The eigenvalues of the Laplacian for the homology of the Lie algebra corresponding to a poset

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Abstract

In this paper we study the spectral resolution of the Laplacian $\mathcal{L}$ of the Koszul complex of the Lie algebras corresponding to a certain class of posets.

Given a poset P on the set $\{1, 2, \ldots, n\}$, we define the nilpotent Lie algebra L_P to be the span of all elementary matrices $z_{x,y}$, such that x is less than y in P. In this paper, we make a decisive step toward calculating the Lie algebra homology of L_P in the case that the Hasse diagram of P is a rooted tree.

We show that the Laplacian $\mathcal{L}$ simplifies significantly when the Lie algebra corresponds to a poset whose Hasse diagram is a tree. The main result of this paper determines the spectral resolutions of three commuting linear operators whose sum is the Laplacian $\mathcal{L}$ of the Koszul complex of L_P in the case that the Hasse diagram is a rooted tree.

We show that these eigenvalues are integers, give a combinatorial indexing of these eigenvalues and describe the corresponding eigenspaces in representation-theoretic terms. The homology of L_P is represented by the nullspace of $\mathcal{L}$, so in future work, these results should allow for the homology to be effectively computed.

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1 Preliminaries

1.1 Definitions

A **partially ordered set** P (or **poset**, for short) is a set (which by abuse of notation we also call P), together with a binary relation denoted $\leq$ (or $\leq_P$ when there is a possibility of confusion), satisfying the following three axioms:

1. For all $x \in P$, $x \leq x$. **(reflexivity)**
2. If $x \leq y$ and $y \leq x$, then $x = y$. **(antisymmetry)**
3. If $x \leq y$ and $y \leq z$, then $x \leq z$. **(transitivity)**

A **chain** (or **totally ordered set** or **linearly ordered set**) is a poset in which any two elements are comparable. A subset C of a poset P is called a **chain** if C is a chain when regarded as a subposet of P.

Definition 1.1 *A poset P is* **linear** *if for any two comparable elements x, $y \in P$, the interval $[x, y]$ is a chain, i.e., if every interval has the structure of a chain.*

The **length** $l(C)$ of a finite chain C is defined by $l(C) = |C| - 1$.

1.2 The homology of a poset

The combinatorial approach to a homology theory for posets was developed by Rota [29], Farmer [8], Lakser [22], Mather [25], Crapo [5] and others (more references can be found in [33]). A systematic development of the relationship between the combinatorial and topological properties of posets was begun by K. Baclawski [1] and A. Björner [2] and continued by J. Walker [33].

Define the set $\mathcal{C}_r(P)$ to be the set of 0-1 chains of length r in the poset P. By abuse of notation we will use the same name for the complex vector space C_r or $\mathcal{C}_r(P)$, with basis the set of r-chains. The C_r's are called **chain spaces**. The map $\partial_r : C_r \to C_{r-1}$, called the **boundary map**, is defined by:

$$\partial_r(\hat{0} < x_1 < \ldots < x_r < \hat{1}) = \sum_{i=1}^{r} (-1)^{i-1} (\hat{0} < x_1 < \ldots < \widehat{x_i} < \ldots < x_r < \hat{1})$$

It is easy to check that:

Lemma 1

$$\partial_{r-1} \circ \partial_r = 0.$$

This allows us now to define the **homology of a poset** to be:

$$H_r(P) = Ker(\partial_r)/Im(\partial_{r+1})$$

Later in this work we will talk about an operator, called the Laplacian of a complex, for which we need to identify the transpose of the boundary map. We are in fact transposing the matrix of the boundary map with respect to the basis of r-chains. In this case - the case of the poset homology, the transpose of the boundary map is not so difficult to evaluate.

Lemma 2 *The transpose of the boundary operator (viewed as a linear map), is given by the following expression:*

$$\begin{aligned} &\partial^t(\hat{0} < x_1 < \ldots < x_r < \hat{1}) \\ &= \sum_{i=0}^{r} \sum_{x_i < y < x_{i+1}} (-1)^i (\hat{0} < x_1 < \ldots < x_i < y < x_{i+1} < \ldots < x_r < \hat{1}), \end{aligned}$$

where $x_0 = \hat{0}$ and $x_{r+1} = \hat{1}$.

1.3 Lie Algebras

In this section we will introduce some basic notions from the theory of Lie algebras, and the homology of Lie algebras.

We will always work over $\mathbb{C}$, the field of complex numbers.

Lie algebras arise "in nature" as vector spaces of linear transformations endowed with an operation which is in general neither commutative nor associative:

$$[x, y] = xy - yx.$$

It is possible to describe this kind of system abstractly in a few axioms.

