Colouring non-even digraphs*

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Abstract

A colouring of a digraph as defined by Neumann-Lara in 1982 is a vertex-colouring such that no monochromatic directed cycles exist. The minimal number of colours required for such a colouring of a loopless digraph is defined to be its *dichromatic number*. This quantity has been widely studied in the last decades and can be considered as a natural directed analogue of the chromatic number of a graph. A digraph D is called *even* if for every 0-1-weighting of the edges it contains a directed cycle of even total weight. We show that every non-even digraph has dichromatic number at most 2 and an optimal colouring can be found in polynomial time. We strengthen a previously known NP-hardness result by showing that deciding whether a directed graph is 2-colourable remains NP-hard even if it contains a feedback vertex set of bounded size.

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1 Introduction

Graphs in this paper are considered simple, that is, without loops and multiple edges, while digraphs have no loops or parallel edges, but are allowed to have antiparallel pairs of edges (digons). An undirected edge with *endpoints* u and v will be denoted by uv, or vu symmetrically, while a directed edge with *tail* u and *head* v will be denoted as (u, v). A digraph D is called *strongly connected* if for every pair of vertices $u, v \in V(D)$ there is a directed path from u to v and from v to u. The *girth* of D is the minimum length of a directed cycle in D. We call a set $X \subseteq V(D)$ *acyclic*, if $D[X]$ is acyclic.

A *colouring* of a digraph D with k colours is a function $c\colon V(D) \to \{0, \ldots, k-1\}$. A colouring is called *proper* if $c^{-1}(i)$ is acyclic for every $i \in \{0, \ldots, k-1\}$. The *dichromatic number* $\vec{\chi}(D)$ is the smallest integer k such that D has a proper colouring with k colours.

One of the arguably most influential problems in graph theory was the Four-Colour-Conjecture, answered positively by Appel and Haken in 1976. As a directed version of this famous theorem, the Two-Colour-Conjecture posed by Erdős and Neumann-Lara and independently by Skrekovski still stands open. A digraph D is called *oriented* if its underlying undirected graph is simple.

Conjecture 1 (Two-Colour-Conjecture[BFJ+04, NL82])**.** Every oriented planar digraph D is 2-colourable.

Although this conjecture has an easy formulation, there seems to be a lack of methods for attacking it. The strongest partial result proved so far is due to Mohar and Li, who showed the following:

Theorem 2 ([LM17])**.** *Every oriented planar digraph of girth at least* 4 *is* 2*-colourable.*

In the undirected case, 2-colourability is very well understood and the class of bipartite graphs can be characterised in many different ways. For one, bipartite graphs are exactly the graphs without cycles of odd length, on the other hand the famous theorem by Kőnig can also be used to characterise bipartite graphs.

Theorem 3 ([Kőn31])**.** *A graph G is bipartite if and only if for all subgraphs $G' \subseteq G$ the size of a maximum matching of G' equals the size of a minimum vertex cover.*

Matchings and vertex covers can be generalised to digraphs as well. A *transversal*, or *feedback vertex set*, in a digraph D is a set T of vertices which intersects every directed cycle in D, i.e., $D - T$ is acyclic. A *cycle packing* is a collection $\mathcal{C}$ of pairwise (vertex-) disjoint cycles. The cardinality of a minimum transversal of D is denoted by $\tau(D)$ and the cardinality of a maximum cycle packing of D is denoted by $\nu(D)$. We say that D has the *Kőnig property* if $\nu(D') = \tau(D')$ for all subdigraphs $D' \subseteq D$.

An edge (u, v) in a digraph D is *butterfly contractible* if it is the only outgoing edge of u or the only incoming edge of v. The *butterfly contraction* of a butterfly contractible edge (u, v) which is the only outgoing edge of u is obtained from D by adding the edge (x, v) for every edge (x, u) in D (if it does not yet exist) and then deleting the vertex u. Analogously, if (u, v) is the only incoming edge of v, we obtain the butterfly contraction of (u, v) by adding the edge (u, x) for every edge (v, x) in D (if it does not yet exist) and then deleting v. A digraph D' is a *butterfly minor* of D if it can be obtained by butterfly contractions from a subdigraph of D.

For an undirected graph G, the digraph obtained from G by replacing every undirected edge xy with the two directed edges (x, y) and (y, x) is called the *bidirected* graph $\overleftrightarrow{G}$. If G is a cycle we call $\overleftrightarrow{G}$ a *bicycle*.

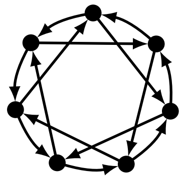


Figure 1: The digraph F_7.

Similar to Theorem 3 the digraphs with the Kőnig-property can be described by forbidding odd bicycles and a single digraph called F_7 (illustrated in Figure 1). Surprisingly, this class turns out to be closed under butterfly minors.

Theorem 4 ([GT11]). *A digraph D has the Kőnig-property if and only if it does not contain F_7 or an odd bicycle as a butterfly minor.*

The odd bicycles also appear in another context. Namely, the so-called *non-even digraphs* extend the class of digraphs described by Theorem 4 and were helpful in the study of structural bipartite matching theory as well as in the solution of the famous *even cycle problem* for digraphs. A digraph D is called *even* if for every edge weighting $w: E(D) \to \{0, 1\}$ there exists a directed cycle of even total weight in D.

Theorem 5 ([ST87]). *A directed graph is non-even if and only if it does not contain an odd bicycle as a butterfly minor.*

Non-even digraphs and their recognition problem naturally correspond to a famous problem from structural matching theory. An undirected graph G is called *matching covered* if G is connected and for every edge $e \in E(G)$ there is some $M \in \mathcal{M}(G)$ with $e \in M$, where $\mathcal{M}(G)$ denotes the set of all perfect matchings of G. A set $S \subseteq V(G)$ of vertices is called *conformal* if $G - S$ has a perfect matching. A subgraph $H \subseteq G$ is *conformal* if $V(H)$ is a conformal set and H has a perfect matching. A cycle C in G is called *M-alternating* if it alternately uses edges from M and $E(G) \setminus M$. Clearly, the conformal cycles of G are exactly the cycles occurring as an alternating cycle in at least one perfect matching.

Counting the number of perfect matchings in a given graph (also known as the *dimer problem*) is an important and well-known task which is known to be $\#P$-hard on general graphs [Val79]. However, there is a rather rich class of graphs for which the number of perfect matchings can be expressed as the permanent of a well-known matrix and can thus be computed in polynomial time [Kas67, Lit75, Tho06a], known as the *Pfaffian* graphs:

A graph G is called *Pfaffian* if there exists an orientation $\vec{G}$ such that every conformal cycle of G contains an odd number of directed edges going in one direction and an odd number of directed edges going in the other direction in $\vec{G}$. Such an orientation is also called *Pfaffian*. It is well-known that any planar graph is Pfaffian (see [Kas67]). Since edges that are not contained in a perfect matching do not contribute to a pfaffian orientation in any way, one usually just considers matching covered graphs in this context. Similar to non-even digraphs, bipartite matching covered Pfaffian graphs can be described by forbidden minors. To state the complete theorem, we need a connection between directed graphs and bipartite graphs with perfect matchings, as well as the definition for minors in the context of matching covered graphs.

Let G be a graph with a perfect matching and let v_0 be a vertex of G of degree two incident to the edges $e_1 = v_0v_1$ and $e_2 = v_0v_2$. Let H be obtained from G by contracting both e_1 and e_2 and deleting all resulting parallel edges and loops. We say that H is obtained from G by *bicontraction* or *bicontracting the vertex* v_0. Note that in case G is matching covered, then so is H. We say that H is a *matching minor* of G if H can be obtained from a conformal subgraph of G by repeatedly bicontracting vertices of degree two. Similar to how topological minors specialise graph minors, there is the following specialisation of matching minors: A *bisubdivision* of an edge is a subdivision, i.e. replacing the edge by a path joining its endpoints, with an even number (possibly 2) of vertices. We call H_2 a *bisubdivision of* H_1 if H_1 is a matching covered graph and H_2 can be obtained by bisubdividing the edges of H_1. If a matching covered graph G contains a conformal bisubdivision of a matching covered graph H, then H is a matching minor of G, but the converse is not true. If G contains no conformal bisubdivision of H, it is called H-*free*.

Definition 6. Let $G = (A \cup B, E)$ be a bipartite graph and let $M \in \mathcal{M}(G)$ be a perfect matching of G. The M-*direction* $\mathcal{D}(G, M)$ of G is a digraph defined as follows. Let $M = \left\{a_1b_1, \ldots, a_{|M|}b_{|M|}\right\}$ with $a_i \in A, b_i \in B$ for $1 \leqslant i \leqslant |M|$. Then,

i) $V(\mathcal{D}(G, M)) \coloneqq \left\{v_1, \ldots, v_{|M|}\right\}$ and

ii) $E(\mathcal{D}(G, M)) \coloneqq \{(v_i, v_j) \mid a_ib_j \in E(G), i \neq j\}$.

Note furthermore that the above operation is reversible and that every digraph D is the M-direction of its bipartite splitting-graph equipped with the canonical perfect matching.

The M-directions of a bipartite matching covered graph G inherit some of the properties of G. Most importantly, the directed cycles in an M-direction are in bijection with the M-alternating cycles of G. Another relation is about connectivity. A graph G is called k-*extendable* if it is connected, has at least $2k + 2$ vertices and every matching of size k is contained in a perfect matching of G. The following statement is folklore, it is mentioned in [RST99] and a proof can be derived from [AHLS03] by using the notion of M-directions.

Theorem 7 (see [RST99, AHLS03])**.** *Let G be a bipartite matching covered graph and M a perfect matching of G. Then G is k-extendable if and only if $\mathcal{D}(G, M)$ is strongly k-(vertex-)connected.*

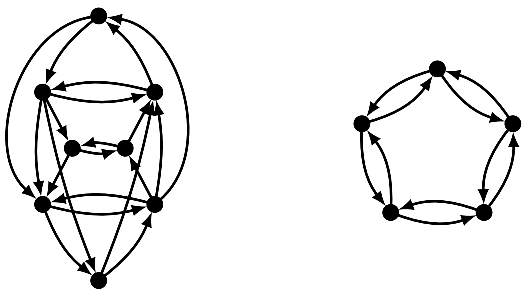
Lemma 8 ([McC00]). *Let G and H be bipartite matching covered graphs. Then H is a matching minor of G if and only if there exist perfect matchings $M \in \mathcal{M}(G)$ and $M' \in \mathcal{M}(H)$ such that $\mathcal{D}(H, M')$ is a butterfly minor of $\mathcal{D}(G, M)$.*

The problem of describing and recognising bipartite Pfaffian graphs has given rise to a wide range of different results. For a good overview on the topic consult the outstanding work by McCuaig [McC04] which also includes a proof of the following collection of results based on Little's characterisation of bipartite Pfaffian graphs [Lit75].

Theorem 9 ([Lit75, ST87, RST99, McC04]). *Let G be a bipartite graph with a perfect matching M. The following statements are equivalent.*

i) G is Pfaffian.
ii) G does not contain $K_{3,3}$ as a matching minor.
iii) $\mathcal{D}(G, M)$ is non-even.
iv) $\mathcal{D}(G, M)$ does not contain an odd bicycle as a butterfly minor.

Please note the huge discrepancy between the single forbidden minor $K_{3,3}$ in the matching setting opposed to the infinite antichain that needs to be excluded for digraphs. We will later encounter a similar phenomenon in the proof of our main theorem.



R $\quad\quad$ $\overset{\leftrightarrow}{C}_5$

Figure 2: The non-planar non-even digraph R and the planar even digraph $\overset{\leftrightarrow}{C}_5$.

Since every matching minor of a graph is also an ordinary minor, from Theorem 9 it becomes clear that every planar, bipartite and matching covered graph is Pfaffian, which was known before. However, there are also non-planar Pfaffian graphs with non-planar M-directions which still are non-even (for an example, consider the graph R in Figure 2). On the other hand, every non-Pfaffian bipartite graph must be non-planar, but the operation of contracting a perfect matching to obtain the M-direction does not preserve non-planarity. In particular, all odd bicycles are indeed planar.

Therefore, a positive answer to the question whether all non-even digraphs are 2-colourable is **no** answer to the Two-Colour-Conjecture. However, the class of non-even digraphs and the class of planar oriented graphs have a non-trivial intersection.

A digraph D is called *strongly planar* if there is a simple, non-crossing topological plane-embedding of D such that for each $x \in V(D)$ the incoming (resp. outgoing) edges incident to x form a consecutive interval in the cyclic ordering around x.

For strongly planar oriented graphs, our main result as stated below yields a proof of Conjecture 1.

Theorem 10. *Let D be a non-even digraph. Then $\vec{\chi}(D) \leqslant 2$.*

The proof of Theorem 10 can be found in Section 2.

Given a matching covered graph G and a perfect matching $M \in \mathcal{M}(G)$, an *M-colouring* of G with k colours is a function $c\colon M \to \{0, \dots, k-1\}$. An M-colouring is called *proper* if there is no M-alternating cycle whose matching edges are all of the same colour, i.e., $c^{-1}(i)$ is the unique perfect matching of the subgraph of G induced by the endpoints of the edges in $c^{-1}(i)$ for all i. The *M-chromatic number* $\chi(G, M)$ of G is the smallest integer k such that G has a proper M-colouring with k colours.

By the correspondence of M-alternating cycles in G and directed cycles in $\mathcal{D}(G, M)$, we have $\chi(G, M) = \vec{\chi}(\mathcal{D}(G, M))$ for any bipartite graph G with a perfect matching M.

From Theorem 9 we immediately derive the following equivalent formulation of Theorem 10.

Corollary 11. *Let G be a bipartite graph with a perfect matching M. If $\chi(G, M) \geqslant 3$, then G contains $K_{3,3}$ as a matching minor.*

Due to their equivalence we refer to both of them by the term 'main theorem' and only specify the digraphic (which is Theorem 10) or the matching theoretic (which is Corollary 11) version when necessary.

