MULTICOLOURED HAMILTON CYCLES

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Abstract

The edges of the complete graph K_n are coloured so that no colour appears more than $\lceil cn \rceil$ times, where $c < 1/32$ is a constant. We show that if n is sufficiently large then there is a Hamiltonian cycle in which each edge is a different colour, thereby proving a 1986 conjecture of Hahn and Thomassen [7]. We prove a similar result for the complete digraph with $c < 1/64$. We also show, by essentially the same technique, that if $t \geq 3$, $c < (2t^2(1+t))^{-1}$, no colour appears more than $\lceil cn \rceil$ times and $t|n$ then the vertices can be partitioned into n/t $t-$sets $K_1, K_2, \ldots, K_{n/t}$ such that the colours of the $n(t-1)/2$ edges contained in the K_i's are distinct. The proof technique follows the lines of Erdős and Spencer's [2] modification of the Local Lemma [1].

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1 Introduction

Let the edges of the complete graph K_n be coloured so that no colour is used more than $k = k(n)$ times. We refer to this as a k-bounded colouring. We say that a subset of the edges of K_n is *multicoloured* if each edge is of a different colour. We say that the colouring is **H-good** if a multi-coloured Hamilton cycle exists i.e., one with a multi-coloured edge-set. Clearly the colouring is H-good if $k = 1$ and may not be if $k \geq n/2$, since then we may only use $n-1$ colours. The main question we address here then is that of how fast can we allow k to grow and still *guarantee* that a k-bounded colouring is H-good.

The problem is mentioned in Erdös, Nestril and Rödl [3]. There they mention it as an Erdös - Stein problem and show that k can be any constant. Hahn and Thomassen [7] were the next people to consider this problem and they showed that k could grow as fast as $n^{1/3}$ and conjectured that the growth rate of k could in fact be linear. In unpublished work Rödl and Winkler [9] in 1984 improved this to $n^{1/2}$. Frieze and Reed [5] showed that there is an absolute constant A such that if n is sufficiently large and k is at most $\lceil n/(A \ln n) \rceil$ then any k-bounded colouring is H-good.

In this paper we remove the $\log n$ factor and prove the conjecture of [7].

Theorem 1 *If n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < 1/32$ then any k-bounded colouring of K_n is H-good.*

We can extend this to the directed case.

Theorem 2 *If n is sufficiently large and k is at most $\lceil cn \rceil$, where $c < 1/64$ then any k-bounded colouring of the edges of the complete digraph DK_n is*

H-good.

As another wrinkle on this problem, we have

Theorem 3 *Suppose the edges of K_n are coloured so that the graphs induced by the edges of a single colour all have maximum degree at most cn, where $c < 1/32$. Then there exists a Hamilton cycle in which each vertex is incident with two edges of a distinct colours.*

We prove Theorem's 1, 2 and 3 as corollaries of the following.

Theorem 4 *Let Γ be a graph whose vertex set is the edge set of K_n. Suppose that Γ has maximum degree bounded above by cn, where $c < 1/32$. Then K_n contains a Hamilton cycle H whose edge set is an independent subset in Γ.*

We finally consider multi-coloured sets of cliques of size t. More precisely, assume that $t \geq 3, t|n$, and let $\mathcal{K} = K_1, K_2, \ldots, K_{n/t}$ be a partition of $[n]$ into subsets of size n/t. We say that $\mathcal{K}$ is multi-coloured if the set of $n(t-1)/2$ edges which have both endpoints in the same t-set is multi-coloured.

Theorem 5 *If n is sufficiently large and k is at most $\lceil cn \rceil$, $c < (2t^2(1+t))^{-1}$, then in any k-bounded colouring of the edges of K_n there is multi-coloured partition $\mathcal{K}$.*

2 Modification of the Lovàsz local lemma

Let $A_1, A_2, \ldots, A_N$ denote events in some probability space. Using $\bar{A}$ to denote the complement of an event A, we are as usual interested in showing that $\mathbf{Pr}(\bigcap_{i=1}^n \bar{A}_i) > 0$.

Suppose that for each i there is a partition of $[N]\setminus\{i\}$ into X_i and Y_i. In the usual version of the local lemma, A_i will be independent of the the events in X_i. Here all of the events will be interdependent, but one can still apply the methodology of the usual proof of the local lemma. We should point out here that this idea is not our own, it is already in Erdős and Spencer [2].