Definition 1.2 *A vector space L over a field $\mathbb{C}$, with an operation $L \times L \to L$, denoted $(x, y) \to [x, y]$ and, called the* **bracket** *or* **commutator** *of x and y, is a* **Lie algebra** *over $\mathbb{C}$ if the following axioms are satisfied:*

(L1) *The bracket operation is bilinear.*

(L2) $[x, x] = 0$ *for all* $x \in L$.

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ (x, y, z \in L)$.

Axiom (L3) is called **Jacobi identity**. The axioms (L1) and (L2) imply (L2'): $[x, y] = -[y, x]$. In the field of complex numbers (L2') implies (L2).

1.4 Homology of a Lie algebra

Suppose L is a Lie algebra and A is a module over L. The space $\Gamma_q(L; A)$ of q-dimensional chains of the Lie algebra L with coefficients in A is defined as $A \otimes \Lambda^q L$. The boundary operator $\partial = \partial_q : \Gamma_q(L; A) \to \Gamma_{q-1}(L; A)$ acts in accordance with the formula

$$\begin{aligned} &\partial(a \otimes (x_1 \wedge \ldots \wedge x_q)) = \\ &= \sum_{1 \le s < t \le q} (-1)^{s+t-1} a \otimes ([x_s, x_t] \wedge x_1 \wedge \ldots \hat{x}_s \ldots \hat{x}_t \ldots \wedge x_q) \\ &+ \sum_{1 \le s \le q} (-1)^{s-1} x_s a \otimes (x_1 \wedge \ldots \hat{x}_s \ldots \wedge x_q) \end{aligned} \tag{1}$$

Lemma 3

$$\partial_{r-1}\partial_r = 0$$

The proof of this lemma is straightforward.

Let θ be the representation of L on $A \otimes \Lambda^q L$. If $y \in L$, we have:

$$\begin{aligned} &\theta(y)(a \otimes x_1 \wedge \ldots \wedge x_q) \\ &= (y \cdot a \otimes x_1 \wedge \ldots \wedge x_q) + \sum_i (a \otimes x_1 \wedge \ldots \wedge [y, x_i] \wedge \ldots \wedge x_q) \end{aligned}$$

It is easy to check:

Lemma 4 *For $y \in L$:*

$$\partial_q \circ \theta(y) = \theta(y) \circ \partial_q$$

The homology of the complex $\{\Gamma_q(L;A), \partial_q\}$ is referred to as **the homology of the Lie algebra L with coefficients in A** and denoted by $H_q(L;A)$; if A is the field of complex numbers viewed as a trivial L-module (as in our case), the second sum in the formula 1 vanishes. In this case the notations $\Gamma_q(L;A)$ and $H_q(L;A)$ are abbreviated to $\Gamma_q(L)$ and $H_q(L)$.

1.5 The Laplacian operator

Suppose that $\{\Gamma_r(L), \partial_r\}$ is a finite dimensional complex. We will first define an orthogonal inner product $\langle \cdot, \cdot \rangle$ on the product $\oplus \Gamma_r$, such that $\langle \Gamma_r, \Gamma_s \rangle = 0$ whenever $r \neq s$. We will restrict our attention to the subspaces of the nilpotent Lie algebra $T_n(\mathbb{C})$ of all strictly upper triangular matrices over the complex numbers, with standard basis $\{z_{i,j} : 1 \leq i < j \leq n\}$, so we can define this product naturally:

Definition 1.3 *Let L be a Lie algebra, $L \subset T_n(\mathbb{C})$. Define an inner product for standard basis elements $v, w \in L$ by:*

$$\langle v, w \rangle = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \\ 0 & \text{if } v \text{ and } w \text{ have different exterior degrees} \end{cases}$$

Extend this to the exterior algebra, i.e., to the complexes mentioned above.

Definition 1.4 *Suppose that $v = v_1 \wedge \cdots \wedge v_k$ and $w = w_1 \wedge \cdots \wedge w_k$. Then define the inner product:*

$$\langle v, w \rangle = det(\langle v_i, w_j \rangle)_{1 \leq i,j \leq k}$$

Note that this can be written also as

$$\langle v, w \rangle = \sum_{\sigma \in S_n} sgn(\sigma) \prod_i \langle v_i, w_{\sigma(i)} \rangle = \begin{cases} sgn(\sigma) & \text{iff } v_i = w_{\sigma(i)} \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

In other words, the product of two pure wedges of basis elements is nonzero if and only if two pure wedges differ only in the order of the elements, and in that case, the product is just the sign of the permutation that changes one into another.