Hadwiger [Had43] conjectured for the undirected chromatic number that, if $\chi(G) \geqslant k$, G would contain K_k as a minor. The case $k = 5$ has been shown by Wagner [Wag37] to be equivalent to the Four-Colour-Theorem and, in this sense, our main theorem might be regarded as a directed and matching theoretic analogue of this case. In other words, Wagner proved that every graph with chromatic number at least five must contain K_5 as a minor and therefore, in particular, the graph cannot be planar. From this angle, Corollary 11 can be seen as a matching theoretic analogue of Wagner's Theorem as $K_{3,3}$ is the smallest non-planar complete bipartite graph and, again, we have a connection between the necessity of a certain number of different colours and the existence of a complete (and particularly non-planar) matching minor. For the setting of digraphs however, this analogue does not work just as smoothly since excluding $K_{3,3}$ as a matching minor in bipartite graphs is equivalent to the exclusion of a whole infinite anti-chain of butterfly minors in digraphs. We discuss this topic in more detail in Section 7.

Since Wagner's Theorem is a generalisation of the Four Colour Theorem to K_5-minor free graphs one could ask: What is the appropriate analogue of the Four-Colour-Theorem for digraphs? One acceptable answer to this is Conjecture 1. However, Conjecture 1 does not include all planar digraphs, it is only concerned with planar oriented graphs and therefore it is not closed under any of the commonly used minor operations for digraphs such as butterfly minors. To see this simply observe that any directed cycle contains $\overleftrightarrow{K}_2$ as a butterfly minor, which is not an oriented graph. Another way of relating the Four-Colour-Theorem to digraphs could be to change the notion of planarity.

The concept of strong planarity is closed under butterfly minors. Moreover, in [RST99] it was shown that a digraph is non-even if and only if it can be constructed from strongly 2-connected strongly planar digraphs and F_7 by means of local sum operations closely resembling the clique sums which appear in the description of K_5-minor free graphs given by Wagner's Theorem. So asking whether every strongly planar digraph has dichromatic number at most two could also be seen as a worthy analogue of the Four-Colour-Theorem. Since, however, every strongly planar digraph is non-even [RST99] and thus a positive answer to this question follows directly from Theorem 10, Conjecture 1 appears to be a more difficult problem. At last recall Figure 2 to see that also the strongly planar digraphs do not contain all planar digraphs. Moreover, there are strongly planar digraphs which are not oriented graphs and there are oriented graphs which are not strongly planar. Hence both problems discussed here, while similar in spirit, are very much distinct.

In addition to Theorem 10 we present further results, which can be divided into four main topics and are summarised below.

An Application: The Forcing Number In the context of M-colourings of graphs one can identify certain subsets of perfect matchings, namely the *forcing sets.* Given a perfect matching M of a graph, a subset $S \subseteq M$ of edges is called *forcing* if M is the unique perfect matching containing S. The *forcing number* $f(G, M)$ of a graph G with a perfect matching M denotes the size of a smallest forcing set for M. This notion arises from resonance theory in chemistry and has attracted wide interest in the last three decades. We refer to [CC11] for a comprehensive survey on this topic.

For any partial matching $S \subseteq M$ of a perfect matching M in a graph G, it is clear that S is forcing if and only if there is no M-alternating cycle with vertices in $V(G) \setminus V(S)$. Consequently, an M-colouring with k colours corresponds to a partition $M = S_1 \cup \cdots \cup S_k$ such that for any i, $M \setminus S_i$ is forcing. We may thus reformulate Corollary 11 as follows:

Corollary 12. *Every perfect matching M of a Pfaffian bipartite graph G with at least one perfect matching can be partitioned into two disjoint forcing sets.*

This directly yields the following corollary.

Corollary 13. *For any Pfaffian bipartite graph G and every perfect matching M of G, we have $f(G, M) \leqslant \frac{|M|}{2} = \frac{|V(G)|}{4}$.*

This generalises Theorem 2.9 in [CC11] from bipartite graphs without $K_{3,3}$ as an ordinary minor to bipartite graphs without $K_{3,3}$ as a matching minor, which is a weaker condition.

Other Notions of Colourings There are several concepts of colourings related to the colourings used to define the dichromatic number. One possible such concept is the idea of *polychromatic colourings* derived from hypergraph colourings as defined by Bollobás et al. [BPRS10]. We consider polychromatic colourings for strongly planar digraphs in Section 3. To the best of our knowledge, polychromatic colourings of digraphs in the above sense have not been investigated before, and we hope that the conjectures proposed in

Section 3 might initiate research in this direction. We also present, as evidence towards our conjectures, that a relaxed version based on the *fractional dichromatic number* does indeed hold for all strongly planar digraphs.

Another concept of colourings which also occurs for undirected graphs as a generalisation of proper graph colourings is the notion of *list colourings*. In Section 4 we give an example of a strongly planar, and therefore non-even, digraph which is not 2-choosable and thereby refute that Theorem 10 can be extended to list colourings of non-even digraphs while still using just two colours. We are, however, able to show that lists of size three suffice for any non-even digraph, which therefore is the optimal upper bound.

Computational Complexity Similar to the undirected case, the problem of deciding whether a given digraph D has dichromatic number at most k is NP-complete for all $k \geqslant 2$ [FHM03], [HSS18]. For the chromatic number however, for example by using Courcelle's Theorem, one can approach colouring on undirected graphs by parametrising with treewidth [Cou90]. While many problems become tractable for fixed parameters in the undirected case (see [DF12] for an introduction to the topic) directed width measures in general do not seem as capable [GHK+10]. In Section 5 we explore the computational complexity of deciding the colourability of digraphs regarding fixed parameters.

We show in Theorem 42 that the positive results for treewidth and colouring of graphs do not carry over to the world of digraphs. More precisely and somewhat surprisingly, we show that deciding whether a digraph is 2-colourable is NP-hard even if $\tau(D) \leqslant 6$, where D is the input digraph. With directed treewidth being bounded in a function of $\tau(D)$ this implies the hardness for bounded directed treewidth. This strengthens the previous hardness reduction due to [BFJ+04].

We then generalise this result in Theorem 44 and show that, under standard assumptions, the trivial brute-force algorithm for Digraph k-Colouring is essentially optimal, even if k, the *out-degeneracy*[1] and $\tau(D)$ are assumed to be constant.

***M*-Colourings of Non-Bipartite Graphs** In Section 6 we consider a possible generalisation of Corollary 11 to non-bipartite matching covered graphs. That is, we consider $\chi(G, M)$ for non-bipartite graphs G with perfect matchings M. As these graphs bare a much more complicated structure than their bipartite cousins, we are not able to extend our colouring results in their full strength. Even in the planar case there exist graphs which have a perfect matching that is not 2-colourable. A smallest example of such a graph is found in the *triangular prism*, which is the complement of C_6. However, we are able to bring down the planar case to exactly this graph in the sense of conformal bisubdivisions and matching minors.

Theorem 14. *Let G be a planar and matching covered graph, and M a perfect matching of G. If $\chi(G, M) \geqslant 3$, then G contains a conformal bisubdivision of $\overline{C_6}$, and thus has $\overline{C_6}$ as a matching minor.*

On non-bipartite graphs, to the best of our knowledge, very little is known on the forcing number (see [CC11]). However, Theorem 14 implies that we can partition every

[1] See Section 5 for a definition.

perfect matching of a planar, $\overline{C_6}$-free matching covered graph into two forcing sets and thus we obtain the following corollary.

Corollary 15. *For any planar matching covered graph G without a $\overline{C_6}$ matching minor and every perfect matching M of G, we have $f(G, M) \leqslant \frac{|M|}{2} = \frac{|V(G)|}{4}$.*

2 2-Colourings of Non-Even Digraphs

This section is dedicated to the proof of Theorem 10. The key idea of our proof is to consider a minimal (with respect to the number of vertices) non-2-colourable non-even digraph. We introduce a number of local reductions of digraphs transporting 2-colourability while ensuring that the reduced digraph is still non-even and prove that for any non-even digraph with at least 3 vertices one of our reductions is applicable.

Each of our reductions can be applied in polynomial time and thus this technique implies a polynomial time algorithm for 2-colouring a non-even digraph.

We start with two splitting operations, reducing the 2-colouring problem to the strongly 2-connected non-even digraphs.

Definition 16. Let D, D_1 and D_2 be digraphs. Then D is called a 0-*sum* of D_1 and D_2 if there is a partition of $V(D)$ into non-empty sets X and Y such that no edge of D has its head in X and its tail in Y, and $D_1 = D[X]$, $D_2 = D[Y]$.

We call a strongly connected digraph D the 1-*sum* of D_1 and D_2 *at a vertex* $v \in V(D)$ if there is a partition of $V(D) \setminus \{v\}$ into non-empty sets X and Y such that no edge in D has its head in X and its tail in Y, and such that D_1 arises from D by identifying $Y \cup \{v\}$ into a single vertex and D_2 arises by identifying $X \cup \{v\}$ into a single vertex. In both cases, we unify possible multiple occurences of parallel edges into single edges.

In the context of perfect matchings in bipartite graphs, the described reduction of D to D_1 and D_2 corresponds to a so-called *tight cut contraction*. Let G be an undirected graph and $X \subseteq V(G)$. The *cut* around X, denoted by $\partial(X)$, is the set of all edges in G with exactly one endpoint in X. If G is matching covered and $|\partial(X) \cap M| = 1$ for every perfect matching $M \in \mathcal{M}(G)$, we call $\partial(X)$ a *tight cut*. If $\partial(X)$ is a tight cut and $|X| \geqslant 2$, it is *non-trivial*. Identifying the *shore* X of a non-trivial tight cut $\partial(X)$ into a single vertex is called a *tight cut contraction* and the resulting graph G' can easily be seen to be matching covered again. Among many other things, tight cut contractions can be used to produce reductions of Pfaffian graphs as shown by Vazirani and Yannakakis.

Theorem 17 ([VY89], Theorem 4.2)**.** *Let G be a matching covered graph, $X \subseteq V(G)$ such that $\partial(X)$ is a non-trivial tight cut and G_1, G_2 the two graphs obtained by the tight cut contractions of X and $\overline{X}$ in G respectively. Then G is Pfaffian if and only if G_1 and G_2 are Pfaffian.*

To combine the theory of tight cuts and digraphs we need to be able to translate between the two more smoothly. Given a bipartite graph $G = (A \cup B, E)$ and a set $X \subseteq V(G)$ such that $|X \cap A| < |X \cap B|$, we call A the *minority* and B the *majority* of X, and analogously if the roles of A and B are reversed. Consider the following characterisation of tight cuts in bipartite graphs.

Lemma 18 ([LdCM15], Proposition 5)**.** *Let $G = (A \cup B, E)$ be a bipartite matching covered graph and $X \subseteq V(G)$ of odd size. Then $\partial(X)$ is tight if and only if $\big||X \cap A| - |X \cap B|\big| = 1$ and no vertex of the minority of X has a neighbour in $\overline{X}$.*

In a digraph D we call (X, Y) a *directed separation* if $X \cup Y = V(D)$ and there is no edge with tail in $Y \setminus X$ and head in $X \setminus Y$. The *order* of the separation is $|X \cap Y|$. The following is folklore, but we provide a proof for completeness.

Lemma 19. *Let $G = (A \cup B, E)$ be a bipartite matching covered graph, M a perfect matching in G and let $X \subseteq V(G)$. Moreover let $M_Y \coloneqq (E(G[Y]) \cup \partial(X)) \cap M$ for $Y \in \{X, \overline{X}\}$ and let v_e for $e \in M$ denote the vertex of the M-direction of G corresponding to the edge e. Then $\partial(X)$ is tight if and only if $(\{v_e \mid e \in M_X\}, \{v_e \mid e \in M_{\overline{X}}\})$ or $(\{v_e \mid e \in M_{\overline{X}}\}, \{v_e \mid e \in M_X\})$ is a directed separation of order 1 in $\mathcal{D}(G, M)$.*

Proof. First suppose $\partial(X)$ is tight. Note that this implies $|X|$ to be odd, since for every perfect matching M there is exactly one vertex of X not matched within X. By Lemma 18 no vertex of the minority of X has a neighbour in $\overline{X}$. By symmetry, we may assume that $B \cap X$ is the minority of X. The M-direction of G must be strongly connected, however there cannot exist an edge in $\mathcal{D}(G, M)$ with head v_e and tail $v_{e'}$ where $e \subseteq X$ and $e' \subseteq \overline{X}$ since such an edge would link a vertex of $X \cap B$ to a vertex of $\overline{X} \cap A$. Hence every directed path from $v_{e'}$ to v_e must contain the vertex v_f where f is the unique edge of M in $\partial(X)$. Thus $(\{v_e \mid e \in M_X\}, \{v_e \mid e \in M_{\overline{X}}\})$ is a directed separation and v_f is the unique vertex in the intersection of the two sets.

For the other direction let $(\{v_e \mid e \in M_X\}, \{v_e \mid e \in M_{\overline{X}}\})$ be a directed separation of order 1 in $\mathcal{D}(G, M)$. The other case follows analogously. Let f be the unique matching edge corresponding to the cut vertex. Then every directed cycle in $\mathcal{D}(G, M)$ must contain v_f and has exactly one edge with endpoints in $\{v_e \mid e \in M_X\} \setminus \{v_f\}$ and $\{v_e \mid e \in M_{\overline{X}}\} \setminus \{v_f\}$. This means that every M-alternating cycle in G contains exactly two edges of $\partial(X)$, namely f and one non-matching edge. We know that $|\partial(X) \cap M| = |\{f\}| = 1$, and so to prove that $\partial(X)$ is tight, we must show that any other perfect matching M' of G has the same number of edges on $\partial(X)$ as M. For this, observe that the symmetric difference $M \Delta M'$ decomposes into a vertex-disjoint union of cycles $C_1, \dots, C_t$ which are simultaneously M- and M'-alternating. Consequently, exchanging matching with non-matching edges for each C_i one after the other ("flipping") transforms M into M'. Clearly, this operation can change the number of matching edges on $\partial(X)$ only if a cycle containing vertices of both X and $\overline{X}$ is flipped, but according to the above, each such cycle must contain f, and so at most one C_j can intersect $\partial(X)$, and $E(C_j) \cap \partial(X) = \{f, f'\}$ for a non-matching edge f'. Flipping C_j now makes f' into a matching and f into a non-matching edge. In any case, after having performed the sequence of flips, we thus obtain that $M' \cap \partial(X)$ consists of a single edge, and, hence, $\partial(X)$ must be tight. □

From Theorem 17 and Lemma 19 we obtain the following corollary.

