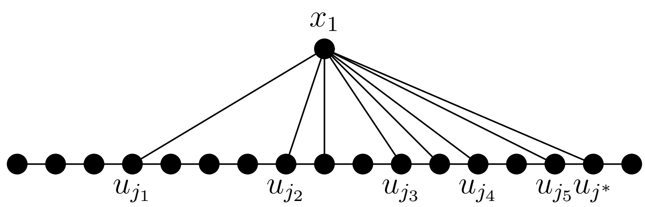

Figure 3: An example for $s = 5$.

For the moment, suppose that $s = 1$. Since $N_G(x_1) \cap \{u_j : 1 \leqslant j \leqslant n_0\} \neq \varnothing$ and $N_G(x_1) \cap (V(P) \setminus X_0) \neq \varnothing$, this forces $N_G(x_1) = \{u_{n_0}, u_{n_0+1}\}$. Then

$$\{x_1, x_2, u_{n_0}, u_{n_0-1}, \ldots, u_{n_0-n+1}, u_{n_0+1}, u_{n_0+2}, \ldots, u_{n_0+n}\}$$

induces a copy of $F^{(2)}_{n,n}$ in G, which is a contradiction. Thus $s \geqslant 2$.

Let $Q_1 = u_1u_2 \cdots u_{j_1}$ and $Q_{s+1} = u_{j^*}u_{j^*+1} \cdots u_m$ be subpaths of P. For p with $2 \leqslant p \leqslant s$, let $Q_p = u_{j_{p-1}+2}u_{j_{p-1}+3} \cdots u_{j_p}$ be a subpath of P. Then $V(P) \setminus (\bigcup_{1 \leqslant p \leqslant s+1} V(Q_p)) = \{u_{j_p+1} : 1 \leqslant p \leqslant s-1\}$, and hence

$$\begin{aligned}
2n_0 + 1 &\leqslant |V(P)| \\
&= \left| V(P) \setminus \left(\bigcup_{1 \leqslant p \leqslant s+1} V(Q_p) \right) \right| + \left| \bigcup_{1 \leqslant p \leqslant s+1} V(Q_p) \right| \\
&\leqslant (s-1) + \sum_{1 \leqslant p \leqslant s+1} |V(Q_p)| \\
&= \sum_{1 \leqslant p \leqslant s+1} (|V(Q_p)| + 1) - 2.
\end{aligned}$$

This implies that $\sum_{1 \leqslant p \leqslant s+1}(|V(Q_p)| + 1) \geqslant 2n_0 + 3 \geqslant 2\lceil \frac{n^2-n-2}{2} \rceil + 3 \geqslant n^2 - n + 1$. If $|V(Q_p)| \leqslant n - 1$ for all p with $1 \leqslant p \leqslant s + 1$, then $n^2 - n + 1 \leqslant \sum_{1 \leqslant p \leqslant s+1}(|V(Q_p)| + 1) \leqslant (s + 1)n \leqslant (n - 1)n$, which is a contradiction. Thus $|V(Q_q)| \geqslant n$ for some q with $1 \leqslant q \leqslant s + 1$.

Note that $|N_G(x_1) \cap V(Q_q)| = 1$. Write $N_G(x_1) \cap V(Q_q) = \{u_j\}$. If $q \neq s + 1$, then $j \in \{j_p : 1 \leqslant p \leqslant s\}$; otherwise, $j = j^*$ $(\in \{j_s, j_{s+1}\})$. Since u_j is an endvertex of Q_q, there exists a subpath Q of Q_q such that u_j is an endvertex of Q and $|V(Q)| = n$. Since $|S| \geqslant 2$, we can take a vertex $v \in S$ as follows: If $q \neq s+1$, let $v \in S \setminus \{u_j\}$; otherwise (i.e., $j = j^*$), let $v = u_{j_1}$. Then by the definition of Q_p $(1 \leqslant p \leqslant s + 1)$, $N_G(v) \cap V(Q_q) = \varnothing$. Since $2n_0 \geqslant 2n \geqslant n + 1$, the vertices x_i with $2 \leqslant i \leqslant n + 1$ have been defined, and hence this implies that $\{x_1, v, x_2, x_3, \ldots, x_{n+1}\} \cup V(Q)$ induces a copy of $F^{(1)}_{n,n}$ in G, which is a contradiction. □

Lemma 10. *Let i be an integer with $n_0 + 1 \leqslant i \leqslant m - n_0$, and let $y \in Y_i$. Then the following hold:*