We consider one of the terms in the expression

$$\mathbf{Pr}\left(\bigcap_{i=1}^{N}\bar{A}_i\right)=\prod_{i=1}^{N}\mathbf{Pr}\left(\bar{A}_i\mid\bigcap_{j=1}^{i-1}\bar{A}_j\right). \tag{1}$$

We want to show that for $1\le i\le N$,

$$\mathbf{Pr}\left(\bar{A}_i\mid\bigcap_{j=1}^{i-1}\bar{A}_j\right)>0. \tag{2}$$

So, we try to prove by induction on $|S|$, $S\subseteq[N]$, that for $i\notin S$,

$$\mathbf{Pr}\left(A_i\mid\bigcap_{j\in S}\bar{A}_j\right)\le\alpha, \tag{3}$$

for some suitable choice of α.

Now,

$$\begin{aligned}
\mathbf{Pr}\left(A_i\mid\bigcap_{j\in S}\bar{A}_j\right) &= \frac{\mathbf{Pr}\left(A_i\cap\bigcap_{k\in S\cap Y_i}\bar{A}_k\mid\bigcap_{j\in S\cap X_i}\bar{A}_j\right)}{\mathbf{Pr}\left(\bigcap_{k\in S\cap Y_i}\bar{A}_k\mid\bigcap_{j\in S\cap X_i}\bar{A}_j\right)}\\
&\le \frac{\mathbf{Pr}\left(A_i\mid\bigcap_{j\in S\cap X_i}\bar{A}_j\right)}{\mathbf{Pr}\left(\bigcap_{k\in S\cap Y_i}\bar{A}_k\mid\bigcap_{j\in S\cap X_i}\bar{A}_j\right)}\\
&\le \frac{\mathbf{Pr}\left(A_i\mid\bigcap_{j\in S\cap X_i}\bar{A}_j\right)}{1-\sum_{k\in S\cap Y_i}\mathbf{Pr}\left(A_k\mid\bigcap_{j\in S\cap X_i}\bar{A}_j\right)} \qquad (4)
\end{aligned}$$

Let now

$$\beta = \max\{\mathbf{Pr}(A_i \mid \bigcap_{j\in T} \bar{A}_j) : i \in [N], T \subseteq X_i\}, \tag{5}$$

and

$$m = \max\{|Y_i| : i \in [N]\}. \tag{6}$$

We will have to prove that given m, β we can choose $0 \leq \alpha < 1$ such that

$$\alpha(1 - m\alpha) \geq \beta. \tag{7}$$

Assume that (7) holds. If $S \subseteq X_i$ then (5) and (7) will imply

$$\begin{aligned} \mathbf{Pr}\left(A_i \mid \bigcap_{j\in S} \bar{A}_j\right) &\leq \beta \\ &\leq \alpha. \end{aligned}$$

On the other hand if $S \not\subseteq X_i$ then we can apply the induction hypothesis to $\mathbf{Pr}(A_k \mid \bigcap_{j\in S\cap X_i} \bar{A}_j)$ in the numerator of (4) and obtain

$$\begin{aligned} \mathbf{Pr}\left(A_i \mid \bigcap_{j\in S} \bar{A}_j\right) &\leq \frac{\beta}{1 - m\alpha} \\ &\leq \alpha, \end{aligned}$$

by (7).

The base case of the induction, $S = \emptyset$, follows from considering $T = \emptyset$ in (5) and using $\beta \leq \alpha$.

So the proof of (2) rests on proving that (7) holds. This is what we do for Theorem 4. The proof of Theorem 5 is slightly different, in that we need to partition $A_1, A_2, \ldots, A_N$ into two types of event.

It may be useful to summarise the above discussion as a lemma in case it can be used in other circumstances.

Lemma 1 *Let $A_1, A_2, \ldots, A_N$ denote events in some probability space. Suppose that for each i there is a partition of $[N] \setminus \{i\}$ into X_i and Y_i. Let $m = \max\{|Y_i| : i \in [N]\}$ and $\beta = \max\{\mathbf{Pr}(A_i \mid \bigcap_{j \in T} \bar{A}_j) : i \in [N], T \subseteq X_i\}$. If there exists $0 \leq \alpha < 1$ such that $\alpha(1 - m\alpha) \geq \beta$ then $\mathbf{Pr}(\bigcap_{i=1}^n \bar{A}_i) > 0$.*

3 Hamilton Cycles

3.1 Proof of Theorems 1 and 2

We show here that Theorems 1 and 2 are corollaries of Theorem 4. Assume n is large and $k \leq cn$, and an arbitrary k-bounded colouring of K_n is given.