Corollary 20. *Let D be a digraph and $i \in \{0,1\}$ such that D is the i-sum of the digraphs D_1 and D_2. Then D is non-even if and only if D_1 and D_2 are non-even.*

Proof. For $i = 0$, this can be seen directly from the definition of an even digraph: D is non-even if and only if there is a subset $A \subseteq E(D)$ of edges intersected an odd number of times by each directed cycle. However, the set of directed cycles in D consists of the directed cycles in $D[X] = D_1$ and $D[Y] = D_2$ for a partition (X, Y) as in Definition 16, because no directed cycle can pass trough X and Y at the same time. Thus, the above is the same as saying that there are edge sets $A_i \subseteq E(D_i)$, $i = 1, 2$, intersecting each directed cycle in D_i an odd number of times, which is the same as saying that D_1, D_2 are non-even.

For $i = 1$, this is a direct consequence of Lemma 19 and Theorems 9 and 17. □

So 0- and 1-sums preserve non-eveness. Next, we need to make sure we can obtain a 2-colouring of D from 2-colourings of its sumands D_1 and D_2.

Lemma 21. *Let D be a non-even digraph and D_1, D_2 digraphs such that D is the i-sum of D_1 and D_2 for $i \in \{0,1\}$. If D_1 and D_2 are 2-colourable, so is D.*

Proof. Assume first that D is the 0-sum of $D_1 = D[X]$, $D_2 = D[Y]$ for a partition X, Y of $V(D)$. Then the directed cycles in D are exactly the directed cycles in D_1 together with the directed cycles in D_2, and thus any proper 2-colouring of D_1 joined with a proper 2-colouring of D_2 yields a proper 2-colouring of D.

Now assume D is the 1-sum of D_1 and D_2 at v, let v_1 be the vertex of D_1 obtained from identifying $Y \cup \{v\}$, and let v_2 be the vertex in D_2 identifying $X \cup \{v\}$. For $i \in \{1,2\}$ let $c_i \colon V(D_i) \to \{0,1\}$ be a proper 2-colouring of D_i. By possibly exchanging 0 and 1 in c_2, we may assume that $c_1(v_1) = c_2(v_2)$. We define a colouring c for D as follows.

$$c(u) \coloneqq \begin{cases} c_1(u)\,, & u \in X \\ c_1(v_1) = c_2(v_2)\,, & u = v \\ c_2(u)\,, & u \in Y. \end{cases}$$

To see that this defines a proper 2-colouring of D, assume towards a contradiction that C is a monochromatic directed cycle in D. If C stays within $X \cup \{v\}$ or $Y \cup \{v\}$, then it also appears as a directed cycle in D_1, or D_2 respectively, contradicting the feasibility of the 2-colourings c_1 and c_2. Otherwise, C traverses vertices of both X and Y and thus, as there are no edges starting in Y and ending in X, C also contains v. Moreover, $C - v$ can be decomposed into exactly two directed paths P_1 and P_2, one contained in X and the other in Y. Hence C corresponds to the directed cycles $C_i = P_i + v_i$ in D_i for each $i \in \{1,2\}$ and both C_i must be monochromatic under their respective colourings c_i. This again violates the feasibility of the c_i. Consequently, c defines a colouring of D as desired. □

Robertson et. al. [RST99] defined in total five different sum operations which they used to prove a generation theorem for non-even digraphs. From this the following result follows.

Theorem 22 ([Tho06b], Corollary 5.4)**.** *Let D be a strongly 2-connected and non-even digraph on at least two vertices. Then $|E(D)| \leqslant 3\,|V(D)| - 4$.*

Corollary 23. *Any strongly* 2*-connected, non-even digraph* D *on at least three vertices contains at least two vertices of out-degree* 2*.*

Proof. Let $n := |V(D)|$. By Theorem 22 we have $|E(D)| < 3(n-1)$. If at most one vertex in D had out-degree less than 3 we would have $|E(D)| = \sum_{v \in V(D)} \deg^{\text{out}}(v) \geqslant 0 + 3(n-1)$, a contradiction, and so there are at least two vertices of out-degree at most two, and thus, because D is strongly 2-connected, exactly two. □

Besides edge deletions, butterfly contractions and 0- and 1-sums, we will use another special operation in order to reduce our digraphs. A bidirected K_2 is called a *digon*. If we encounter an out-degree 2 vertex v in a digraph D such that v is contained in at most one digon, we will need to delete some edges incident with v in order to create a butterfly contractible edge. However, if v is contained in two different digons, we will directly contract the three digon vertices, namely v and the two vertices with which v forms a digon each, into a single vertex. While this is not a standard butterfly contraction, it is natural in the context of our proof and it preserves the property of being non-even, which we show later by using matching theory.

Note that bicontractions in matching covered graphs are a special case of tight cut contractions. To see this, consider X as the set of size 3 containing a degree 2 vertex v together with its two neighbours. Then $\partial(X)$ is tight since every perfect matching must match v to one of its neighbours and thus exactly one matching edge can and must leave X. Thus one can derive the following corollary from Theorem 17 or, alternatively, Theorem 9.

Corollary 24. *Let* G *be a Pfaffian matching covered graph. Then every matching minor of* G *is Pfaffian.*

Lemma 25. *Let* D *be a non-even digraph with a vertex* $v \in V(D)$ *with* $N^{out}(v) = \{v_1, v_2\}$ *such that* v *induces a digon together with* v_i *for both* $i \in \{1, 2\}$*. Then the digraph* D^**, obtained by first deleting all edges of the form* (u, v) *with* $u \notin \{v_1, v_2\}$ *as well as all edges between the vertices* v, v_1, v_2*, and then identifying* v_1*,* v *and* v_2 *into a single vertex (and identifying occurring parallel edges into single edges afterwards), is non-even as well.*

Proof. Let D be the digraph together with the vertices v, v_1, and v_2 as in the assertion. By Theorem 5, when deleting all incoming edges of v with tails other than v_1 or v_2 we obtain a subdigraph D' which is non-even as well. Moreover, by Corollary 20, D' is non-even if and only if every strongly connected component of D' is non-even. Since v, v_1 and v_2 are contained in two digons sharing a vertex, they all must appear in the same strong component of D', say, D'_0. Let H be the strong component of D^* containing the vertex obtained from identifying v, v_1, and v_2 into a single vertex. Observe that, by definition of D^*, H is obtained from D'_0 by all edges between the vertices v, v_1, v_2, and then identifying v_1, v and v_2 into a single vertex (and identifying occurring parallel edges into single edges afterwards). Hence, it suffices to show that the contraction of the three vertices into one in D'_0 preserves non-eveness.

With D'_0 being strongly connected, there exists a bipartite matching covered graph G together with a perfect matching $M \in \mathcal{M}(G)$ such that $D'_0 = \mathcal{D}(G, M)$. We identify the vertices v, v_1 and v_2 of D_0' as the edges e_v, e_{v_1} and e_{v_2}, respectively, in M. Additionally let A and B be the two colour classes of G. Then a_x is the vertex of e_x in A and b_x the vertex in B for all $x \in \{v, v_1, v_2\}$. Since v and v_1 form a digon in D_0', the edges $a_v b_{v_1}$ and $a_{v_1} b_v$ exist in G and, thus, together with e_v and e_{v_1} they form a conformal cycle of length 4. Therefore we can obtain a new perfect matching from M as follows.

$$M' := (M \setminus \{e_v, e_{v_1}\}) \cup \{a_v b_{v_1}, a_{v_1} b_v\}$$

Now consider $G - e_v$ and note that it still has M' as a perfect matching and that it is a matching minor of G (see Figure 3 for an illustration). By our assumptions, v has exactly two out- and two in-neighbours in D_0' and therefore the two vertices a_v and b_v must be of degree 2 in $G - e_v$. Hence we can bicontract these two vertices and identify b_{v_1}, a_v, and b_{v_2} into $b_{v_1 v v_2}$ and the other three vertices into $a_{v_1 v v_2}$ respectively. Let us call the resulting graph G^* and denote the edge $a_{v_1 v v_2} b_{v_1 v v_2}$ by $e_{v_1 v v_2}$. One can easily check that G^* still is matching covered and since it is a matching minor of G it must be Pfaffian by Corollary 24. Moreover, the strongly connected digraph $D_0^* := \mathcal{D}(G^*, M^*)$ must be non-even. Let $M^* := (M \setminus e_{v_1}, e_v, e_{v_2}) \cup \{e_{v_1} v v_2\}$. Since $M^* \setminus \{e_{v_1 v v_2}\} = M' \setminus \{a_v b_{v_1}, a_{v_1} b_v, e_{v_2}\} = M \setminus \{e_{v_1}, e_v, e_{v_2}\}$ and the two edges e_{v_1} and $e_{v_1 v v_2}$ can be identified (again see Figure 3) D_0^* is isomorphic to the digraph obtained from D_0' identifying the three vertices v, v_1, and v_2 into one, and so the latter has to be non-even as well. From this we deduce that all strong components of D^* are non-even, proving the assertion. □

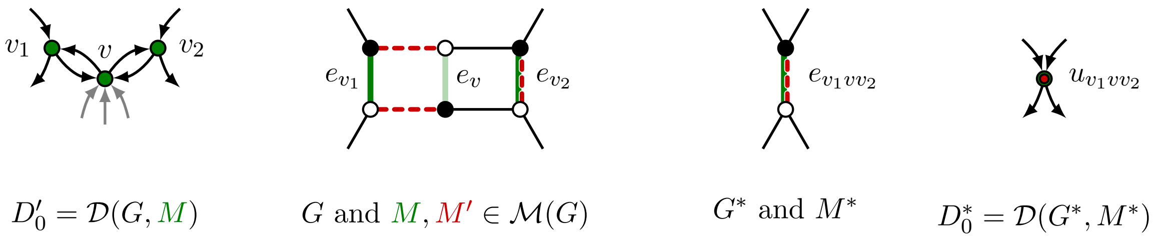


Figure 3: The four steps of the contraction of v, v_1, and v_2 in Lemma 25. The matching M' is given by dashed edges while the edges of M are thicker.

We are now ready to prove our main theorem, concluding this section.

Proof (of Theorem 10). Assume towards a contradiction that there is a non-even digraph D that is not 2-colourable. Furthermore, let us assume D to be minimal (with respect to $|V(D)|$) with this property. Clearly $|V(D)| \geqslant 3$.

First observe that, due to Lemma 21, D is neither a 0-sum nor a 1-sum of some other non-even digraphs D_1 and D_2. Hence, D does not have a directed cut or a cut vertex, and must therefore be strongly 2-connected. By Corollary 23 there exists a vertex $v \in V(D)$ with $\deg^{\mathsf{out}}(v) = 2$. Let $e_1 = (v, v_1)$ and $e_2 = (v, v_2)$ be the two outgoing edges of v. We now distinguish two cases:

Case 1: Both edges e_1 and e_2 are contained in digons.

If e_1 and e_2 are contained in digons, we can construct a non-even digraph D^* from D by applying the operation from Lemma 25 on v and its two out-neighbours. First, we delete all incoming edges of v except (v_1, v) and (v_2, v) from the graph and then contract v_1, v, and v_2 into a single vertex. Since $|V(D^*)| = |V(D)| - 2$ and D^* is non-even, by the minimality of D, D^* admits a proper 2-colouring $c^*\colon V(D^*) \to \{0, 1\}$. Denote by $u_{v_1vv_2}$ the vertex of D^* into which v_1, v and v_2 were identified. We now define a 2-colouring for the vertices $x \in V(D)$ as follows.

$$c(x) \coloneqq \begin{cases} c^*(u_{v_1vv_2})\,, & x \in \{v_1, v_2\} \\ 1 - c^*(u_{v_1vv_2})\,, & x = v \\ c^*(x)\,, & \text{otherwise.} \end{cases}$$

By assumption, D is not 2-colourable and thus there must be a directed cycle C whose vertices receive the same colour from c. Moreover, C must avoid v, since any directed cycle in D containing v must either contain v_1 or v_2 and thus, by the definition of c, cannot be monochromatic. Consequently, C must be contained in $D - v$. By identifying possible occurrences of v_1 or v_2 with $u_{v_1vv_2}$, the existence of a closed directed monochromatic walk C^* in D^* follows. Note that v_1 and v_2 do not form a digon, as otherwise v, v_1 and v_2 would be an odd bicycle in D, contradicting the assumption that D is non-even. Hence, the walk C^* must contain a directed cycle which, in turn, must also be monochromatic with respect to c^*. However, the existence of such a cycle contradicts the choice of c^*.

Case 2: At least one of the edges e_1 or e_2 is not contained in a digon.

Without loss of generality assume e_1 to not be part of a digon in D. We now delete all edges with endpoints v and v_2, thereby obtaining a non-even digraph in which v has a single out-going edge, which is e_1. With this, e_1 is now butterfly contractible. Let D' be the digraph obtained by contracting e_1 and let w be the contraction vertex. Butterfly contractions are very special cases of 1-sums, where one of the two digraphs D_1 and D_2 is a digraph on two vertices and the other one is D'. Therefore, Corollary 20 yields that D' is again non-even, alternatively, this follows from Theorem 5. Moreover, as $|V(D')| = |V(D)| - 1$, D' must admit a proper 2-colouring $c'\colon V(D') \to \{0, 1\}$ by the minimality of D. Similar to the first case we use c' to define a 2-colouring c for the vertices $x \in V(D)$.

$$c(x) \coloneqq \begin{cases} c'(w)\,, & x = v_1 \\ 1 - c'(v_2)\,, & x = v \\ c'(x)\,, & \text{otherwise.} \end{cases}$$

Again, we assumed D to not be 2-colourable and thus there must be a monochromatic (with respect to c) directed cycle C in D. If C contains v, it cannot contain v_2 as $c(v_2) \neq c(v)$. Therefore, it must contain the edge e_1. Since e_1 is not contained in a digon we have $|V(C)| \geqslant 3$ and thus there exists a cycle C' in D' with $V(C') \setminus \{w\} = V(C) \setminus \{v, v_1\}$. By definition of c, C' must be monochromatic with respect to c' which yields the desired contradiction in this case. Otherwise, C does not contain v. Then, possibly after replacing v_1 with w, C again corresponds to a directed cycle in D' which, again, has to be monochromatic with respect to c', contradicting our choice of c'. $\square$

The proof of Theorem 10 yields a polynomial time algorithm to find a proper 2-colouring of a non-even digraph. One first reduces a digraph D into its strong components, then finds the cut vertices and decomposes D into strongly 2-connected digraphs. Then, one either finds a out-degree 2 vertex contained in two digons, which can be dealt with by Case 1 of the proof, or Case 2 of the proof can be applied. Afterwards, these reduction steps are reiterated until every such graph is reduced to a digraph on one or two vertices, which is trivially 2-colourable. Then, by reversing the reductions step by step, we can extend these 2-colourings until all of D is coloured. Additionally, the work of Robertson et. al. and McCuaig [RST99, McC04] imply polynomial time algorithms to recognise non-even digraphs. Hence, given a digraph D we can decide whether it is non-even and then find a proper 2-colouring in polynomial time.