(i) *If $yu_{i+1} \notin E(G)$, then $N_G(y) \cap V(P) = \{u_i, u_{i+2}\}$.*

(ii) *We have $Y_{m-n_0} = \varnothing$.*

(iii) *If G is $F^{(3)}_{n,n}$-free, then $yu_{i+1} \in E(G)$.*

Proof. (i) Suppose that $yu_{i+1} \notin E(G)$ and $N_G(y) \cap V(P) \neq \{u_i, u_{i+2}\}$. Let $k = \max\{j : n_0 + 1 \leqslant j \leqslant m - n_0,\ yu_j \in E(G)\}$. If $k = i$ (i.e., $N_G(y) \cap V(P) = \{u_i\}$), then

$$\{u_i, y, u_{i-1}, u_{i-2}, \ldots, u_{i-n}, u_{i+1}, u_{i+2}, \ldots, u_{i+n}\}$$

induces a copy of $F^{(1)}_{n,n}$ in G, which is a contradiction. Since $yu_{i+1} \notin E(G)$ and $N_G(y) \cap V(P) \neq \{u_i, u_{i+2}\}$, this forces $k \geqslant i + 3$. Then

$$\{u_i, u_{i+1}, u_{i-1}, u_{i-2}, \ldots, u_{i-n}, y, u_k, u_{k+1}, \ldots, u_{k+n-2}\}$$

induces a copy of $F^{(1)}_{n,n}$ in G, which is a contradiction.

(ii) By (i), if there exists a vertex $y \in Y_{m-n_0}$, then it follows that $yu_{m-n_0+1} \in E(G)$ or $N_G(y) \cap V(P) = \{u_{m-n_0}, u_{m-n_0+2}\}$, and in particular, $N_G(y) \cap X_0 \neq \varnothing$, which contradicts the definition of Y. Thus we have $Y_{m-n_0} = \varnothing$.

(iii) Suppose that G is $F^{(3)}_{n,n}$-free and $yu_{i+1} \notin E(G)$. Then it follows from (i) that $N_G(y) \cap V(P) = \{u_i, u_{i+2}\}$, and hence

$$\{y, u_{i+1}, u_i, u_{i-1}, \dots, u_{i-n+1}, u_{i+2}, u_{i+3}, \dots, u_{i+n+1}\}$$

induces a copy of $F^{(3)}_{n,n}$ in G, which is a contradiction. □

Lemma 11. *We have* $V(G) = V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leqslant i \leqslant 2n_0-1} X_i)$.

Proof. Suppose that $V(G) \neq V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leqslant i \leqslant 2n_0-1} X_i)$. Since G is connected, there exists a vertex $z \in V(G) \setminus (V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leqslant i \leqslant 2n_0-1} X_i))$ adjacent to a vertex $y \in V(P) \cup N_G(V(P)) \cup (\bigcup_{2 \leqslant i \leqslant 2n_0-1} X_i)$ in G. By Lemma 9 and the definition of X_i and Y, this implies that $y \in Y$. Let i be the integer such that $y \in Y_i$. Then by Lemma 10(ii), $n_0 + 1 \leqslant i \leqslant m - n_0 - 1$. Let $k = \max\{j : n_0 + 1 \leqslant j \leqslant m - n_0,\ yu_j \in E(G)\}$. By Lemma 10(i), $k \geqslant i + 1$. If $k = i + 1$, then

$$\{y, z, u_i, u_{i-1}, \dots, u_{i-n+1}, u_{i+1}, u_{i+2}, \dots, u_{i+n}\}$$

induces a copy of $F^{(2)}_{n,n}$ in G; if $k \geqslant i + 2$, then

$$\{y, z, u_i, u_{i-1}, \dots, u_{i-n+1}, u_k, u_{k+1}, \dots, u_{k+n-1}\}$$

induces a copy of $F^{(1)}_{n,n}$ in G. In either case, we obtain a contradiction. □

Now we recursively define the values α_i ($i \geqslant 0$) as follows: Let $\alpha_0 = 2\lceil \frac{n_0}{2} \rceil$, and for i with $i \geqslant 1$, let $\alpha_i = (n-1)R(n, \alpha_{i-1} + 1) - 1$.