To prove Theorem 1 we define Γ as follows. Two edges e, f of K_n correspond to the endpoints of an edge of Γ if and only if they have the same colour. Thus a set of vertices of Γ is independent if and only it corresponds to a multicoloured set of edges of K_n. Clearly the maximum degree of Γ is at most $k - 1$ and so we can apply Theorem 4 to obtain Theorem 1.

To prove Theorem 2 we need a slight change in the definition of Γ. Two edges $e = \{e_0, e_1\}, f = \{f_0, f_1\}$ of K_n define an edge of Γ if and only if the colours of the four directed edges $(e_0, e_1), (e_1, e_0), (f_0, f_1), (f_1, f_0)$ are not all distinct. Thus a set S of vertices of Γ is independent if and only the set of edges obtained by taking, each $e = \{e_0, e_1\} \in S$ and replacing it by $(e_0, e_1), (e_1, e_0)$ (giving $2|S|$ directed edges) is multicoloured. Clearly the maximum degree of Γ is at most $2(k - 1)$ and so we can apply Theorem 4 to obtain a slight strengthening of Theorem 1 viz. there is a Hamilton cycle and its reversal spanning a multicoloured set of directed edges.

To prove Theorem 3 we let two edges e, f of K_n correspond to the endpoints of an edge of Γ if and only if they have the same colour and are incident with a common vertex.

3.2 Proof of Theorem 4

Let H be a Hamilton cycle chosen uniformly at random from the set of $(n-1)!/2$ Hamilton cycles of K_n. Let $\{(e_i, f_i) : 1 \leq i \leq N\}$ be an enumeration of the edges of Γ. Let

$$A_i = \{H : e_i, f_i \text{ are both edges of } H\}.$$

We will prove Theorem 1 by using the argument of Section 2 to show that $\mathbf{Pr}\left(\bigcap_{i=1}^N \bar{A}_i\right) > 0$. We use the notation of that section.

For $1 \leq i \leq N$ let

$$Y_i = \{j \neq i : (e_j \cup f_j) \cap (e_i \cup f_i) \neq \emptyset\}.$$

Thus $j \in Y_i$ if in K_n, one of e_j, f_j shares a vertex with one of e_i, f_i. Let $X_i = [N] \setminus (Y_i \cup \{i\})$. Clearly, $|Y_i| \leq 4cn^2$ and so

$$m \leq 4cn^2.$$

We will show that

$$\beta \leq \frac{2}{n^2 - 15n + 56} \tag{8}$$

and Theorem 1 follows on choosing

$$\alpha = \frac{1}{8c}\left(1 - \sqrt{1 - (32+\epsilon)c}\right) n^{-2}$$

and checking that (7) holds for $\epsilon > 0$ sufficiently small and n sufficiently large.

[Now $m\alpha \leq (1-\sqrt{1-(32+\epsilon)})/2$ and so $1-m\alpha \geq (1+\sqrt{1-(32+\epsilon)})/2$. Thus $\alpha(1-m\alpha) > (2+\epsilon/16)n^{-2}$.]

Equation (8) follows from the following Lemma:

Lemma 2 *Let e, f be edges of K_n and $X \subseteq E(K_n)$ be such that no edge in X shares an endpoint with either e or f. Then we can find, for each Hamilton cycle C containing both e and f and no edges of X, a set $S(C)$ of $(n-6)(n-9)/2$ Hamilton cycles containing neither e, f or any edge in X, in such a way that if $C \neq C'$ then $S(C) \cap S(C') = \emptyset$.*

Proof Let e_0, e_1 and f_0, f_1 be the endpoints of e and f respectively, chosen so that e_0 has the smallest index of e_0, e_1, f_0, f_1. Let

$$C = e_0, e_1 \longrightarrow f_0, f_1 \longrightarrow e_o.$$

[It is possible that $e_1 = f_0$ here.]