3 Polychromatic Colourings and Cycle Packings of Strongly Planar Digraphs

In this section, we study colouring properties of so-called *strongly planar digraphs*. These form a canonical class of planar non-even digraphs (however, there are many others). To motivate their definition, consider an arbitrary bipartite, matching-covered planar graph G with bipartition (A, B). Because G is planar, it must be Pfaffian. Choose some perfect matching M of G. Considering the orientation $\vec{G}$ of G orienting all edges from A to B, we can view $\mathcal{D}(G, M)$ as being obtained from $\vec{G}$ by contraction of all edges in M. It is now clear that the digraph $\mathcal{D}(G, M)$ inherits a natural plane-embedding from G in which for each vertex, the incident incoming and outgoing edges are separated into two intervals in the cyclic ordering. It is not hard to reverse the described relationship to see that any digraph D admitting such an embedding is isomorphic to $\mathcal{D}(G, M)$ for some planar bipartite graph and a perfect matching M. In other words, a digraph D is springly planar if and only if $D \cong \mathcal{D}(G, M)$ for a planar bipartite graph G and a perfect matching M.

An example of a strongly planar digraph is given in Figure 4.

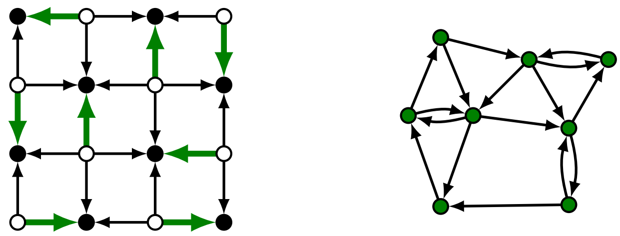


Figure 4: Left: An oriented grid equipped with a perfect matching. Right: The arising M-direction, a strongly planar digraph.

By Theorem 9, every strongly planar digraph is non-even and so, according to Theorem 10, it is 2-colourable.

In this section, we seek a strengthening of 2-colourability for strongly planar digraphs of large girth. While $\vec{\chi}(D) \leqslant 2$ for all strongly planar digraphs can be rephrased as the existence of a packing of two disjoint feedback vertex sets in any strongly planar digraph, we conjecture the following generalisation.

Conjecture 26. For any strongly planar digraph D of girth g, there exists a packing of g pairwise disjoint feedback vertex sets. In other words, D can be vertex g-coloured such that every directed cycle uses each colour at least once.

Clearly, the directed cycle $\vec{C}_g$ of length g admits a packing of g and no more disjoint feedback vertex sets, and consequently, this conjecture, if true, is best-possible.

For an arbitrary bipartite planar graph G with a perfect matching M, a feedback vertex set in $\mathcal{D}(G, M)$ corresponds to a partial matching $S \subseteq M$ with the property that every M-alternating cycle uses an edge in S, which is the same as saying that S is forcing. Consequently, in the language of perfect matchings, the above translates to:

Conjecture 27. Let G be a bipartite planar graph with a perfect matching M and let $2g$ be the length of a shortest M-alternating cycle. Then M can be decomposed into g pairwise disjoint forcing sets.

The type of colouring as described for cycles in digraphs was investigated more generally for hypergraphs by Bollobás et al. [BPRS10]. Given a hypergraph H, a *polychromatic k-colouring* of $\mathcal{H}$ is defined to be a vertex-colouring $c\colon V(H) \to \{0, \dots, k-1\}$ such that every hyperedge $e \in E(H)$ contains at least one vertex of each colour. The *polychromatic number* of H then is defined as the maximal k for which a polychromatic k-colouring of H exists. Clearly, the polychromatic number of a hypergraph H is upper bounded by its *rank*, that is, the size of a smallest hyperedge.

Given a digraph D, we may associate with it the *cycle hypergraph* $\mathcal{C}(D)$ having $V(D)$ as vertex set and containing the vertex sets of all directed cycles in D as hyperedges. It is now clear that Conjecture 26 claims that the cycle hypergraph $\mathcal{C}(D)$ of any strongly planar digraph D has the very special property that the polychromatic number matches its rank.

Looking at general planar digraphs, for any $g \geqslant 2$, there are examples of planar digraphs with girth g which do not admit a packing of g disjoint feedback vertex sets (cf. [HS18]). However, the following statement, which contains the 2-Colour-Conjecture (Conjecture 1) as the subcase $g = 3$, might still be true.

Conjecture 28 (Hochstättler and S. [HS18])**.** For any planar digraph of girth $g \geqslant 3$, there exists a packing of $g - 1$ disjoint feedback vertex sets.

The rest of this section is devoted to partial results towards Conjecture 26 using the concept of fractional colourings.

Given a fixed natural number $b \geqslant 1$ and some $k \in \mathbb{N}, k \geqslant b$, a *$b$-tuple k-colouring* of a digraph D is defined to be an assignment of subsets of $\{0, \dots, k-1\}$ of size b to the vertices of D in such a way that for any $i \in \{0, \dots, k-1\}$, the subdigraph of D induced by those vertices whose colour-set contains i is acyclic. The *b-dichromatic number* $\vec{\chi}_b(D)$ of a digraph is then defined to be the least k for which a b-tuple k-colouring of D exists.

The *fractional dichromatic number* of a digraph defined as $\vec{\chi}_f(D) := \inf_{b\geqslant 1} \frac{\vec{\chi}_b(D)}{b} \in [1,\infty)$ is always a lower bound for the dichromatic number.

It has been proved in [Sev], Chapter 5 that $\vec{\chi}_f(D)$ is always a rational number and can be alternatively represented as the optimal value of the following linear relaxation of a natural integer program formulation of the dichromatic number:

Theorem 29 (Severino [Sev])**.** *Let D be a digraph. Then there is an integer $b \geqslant 1$ such that $\vec{\chi}_f(D) = \frac{\vec{\chi}_b(D)}{b}$. Denote the collection of acyclic vertex sets in D by $\mathcal{A}(D)$ and for any $v \in V(D)$ let $\mathcal{A}(D,v) \subseteq \mathcal{A}(D)$ consist of only those acyclic sets containing v. Then $\vec{\chi}_f(D)$ is the optimal value of*

$$\begin{aligned} &\min \sum_{A\in\mathcal{A}(D)} x_A \\ \textit{subj. to} &\sum_{A\in\mathcal{A}(D,v)} x_A \geqslant 1, \ \textit{for all } v \in V(D) \\ &x \geqslant 0. \end{aligned} \tag{1}$$

The fractional dichromatic number has turned out to be a useful concept. For instance, it was used in [MW16] to prove a fractional version of the so-called *Erdős-Neumann-Lara-Conjecture*.

To make the statement of our results clearer, we reformulate Conjecture 26 in the setting of *circular colourings* of digraphs. The *star dichromatic number* $\vec{\chi}^*(D)$ of a digraph was recently introduced in [HS18] as a refined measure of the dichromatic number of a digraph which, similar to the circular or fractional chromatic number of a graph (cf. [Vin88] and [SU11]), can take on rational values. Instead of a finite colour set, for any $p \in \mathbb{R}, p \geqslant 1$, in an *acyclic p-colouring* of a digraph D, vertices are coloured with points on a plane circle S_p with perimeter p such that for any open cyclic subinterval $I \subseteq S_p$ of length 1, the set of vertices mapped to this interval is acyclic. The star dichromatic number $\vec{\chi}^*(D)$ is now defined as the infimal value of p for which an acyclic p-colouring of D exists. It was proved in [[HS18], Proposition 5] that this infimum is attained and thus may be written as a minimum.

Intuitively, having fractional or star dichromatic number close to 1 captures the property of a digraph being "close" to acyclic.

We restate the following basics.

Proposition 30 (Hochstättler and S. [HS18])**.** *Let D be a digraph, then the following statements hold.*

i) The star dichromatic number $\vec{\chi}^*(D)$ is a fraction with numerator at most $|V(D)|$ satisfying $\lceil\vec{\chi}^*(D)\rceil = \vec{\chi}(D)$.

ii) For any pair of integers $k \geqslant d \geqslant 1$, we have $\vec{\chi}^*(D) \leqslant \frac{k}{d}$ if and only if there is a colouring $c\colon V(D) \to \mathbb{Z}_k \simeq \{0,1,\dots,k-1\}$ of the vertices of D such that $c^{-1}(\{i,i+1,\dots,i+d-1\})$ (sums taken modulo k) is an acyclic vertex set for every $i \in \mathbb{Z}_k$.

iii) $\vec{\chi}_f(D) \leqslant \vec{\chi}^*(D)$.

iv) For every $n \in \mathbb{N}, n \geqslant 2$ *we have* $\vec{\chi}_f\left(\vec{C}_n\right) = \vec{\chi}^*\left(\vec{C}_n\right) = \frac{n}{n-1}$.

As a consequence of the second point, we obtain that, for a digraph D of girth g, $\vec{\chi}^*(D) \leqslant \frac{g}{g-1}$ if and only if $V(D)$ can be coloured with the elements of $\mathbb{Z}_g$ such that for any $i \in \mathbb{Z}_g$, the vertices mapped to $\mathbb{Z}_g \setminus \{i\}$ form an acyclic set. However, this is the same as saying that D can be vertex-coloured with g colours such that each colour class is a feedback vertex set of D. Therefore, the following is an equivalent reformulation of Conjecture 26 (we use (iv) in Proposition 30 to conclude that $\vec{\chi}^*(D) \geqslant \vec{\chi}^*(\vec{C}_g) = \frac{g}{g-1}$ for every planar digraph of directed girth g).

Conjecture 31. For any strongly planar digraph D of directed girth $g \geqslant 2$, we have $\vec{\chi}^*(D) = \frac{g}{g-1}$.

For planar digraphs, the fractional and the star dichromatic number often coincide or are closely tied to each other. Thus, the following result can be seen as a source of evidence for Conjecture 26.

Theorem 32. *For any strongly planar digraph* D *of girth* $g \geqslant 2$*, we have* $\vec{\chi}_f(D) = \frac{g}{g-1}$.

To prove this result, we use insights from the theory of clutters. A *clutter* is defined to be a collection $\mathcal{C}$ of subsets of a finite ground set S such that $C_1 \nsubseteq C_2$ for any $C_1 \neq C_2 \in \mathcal{C}$. We refer to the first chapter of [Cor01] for a short and comprehensible introduction to the topic.

Associated with any clutter $\mathcal{C}$ over the ground set S we have a *clutter matrix* $M_{\mathcal{C}}$ whose columns are indexed by the elements of S and whose rows correspond to the characteristic vectors of the members of $\mathcal{C}$ with respect to S. The following primal-dual pair of linear optimisation programs resembles natural covering and packing problems related to clutters. Here, $w \geqslant 0$ denotes a row vector whose entries are non-negative real numbers or possibly ∞, and $\mathbf{1}$ denotes the vector with all entries equal to 1 (with a slight abuse of notation we use it both as a column and row-vector, it should always be clear from context what is meant). Vector-inequalities are to be understood component-wise.

$$\min \{wx \mid x \geqslant 0,\ M_{\mathcal{C}}x \geqslant \mathbf{1}\} \tag{2}$$
$$= \max \{y\mathbf{1} \mid y \geqslant 0,\ yM_{\mathcal{C}} \leqslant w\} \tag{3}$$

In the following, we introduce a number of important notions for clutters related to integral solutions of the linear programs (2) and (3).

Given a clutter $\mathcal{C}$, we will say that it admits the *Max-Flow-Min-Cut-Property* (MFMC for short) if, for any non-negative w with integral entries, there exists a primal-dual pair of integral optimal solutions to the linear programs (2) and (3).

We say that $\mathcal{C}$ *packs* if the same holds true at least for $w = \mathbf{1}$. If such an integral primal-dual solution exists for all vectors w with entries 0, 1 or ∞, we say that the clutter is *packing*.

It is not hard to see that if a clutter has the MFMC-property, it is packing, and, clearly, any packing clutter also packs. While there are examples of clutters that pack but do not have the packing property, it is a famous open problem due to Conforti and Cornuejols to show that in fact, the packing property and the MFMC-property are equivalent.

Conjecture 33 (Conforti and Cornuejols)**.** A clutter has the packing property if and only if it has the MFMC property.

For the following, we will furthermore need the notion of *idealness* for clutters. A clutter is said to be *ideal* if, for any real-valued vector $w \geqslant 0$, the primal linear program (2) has an integral optimal solution vector x. It is not hard to show that the MFMC-property implies idealness of a clutter, we refer to the paragraph after Definition 1.5 in [Cor01] for detailed explanation of this fact.

A famous example of a clutter related to digraphs is the clutter of all minimal directed cuts of a fixed directed graph D. The following well-known result of Lucchesi and Younger can be rephrased as the fact that the clutter of minimal directed cuts of any digraph has the MFMC-property. To formulate the theorem, we need the following terminology: A *dijoin* of a digraph D is a subset of $E(D)$ intersecting every directed cut in at least one edge.

Theorem 34 (Lucchesi and Younger [LY78])**.** *Let D be a digraph and $w\colon E(D) \to \mathbb{N}_0$ a non-negative integral edge-weighting. Then the minimal weight of a dijoin in D equals the maximal size of a collection of (minimal) directed cuts in D so that any edge $e \in E(D)$ is contained in at most $w(e)$ of them.*

Using planar duality of digraphs, the above theorem restricted to planar digraphs reformulates as follows.

Corollary 35. *Let D be a planar digraph and $w\colon E(D) \to \mathbb{N}_0$ a non-negative integral edge-weighting. Then the minimal weight of a directed cycle in D equals the maximal number of feedback arc sets containing any edge $e \in E(D)$ at most $w(e)$ times.*

We now use the above result to prove that given a strongly planar digraph D, the associated clutter containing the vertex sets of all induced directed cycles in D admits the MFMC-property. This result has already been observed for instance in [Gue01], we provide its proof for completeness.