Lemma 12. *For an integer i with $i \geqslant 0$, $\alpha(G[X_i]) \leqslant \alpha_i$.*

Proof. We proceed by induction on i. If $|V(P)| \leqslant 2n_0$, then $G[X_0]$ equals to P, and hence $\alpha(G[X_0]) = \alpha(P) = \lceil \frac{|V(P)|}{2} \rceil \leqslant n_0 \leqslant \alpha_0$; if $|V(P)| \geqslant 2n_0 + 1$, then $G[X_0]$ consists of two components each of which is a path of order n_0, and hence $\alpha(G[X_0]) = 2\lceil \frac{n_0}{2} \rceil = \alpha_0$. In either case, we have $\alpha(G[X_0]) \leqslant \alpha_0$. Thus we may assume that $i \geqslant 1$, and suppose that $\alpha(G[X_{i-1}]) \leqslant \alpha_{i-1}$. Since $X_i \subseteq N_G(X_{i-1})$, it follows from Lemma 8 that $\alpha(G[X_i]) \leqslant \alpha(G[N_G(X_{i-1})]) \leqslant (n-1)R(n, \alpha_{i-1} + 1) - 1 = \alpha_i$, as desired. □

Note that the value $\sum_{1 \leqslant i \leqslant 2n_0-1} \alpha_i$ is a constant depending on n only. Thus, considering Lemmas 7, 11 and 12, it suffices to show that

- $\mathrm{pc}(G[V(P) \cup Y])$ is bounded by a constant depending on n only, and
- if G is $\{F^{(3)}_{n,n}, F^{(4)}_{n,n}\}$-free, then $\mathrm{pp}(G[V(P) \cup Y])$ is bounded by a constant depending on n only.

Hence the following lemma completes the proof of Theorem 5.

Lemma 13. (i) *We have* $\mathrm{pc}(G[V(P)\cup Y]) \leqslant \max\{3n-6, 1\}$.

(ii) *If* G *is* $\{F^{(3)}_{n,n}, F^{(4)}_{n,n}\}$*-free, then there exists a Hamiltonian path of* $G[V(P)\cup Y]$*, i.e.,* $\mathrm{pp}(G[V(P)\cup Y]) = 1$.

Proof. If $Y = \varnothing$, then P is a Hamiltonian path of $G[V(P)\cup Y]$, and hence $\mathrm{pc}(G[V(P)\cup Y]) = \mathrm{pp}(G[V(P)\cup Y]) = 1$. Thus we may assume that $Y \neq \varnothing$. By Lemma 10(ii), $Y_{m-n_0} = \varnothing$.

We first prove (i). Fix an integer i with $n_0+1 \leqslant i \leqslant m-n_0-1$. Let $Y_{i,1} = \{y \in Y_i : yu_{i+1} \in E(G)\}$ and $Y_{i,2} = Y_i \setminus Y_{i,1}$. Then by Lemma 10(i), $N_G(y) \cap V(P) = \{u_i, u_{i+2}\}$ for all $y \in Y_{i,2}$. Let $j \in \{1,2\}$. If there exists an independent set $U \subseteq Y_{i,j}$ of G with $|U| = n-1$, then $\{u_{i-1}, u_i\} \cup U$ induces a copy of $K_{1,n}$ in G, which is a contradiction. Thus $\alpha(G[Y_{i,j}]) \leqslant n-2$. Since $Y \neq \varnothing$, i.e., $Y_{p,q} \neq \varnothing$ for some p and q with $n_0+1 \leqslant p \leqslant m-n_0-1$ and $q \in \{1,2\}$, this implies that $n \geqslant 3$. By Lemma 7, there exists a path partition $\mathcal{P}_{i,j} = \{Q^{(1)}_{i,j}, Q^{(2)}_{i,j}, \dots, Q^{(s_{i,j})}_{i,j}\}$ of $G[Y_{i,j}]$ with $s_{i,j} \leqslant n-2$, where $\mathcal{P}_{i,j} = \varnothing$ and $s_{i,j} = 0$ if $Y_{i,j} = \varnothing$. For an integer t with $1 \leqslant t \leqslant n-2$, if $t \leqslant s_{i,j}$, let $R^{(t)}_{i,j}$ be the path $u_i v Q^{(t)}_{i,j} w u_{i+j}$, where $\{v, w\}$ is the set of endvertices of $Q^{(t)}_{i,j}$; otherwise, let $R^{(t)}_{i,j}$ be the path between u_i and u_{i+j} on P (i.e., $R^{(t)}_{i,1} = u_i u_{i+1}$ and $R^{(t)}_{i,2} = u_i u_{i+1} u_{i+2}$). We define the value ξ_2 (resp. ξ_3) with $\xi_2 = m-n_0$ or $\xi_2 = m-n_0-1$ (resp. $\xi_3 = m-n_0-1$ or $\xi_3 = m-n_0$) according as m is odd or even. Let