Define δ_r mapping Γ_r into Γ_{r+1} by

$$\langle \delta_r v, w \rangle = \langle v, \partial_{r+1} w \rangle$$

over all $v \in \Gamma_r$, and all $w \in \Gamma_{r+1}$. It is enough to calculate δ on pure wedges (as in our definitions), since the inner product and δ are both linear functions.

Lemma 5 *The map δ is given by*

$$\begin{aligned}&\delta_r(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \ldots \wedge z_{x_r,y_r})\\ &= \sum_{s=1}^{r}(-1)^{s-1} \sum_{x_s<l<y_s} z_{x_1,y_1} \wedge \ldots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \ldots \wedge z_{x_r,y_r}\end{aligned}$$

Note: It is easy to check that $\delta_{r+1}\delta_r = 0$, thus δ_* defines a coboundary operator, and so we can define the cohomology to be

$$H^r(L) = Ker(\delta_r)/Im(\delta_{r-1})$$

Proof: But to prove that, it is enough to show that the coefficient of the pure wedge $z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \ldots \wedge z_{x_r,y_r}$ in $\partial(z_{x_1,y_1} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r})$ is $(-1)^{s-1}$ for any $l \in (x_s, y_s)$, i.e.,

$$\begin{aligned}&\partial(z_{x_1,y_1} \wedge \ldots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \ldots \wedge z_{x_r,y_r})\\ &= \ldots + (-1)^{s-1}(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \ldots \wedge z_{x_r,y_r}) + \ldots\end{aligned}$$

and this is not difficult by the definition of ∂.

Note that we can change the order of the elements in the pure wedges, and obtain a slightly different form for δ:

$$\begin{aligned}&\delta_r(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \ldots \wedge z_{x_r,y_r})\\ &= \sum_{s=1}^{r}(-1)^{s-1} \sum_{x_s<l<y_s} z_{x_1,y_1} \wedge \ldots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \ldots \wedge z_{x_r,y_r}\\ &= \sum_{m} \sum_{x_m<l<y_m} (z_{x_m,l} \wedge z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k})\end{aligned}$$

This is the form for the $\delta = \partial^t$ we will use.

Definition 1.5 *Define the* **Laplacian operator** $L_r : \Gamma_r \to \Gamma_r$ *by*

$$L_r = \delta_{r-1}\partial_r + \partial_{r+1}\delta_r$$

Theorem 6 (Kostant, [19]) *Let $B = \{\beta_1, \ldots, \beta_d\}$ be a basis for $Ker(L_r)$. Then B is simultaneously a complete set of representatives of $H^r(L)$ and $H_r(L)$. In particular* $\dim(H^r(L)) = \dim(H_r(L)) = \dim(Ker(L_r))$.

Sometimes, the Laplacian L_r will turn out to be very simple. In these cases, Theorem 6 is a very efficient method for evaluating the homology and cohomology of a Lie algebra. One famous result obtained in this way is given by Kostant [19].

1.6 Kostant's Theorem

We need some preliminary definitions. Suppose $\mathcal{G}$ is a semisimple Lie algebra, with the root system R, whose basis is Δ. Thus $\mathcal{G} = H \oplus (\oplus_{\alpha\in R}\langle z_\alpha\rangle)$, where H is the torus. Suppose that $S \subset \Delta$, and let R_S be the set of roots in the $\mathbb{Z}$ (integer) module spanned by elements of S. Define $\mathcal{G}_S$ to be $\mathcal{G}_{\mathcal{S}} = H \oplus \langle z_\alpha : \alpha \in R_S\rangle$. Define a $\mathcal{G}_S$ module N_S to be $N_S = \langle z_\alpha : \alpha \in R^+ \setminus R_S^+\rangle$.

We will state a couple of facts without proof:

- N_S is a nilpotent subalgebra of $\mathcal{G}$.
- Let W be a $\mathcal{G}$-module. Then W is also a N_S-module and a $\mathcal{G}_{\mathcal{S}}$-module.
- Thus we can compute $H(N_S; W^\mu)$ as $\mathcal{G}_{\mathcal{S}}$-module, where W^μ is an irreducible $\mathcal{G}$-module. Kostant used the Laplacian operator to prove the following theorem:

Theorem 7 (Kostant, Theorem 5.7,[19]) *Let λ be a dominant weight for $\mathcal{G}$, and let μ be a minimal weight for $\mathcal{G}_{\mathcal{S}}$. Let V be a $\mathcal{G}_{\mathcal{S}}$-invariant subspace of $W^\lambda \otimes \bigwedge^r N_S$ isomorphic to the $\mathcal{G}_{\mathcal{S}}$-irreducible (indexed by μ) with minimal weight μ.*

- *The Laplacian $L = \delta\partial + \partial\delta$ preserves V.*
- *Then, $L|_V$ is a scalar, and the scalar is given by*

$$\frac{1}{2}(|\rho+\lambda|^2 - |\rho-\mu|^2)$$

where ρ is half of the sum of the positive roots of $\mathcal{G}$.