Theorem 36. *Let D be strongly planar. Then for any non-negative integral vertex-weighting $w\colon V(D) \to \mathbb{N}_0$, the minimal weight of a feedback vertex set in D equals the maximal number of (induced) directed cycles in D which together contain any vertex $x \in V(D)$ at most $w(x)$ times.*

Proof. We construct an auxiliary splitting-digraph D' by replacing each vertex $x \in V(D)$ by a directed edge $e_x \in E(D')$ in such a way that all the incoming edges incident to x in D are now incident to $\mathsf{tail}(e_x)$ while all the outgoing edges of x in D are now emanating from $\mathsf{head}(e_x)$. By contracting the edge e_x for each $x \in V(D)$, it is clear that the directed cycles in D' are in one-to-one correspondence with the directed cycles of D. Moreover, the vertex-intersection of a pair of directed cycles in D yields a subset of the edge-intersection of the corresponding directed cycles in D'. It is furthermore easy to see from the fact that the outgoing and incoming edges incident to any vertex in D are separated in the cyclic ordering, that D' indeed admits a planar embedding. We now define a corresponding weighting of the edges of D' by setting $w'(e_x) := w(x)$ for any $x \in V(D)$ and $w'(e) := M$ for a large natural number $M \in \mathbb{N}$ for any other edge of D'. If we choose M large enough, we find that the minimal edge-weight of a feedback edge set in D' is exactly the minimal vertex-weight of a feedback vertex set in D. Corollary 35 now tells us that the latter is the same as the maximal size of a collection of directed cycles in D' in which any e_x is contained at most $w(x)$ times while any other edge is contained at most M times. As the latter condition becomes redundant for M large enough, this again is the same as the maximal size of a collection of directed cycles in D in which any vertex $x \in V(D)$ is contained at most $w(x)$ times. As we may assume all the directed cycles in an optimal collection to be induced, this implies the claim. $\square$

As a consequence, the clutter of vertex sets of induced directed cycles of a strongly planar digraph D is MFMC and, thus, also ideal.

Given any clutter $(S, \mathcal{C})$, we may define a corresponding dual clutter (called *blocking clutter* and denoted by $(S, \mathcal{C}^*)$) which contains all the inclusion-wise minimal subsets $X \subseteq S$ with the property that $X \cap C \neq \varnothing$ for all $C \in \mathcal{C}$. It is clear that the blocking clutter of the clutter of vertex sets of induced directed cycles of a digraph is just the clutter of inclusion-wise minimal feedback vertex sets. To proceed, we will need the following theorem of Lehman.

Theorem 37 (Lehman, [Leh79], and [Cor01], Theorem 1.17)**.** *A clutter is ideal if and only if its blocking clutter is.*

In our case, this implies that, for strongly planar digraphs, the clutter of minimal feedback vertex sets is ideal and, consequently, the corresponding linear optimisation problem (2) admits an integer optimal solution $x \geqslant 0$ for any real-valued vector $w \geqslant 0$. By setting $w := \mathbf{1}^T$, we obtain the following.

Lemma 38. *Let D be a strongly connected strongly planar and let g be the girth of D. Then there is a collection $F_1, \dots, F_m$ of feedback vertex sets of D equipped with a weighting $y_1, \dots, y_m \in \mathbb{R}_{\geqslant 0}$ such that $y_1 + \dots + y_m = g$ and for any vertex $v \in V(D)$, we have $\sum_{\{j \mid v \in F_j\}} y_j \leqslant 1$.*

Proof. Let $x \geqslant 0$ be an integer-valued optimal solution of the linear program (2) corresponding to the clutter of inclusion-wise minimal feedback vertex sets of D and $w = \mathbf{1}^T$. It is easy to see from the definition of the linear program (2) that, in any optimal solution, we have $x \leqslant \mathbf{1}$ (component-wise), as otherwise one could replace x with $\min\{x, \mathbf{1}\}$, obtaining a better solution to the linear program, contradicting the optimality. Consequently, we know that x has only 0 and 1 as entries and is thus determined by its support $X := \text{supp}(x) \subseteq V(D)$. From the conditions in the program (2) we derive that X has a non-empty intersection with every feedback vertex set of D and thus must contain a directed cycle (as $V(D) \setminus X$ cannot be a feedback vertex set). Hence $wx = |X| \geqslant g$. On the other hand, the $(0,1)$-vector whose support is given by the vertex set of some directed cycle of length of g clearly has value g and also satisfies the conditions of the program and thus is an optimal solution. Consequently, also the optimal value of the dual program

(3) is g and thus there is an optimal solution vector $y \geqslant 0$ with $y\mathbf{1} = g$. This implies the claim. □

We are now ready to give a proof of Theorem 32 which will conclude this section.

Proof (of Theorem 32). Let D be strongly planar and let $g \geqslant 2$ denote the girth of D. We show that $\vec{\chi}_f(D) = \frac{g}{g-1}$. First of all, the fractional dichromatic number cannot increase by taking subdigraphs, and so we have $\vec{\chi}_f(D) \geqslant \vec{\chi}_f\left(\vec{C}_g\right) = \frac{g}{g-1}$. It remains to prove $\vec{\chi}_f(D) \leqslant \frac{g}{g-1}$. For this purpose we construct a feasible instance of the linear optimisation program (1) with value at most $\frac{g}{g-1}$. To do so, let $F_1, \ldots, F_m$ be a collection of feedback vertex sets as given by Lemma 38 with a corresponding weighting $y_1, \ldots, y_m \geqslant 0$. The complements $V(D) \setminus F_i$ are clearly acyclic for any $j \in \{1, \ldots, m\}$. For any acyclic vertex $A \in \mathcal{A}(D)$ we now define the value of the corresponding variable to be

$$x_A := \frac{1}{g-1} \sum_{\{j | A = V(D) \setminus F_j\}} y_j \geqslant 0.$$

We then have for any vertex $v \in V(D)$:

$$\sum_{A \in \mathcal{A}(D,v)} x_A = \frac{1}{g-1} \sum_{\{j | v \notin F_j\}} y_j = \frac{1}{g-1} \left(\underbrace{\sum_{j=1}^{m} y_j}_{=g} - \underbrace{\sum_{\{j | v \in F_j\}} y_j}_{\leqslant 1} \right) \geqslant \frac{g-1}{g-1} = 1,$$

so this is indeed a feasible instance of the program (1) and we obtain

$$\vec{\chi}_f(D) \leqslant \sum_{A \in \mathcal{A}(D)} x_A = \frac{1}{g-1} \sum_{j=1}^{m} y_j = \frac{g}{g-1}$$

as desired. □

4 List Colourings of Non-Even Digraphs

List colourings naturally generalise several types of colourings of graphs and have been widely investigated. While a lot of progress has been made in the last decades, many important questions, such as the *list colouring conjecture*, still remain open.

Given some collection $\mathcal{P}$ of subsets of a ground sets P, we denote by $\bigcup \mathcal{P}$ the set $\bigcup_{X \in \mathcal{P}} X$.

It is natural to apply the concept of list colouring also to colourings of digraphs. Indeed, such a notion was investigated in [BHKL18]. Therein, for a given digraph D equipped with an assignment of finite *colour lists* $\mathcal{L} = \{L(v) | v \in V(D)\}$ to the vertices, an *$\mathcal{L}$-list-colouring* of D is defined to be a choice function $c : V(D) \to \bigcup \mathcal{L}$ such that for any vertex $v \in V(D)$, we have $c(v) \in L(v)$, and moreover, c defines a proper digraph colouring, that is, $D[c^{-1}(i)]$ is acyclic for all $i \in \bigcup \mathcal{L}$.

Putting $L(v) := \{1, \ldots, k\}$ for each vertex simply yields the definition of a usual digraph k-colouring. In [BHKL18], a digraph D is called *k-list colourable* (also *k-choosable*) if for any list assignment $\mathcal{L}$, where $|L(v)| \geqslant k$ for every $v \in V(D)$, there is an $\mathcal{L}$-list colouring of D. The smallest integer $k \geqslant 1$ for which a digraph D is k-choosable now is defined to be the *list dichromatic number* (also *choice number*) $\vec{\chi}_\ell(D)$. Clearly, we have $\vec{\chi}(D) \leqslant \vec{\chi}_\ell(D)$ for every digraph. However, as pointed out in [BHKL18], this estimate can be arbitrarily bad in general.

It is therefore desirable to identify classes of digraphs with bounded choice number. In the context of Conjecture 1, the authors of [BHKL18] observed that every oriented planar digraph is 3-choosable and posed the question whether all oriented planar digraphs are 2-choosable.

We have shown in Section 2 that all non-even digraphs are 2-colourable, and so it is natural to ask whether they are even 2-choosable. This question can rather easily be answered in the negative, see Figure 5 for an example of a strongly planar digraph with choice number 3. In the remainder of this section, we show that 3 is the (best possible) upper bound for the choice number of non-even digraphs.

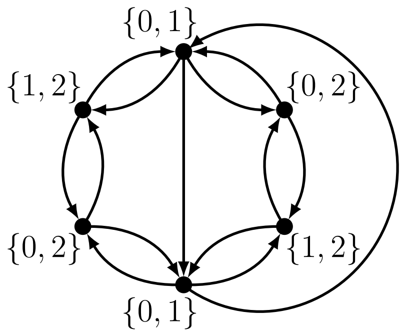


Figure 5: A non-2-choosable strongly planar digraph.

Theorem 39. *Let D be a non-even digraph. Then $\vec{\chi}_\ell(D) \leqslant 3$. Moreover, for any choice of a designated vertex $v_0 \in V(D)$, D is $\mathcal{L}$-list colourable for every list assignment $\mathcal{L} = \{L(v) | v \in V(D)\}$ fulfilling $|L(v_0)| = 1$ and $|L(v)| \geqslant 3$ for all $v \in V(D) \setminus \{v_0\}$.*

Proof. We show the second (stronger) assertion. Assume towards a contradiction that there is a non-even digraph D which does not satisfy the assertion, and assume D to be chosen minimal with respect to the number of vertices. Let in the following $\mathcal{L}$ be a fixed list assignment for D, where $|L(v_0)| = 1$ for some designated $v_0 \in V(D)$, $|L(v)| \geqslant 3$ for all $v \in V(D) \setminus \{v_0\}$, and such that D is not $\mathcal{L}$-choosable. Clearly, we have $|V(D)| \geqslant 3$.

We first show that D must be strongly 2-connected: Assume for a contradiction that there is a directed separation of order $i \in \{0, 1\}$ in D. By Corollary 20, we find that there are non-even digraphs D_1 and D_2 with fewer vertices than D such that D is the i-sum of

D_1 and D_2. By the assumed minimality of D, we know that D_1 and D_2 both satisfy the assertion.

If $i = 0$, consider a partition (X, Y) of $V(D)$ such that $D_1 = D[X], D_2 = D[X]$ and the edges with exactly one endpoint in X and exactly one endpoint in Y form a directed cut in D. Restricting $\mathcal{L}$ to X resp. Y defines list assignments for D_1 and D_2 (each with at most one list of size less than 3), and we find that D_j admits a choice function c_j for $j = 1, 2$ that defines a valid digraph colouring and satisfies $c_j(x) \in L(x)$ for all $x \in V(D_j)$. Putting

$$c(x) := \begin{cases} c_1(x)\,, & x \in X \\ c_2(x)\,, & x \in Y \end{cases}$$

now defines a valid choice of colours for D without a monochromatic directed cycle, proving that D is $\mathcal{L}$-choosable. This is a contradiction to our initial assumption.

If $i = 1$, let $w \in V(D)$ be such that D is the 1-sum of D_1 and D_2 along w. Consider a partition (X, Y) of $V(D) \setminus \{w\}$ such that no edge in D has its head in X and its tail in Y, and such that D_1 arises from D by identification of $Y \cup \{w\}$ into a single vertex v_1, and D_2 by identification of $X \cup \{w\}$ into a vertex v_2.

We have that $v_0 \in X \cup \{w\}$ or $v_0 \in Y \cup \{w\}$. Assume for the following that $v_0 \in X \cup \{w\}$, the other case works symmetrically. Define an assignment $\mathcal{L}_1$ of lists to the vertices of D_1 according to $L_1(x) := L(x)$ for all $x \in X$ and $L_1(v_1) := L(w)$. Because D_1 satisfies the assertion, we find a choice function c_1 which defines a proper digraph colouring of D_1 while satisfying $c_1(x) \in L(x), x \in X$, and $\widetilde{c} := c_1(v_1) \in L(w)$. Now define a list assignment $\mathcal{L}_2$ for D_2 according to $L_2(x) := L(x)$ for $x \in Y$ and $L_2(v_2) := \{\widetilde{c}\}$. Because we have $|L_2(x)| = |L(x)| \geqslant 3$ for all $x \in Y = V(D_2) \setminus \{v_2\}$, we can apply the assertion to D_2 and thus find a choice function c_2 on $V(D_2)$ satisfying $c_2(x) \in L(x)$ for all $x \in Y$ and $c_2(v_2) = \widetilde{c} = c_1(v_1)$. Now define a choice function c on $V(D)$ by

$$c(x) := \begin{cases} c_1(x)\,, & x \in X \\ \widetilde{c}, & x = w \\ c_2(x)\,, & x \in Y. \end{cases}$$

By the above it is clear that we have $c(x) \in L(x)$ for all $x \in V(D)$. Because D is not $\mathcal{L}$-choosable, this implies that there is a directed cycle C in D which is monochromatic under c. Because c_1 and c_2 are valid digraph colourings of D_1 and D_2, C must contain vertices of both X and Y and therefore must visit w as well as exactly one edge with tail in X and head in Y. Therefore, identifying all vertices in $Y \cup \{v\}$ on C into a single vertex results in a directed cycle in D_1, which has to be monochromatic as well. This finally is a contradiction to the definition of c_1.