$$\begin{aligned}
R^{(t)}_1 &= u_1 u_2 \cdots u_{n_0+1} R^{(t)}_{n_0+1,1} u_{n_0+2} R^{(t)}_{n_0+2,1} u_{n_0+3} \cdots u_{m-n_0-1} R^{(t)}_{m-n_0-1,1} u_{m-n_0} u_{m-n_0+1} \cdots u_m,\\
R^{(t)}_2 &= u_1 u_2 \cdots u_{n_0+1} R^{(t)}_{n_0+1,2} u_{n_0+3} R^{(t)}_{n_0+3,2} u_{n_0+5} \cdots u_{\xi_2-2} R^{(t)}_{\xi_2-2,2} u_{\xi_2} u_{\xi_2+1} \cdots u_m, \text{ and}\\
R^{(t)}_3 &= u_1 u_2 \cdots u_{n_0+2} R^{(t)}_{n_0+2,2} u_{n_0+4} R^{(t)}_{n_0+4,2} u_{n_0+6} \cdots u_{\xi_3-2} R^{(t)}_{\xi_3-2,2} u_{\xi_3} u_{\xi_3+1} \cdots u_m.
\end{aligned}$$

Then we easily verify that $\{R^{(t)}_a : a \in \{1,2,3\},\ 1 \leqslant t \leqslant n-2\}$ is a path cover of $G[V(P)\cup Y]$ having cardinality at most $3(n-2)$, which proves (i).

Next we prove (ii). Suppose that G is $\{F^{(3)}_{n,n}, F^{(4)}_{n,n}\}$-free. We start with the following claim.

Claim 14. *For an integer* i *with* $n_0+1 \leqslant i \leqslant m-n_0-1$, $\{u_i, u_{i+1}\} \cup Y_i$ *is a clique of* G.

Proof. Suppose that there exist two vertices $y, y' \in \{u_i, u_{i+1}\} \cup Y_i$ with $yy' \notin E(G)$. By the definition of Y_i and Lemma 10(iii), every vertex in Y_i is adjacent to both u_i and u_{i+1} in G. Thus $y, y' \in Y_i$. Recall that $N_G(Y) \cap V(P) \subseteq \{u_j : n_0+1 \leqslant j \leqslant m-n_0\}$. Let $k = \max\{j : n_0+1 \leqslant j \leqslant m-n_0,\ N_G(u_j) \cap \{y, y'\} \neq \varnothing\}$. We may assume that $yu_k \in E(G)$. Note that $k \geqslant i+1$. If $k = i+1$, then

$$\{y, y', u_i, u_{i-1}, \dots, u_{i-n+1}, u_{i+1}, u_{i+2}, \dots, u_{i+n}\}$$

induces a copy of $F^{(4)}_{n,n}$ in G, which is a contradiction. Thus $k \geqslant i+2$. If $y'u_k \in E(G)$, then

$$\{y, y', u_i, u_{i-1}, \dots, u_{i-n+1}, u_k, u_{k+1}, \dots, u_{k+n-1}\}$$

induces a copy of $F^{(3)}_{n,n}$ in G; if $y'u_k \notin E(G)$, then

$$\{u_i, y', u_{i-1}, u_{i-2}, \ldots, u_{i-n}, y, u_k, u_{k+1}, \ldots, u_{k+n-2}\}$$

induces a copy of $F^{(1)}_{n,n}$ in G. In either case, we obtain a contradiction. □

For an integer i with $n_0 + 1 \leqslant i \leqslant m - n_0 - 1$, it follows from Claim 14 that there exists a Hamiltonian path R_i of $G[\{u_i, u_{i+1}\} \cup Y_i]$ with the endvertices u_i and u_{i+1}. Then

$$u_1u_2 \cdots u_{n_0+1}R_{n_0+1}u_{n_0+2}R_{n_0+2}u_{n_0+3} \cdots u_{m-n_0-1}R_{m-n_0-1}u_{m-n_0}u_{m-n_0+1} \cdots u_m$$

is a Hamiltonian path of $G[V(P) \cup (\bigcup_{n_0+1 \leqslant i \leqslant m-n_0-1} Y_i)]$ $(= G[V(P) \cup Y])$, as desired. □

2.2 The "only if" parts of Theorem 2

Let $s \geqslant 2$ and $t \geqslant 3$ be integers, and let $Q_i = u^{(1)}_i u^{(2)}_i \cdots u^{(t)}_i$ $(1 \leqslant i \leqslant s)$ be s pairwise vertex-disjoint paths. We define four graphs.