Consider two disjoint edges $x = (x_0, x_1), y = (y_0, y_1)$ of C sharing no endpoint with e or f. There are at least $(n-6)(n-9)/2$ choices for x, y. There are now two possibilities:

$$C = e_0, e_1 \longrightarrow x_0, x_1 \longrightarrow f_0, f_1 \longrightarrow y_0, y_1 \longrightarrow e_o.$$

or

$$C = e_0, e_1 \longrightarrow f_0, f_1 \longrightarrow x_0, x_1 \longrightarrow y_0, y_1 \longrightarrow e_o.$$

In the first case define:

$$\hat{C}_{x,y} = e_0, x_0 \longrightarrow e_1, y_0 \longrightarrow f_1, x_1 \longrightarrow f_0, y_1 \longrightarrow e_o.$$

In the second case define:

$$\hat{C}_{x,y} = e_0, x_1 \longrightarrow y_0, f_1 \longrightarrow x_0, e_1 \longrightarrow f_0, y_1 \longrightarrow e_o.$$

In both cases we delete the edges e, f, x, y from C and add edges that are incident with one of e_0, e_1, f_0, f_1, so that $\hat{C}_{x,y}$ does not contain an edge of X.

It is important to realise that in both cases the procedure is reversible in that C can be reconstructed from $\hat{C}_{x,y}$. We can recognise which case we are in from the relative order of the e's and f's and then identify the x's and y's from their positions.

Thus taking $S(C) = \{\hat{C}_{x,y} : x, y \text{ as above }\}$, we obtain $|S(C)| \geq (n-6)(n-9)/2$ and $S(C) \cap S(C') = \emptyset$ for $C \neq C'$. □

To prove (8) we apply Lemma 2 with $i \in [N]$, $\{e, f\} = \{e_i, f_i\}$ and $X \subseteq X_i$. Let $\mathcal{C}$ denote the set of Hamilton cycles containing e_i and f_i. Then

$$\begin{aligned}
\mathbf{Pr}(A_i \mid \bigcap_{j \in X} \bar{A}_j) &= \sum_{C \in \mathcal{C}} \mathbf{Pr}(H = C \mid \bigcap_{j \in X} \bar{A}_j) \\
&\leq \frac{2}{n^2 - 15n + 56} \sum_{C \in \mathcal{C}} \mathbf{Pr}(H \in \{C\} \cup S(C) \mid \bigcap_{j \in X} \bar{A}_j) \\
&\leq \frac{2}{n^2 - 15n + 56}.
\end{aligned}$$

This completes the proof of Theorem 4.

4 Partition into cliques

Assume n is large and $k \leq cn$, $c < (2t^2(1+t))^{-1}$, and an arbitrary k-bounded colouring of K_n is given. Let $\{(S_1, T_1), (S_2, T_2), \ldots, (S_N, T_N)\}$ be an enumeration of the pairs of t-subsets of $[n]$ such that for each $1 \leq i \leq N$ either (a) $S_i = T_i$ and S_i contains a pair of edges e, f of the same colour, or

(b) $S_i \cap T_i = \emptyset$ and there are edges e, f of the same colour, $e \subseteq S_i, f \subseteq T_i$. In either case we say that (S_i, T_i) contains e, f.

Let $I_a = \{i \in [N] : S_i = T_i\}$ and $I_b = [N] \setminus I_a$. Now let $\mathcal{K}$ be chosen randomly from the set of possible partitions and define the events

$$A_i = \{S_i \text{ and } T_i \text{ are both members of } \mathcal{K}\}.$$

Once again, we prove that $\mathbf{Pr}(\bigcap_{i=1}^n \bar{A}_i) > 0$.

We now define $Y_i = \{j \neq i :$ the following three conditions hold:

1. $\max\{|S_j \cap S_i|, |S_j \cap T_i| \geq t-1$.
2. $\max\{|T_j \cap S_i|, |T_j \cap T_i| \geq t-1$.
3. (S_j, T_j) contains a pair of identically coloured edges e, f which are *not* contained in $(S_i, T_i)\}$.

Naturally, $X_i = [N] \setminus (Y_i \cup \{i\})$.