As both cases led to a contradiction, for the rest of the proof we may assume that $|V(D)| \geqslant 3$ and D is strongly 2-connected. Applying Corollary 23 we find that there is a vertex $u \in V(D) \setminus \{v_0\}$ of out-degree two. Clearly, $D - u$ is non-even as well and has less vertices, so the minimality of D implies that for the induced assignment $\mathcal{L}' := \{L(x) | x \in V(D) \setminus \{u\}\}$ of lists, there is a choice function c' which defines a valid digraph colouring of $D - u$. Let u_1, u_2 be the two out-neighbours of u. Since

$|L(u) \setminus \{c'(u_1), c'(u_2)\}| \geqslant 1$, we can extend c' to a choice function c on $V(D)$ such that $c(x) = c'(x) \in L(x)$ for all $x \in V(D) \setminus \{u\}$ and $c(u) \in L(u) \setminus \{c(u_1), c(u_2)\}$. Because D is by initial assumption not $\mathcal{L}$-choosable, this implies that there is a directed cycle in D which is monochromatic with respect to c. Since c' defined a valid digraph colouring, this is only possible if the cycle traverses u and thus one of the edges (u, u_1) or (u, u_2). However, this gives a contradiction to the fact that both of these edges are bi-coloured.

This final contradiction shows that our initial assumption was false and concludes the proof of the Theorem. □

5 Computational Hardness

Formally, we consider the following decision problem.
Digraph k-Colouring

Input A digraph D.
Question Does there exist a proper k-colouring for D?

Our hardness results bounds not only $\tau(D)$, but also the *out-degeneracy* of D.

Definition 40. Let D be a digraph. The *out-degeneracy* of D (written $d(D)$) is the minimum x such that a linear ordering $\preceq$ of $V(D)$ exists with the property that

$$|\{u \in N^{\mathsf{out}}(v)\,|u \preceq v\}| \leqslant x$$

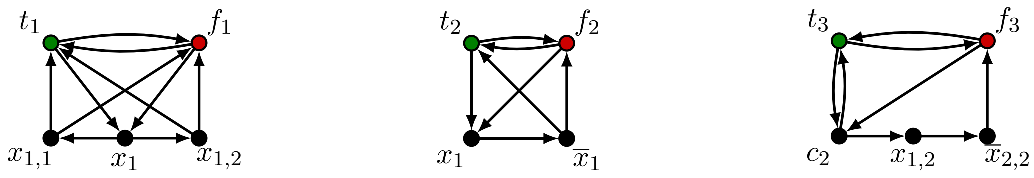
for each $v \in V(D)$.

The hardness result presented below is relatively tight with respect to $\tau(D)$ and $d(D)$: If $\tau(D) \leqslant k-1$, we can find a feedback vertex set S in time $f(k)n^{\mathcal{O}(1)}$ [CLL+08], assign each vertex of S a different colour in $[k-1]$ and the remaining vertices the remaining colour k. Further, one can easily find a proper $(d(D)+1)$-colouring of a digraph by greedily assigning each vertex a colour which does not appear in its smaller outneighbours. Hence, if $d(D) \leqslant k-1$ or $\tau(D) \leqslant k-1$, finding a proper k-colouring for D can be done in $f(k)n^{\mathcal{O}(1)}$ time. In contrast, our hardness result excludes the existence of an $n^{f(k)}$-time algorithm if we only assume $\tau(D) \leqslant k+4$ and $d(D) \leqslant k+1$ instead, leaving only the cases $k \leqslant \tau(D) \leqslant k+3$ and $d(D) = k$ open.

In what follows, for a natural number $n \in \mathbb{N}$ we denote the set $\{1, \ldots, n\}$ by $[n]$.

Lemma 41. Digraph *2*-Colouring *is* NP*-hard even if* $\tau(D) \leqslant 6$ *and* $d(D) \leqslant 3$*, where* D *is the input digraph.*

Proof. We provide a reduction from SAT to Digraph 2-Colouring. Let $C_1, C_2, \ldots, C_m$ denote the clauses and $X_1, X_2, \ldots, X_n$ the variables in the SAT instance. We construct a digraph D which is 2-colourable if and only if there is a satisfying assignment for the SAT instance. For each clause C_i we add the vertex c_i to D, and for each literal $L_j \in C_i$ we add the vertex $l_{j,i}$. That is, we add the vertex $x_{j,i}$ if $X_j \in C_i$ and the vertex $\overline{x}_{j,i}$ if $\overline{X}_j \in C_i$. To simplify our notation, we assume that a literal L_j is associated with the variable X_j, that is $L_j = X_j$ or $L_j = \overline{X}_j$, and that l_j corresponds to the lower-case variant of L_j, that is $l_j = x_j$



(a) Literal gadget. (b) Variable gadget. (c) Clause gadget.

Figure 6: Variable, literal and clause gadgets of the proof of Lemma 41 for the variable X_1 and the clause $(X_1 \vee \overline{X}_2)$ in the SAT formula $(X_1 \vee X_2) \wedge (X_1 \vee \overline{X_2}) \wedge (\overline{X_1} \vee X_2)$.

if $L_j = X_j$ and $l_j = \overline{x}_j$ if $L_j = \overline{X}_j$. We want the colour of a vertex $x_{j,i}$ to correspond to an assignment of the variable X_i. To this end, we add a set $S = \{t_1, t_2, t_3, f_1, f_2, f_3\}$ of vertices which will correspond to a feedback vertex set in D. Furthermore, for each literal L_j we add a vertex l_j. We now add cycles to D in such a way that any proper colouring $c\colon V(D) \to \{0, 1\}$ must have the following properties.

(i) $c(l_{j,i}) = c(l_{j,h})$ for all $j \in [n]$ and $i, h \in [m]$, and

(ii) $c(\overline{x}_{j,h}) \neq c(x_{j,i})$ for all $j \in [n]$ and $i, h \in [m]$.

Clearly, these properties allow us to obtain a variable assignment from any proper 2-colouring of D.

To ensure (i), we construct a *literal gadget* (illustrated in Figure 6a). First, we add the cycle t_1, f_1. Then, for each literal L_j and each clause C_i with $L_j \in C_i$ we add the cycles $l_j, l_{j,i}, t_1$ and $l_j, l_{j,i}, f_1$. If there are $i, h \in [m]$ such that $l_{j,i}$ and $l_{j,h}$ have different colours, one of them, say, $l_{j,i}$, must have the same colour as l_j. Since t_1, f_1 forms a cycle, they must have different colours in any solution of the 2-colouring problem. Hence, the cycle $l_j, l_{j,i}, t_1$ or the cycle $l_j, l_{j,i}, f_1$ is monochromatic if $l_{i,j}$ and $l_{i,h}$ have different colours. This proves (i).

For (ii), we construct a *variable gadget* (illustrated in Figure 6b). First, we add the cycle t_2, f_2. Then, we add the cycles $x_j, \overline{x}_j, t_2$ and $x_j, \overline{x}_j, f_2$ for each $j \in [n]$ where both X_j and $\overline{X}_j$ appear in the formula. If x_j and $\overline{x}_j$ receive the same colour, then one of the added cycles is monochromatic as t_2 and f_2 must receive different colours. Because of the literal gadgets, we know that l_j and $l_{j,i}$ have different colours for all $j \in [n]$ and $i \in [m]$. As x_j and $\overline{x}_j$ have different colours, it follows from (i) that $x_{j,i}$ and $\overline{x}_{j,h}$ have different colours for all $j \in [n]$ and all $i, h \in [m]$. This implies (ii).

We now construct a *clause gadget* (illustrated in Figure 6c) that ensures that each clause is satisfied by at least one of its literals. We first add the cycle t_3, f_3. Then, for each clause C_i we add the cycle c_i, t_3. Finally, we add the cycle $c_i, l_{j_1,i}, l_{j_2,i}, \ldots, l_{j_h,i}, f_3$, where $l_{j_1,i}, l_{j_2,i}, \ldots, l_{j_h,i}$ are the literals of C_i. We sort the literals in such a way that $j_1 < j_2 < \cdots < j_h$ and such that X_j comes before $\overline{X}_j$. This concludes the construction of D.

We first show that $\tau(D) \leqslant 6$. We claim that the set $S = \{t_1, t_2, t_3, f_1, f_2, f_3\}$ is a feedback vertex set of D. We prove that $D - S$ is acyclic by finding a topological ordering of its vertices. We first take the positive literal vertices x_j and the clause vertices c_i into the ordering, as these are sources in $D - S$. Removing these vertices, all negative literal vertices $\overline{x}_j$ become sources, which we then add to the end of the current topological ordering. The only remaining vertices are the variable vertices $l_{j,i}$. It follows from the construction of the clause gadget that ordering the $l_{j,i}$ monotonically in j, with positive literals preceding corresponding negative literals, completes the topological ordering of $D - S$.

To show that the degeneracy of D is 3, we construct a linear ordering of the vertices as follows. The first vertices of the ordering are t_1, f_1, t_2, f_2, t_3 and f_3. These have at most one outgoing arc to vertices which are smaller. Afterwards come all positive literal vertices x_j, then all negative literal vertices $\overline{x}_j$, followed by the variable vertices $l_{j,i}$. The vertices $\overline{x}_j$ have arcs to t_2 and f_2, and x_j has no arc to smaller vertices. Hence, they have at most two arcs to smaller vertices. The vertices $l_{j,i}$ have arcs to t_1, f_1 and potentially to some other $l_{h,i}$ or to f_3, but never both. Hence, they have at most 3 arcs to smaller vertices. The last vertices in the ordering are the clause vertices c_i. These have an arc to t_3 and another to some $l_{j,i}$. Hence, the directed degeneracy of D is at most 3.

We now prove that D is 2-colourable if there is a truth assignment of the variables satisfying all clauses.

Let $\beta : \{X_j \mid j \in [n]\} \to \{0, 1\}$ be a satisfying truth assignment of the variables. We construct a colouring $c\colon V(D) \to \{0, 1\}$ as follows.

i) $c(f_i) \coloneqq 0$ and $c(t_i) \coloneqq 1$ for $i \in [3]$.
ii) $c(c_i) \coloneqq 0$ for $i \in [m]$.
iii) $c(x_{j,i}) \coloneqq \beta(X_j)$ for all $j \in [n]$ and $i \in [m]$ with $X_j \in C_i$.
iv) $c(\overline{x}_{j,i}) \coloneqq 1 - \beta(X_j)$ for all $j \in [n]$ and $i \in [m]$ with $\overline{X}_j \in C_i$.
v) $c(x_j) \coloneqq 1 - \beta(X_j)$ and $c(\overline{x}_j) \coloneqq \beta(X_j)$ for all $j \in [n]$.

This concludes the construction of c. We now argue that each colour class induces an acyclic digraph in D.

Let $d \in \{0, 1\}$ be some colour. Note that either $t_1, t_2, t_3 \in c^{-1}(d)$ or $f_1, f_2, f_3 \in c^{-1}(d)$, as these vertices receive different colours. Since S is a feedback vertex set of D, it suffices to show that there are no cycles using vertices of $S_d \coloneqq c^{-1}(d) \cap S$ in $D[c^{-1}(d)]$.

Assume, without loss of generality, that $t_1, t_2 \in c^{-1}(d)$. The case $f_1, f_2 \in c^{-1}(d)$ follows analogously. We prove that no cycle contains t_1 or t_2 by progressively identifying and removing sinks from $D[c^{-1}(d)]$. As for all $j \in [n]$ and $i \in [m]$ we have $c(x_j) \neq c(\overline{x}_j) = c(x_{j,i})$, it follows that all x_j are sinks in $D[c^{-1}(d)]$. Removing all x_j, we can see that t_2 is now a sink. Hence, no directed cycle in $D[c^{-1}(d)]$ contains t_2. As $c(\overline{x}_j) \neq c(\overline{x}_{j,i})$, it follows that $\overline{x}_j$ is now a sink and we can remove it. Without literal vertices, t_1 becomes a sink, implying no cycle goes through t_1 in $D[c^{-1}(d)]$, as desired. Consequently, for any $d \in \{0, 1\}$, no directed cycle in $D[c^{-1}(d)]$ can possibly use one of the vertices t_1, t_2, f_1, f_2 and therefore must either contain t_3 or f_3.

If $t_3 \in c^{-1}(d)$, then $c_i \notin c^{-1}(d)$ for all $i \in [m]$, as $c(t_3) = 1$ and $c(c_i) = 0$. Hence, t_3 has no neighbours in $D[c^{-1}(d)]$ and cannot be in any cycle. If $f_3 \in c^{-1}(d)$, assume towards a contradiction that there is a cycle C in $D[c^{-1}(d)]$ containing f_3. Note that this cycle must also contain c_i for some $i \in [m]$, as these are the only out-neighbours of f_3 in $D[c^{-1}(d)]$. Furthermore, the out-neighbour of c_i in C is some $l_{j,i}$, and the only out-neighbours of $l_{j,i}$ are t_1 and potentially some $l_{h,i}$ or f_3, as these were the arcs added in the clause gadgets.

The vertices $l_{j,i}$ in C correspond to the literals in c_i. In order to form a cycle, all literals in c_i must be in C. However, this means that $c(x_{j,i}) = 0$ for all X_j in clause C_i and $c(\overline{x}_{j,i}) = 0$ for all $\overline{X}_j$ in clause C_i. By construction of c, this implies that all literals in C_i are set to false, which means that the clause is not satisfied, a contradiction to our initial assumption. Hence, the digraph $D[c^{-1}(d)]$ is acyclic, and D is 2-colourable.

We now show that the formula is satisfiable if $\vec{\chi}(D) \leqslant 2$ by constructing a satisfying variable assignment β from a proper 2-colouring of D. Let $c\colon V(D) \to \{0,1\}$ be a proper colouring of D. Without loss of generality, we assume that $c(t_3) = 1$, which implies that $c(f_3) = 0$. We set $\beta(X_j)$ to true if $c(x_j) = 0$ and to false if $c(x_j) = 1$.

Assume towards a contradiction that there is some clause C_i which is not satisfied by β. By simply renaming the variables, we can assume without loss of generality that the literals of C_i are $L_1, L_2, \ldots, L_a$. As C_i is not satisfied, it follows that all L_j evaluate to false with β. By construction of the literal gadget, $c(l_j) \neq c(l_{j,i})$ for all $i \in [m]$ with $L_j \in C_i$. From (i) and (ii), for all $j \in [n]$ it follows that $c(l_{j,i}) = 1$ if the literal L_j is true, and that $c(l_{j,i}) = 0$ if the literal L_j is false. As C_i is not satisfied, $c(c_i) = c(f_3) = c(l_{j,i}) = 0$ for all $j \in [a]$. Hence, the cycle $C = c_i, l_1, l_2, \ldots l_a, f_3$ is monochromatic, contradicting our assumption that c is a proper colouring. This implies that β is a satisfying variable assignment, concluding our proof. □

With a simple self-reduction, we can extend the previous result to all $k \geqslant 2$.