- Let $H^{(1)}_{s,t}$ be the graph obtained from the union of the paths $Q_1, \ldots, Q_s$ by adding $2(s-1)$ vertices v_i, w_i $(1 \leqslant i \leqslant s-1)$ and $3(s-1)$ edges $v_iw_i, v_iu^{(t)}_i, v_iu^{(1)}_{i+1}$ $(1 \leqslant i \leqslant s-1)$.
- Let $H^{(2)}_{s,t}$ be the graph obtained from $H^{(1)}_{s,t}$ by adding $s-1$ edges $u^{(t)}_i u^{(1)}_{i+1}$ $(1 \leqslant i \leqslant s-1)$.
- Let $H^{(3)}_{s,t}$ be the graph obtained from the union of the paths $Q_1, \ldots, Q_s$ by adding $2(s-1)$ vertices v_i, w_i $(1 \leqslant i \leqslant s-1)$ and $4(s-1)$ edges $v_iu^{(t)}_i, v_iu^{(1)}_{i+1}, w_iu^{(t)}_i, w_iu^{(1)}_{i+1}$ $(1 \leqslant i \leqslant s-1)$.
- Let $H^{(4)}_{s,t}$ be the graph obtained from $H^{(3)}_{s,t}$ by adding $s-1$ edges $u^{(t)}_i u^{(1)}_{i+1}$ $(1 \leqslant i \leqslant s-1)$.

Lemma 15. *We have* $\mathrm{pc}(H^{(1)}_{s,t}) = \mathrm{pc}(H^{(2)}_{s,t}) = \lceil \frac{s+1}{2} \rceil$.

Proof. Note that $u^{(1)}_1, u^{(t)}_s, w_i$ $(1 \leqslant i \leqslant s-1)$ have degree one in $H^{(2)}_{s,t}$. Since a path contains at most two vertices of degree at most one, $\mathrm{pc}(G) \geqslant \lceil \frac{l}{2} \rceil$ for every graph G where l is the number of the vertices of G having degree one. In particular, we have

$$\mathrm{pc}(H^{(2)}_{s,t}) \geqslant \left\lceil \frac{s+1}{2} \right\rceil. \tag{1}$$

If s is odd, let

$$\mathcal{P} = \left\{ H^{(1)}_{s,t} - \{w_j : 1 \leqslant j \leqslant s-1\},\ w_{2i-1}v_{2i-1}u^{(1)}_{2i}Q_{2i}u^{(t)}_{2i}v_{2i}w_{2i} : 1 \leqslant i \leqslant \frac{s-1}{2} \right\};$$

if s is even, let

$$\mathcal{P} = \left\{ H^{(1)}_{s,t} - \{w_j : 1 \leqslant j \leqslant s-1\},\ H^{(1)}_{s,t}[\{w_{s-1}\}],\ w_{2i-1}v_{2i-1}u^{(1)}_{2i}Q_{2i}u^{(t)}_{2i}v_{2i}w_{2i} : 1 \leqslant i \leqslant \frac{s-2}{2} \right\}.$$

Then we verify that $\mathcal{P}$ is a path cover of $H^{(1)}_{s,t}$ with $|\mathcal{P}| = \lceil \frac{s+1}{2} \rceil$. Furthermore, since $H^{(1)}_{s,t}$ is a spanning subgraph of $H^{(2)}_{s,t}$, a path cover of $H^{(1)}_{s,t}$ is also a path cover of $H^{(2)}_{s,t}$, and hence $\mathrm{pc}(H^{(2)}_{s,t}) \leqslant \mathrm{pc}(H^{(1)}_{s,t}) \leqslant \lceil \frac{s+1}{2} \rceil$. This together with (1) leads to the desired conclusion. □

Lemma 16. *We have* $\mathrm{pp}(H_{s,t}^{(3)}) = \mathrm{pp}(H_{s,t}^{(4)}) = s$.

Proof. We first prove that

$$\mathrm{pp}(H_{s,t}^{(4)}) \geqslant s. \tag{2}$$

Let $\mathcal{P}$ be a path partition of $H_{s,t}^{(4)}$. It suffices to show that $|\mathcal{P}| \geqslant s$. For each i with $1 \leqslant i \leqslant s$, let R_i be the unique element of $\mathcal{P}$ containing $u_i^{(2)}$. We remark that R_i might equal to R_j for some $1 \leqslant i < j \leqslant s$. Let $I = \{i : 1 \leqslant i \leqslant s-1,\ R_i = R_{i+1}\}$, and write $I = \{i_1, i_2, \ldots, i_h\}$ with $i_1 < i_2 < \ldots < i_h$ where $h = 0$ if $I = \varnothing$. For integers i and i' with $1 \leqslant i < i' \leqslant s$, any paths of $H_{s,t}^{(4)}$ joining $u_i^{(2)}$ and $u_{i'}^{(2)}$ contain every vertex in $\{u_j^{(2)} : i < j < i'\}$. This implies that if $R_i = R_{i'}$ with $1 \leqslant i < i' \leqslant s$, then $i' - i + 1$ paths R_j $(i \leqslant j \leqslant i')$ are equal. In particular, we have $|\{R_i : 1 \leqslant i \leqslant s\}| = s - h$.