1.7 The Lie Algebra corresponding to a Poset

Definition 1.6 *A* **standard labeling** *of the poset P is a total ordering of the elements of P such that whenever $x <_P y$, x precedes y in that total ordering.*

Since P is a partial order, i.e. transitive , there always is such labeling. Fix a standard labeling of the poset P.

We can define a Lie algebra L_P corresponding to the poset P in the following way. First, for every relation $x <_P y$ in the poset P, i.e., for every two elements $x, y \in P$ such that $x <_P y$ we can define the matrix $z_{x,y}$ having all entries equal to zero, except for exactly one entry equal to 1, namely the entry at the position x, y in the standard labeling of the poset P.

All matrices $z_{x,y}$ are strictly upper triangular because of our labeling. So L_P is a subalgebra of T_n. The Lie algebras L_P obtained from distinct labellings are isomorphic – the labeling only specifies embedding of L_P in the $n \times n$ matrices.

2 The Formula for Laplacian of a Linear Poset

In this section we will present a significant simplification of the Lie algebra Laplacian in the case of linear posets. That will allow us to prove our main result on the eigenvalues of those Laplacians.

2.1 Simplification

Recall the Lie algebra boundary map:

$$\begin{aligned}
&\partial(z_{x_1,y_1} \wedge \ldots \wedge z_{x_k,y_k}) \\
&\quad = \sum_{i<j} (-1)^{i+j-1} [z_{x_i,y_i}, z_{x_j,y_j}] \wedge z_{x_1,y_1} \wedge \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \ldots \wedge \widehat{z_{x_j,y_j}} \wedge \ldots \wedge z_{x_k,y_k}
\end{aligned}$$

The transpose, ∂^t, is given by the following formula:

$$\begin{aligned}
&\partial^t_r(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \ldots \wedge z_{x_r,y_r}) \\
&\quad = \sum_{s=1}^{r} (-1)^{s-1} \sum_{x_s<l<y_s} z_{x_1,y_1} \wedge \ldots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \ldots \wedge z_{x_r,y_r} \\
&\quad = \sum_{m} \sum_{x_m<l<y_m} (z_{x_m,l} \wedge z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k})
\end{aligned}$$

To compute the action of L on a basis vector $z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k}$ of $\Gamma_k(L_P)$ we begin with the action of $\partial\partial^t$. We have,

$$\begin{aligned}
&\partial\partial^t(z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k}) \\
&\quad = \sum_{m} \sum_{x_m<l<y_m} \partial(z_{x_m,l} \wedge z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k}) \\
&\quad = \sum_{i<j} \sum_{m\neq i,j} \sum_{x_m<l<y_m} (-1)^{i+1+j} ([z_{x_i,y_i}, z_{x_j,y_j}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
&\qquad \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) \\
&\quad + \sum_{m} \sum_{j\neq m} \sum_{x_m<l<y_m} (-1)^{1+j+1-1} ([z_{x_m,l}, z_{x_j,y_j}] \wedge z_{x_1,y_1} \wedge \ldots \\
&\qquad \ldots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k}) \\
&\quad + \sum_{i<m} \sum_{x_m<l<y_m} (-1)^{i+1+m+1-1} ([z_{x_i,y_i}, z_{l,y_m}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
&\qquad \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge \widehat{z_{x_m,y_m}} \wedge \cdots \wedge z_{x_k,y_k}) \\
&\quad + \sum_{m<j} \sum_{x_m<l<y_m} (-1)^{m+1+j+1-1} ([z_{l,y_m}, z_{x_j,y_j}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
&\qquad \ldots \wedge \widehat{z_{x_m,y_m}} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) \\
&\quad + \sum_{m=1}^{k} |(x_m, y_m)| (z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k})
\end{aligned}$$

which is equal to:

$$
\begin{aligned}
= & \sum_{i<j} \sum_{m \neq i,j} \sum_{x_m<l<y_m} (-1)^{i+j-1}([z_{x_i,y_i}, z_{x_j,y_j}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
& \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{i<m} \sum_{x_m<l<y_m} (-1)^{i+1}([z_{x_m,l}, z_{x_i,y_i}] \wedge z_{x_1,y_1} \wedge \ldots \\
& \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{m<j} \sum_{x_m<l<y_m} (-1)^{j+1}([z_{x_m,l}, z_{x_j,y_j}] \wedge z_{x_1,y_1} \wedge \ldots \\
& \ldots \wedge z_{l,y_m} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{i<m} \sum_{x_m<l<y_m} (-1)^{i+m+1}([z_{x_i,y_i}, z_{l,y_m}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
& \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge \widehat{z_{x_m,y_m}} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{m<j} \sum_{x_m<l<y_m} (-1)^{m+j+1}([z_{l,y_m}, z_{x_j,y_j}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
& \ldots \wedge \widehat{z_{x_m,y_m}} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{m=1}^{k} |(x_m, y_m)| (z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k})
\end{aligned}
$$

Now use the definition of bracket in this Lie algebra:

$$
[z_{x_i,y_i}, z_{x_j,y_j}] = \delta_{y_i,x_j} z_{x_i,y_j} - \delta_{x_i,y_j} z_{x_j,y_i}
$$

and we have the following:

$$
\begin{aligned}
& \partial\partial^t(z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k}) \\
= & \sum_{i<j} \sum_{m \neq i,j} \sum_{x_m<l<y_m} (-1)^{i+j-1}([z_{x_i,y_i}, z_{x_j,y_j}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \cdots \wedge \widehat{z_{x_i,y_i}} \wedge \ldots \\
& \ldots \wedge z_{l,y_m} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{i<m} \sum_{x_m<l<y_m} \delta_{l,x_i}(z_{x_1,y_1} \wedge \cdots \wedge z_{x_m,y_i} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k}) \\
- & \delta_{x_m,y_i}(z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,l} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{m<j} \sum_{x_m<l<y_m} \delta_{l,x_j}(z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_m,y_j} \wedge \cdots \wedge z_{x_k,y_k}) \\
- & \delta_{x_m,y_j}(z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge z_{x_j,l} \wedge \cdots \wedge z_{x_k,y_k}) \\
+ & \sum_{i<m} \sum_{x_m<l<y_m} \delta_{l,y_i}(z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,y_j} \wedge \cdots \wedge z_{x_m,l} \wedge \cdots \wedge z_{x_k,y_k}) \\
- & \delta_{x_i,y_m}(z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_i} \wedge \cdots \wedge z_{x_m,l} \wedge \cdots \wedge z_{x_k,y_k})
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{m<j} \sum_{x_m<l<y_m} \delta_{l,y_j} (z_{x_1,y_1} \wedge \cdots \wedge z_{x_m,l} \wedge \cdots \wedge z_{x_j,y_m} \wedge \cdots \wedge z_{x_k,y_k}) \\
&- \delta_{x_j,y_m} (z_{x_1,y_1} \wedge \cdots \wedge z_{x_m,l} \wedge \cdots \wedge z_{l,y_j} \wedge \cdots \wedge z_{x_k,y_k}) \\
&+ \sum_{m=1}^{k} |(x_m, y_m)| (z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k})
\end{aligned}
$$

Note that every sum over $x_m < l < y_m$ which has an occurrence of $\delta_{l,*}$ has only one summand if $*$ really is between x_m and y_m, and is zero otherwise. We will use the symbol χ for denoting the truth of some statement, i.e.,

$$
\chi(*) = \begin{cases} 1, & \text{if } * \text{ is true} \\ 0, & \text{if } * \text{ is false} \end{cases}
$$

We label some of the resulting sums:

$$
\begin{aligned}
&\partial\partial^t (z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k}) && (2)\\
&= \sum_{i<j} \sum_{m \neq i,j} \sum_{x_m<l<y_m} (-1)^{i+j-1} ([z_{x_i,y_i}, z_{x_j,y_j}] \wedge z_{x_m,l} \wedge z_{x_1,y_1} \wedge \ldots \\
&\quad \ldots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge z_{l,y_m} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}) && (3)\\
&- \sum_{i<j} \chi(x_j < x_i < y_j) (z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,y_j} \wedge \cdots \wedge z_{x_j,y_i} \wedge \cdots \wedge z_{x_k,y_k}) \\
&- \sum_{i<j} \sum_{x_j<l<y_j} \delta_{x_j,y_i} (z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,l} \wedge \cdots \wedge z_{l,y_j} \wedge \cdots \wedge z_{x_k,y_k}) && (4)\\
&- \sum_{i<j} \chi(x_i < x_j < y_i) (z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,y_j} \wedge \cdots \wedge z_{x_j,y_i} \wedge \cdots \wedge z_{x_k,y_k}) \\
&- \sum_{i<j} \sum_{x_i<l<y_i} \delta_{x_i,y_j} (z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_i} \wedge \cdots \wedge z_{x_j,l} \wedge \cdots \wedge z_{x_k,y_k}) && (5)\\
&+ \sum_{i<j} \chi(x_j < y_i < y_j) (z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,y_j} \wedge \cdots \wedge z_{x_j,y_i} \wedge \cdots \wedge z_{x_k,y_k}) \\
&- \sum_{i<j} \sum_{x_j<l<y_j} \delta_{x_i,y_j} (z_{x_1,y_1} \wedge \cdots \wedge z_{l,y_i} \wedge \cdots \wedge z_{x_j,l} \wedge \cdots \wedge z_{x_k,y_k}) && (6)\\
&- \sum_{i<j} \sum_{x_i<l<y_i} \delta_{x_j,y_i} (z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,l} \wedge \cdots \wedge z_{l,y_j} \wedge \cdots \wedge z_{x_k,y_k}) && (7)\\
&+ \sum_{i<j} \chi(x_i < y_j < y_i) (z_{x_1,y_1} \wedge \cdots \wedge z_{x_i,y_j} \wedge \cdots \wedge z_{x_j,y_i} \wedge \cdots \wedge z_{x_k,y_k}) \\
&+ \sum_{m=1}^{k} |(x_m, y_m)| (z_{x_1,y_1} \wedge \cdots \wedge z_{x_k,y_k})
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
&\partial^t\partial(z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k})\\
&= \sum_{i<j}(-1)^{i+j-1}\partial^t([z_{x_i,y_i},z_{x_j,y_j}]\wedge z_{x_1,y_1}\ldots\wedge\widehat{z_{x_i,y_i}}\wedge\cdots\wedge\widehat{z_{x_j,y_j}}\wedge\cdots\wedge z_{x_k,y_k})\\
&= \sum_{i<j}\sum_{m\neq i,j}\sum_{x_m<l<y_m}(-1)^{i+j-1}(z_{x_m,l}\wedge[z_{x_i,y_i},z_{x_j,y_j}]\wedge z_{x_1,y_1}\wedge\ldots\\
&\qquad \ldots\wedge\widehat{z_{x_i,y_i}}\wedge\cdots\wedge z_{l,y_m}\wedge\cdots\wedge\widehat{z_{x_j,y_j}}\wedge\cdots\wedge z_{x_k,y_k})\\
&+ \sum_{i<j}\sum_{x_m<l<y_m}(-1)^{i+j-1}\delta_{x_j,y_i}(z_{x_i,l}\wedge z_{l,y_j}\wedge z_{x_1,y_1}\wedge\ldots\\
&\qquad \ldots\wedge\widehat{z_{x_i,y_i}}\wedge\cdots\wedge\widehat{z_{x_j,y_j}}\wedge\cdots\wedge z_{x_k,y_k})\\
&- \sum_{i<j}\sum_{x_m<l<y_m}(-1)^{i+j-1}\delta_{x_i,y_j}(z_{x_j,l}\wedge z_{l,y_i}\wedge z_{x_1,y_1}\wedge\ldots\\
&\qquad \ldots\wedge\widehat{z_{x_i,y_i}}\wedge\cdots\wedge\widehat{z_{x_j,y_j}}\wedge\cdots\wedge z_{x_k,y_k})
\end{aligned}
$$

Now use the fact that we are dealing with a linear poset. This implies that for every interval (x_m,y_m) and every l , $x_m<l<y_m$ we have