Theorem 42. *For each $k \geqslant 2$,* Digraph k-Colouring *is* NP*-hard even if $\tau(D) \leqslant k + 4$ and $d(D) \leqslant k + 1$, where D is the input digraph.*

Proof. We prove the statement by induction on k. The case $k = 2$ follows from Lemma 41. We provide a reduction from Digraph $(k-1)$-Colouring to Digraph k-Colouring such that $\tau(D') \leqslant \tau(D) + 1$ and $d(D') \leqslant d(D) + 1$, where D is the input instance and D' is the reduced instance. We obtain D' be adding a vertex x to D, together with the edges $\{(x, v), (v, x) \mid v \in V(D)\}$. If D is $(k-1)$-colourable, then setting the colour of x to k gives a proper k-colouring for D'. If D' is k-colourable, then no vertex in D has the same colour as x. Hence, D is $(k-1)$-colourable. Furthermore, all new cycles created by adding x go through x. If $D - S$ is acyclic for some vertex set S, then $D' - (S \cup \{x\}) = D - S$ is also acyclic. Hence, $\tau(D') \leqslant \tau(D) + 1 = k + 4$. To show that the degeneracy of D' increased by at most one, we consider some ordering of D with degeneracy $d(D) = k$. By placing x as the smallest vertex with respect to the ordering, we increase the outdegree of the vertices in D by one. Hence, the degeneracy of D' is at most $d(D) + 1 = k + 1$, as desired. □

As an immediate consequence of the above theorem arises the following corollary.

Corollary 43. *There is no $n^{f(k,x,y)}$-time algorithm deciding* Digraph k-Colouring *where $x = \tau(D)$, $y = d(D)$ and f is some function, unless* P=NP*.*

A finer analysis of the reduction provided in Lemma 41 gives us stronger hardness results under a different assumption. Similar to how no polynomial-time algorithms for NP-complete problems are known, no $2^{o(n)} n^{\mathcal{O}(1)}$-time algorithm for k-SAT is known, where

n is the number of variables in the input formula (which contains at most k literals in each clause). An algorithm with such a running time is called a *subexponential-time* algorithm. Impagliazzo and Paturi [IP01] provided evidence that no such algorithm for k-SAT exists, and formulated the following hypothesis (often referred to as *ETH*).

Hypothesis (Exponential Time Hypothesis [IP01])**.** For each $k \geqslant 3$ there is some $s_k > 0$ such that no $2^{s_k n} n^{\mathcal{O}(1)}$-time algorithm for k-SAT exists.

Note that the ETH only considers the running time with respect to the number of variables in the input formula, not the number of clauses. In several reductions, however, it is difficult to ensure that the size of the reduced instance depends only on the number of variables. For example, the reduction in Lemma 41 contains one vertex for each clause. This would prevent us from directly applying the ETH. Fortunately, [IPZ01] showed that it is possible to assume that $m \in \mathcal{O}(n)$, where m is the number of clauses, by proving the following lemma.

Lemma (Sparsification Lemma, Impagliazzo, Paturi and Zane [IPZ01])**.** For all $\epsilon > 0$ and $k > 0$ there is a constant C so that any k-SAT formula Φ with n variables can be expressed as $\Phi' = \bigvee_{i=1}^{t} \Psi_i$, where $t \leqslant 2^{\epsilon n}$ and each Ψ_i is a k-SAT formula with at most Cn clauses such that each variable appears in constantly many clauses. Moreover, this disjunction can be computed by an algorithm running in time $2^{\epsilon n} n^{\mathcal{O}(1)}$.

By first applying the sparsification lemma to the input formula and then the reduction from Theorem 42, we can show the following.

Theorem 44. *For each $k \geqslant 2$ there is some $\epsilon > 0$ such that no $2^{\epsilon n} n^{f(x,y)}$ algorithm for* Digraph k-Colouring *exists, where D is the input digraph, $x = \tau(D)$, $y = d(D)$ and f is some function, unless the ETH is false.*

Proof. First note that the reduction from Digraph $(k-1)$-Colouring to Digraph k)-Colouring from Theorem 42 increases the input instance by one vertex. Hence, it suffices to show the statement for $k = 2$, as the remaining cases follow by induction. We first use the sparsification lemma to obtain at most $2^{\epsilon n}$ many 3-SAT instances where each variable appears in constantly many clauses. Applying the reduction from Lemma 41 to each instance, we obtain at most $2^{\epsilon n}$ many digraphs where for each variable we have constantly many vertices and for each clause we have one vertex. This means that the number of vertices on the reduced instances is linear in the number of variables of the formula. Hence, a subexponential-time algorithm for Digraph 2-Colouring implies a subexponential-time algorithm for 3-SAT, which would contradict the ETH. $\square$

Note that an algorithm with running time $\mathcal{O}(k^n \cdot (n + m))$ is trivial: test all k^n colourings of the vertices of D, and then check if each colour class is a DAG in linear time by computing a topological ordering.

6 Non-Bipartite Graphs

In the previous sections we were concerned with digraphs, which correspond exactly to the bipartite graphs with perfect matchings. However, a matching covered graph does not need to be bipartite. In fact, most parts of (bipartite) matching theory directly translate into the world of general matching covered graphs. This includes, especially, tight cuts, their contractions, and pfaffian orientations.

In particular, the M-chromatic number is defined on all matching covered graphs. By Corollary 11 and Theorem 9 every bipartite Pfaffian graph has M-chromatic number at most 2 for every perfect matching. A natural question to ask would be whether this generalises to all Pfaffian graphs. To this question there exists a rather easy negative answer. The *triangular prism* is the complement $\overline{C_6}$ of the 6-cycle.

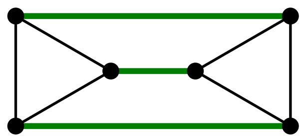


Figure 7: The triangular prism $\overline{C_6}$ together with a perfect matching M.

It is planar and therefore Pfaffian, but when considering the perfect matching M from Figure 7, one can see that any two of the three edges in M lie together on a 4-cycle. Hence no two of the three edges may receive the same colour and therefore $\chi\big(\overline{C_6}, M\big) = 3$.

In Corollary 11 we went for a class closed under matching minors, so a next step would be to consider a subclass of the $\overline{C_6}$-matching minor-free graphs. The triangular prism is one of two graphs appearing in a fundamental theorem by Lovász on non-bipartite matching covered graphs.

Theorem 45 (Lovász [Lov87])**.** *Every non-bipartite matching covered graph contains a conformal bisubdivision of K_4 or $\overline{C_6}$.*

A matching covered graph without a non-trivial tight cut is called a *brace* if it is bipartite and a *brick* otherwise. In his seminal paper [Lov87], Lovász introduced a decomposition procedure, known under the name *tight cut decomposition*, which, given a matching covered graph, searches for non-trivial tight cuts, computes both tight cut contractions, and iterates this for both reduced matching covered graphs, until a list of bricks and braces, which are not reducible any more, is obtained. Among many other things, Lovász proved that the list of bricks and braces does not depend on the chosen order in which the tight cuts are contracted. As the following theorem shows, braces correspond exactly to the strongly 2-connected digraphs.

Theorem 46 (Lovász and Plummer [LP86])**.** *A bipartite graph G is a brace if and only if it is 2-extendable.*

Bricks have a more complicated structure and although every 2-extendable graph is either a brick or a brace as seen in Theorem 47, there are bricks that are not 2-extendable. For an example of such a brick consider the triangular prism.

Theorem 47 (Plummer [Plu80]). *Let G be a 2-extendable graph. Then, G is either a brace or a brick.*

There exists a generalisation of tight cuts that crosses the border towards bricks. Given a matching covered graph G and a set $X \subseteq V(G)$ we call the graph G_X obtained from G by identifying X into a single vertex, removing all loops, and identifying parallel edges the *X-contraction* of G. Now a cut $\partial(X)$ is called *separating* if both G_X and $G_{\overline{X}}$ are matching covered.

Theorem 48 (de Carvalho, Lucchesi and Murty [dCLM02]). *Let G be a matching covered graph and $X \subseteq V(G)$. The cut $\partial(X)$ is separating if and only if for every edge $e \in E(G)$ there is a perfect matching M_e of G containing e such that $|\partial(X) \cap M_e| = 1$.*

We call a matching covered graph *solid* if every non-trivial separating cut is already tight.

One can easily check the following lemma on bipartite graphs, showing that any bipartite matching covered graph is solid.

Lemma 49 (de Carvalho, Lucchesi, Kothari and Murty [LDCKM18]). *Let G be a bipartite matching covered graph. Then $\partial(X)$ is separating if and only if it is tight.*

Moreover, being solid is preserved by tight cut contractions (cf. [dCLM02]) and thus a matching covered graph is solid of and only if all of its bricks and braces are solid. Please note that braces are bipartite and thus, by Lemma 49, it further follows that a matching covered graph is solid if and only if all of its bricks are solid.

Please note that bricks may contain non-trivial separating cuts. Again consider the triangular prism from Figure 7 and take a cut around one of the two triangles. Such a cut is separating. In fact, the existence of a prism as a conformal bisubdivision immediately implies the existence of a non-trivial and non-tight separating cut.

Lemma 50 (de Carvalho, Lucchesi, Kothari and Murty [LDCKM18]). *Every solid graph is $\overline{C_6}$-free.*

The goal of this section is to establish an extension of Corollary 11 to non-bipartite matching covered graphs in the form of a conjecture.

Conjecture 51. Let G be a solid and Pfaffian graph and M a perfect matching of G. Then $\chi(G, M) \leqslant 2$.

To provide some evidence towards Conjecture 51, the remainder of this section is dedicated to settle the planar case. For this we first establish a more general version of Lemma 21 by proving it directly for tight cut contractions. We will need a bit of notation here. If G is matching covered, M a perfect matching, and G_X is a tight cut contraction of $\partial(X)$ with contraction vertex v_X, we denote by M_X the perfect matching $\{e \in M \mid e \subseteq V(G - X)\} \cup \{uv_X\}$ where u is the unique vertex of X covered by the edge of M in $\partial(X)$.

Lemma 52. *Let G be a matching covered graph, $\partial(X)$ a non-trivial tight cut in G and M a perfect matching. If $\chi(G_X, M_X) \leqslant 2$ and $\chi(G_{\overline{X}}, M_{\overline{X}}) \leqslant 2$, then $\chi(G, M) \leqslant 2$.*

Proof. For $Y \in \left\{X, \overline{X}\right\}$ let c_Y be a proper 2-colouring of M_Y in G_Y. Let $e_Y \in M_Y$ be the edge covering the contraction vertex. Then we can rename the colours for c_X and $c_{\overline{X}}$ such that $c_X(e_X) = c_{\overline{X}}(e_{\overline{X}})$ and we define a colouring for M as follows.

$$c(e) \coloneqq \begin{cases} c_X(e)\,, & e \in M_X \\ c_X(e_X) = c_{\overline{X}}(e_{\overline{X}})\,, & e \in \partial(X) \cap M \\ c_{\overline{X}}(e)\,, & e \in M_{\overline{X}}. \end{cases}$$

Suppose G contains an M-alternating cycle C that is monochromatic with respect to c. If $V(C)$ is a subset of either X or $\overline{X}$, by definition of c, C must be a monochromatic cycle in either G_X or $G_{\overline{X}}$ and, thus, C must cross $\partial(X)$. Since $\partial(X)$ is tight, C has exactly two edges in $\partial(X)$, one of them belonging to M and the other one not. To see this note that $M \Delta E(C)$ is also a perfect matching of G and thus $|(E(C) \setminus M) \cap \partial(X)| = 1$, but since C is a cycle it must have an even number of edges in $\partial(X)$. Therefore $C - (\partial(X) \cap E(C))$ contains exactly 2 components. Each of them is a path of even length and M covers all vertices but exactly one endpoint. Moreover, each of these paths forms, together with the corresponding edges in $\partial(X)$, an M_Y-alternating cycle in their respective contraction G_Y. By definition of c, these two cycles must also be monochromatic which ultimately contradicts the choice of the c_Y and completes the proof. □

Both shores of a tight cut must be connected graphs [Lov87]. Hence, tight cut contractions of a graph G are minors of G, and therefore preserve planarity. Using the tight cut decomposition, the above Theorem, and this observation, it suffices to show that every perfect matching of a solid planar brick or planar brace is 2-colourable. The brace case is of course taken care of by Corollary 11 and thus our only concern are the solid planar bricks. By Lemma 50 we only have to consider $\overline{C_6}$-free planar bricks. A theorem of Kothari and Murty (cf. [KM16]) gives a precise description of these bricks.

A graph W_k consisting of a cycle of length k and a single vertex adjacent to every vertex on the cycle is called a *wheel*. If k is odd, we call W_k an odd wheel; every odd wheel is a brick.

Let $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ be two disjoint paths with $k \geqslant 2$. The graph S_{2k+2} obtained from the union of these paths by adding the edges u_iv_i for all $i \in \{1, \ldots, k\}$, two new vertices x and y joined by an edge and the edges xu_1, xv_1, yu_k, yv_k, is called a *staircase of order* $2k+2$. Every S_{2k+2} is a brick and S_6 is isomorphic to the triangular prism.

Theorem 53 (Kothari and Murty [KM16])**.**

i) A matching-covered graph is $\overline{C_6}$-free if and only if all the bricks and braces in its tight cut decomposition are $\overline{C_6}$-free.

ii) The only planar $\overline{C_6}$-free bricks are the odd wheels, the staircases of order $4k$ and the tricorn (see Figure 8).

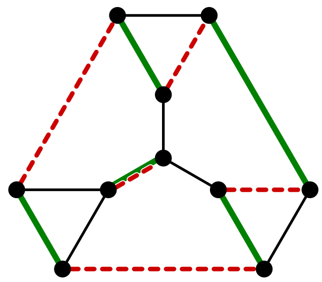


Figure 8: The *tricorn* together with a perfect matching of type I and a perfect matching of type II.