Fix an integer l with $1 \leqslant l \leqslant h$. Then for every path R of $H_{s,t}^{(4)}$ joining $u_{i_l}^{(2)}$ and $u_{i_l+1}^{(2)}$, we easily verify that

- $\{u_{i_l}^{(t)}, u_{i_l+1}^{(1)}\} \subseteq V(R)$, and
- $v_{i_l} \notin V(R)$ or $w_{i_l} \notin V(R)$.

Since $v_{i_l} w_{i_l} \notin E(H_{s,t}^{(4)})$, this implies that there exists an element R'_{i_l} of $\mathcal{P}$ such that either $V(R'_{i_l}) = \{v_{i_l}\}$ or $V(R'_{i_l}) = \{w_{i_l}\}$. Therefore

$$\begin{aligned}|\mathcal{P}| &\geqslant |\{R_i : 1 \leqslant i \leqslant s\} \cup \{R'_{i_j} : 1 \leqslant j \leqslant h\}| \\ &= |\{R_i : 1 \leqslant i \leqslant s\}| + |\{R'_{i_j} : 1 \leqslant j \leqslant h\}| \\ &= (s-h) + h \\ &= s,\end{aligned}$$

which proves (2).

Since

$$\mathcal{P}' = \{H_{s,t}^{(3)} - \{w_j : 1 \leqslant j \leqslant s-1\},\ H_{s,t}^{(3)}[\{w_i\}] : 1 \leqslant i \leqslant s-1\}$$

is a path partition of $H_{s,t}^{(3)}$ with $|\mathcal{P}'| = s$. Furthermore, since $H_{s,t}^{(3)}$ is a spanning subgraph of $H_{s,t}^{(4)}$, a path partition of $H_{s,t}^{(3)}$ is also a path partition of $H_{s,t}^{(4)}$, and hence $\mathrm{pp}(H_{s,t}^{(4)}) \leqslant \mathrm{pp}(H_{s,t}^{(3)}) \leqslant s$. This together with (2) leads to the desired conclusion. □

Now we prove the following proposition, which gives the "only if" parts of Theorem 2.

Proposition 17. *Let* $\mathcal{H}$ *be a finite family of connected graphs.*

(i) *If* $\mathcal{H}$ *satisfies (A1), then* $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}\}$ *for an integer* $n \geqslant 2$.

(ii) *If* $\mathcal{H}$ *satisfies (A2), then* $\mathcal{H} \leqslant \{K_{1,n}, K_n^*, F_{n,n}^{(1)}, F_{n,n}^{(2)}, F_{n,n}^{(3)}, F_{n,n}^{(4)}\}$ *for an integer* $n \geqslant 2$.

Proof. Since $\mathcal{H}$ is a finite family, the value $p = \max\{|V(H)| : H \in \mathcal{H}\}$ is well-defined. If $p \leqslant 2$, then the desired conclusions trivially hold. Thus we may assume that $p \geqslant 3$.

We first suppose that $\mathcal{H}$ satisfies (A1), and show that (i) holds. There exists a constant $c_1 = c_1(\mathcal{H})$ such that $\mathrm{pc}(G) \leqslant c_1$ for every connected $\mathcal{H}$-free graph G. Since $\mathrm{pc}(K_{1,2c_1+1}) = c_1 + 1$ and $\mathrm{pc}(K^*_{2c_1+1}) = c_1 + 1$, neither $K_{1,2c_1+1}$ nor $K^*_{2c_1+1}$ is $\mathcal{H}$-free. This implies that

$$\mathcal{H} \leqslant \{K_{1,2c_1+1}, K^*_{2c_1+1}\}. \tag{3}$$

For each $i \in \{1, 2\}$, it follows from Lemma 15 that $\mathrm{pc}(H^{(i)}_{2c_1,p}) = \lceil \frac{2c_1+1}{2} \rceil = c_1 + 1$, and hence $H^{(i)}_{2c_1,p}$ is not $\mathcal{H}$-free, i.e., $H^{(i)}_{2c_1,p}$ contains an induced subgraph A_i isomorphic to an element of $\mathcal{H}$. Since $|V(A_i)| \leqslant p$, we have

- $|\{j : 1 \leqslant j \leqslant 2c_1,\ V(A_i) \cap V(Q_j) \neq \varnothing\}| \leqslant 2$, and
- $|\{j : 1 \leqslant j \leqslant 2c_1 - 1,\ V(A_i) \cap \{v_j, w_j\} \neq \varnothing\}| \leqslant 1$.