We elaborate the argument of Section 2. We prove the existence of $0 < \alpha_a, \alpha_b < 1$ such that if $S \subseteq [N]$ and $i \in I_x \setminus S$, where $x = a$ or b, then

$$\mathbf{Pr}\left(A_i \mid \bigcap_{j \in S} \bar{A}_j\right) \leq \alpha_x.$$

To do this, we define, for $x = a$ or b,

$$\beta_x = \max\{\mathbf{Pr}(A_i \mid \bigcap_{j \in T} \bar{A}_j) : i \in I_x, T \subseteq X_i\}$$

and

$$m_x = \max\{|Y_i| : i \in I_x\}.$$

We will then, in analogy with (7), only need to show that for $x = a$ or b,

$$\alpha_x(1 - m_a\alpha_a - m_b\alpha_b) \geq \beta_x. \tag{9}$$

Consider first the case where $i \in I_a$. Then $j \in X_i$ implies $j \in I_a$.

$[|S_j \cap T_j| \geq |S_j \cap S_i \cap T_j \cap S_i| \geq |S_j \cap S_i| + |T_j \cap S_i| - |S_i| \geq 2(t-1) - t > 0.]$

Now

$$m_a \leq t\binom{t-1}{2}k. \tag{10}$$

Explanation: We choose S_j by (i) choosing $x \in S_i$, (ii) $e = \{x_1, x_2\} \subseteq S_i \setminus \{x\}$ and then $y \notin S_i$ such that the colour of e is the same as that of $\{x_1, y\}$ or $\{x_2, y\}$. We then take $S_j = (S_i \setminus \{x\}) \cup \{y\}$.

We argue next that

$$\beta_a \leq \frac{1}{t(n-t+1)}. \tag{11}$$

Explanation: Given $\mathcal{K} \in A_i \cap \bigcap_{j \in T} \bar{A}_j$, $T \subseteq X_i$, we can obtain $t(n-t)$ distinct partitions which are in $\bar{A}_i \cap \bigcap_{j \in T} \bar{A}_j$ as follows: Choose $x \in S_i$ and $y \in S$, where S is another t-set of $\mathcal{K}$. Replace S_i by $(S_i \cup \{y\}) \setminus \{x\}$ and S by $(S \cup \{x\}) \setminus \{y\}$ to obtain $\mathcal{K}'$. Note that given $\mathcal{K}'$ we can re-construct $\mathcal{K}$: x is the unique element of S_i which is in a set with elements not in S_i and $y \notin S_i$ is the unique such element which in a set with $t-1$ members of S_i.

Now let us consider the case $i \in I_b$. Now $j \in X_i$ implies that $j \in I_b$. Also,

$$m_b \leq t^2\binom{t-1}{2}(t-1)(n-t)k. \tag{12}$$

Explanation: We choose S_j, T_j by (i) choosing $x \in S_i, y \in T_i$, (ii) $e = \{x_1, x_2\} \subseteq S_i \setminus \{x\}$, (iii) $z \notin S_i$, (iv) $w \in T_i \setminus \{y\}$, (v) $v \in [n]$ such that the colour of $\{v, w\}$ is the same as that of e. Then take $S_j = (S_i \setminus \{x\}) \cup \{z\}$

and $T_j = (T_i \setminus \{y\}) \cup \{w\}$. There are some restrictions on the choices of x, y, z, v, w which are ignored for the purposes of getting an upper bound.

Finally,

$$\beta_b \leq \frac{1}{t^2(n-2t)(n-3t)+1}. \tag{13}$$

Explanation: Given $\mathcal{K} \in A_i \cap \bigcap_{j\in T} \bar{A}_j$, $T \subseteq X_i$, we can obtain $t^2(n-2t)(n-3t)$ distinct partitions which are in $\bar{A}_i \cap \bigcap_{j\in T} \bar{A}_j$ as follows: Choose $x \in S_i$ and $y \in T_i$. Then choose $x', y' \notin S_i \cup T_i$ in distinct subsets S, S' of $\mathcal{K}$. Replace S_i by $(S_i \cup \{x'\}) \setminus \{x\}$, T_i by $(T_i \cup \{y'\}) \setminus \{y\}$, S by $(S \cup \{x\}) \setminus \{x'\}$ and T by $(T \cup \{y\}) \setminus \{y'\}$. to obtain $\mathcal{K}'$. Note once again, that given $\mathcal{K}'$ we can re-construct $\mathcal{K}$.

With these values for $m_a, m_b, \beta_a, \beta_b$ we can enforce (9) by choosing

$$\alpha_a = 2/t \text{ and } \alpha_b = 2/t^2.$$

This completes the proof of Theorem 5.

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