$$(x_m,y_m)=(x_m,l)\cup\{l\}\cup(l,y_m)$$

Hence

$$
\begin{aligned}
&\partial^t\partial(z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k}) && (8)\\
&= \sum_{i<j}\sum_{m\neq i,j}\sum_{x_m<l<y_m}(-1)^{i+j-1}(z_{x_m,l}\wedge[z_{x_i,y_i},z_{x_j,y_j}]\wedge z_{x_1,y_1}\wedge\ldots\\
&\qquad \ldots\wedge\widehat{z_{x_i,y_i}}\wedge\cdots\wedge z_{l,y_m}\wedge\cdots\wedge\widehat{z_{x_j,y_j}}\wedge\cdots\wedge z_{x_k,y_k}) && (9)\\
&+ \sum_{i<j}\sum_{x_i<l<y_i}\delta_{x_j,y_i}(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,l}\wedge\cdots\wedge z_{l,y_j}\wedge\cdots\wedge z_{x_k,y_k}) && (10)\\
&+ \sum_{i<j}\sum_{l=x_j=y_i}\delta_{x_j,y_i}(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,l}\wedge\cdots\wedge z_{l,y_j}\wedge\cdots\wedge z_{x_k,y_k})\\
&+ \sum_{i<j}\sum_{x_j<l<y_j}\delta_{x_j,y_i}(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,l}\wedge\cdots\wedge z_{l,y_j}\wedge\cdots\wedge z_{x_k,y_k}) && (11)\\
&+ \sum_{i<j}\sum_{x_j<l<y_j}\delta_{x_i,y_j}(z_{x_1,y_1}\wedge\cdots\wedge z_{l,y_i}\wedge\cdots\wedge z_{x_j,l}\wedge\cdots\wedge z_{x_k,y_k}) && (12)\\
&+ \sum_{i<j}\sum_{l=x_i=y_j}\delta_{x_i,y_j}(z_{x_1,y_1}\wedge\cdots\wedge z_{l,y_i}\wedge\cdots\wedge z_{x_j,l}\wedge\cdots\wedge z_{x_k,y_k})\\
&+ \sum_{i<j}\sum_{x_i<l<y_i}\delta_{x_i,y_j}(z_{x_1,y_1}\wedge\cdots\wedge z_{l,y_i}\wedge\cdots\wedge z_{x_j,l}\wedge\cdots\wedge z_{x_k,y_k}) && (13)
\end{aligned}
$$

Then we have :

$$
\begin{aligned}
(9)+(3) &= 0\\
(10)+(7) &= 0\\
(11)+(4) &= 0\\
(12)+(6) &= 0\\
(13)+(5) &= 0
\end{aligned}
$$

After these cancellations we obtain the following expression for the action of the Laplacian L:

$$
\begin{aligned}
&L(z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k}) = (\partial\partial^t+\partial^t\partial)(z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k})\\
&= \sum_{m=1}^{k}|(x_m,y_m)|(z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k})\\
&+ \sum_{i<j}(\delta_{x_i,y_j}+\delta_{x_j,y_i})(z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k})\\
&+ \sum_{i<j}\chi(x_i<y_j<y_i)(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,y_j}\wedge\cdots\wedge z_{x_j,y_i}\wedge\cdots\wedge z_{x_k,y_k})\\
&+ \sum_{i<j}\chi(x_j<y_i<y_j)(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,y_j}\wedge\cdots\wedge z_{x_j,y_i}\wedge\cdots\wedge z_{x_k,y_k})\\
&- \sum_{i<j}\chi(x_j<x_i<y_j)(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,y_j}\wedge\cdots\wedge z_{x_j,y_i}\wedge\cdots\wedge z_{x_k,y_k})\\
&- \sum_{i<j}\chi(x_i<x_j<y_i)(z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,y_j}\wedge\cdots\wedge z_{x_j,y_i}\wedge\cdots\wedge z_{x_k,y_k})
\end{aligned}
$$

2.2 The Formula

To further simplify our expressions we will introduce some notation. Define

$$
\begin{aligned}
\zeta &= z_{x_1,y_1}\wedge\cdots\wedge z_{x_k,y_k}\\
\zeta_{i,j} &= z_{x_1,y_1}\wedge\cdots\wedge z_{x_i,y_j}\wedge\cdots\wedge z_{x_j,y_i}\wedge\cdots\wedge z_{x_k,y_k}\\
w(\zeta) &= \sum_{m=1}^{k}|(x_m,y_m)|\\
\Delta(\zeta) &= \sum_{i<j}(\delta_{x_i,y_j}+\delta_{x_j,y_i}) = \sum_{i,j}\delta_{x_i,y_j}
\end{aligned}
$$

Thus, we can reformulate the calculations from the previous section into:

Theorem 8 (The Formula) *Let P be a linear poset and let L_P be the corresponding Lie algebra. The action of the Laplacian L on an element*

$$\zeta = z_{x_1,y_1}\wedge z_{x_2,y_2}\wedge\cdots\wedge z_{x_k,y_k}$$

is given by the following formula:

$$
\begin{aligned}
L(\zeta) &= (w(\zeta)+\Delta(\zeta))\zeta \\
&+ \sum_{i<j}(\chi(x_i<y_j<y_i)+\chi(x_j<y_i<y_j)-\chi(x_j<x_i<y_j)-\chi(x_i<x_j<y_i))\zeta_{i,j}
\end{aligned}
$$

Note that $\zeta_{i,j}$ is obtained from ζ by transposing a comparable pair of y's or a comparable pair of x's.