This implies that A_i is an induced copy of $F^{(i)}_{p,p}$, and hence

$$\mathcal{H} \leqslant \{F^{(1)}_{p,p}, F^{(2)}_{p,p}\}. \tag{4}$$

Let $n = \max\{2c_1 + 1, p\}$. Then by (3) and (4), $\mathcal{H} \leqslant \{K_{1,n}, K^*_n, F^{(1)}_{n,n}, F^{(2)}_{n,n}\}$, which proves (i).

Next we suppose that $\mathcal{H}$ satisfies (A2), and show that (ii) holds. There exists a constant $c_2 = c_2(\mathcal{H})$ such that $\mathrm{pp}(G) \leqslant c_2$ for every connected $\mathcal{H}$-free graph G. Since $\mathrm{pp}(G) \geqslant \mathrm{pc}(G)$ for all graphs G, $\mathcal{H}$ also satisfies (A1). Hence by (i), there exists an integer $m \geqslant 2$ such that

$$\mathcal{H} \leqslant \{K_{1,m}, K^*_m, F^{(1)}_{m,m}, F^{(2)}_{m,m}\}. \tag{5}$$

For each $i \in \{3, 4\}$, it follows from Lemma 16 that $\mathrm{pp}(H^{(i)}_{c_2+1,p}) = c_2 + 1$, and hence $H^{(i)}_{c_2+1,p}$ is not $\mathcal{H}$-free, i.e., $H^{(i)}_{c_2+1,p}$ contains an induced subgraph B_i isomorphic to an element of $\mathcal{H}$. Since $|V(B_i)| \leqslant p$, we have

- $|\{j : 1 \leqslant j \leqslant c_2 + 1,\ V(B_i) \cap V(Q_j) \neq \varnothing\}| \leqslant 2$, and
- $|\{j : 1 \leqslant j \leqslant c_2,\ V(B_i) \cap \{v_j, w_j\} \neq \varnothing\}| \leqslant 1$.

This implies that B_i is an induced copy of $F^{(i)}_{p,p}$, and hence

$$\mathcal{H} \leqslant \{F^{(3)}_{p,p}, F^{(4)}_{p,p}\}. \tag{6}$$

Let $n' = \max\{m, p\}$. Then by (5) and (6), $\mathcal{H} \leqslant \{K_{1,n'}, K^*_{n'}, F^{(1)}_{n',n'}, F^{(2)}_{n',n'}, F^{(3)}_{n',n'}, F^{(4)}_{n',n'}\}$, which proves (ii). □

3 Proof of Theorem 3

In this section, we prove Theorem 3. The following lemma was implicitly proved in [1]. (To keep the paper self-contained, we give its proof.)

Lemma 18 (Choi et al. [1])**.** *Let $n \geqslant 2$ be an integer. There exists a constant $c = c(n)$ depending on n only such that $\alpha(G) \leqslant c$ for every connected $\{K_{1,n}, K_n^*, P_n\}$-free graph G.*

Proof. Let x be a vertex of G, and for an integer i with $i \geqslant 0$, let X_i be the set of vertices y of G such that the distance between x and y in G is exactly i. Note that $X_0 = \{x\}$ and $X_1 = N_G(x)$. Since G is P_n-free, $X_i = \varnothing$ for all $i \geqslant n-1$. Since G is connected, this implies that

$$V(G) = \bigcup_{0 \leqslant i \leqslant n-2} X_i. \tag{7}$$

We recursively define the values α_i ($i \geqslant 0$) as follows: Let $\alpha_0 = 1$, and for i with $i \geqslant 1$, let $\alpha_i = (n-1)R(n, \alpha_{i-1}+1) - 1$.

We prove that

$$\alpha(G[X_i]) \leqslant \alpha_i \text{ for an integer } i \text{ with } 0 \leqslant i \leqslant n-2. \tag{8}$$

We proceed by induction on i. Since $\alpha(G[X_0]) = 1 = \alpha_0$, we may assume that $i \geqslant 1$ and $\alpha(G[X_{i-1}]) \leqslant \alpha_{i-1}$. Since $X_i \subseteq N_G(X_{i-1})$, it follows from Lemma 8 that $\alpha(G[X_i]) \leqslant \alpha(G[N_G(X_{i-1})]) \leqslant (n-1)R(n, \alpha_{i-1}+1) - 1 = \alpha_i$, as desired.

By (7) and (8), we have $\alpha(G) \leqslant \sum_{0 \leqslant i \leqslant n-2} \alpha(G[X_i]) \leqslant \sum_{0 \leqslant i \leqslant n-2} \alpha_i$. Since the value $\sum_{0 \leqslant i \leqslant n-2} \alpha_i$ is a constant depending on n only, we obtain the desired conclusion. □

Proof of Theorem 3. By the definition of cycle cover and cycle partition, "(ii) $\Longrightarrow$ (i)" clearly holds.