If we have a planar and matching covered graph G that does not contain a conformal bisubdivision of $\overline{C_6}$, by Theorem 53 the only bricks G can have are odd wheels, staircases of orders divisible by 4 and tricorns. Along with these bricks, G can have any planar brace. Planar braces are Pfaffian and thus by Corollary 11 2-colourable. While it is our goal to provide evidence towards the 2-colourability of solid Pfaffian graphs, for the planar case we can prove a stronger statement, namely Theorem 14.

Proof (of Theorem 14). Recall that we aim to prove that every perfect matching M of a planar matching covered graph that is $\overline{C_6}$-free can be coloured with two colours without producing monochromatic M-alternating cycles.

Observe that every shore of a tight cut must be a connected graph [Lov87], thus tight cut contractions are special cases of minors and therefore, in particular, tight cut contractions preserve planarity. Hence, by applying Lemma 52, it suffices to consider planar and $\overline{C_6}$-free bricks and planar braces. Since planar braces G all satisfy $\chi(G, M) \leqslant 2$ for all perfect matchings M by Corollary 11 the only case left is where G is a planar and $\overline{C_6}$-free brick. So with Theorem 53 we have to show that the perfect matchings of the odd wheels, staircases of order $4k$ and the tricorn are 2-colourable.

Odd Wheels

For $K_4 = W_3$ we have exactly two edges in every perfect matching and thus are done. Let $k \geqslant 4$ be any odd number. For the odd wheel W_k on $k+1$ vertices, let x be the unique vertex of degree k. Clearly every perfect matching M has to cover x with an edge, say, e_x^M, and every other matching edge lies on the cycle induced by the neighbourhood of x. Consider the graph induced by $\bigcup M \setminus \{e_x^M\}$. Since $N(x)$ induces a cycle, this graph is a path and thus every M-alternating cycle in W_k must contain e_x^M. Hence, by colouring e_x^M with 0 and every other edge of M with 1 we have found a proper 2-colouring for M in W_k.

Staircases of Order $4k$

For the staircases S_{4k} we give a 2-colouring $c\colon E(S_{4k}) \to \{0, 1\}$ of the edges that induces a proper 2-colouring for every perfect matching. Let xy be the unique edge with endpoints in two disjoint triangles. Let $(u_1, \ldots, u_{2k-1})$ be the path from the construction of S_{4k} not on the outer face and assume xu_1 to be an edge of S_{4k}. We colour xy with 0. Then, going counter-clockwise around the outer face, we assign 0 as the colour of the edges xv_1 and v_1v_2, the next two edges receive the colour 1, then two times colour 0 and so forth until the edge $v_{2k-1}y$ is coloured. Since S_{4k} is of order $4k$ we colour $2k$ edges this way and the last two edges receive colour 1. With this the path $(x, v_1, \ldots, v_{2k-1}, y)$ on the outer face is coloured. We set $c(u_iu_{i+1}) := 1 - c(v_iv_{i+1})$ for $i = 1, \ldots, 2k-2$ and $c(xu_1) := 1$ while $c(yu_{2k-1}) = c(u_{2k-2}u_{2k-1})$. At last we need to colour the spokes. Let $c(v_iu_i) := i \mod 2$ for $i = 1, \ldots, 2k-1$. For an illustration consider Figure 9. To show that c induces a proper 2-colouring for every perfect matching, we must show that there is no conformal cycle C such that every second edge has the same colour.

Assume for a contradiction that $S_{4k}[c^{-1}(0)]$ contains C. However, this graph contains a single cycle and this cycle contains exactly the vertices incident with at most one edge of colour 1 in G. Moreover, if k is odd, then the unique cycle in $S_{4k}[c^{-1}(0)]$ is of odd length, hence we may assume k to be even. By these arguments, $V(G) \setminus V(C)$ is a stable set and thus C is not conformal, a contradiction.

Thus C must contain an edge of colour 1 and therefore, by construction, also two consecutive such edges. Consequently, by choice of C, every second edge of C must be of colour 1. There does not exist a path of length 5 in S_{4k} such that the first, third, and fifth edge are coloured with 1, hence C must have length 4. Clearly none of the 4-cycles contains two disjoint edges of the same colour and thus C cannot exist.

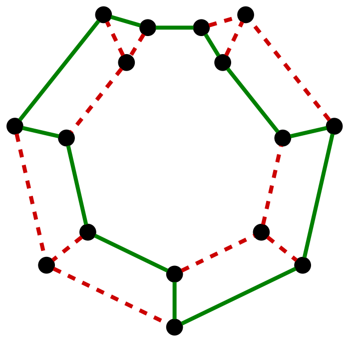


Figure 9: The staircase of order 16 together with a 2-colouring of the edges inducing a proper 2-colouring for every perfect matching. The solid edges are considered to be of colour 0, while the dashed ones are of colour 1.

Tricorn

For the tricorn we first observe that we can classify its perfect matchings into two types. Any perfect matching either contains exactly one edge on the outer face that belongs to a triangle or none (compare Figure 8). If we fix such an edge e on the outer face belonging to a triangle for our perfect matching M_1, the remaining edges of M_1 are uniquely determined. This can be seen as follows: Taking an edge from one of the triangles forces us to match the remaining vertex of said triangle to the middle vertex. Then the remaining neighbours of the middle vertex have to be matched within their respective triangles in such a way that the remaining two vertices are adjacent. There is only one way to do this after e has been chosen and thus $\{e\}$ is in fact a forcing set for M_1. Hence colouring e with 0 and all other edges of M_1 with 1 yields the desired colouring. We call such a matching *type I*.

A matching of *type II* is a matching not containing any edge on the outer face belonging to a triangle. Note that any perfect matching must contain two edges of the outer face. So let e_1 and e_2 be these two edges. One of the three triangles contains an endpoint from both e_1 and e_2, and its third vertex has to be matched to the middle one. This is already enough to determine the last two edges and we obtain M_2. Hence $\{e_1, e_2\}$ is a forcing set of M_2 and, since the tricorn contains no 4-cycle, by colouring e_1 and e_2 with 0 and the rest of M_2 with 1 we are done.

By the above discussion it is clear that any perfect matching of the tricorn is either of type I or II and this concludes the proof. □

By applying Lemma 50, we obtain the special case of Theorem 14 for planar and solid graphs.

Corollary 54. *Let G be a planar solid graph and M a perfect matching of G, then $\chi(G, M) \leqslant 2$.*

Please note that the proof of Theorem 14 works for every Pfaffian matching covered graph whose bricks are planar and $\overline{C_6}$-free. If one was able to show that the number of edges in a solid Pfaffian brick is linearly bounded in the number of vertices, an approach similar to the one for Theorem 10 would likely be successful. It does not seem very likely that solid bricks in general can be very dense, as they cannot contain conformal bisubdivisions of the triangular prism, however, no linear bound on the number of edges is known.

7 Concluding Remarks

In this paper, we initiated the study of relationships between butterfly-minor closed classes of digraphs and the dichromatic number by characterising the largest butterfly-minor closed class of 2-colourable digraphs. Since odd bicycles have dichromatic number 3, one direction of the following is Theorem 5, the reverse follows from Theorem 10.

Corollary 55. *The non-even digraphs form the unique inclusion-wise largest class $\mathcal{D}_2$ of 2-colourable digraphs which is closed under butterfly-minors.*

In the undirected case, Hadwiger's Conjecture claims a characterisation of the largest minor-closed class of k-colourable graphs. In view of Corollary 55, the following is a natural directed analogue.

Question 56. Given a natural $k \geqslant 3$, what is the largest butterfly-minor closed subclass $\mathcal{D}_k$ of the k-colourable digraphs?

Due to the existence of infinte antichains (such as the odd bicycles) in the butterfly-minor order of digraphs, we believe that for larger values of k, possibly no very simple description of the forbidden butterfly minors for $\mathcal{D}_k$ can be obtained. This is already illustrated by the fact that $\mathcal{D}_2$ excludes all odd bicycles, for larger values of k one could expect more complicated anti-chains. Looking at the case $k = 2$, this drastically changed when moving from digraphs to the corresponding bipartite graphs, where we only needed to exclude $K_{3,3}$ as a matching minor. While by now the $K_{3,3}$-matching minor-free bipartite graphs (that is, the Pfaffian bipartite graphs) have many equivalent characterisations and can be recognised in polynomial time, not much is known about the classes of $K_{k,k}$-matching minor-free graphs with $k \geqslant 4$. Clearly, the complete bipartite graph $K_{k,k}$ has M-chromatic number k for any perfect matching. Concerning Corollary 11, we think that the following analogue of Hadwiger's Conjecture for M-colourings of bipartite graphs could be true.

Conjecture 57. Let $k \in \mathbb{N}$, G be a bipartite graph and M an arbitrary perfect matching of G, such that $\chi(G, M) \geqslant k$. Then G contains $K_{k,k}$ as a matching minor.

While for $k = 1, 2$, the statement is trivial, the case $k = 3$ amounts to Corollary 11. At the current state, we do not have a good approach for proving this conjecture even in the first open case of $k = 4$, which is mostly due to the fact that our proof for $k = 3$ relied on a certain sparsity of Pfaffian bipartite graphs, which has not yet been established for classes excluding larger complete bipartite graphs as matching minors. Please note that, if one were only interested in a qualitative description of $\mathcal{D}_k$, a relaxed analogue of Conjecture 57 might still be true.

Conjecture 58. There exists a constant $c \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every digraph D, if $\vec{\chi}(D) \geqslant ck$, then D contains $\overleftrightarrow{K}_k$ as a butterfly minor.

Note that we cannot hope for this conjecture to hold with $c = 1$, since for every integer $k \geqslant 3$ the digraph obtained from $\overleftrightarrow{K}_{k+2}$ by removing the arc-set of a $\overleftrightarrow{C}_5$ has dichromatic number k but does not contain $\overleftrightarrow{K}_k$ as a butterfly-minor.

A related question regarding the existence of complete bipartite graphs as matching minors is, whether high extendibility forces the existence of large minors.

Question 59. Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every $(k-1)$-extendable bipartite graph G without a $K_{k,k}$-matching minor on n vertices has at most $f(k)\,n$ edges? In other words, is the average degree of these graphs bounded in terms of k?

The following observation, which is a direct consequence of a result of Aboulker et al. [ACH+19], provides some evidence towards Conjecture 57.

Theorem 60 (Theorem 32 in [ACH+19])**.** *Let D and F be digraphs, $m := |E(F)|, n := |V(F)|$. If $\vec{\chi}(D) \geqslant 4^m(n-1)+1$, then D contains a subdivision of F as a subdigraph.*

Corollary 61. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for any $k \in \mathbb{N}$, every bipartite graph G with a perfect matching M satisfying $\chi(G, M) \geqslant f(k)$ contains $K_{k,k}$ as a matching minor.*

Proof. Set $f(k) := 4^{k^2-k}(k-1)+1$ and let G be a bipartite graph with a perfect matching M such that $\chi(G, M) \geqslant f(k)$. As the complete bioriented digraph $\overleftrightarrow{K}_k$ has $k^2 - k$ edges and k vertices, we deduce from Theorem 60 that $\vec{\chi}(\mathcal{D}(G,M)) = \chi(G,M) \geqslant f(k)$ implies the existence of a subdivision of $\overleftrightarrow{K}_k$ as a subdigraph of $\mathcal{D}(G,M)$. Clearly, this implies that $\overleftrightarrow{K}_k$, which is the unique perfect matching-direction of $K_{k,k}$, is a butterfly minor of $\mathcal{D}(G,M)$. The claim now follows from Lemma 8. $\square$

It would be interesting to see whether Conjecture 57 would already imply Hadwiger's Conjecture for graphs. While we do not have a proof of this implication yet, it does seem quite likely that a relation exists. For this, note that the chromatic number of a graph can be expressed as the dichromatic number of its bidirection, and that the matching minors of the corresponding bipartite graph to some extent resemble the ordinary minors of the original graph. Here, the complete graph K_k yields the bidirected k-clique, which in the matching context corresponds to $K_{k,k}$.

An additional line of future research could be to investigate colouring properties of classes of digraphs which are closed under different notions of digraph minors. One such candidate are the *topological minors*, which are defined similarly to the undirected case: A digraph D_1 is called a directed topological minor of another digraph D_2 if D_2 contains a subdivision of D_1 (that is, replacing directed edges by directed paths of positive length) as a subdigraph. It is easily seen that topological minors are always butterfly-minors, but that the converse fails in general. In any class of 2-colourable digraphs which is closed under topological minors, the odd bicycles must form a set of forbidden minors. So far, we have been unable to decide the following question. If true, this statement would be a proper generalisation of Theorem 10.

Question 62. Let D be a digraph with $\vec{\chi}(D) \geqslant 3$. Must D contain a subdivision of an odd bicycle?

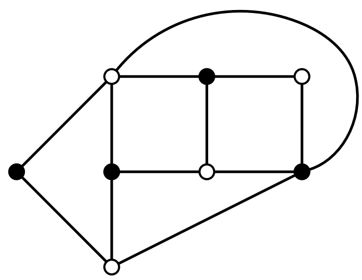


Figure 10: A planar bipartite graph such that for any 2-colouring of its edges, there is a perfect matching with a monochromatic alternating cycle.

Considering the notion of M-colourings, it is natural to ask whether it is necessary to have different colourings of the matching edges for every perfect matching, or whether one might strengthen Corollary 11 by finding a single 2-colouring of *all* edges in a bipartite Pfaffian graph, such that for any perfect matching M the induced 2-colouring on the matching edges yields a proper M-colouring. For an example consider the 2-colouring of the staircase in Figure 9. Although it seems to be possible to find such a "super"-colouring for many bipartite Pfaffian graphs such as the Heawood graph or square grids, there are small examples of (even planar) Pfaffian bipartite graphs without such a colouring (cf. Figure 10).

Furthermore, all stated results and Conjectures are worthy to consider in the more general setting of solid graphs. However, here, even more fundamental questions concerning the structure of these graphs are left widely open, see Section 6.

The questions raised in Section 3 concerning the relationship of girth and disjoint packings of feedback vertex sets might also apply to non-planar digraphs excluding certain butterfly-minors; in fact, Theorem 32 easily extends to the class of so-called *mengerian digraphs* generalising the strongly planar digraphs ([Gue01]), with very similar properties. To conclude, we want to mention the similarity of the treated problems with the following open subcase of a Conjecture of Woodall.

Conjecture 63 ([Egr17]). In every planar digraph D of girth $g \geqslant 3$, there exists a packing of g disjoint feedback arc sets.

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