We show that "(iii) $\Longrightarrow$ (ii)" holds. Let $n \geqslant 2$ be an integer, and let $c = c(n)$ be the constant as in Lemma 18. It suffices to show that there exists a constant $c_1 = c_1(n)$ depending on n only such that $\mathrm{cp}(G) \leqslant c_1$ for every connected $\{K_{1,n}, K_n^*, P_n\}$-free graph G. By the definition of $c(n)$, we have $\alpha(G) \leqslant c$. This together with Lemma 6 leads to $\mathrm{cp}(G) \leqslant \alpha(G) \leqslant c$. Since c is a constant depending on n only, we obtain the desired conclusion.

Finally, we show that "(i) $\Longrightarrow$ (iii)" holds, which completes the proof of Theorem 3. Suppose that a family $\mathcal{H}$ of connected graphs satisfies (A'1). Then there exists a constant $c_1 = c_1(\mathcal{H})$ such that $\mathrm{cc}(G) \leqslant c_1$ for every connected $\mathcal{H}$-free graph G. Since $\mathrm{cc}(K_{1,c_1+1}) = c_1 + 1$, $\mathrm{cc}(K^*_{c_1+1}) = c_1 + 1$ and $\mathrm{cc}(P_{2c_1+1}) = \lceil \frac{2c_1+1}{2} \rceil = c_1 + 1$, none of K_{1,c_1+1}, $K^*_{c_1+1}$ and P_{2c_1+1} is $\mathcal{H}$-free. This implies that $\mathcal{H} \leqslant \{K_{1,c_1+1}, K^*_{c_1+1}, P_{2c_1+1}\}$, and hence $\mathcal{H} \leqslant \{K_{1,2c_1+1}, K^*_{2c_1+1}, P_{2c_1+1}\}$, which leads (iii). □

References

[1] I. Choi, M. Furuya, R. Kim and B. Park. A Ramsey-type theorem for the matching number regarding connected graphs. *Discrete Math.* **343** (2020), 111648.

[2] R. Diestel. *Graph Theory* (5th edition), Graduate Texts in Mathematics **173**, Springer, Berlin (2017).

[3] D. Duffus, R.J. Gould and M.S. Jacobson. Forbidden subgraphs and the hamiltonian theme. *The Theory and Applications of Graphs*, pp. 297–316, Wiley, New York (1981).

[4] H. Enomoto and H. Li. Partition of a graph into cycles and degenerated cycles. *Discrete Math.* **276** (2004), 177–181.

[5] R.J. Faudree and R.J. Gould. Characterizing forbidden pairs for Hamiltonian properties. *Discrete Math.* **173** (1997), 45–60.

[6] S. Fujita. Degree conditions for the partition of a graph into cycles, edges and isolated vertices. *Discrete Math.* **309** (2009), 3534–3540.

[7] S. Fujita, K. Kawarabayashi, C. L. Lucchesi, K. Ota, M. Plummer and A. Saito. A pair of forbidden subgraphs and perfect matchings. *J. Combin. Theory Ser. B* **96** (2006), 315–324.

[8] M. Furuya. Forbidden subgraphs for constant domination number. *Discrete Math. Theor. Comput. Sci.* **20(1)** (2018), Paper No. 19.

[9] R.J. Gould and J.M. Harris. Forbidden triples of subgraphs and traceability, *Congr. Numer.* **108** (1995), 183–192.

[10] R.J. Gould and J.M. Harris, Traceability in graphs with forbidden triples of subgraphs. *Discrete Math.* **189** (1998), 123–132.

[11] R.J. Gould and J.M. Harris. Forbidden triples and traceability: a characterization. *Discrete Math.* **203** (1999), 101–120.

[12] J. Han. On vertex-disjoint paths in regular graphs. *Electron. J. Combin.* **25** (2018), #P2.12.

[13] S. Ishizuka. Closure, path-factors and path coverings in claw-free graphs. *Ars Combin.* **50** (1998), 115–128.

[14] C. Magnant and D.M. Martin. A note on the path cover number of regular graphs. *Australas. J. Combin.* **43** (2009), 211–217.

[15] L. Pósa. On the circuits of finite graphs. *Magyar Tud. Akad. Mat. Kutato Int. Kozl.* **8** (1963), 355–361.

[16] B. Reed. Paths, stars and the number three. *Combin. Probab. Comput.* **5** (1996), 277–295.

[17] G. Yu. Covering 2-connected 3-regular graphs with disjoint paths. *J. Graph Theory* **88** (2018), 385–401.