3 Linear poset with a $\hat{0}$

Suppose now that the poset P has a $\hat{0}$, the minimum element. That is the assumption under which we will work in the future. In that case, we can further simplify our notation:

Lemma 9

$$
\begin{aligned}
L(\zeta) &= (w(\zeta)+\Delta(\zeta))\zeta \\
&+ \sum_{i<j}(\chi(x_i<y_j<y_i)+\chi(x_j<y_i<y_j)-\chi(x_j<x_i<y_j)-\chi(x_i<x_j<y_i))\zeta_{i,j} \\
&= (w(\zeta)+\Delta(\zeta))\zeta+\sum_{y_i<_P y_j}\zeta_{i,j}-\sum_{x_i<_P x_j}\zeta_{i,j}
\end{aligned}
$$

Proof: We need to prove that we can write

$$\chi(y_i<y_j)+\chi(y_j<y_i)-\chi(x_i<x_j)-\chi(x_j<x_i)$$

instead of

$$\chi(x_i<y_j<y_i)+\chi(x_j<y_i<y_j)-\chi(x_j<x_i<y_j)-\chi(x_i<x_j<y_i)$$

in the expression for the Laplacian above.

Let y_i and y_j be two comparable distinct y's. Without loss of generality, assume that $y_i<y_j$. Thus $x_i<y_i<y_j$. The existence of $\hat{0}$ and linearity of the poset implies that the interval $[\hat{0},y_j]$ must be a chain, and since $x_i,x_j\in[\hat{0},y_j]$, x_i and x_j must be comparable. There are several possibilities:

1. $x_j<x_i<y_i<y_j$
2. $x_i<x_j<y_i<y_j$
3. $x_i<y_i<x_j<y_j$

In all three possibilities,

$$\chi(x_i<y_j<y_i)+\chi(x_j<y_i<y_j)-\chi(x_j<x_i<y_j)-\chi(x_i<x_j<y_i)=0,$$

and at the same time

$$\chi(y_i<y_j)+\chi(y_j<y_i)-\chi(x_i<x_j)-\chi(x_j<x_i)=0.$$

On the other hand, if y_i and y_j are incomparable, then we have one of:

1. $x_j < x_i < y_i, y_j$
2. $x_i < x_j < y_i, y_j$
3. $x_i < y_i, x_j < y_j$, x_i and x_j are incomparable.

Now in the first two cases

$$\chi(x_i < y_j < y_i) + \chi(x_j < y_i < y_j) - \chi(x_j < x_i < y_j) - \chi(x_i < x_j < y_i) = -1,$$

with

$$\chi(y_i < y_j) + \chi(y_j < y_i) - \chi(x_i < x_j) - \chi(x_j < x_i) = -1$$

too. In the last remaining case both expressions are zero.

Hence, the expression for the Laplacian above can be rewritten in the following form:

$$L(\zeta) = (w(\zeta) + \Delta(\zeta))\zeta + \sum_{y_i <_P y_j} \zeta_{i,j} - \sum_{x_i <_P x_j} \zeta_{i,j}. \blacksquare$$

In other words, the meaning of the theorem above is that the Laplacian only transposes comparable labels of the element $z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_k,y_k}$, without introducing any new indices. This is the key observation for next section.

Lemma 10 *Let $\zeta = z_{x_1,y_1} \wedge \cdots \wedge z_{x_n,y_n}$, and let $\zeta_\sigma = z_{x_1,y_{\sigma(1)}} \wedge z_{x_2,y_{\sigma(2)}} \wedge \cdots \wedge z_{x_n,y_{\sigma(n)}}$. If $\zeta_\sigma \neq 0$, i.e., if $x_i <_P y_{\sigma(i)}$ for all i, then*

1. *w does not depend on σ, i.e.,*
$$w(\zeta) = w(\zeta_\sigma)$$
2. *Δ does not depend on σ, i.e.,*
$$\Delta(\zeta) = \Delta(\zeta_\sigma)$$

This lemma actually proves that w and Δ are dependent only on the choice of the (multi–)sets $X = \{x_1, x_2, \ldots, x_k\}, Y = \{y_1, y_2, \ldots, y_k\}$ (and a poset P), and not on the specific pure wedge constructed from those sets.

Proof: First we will check the claim for w.

$$w(\zeta) \;=\; \sum_i |(x_i, y_i)| = \sum_i ht(y_i) - ht(x_i) - 1,$$

where the $ht(v)$ is the size of the interval $[\hat{0}, v]$. The sum on the right does not depend on σ, so we can write $w(X, Y)$ instead of $w(\zeta)$.

Now we will check the claim for Δ.

$$\begin{aligned} \Delta(\zeta) &= \sum_{i<j} (\delta_{x_i,y_j} + \delta_{x_j,y_i}) \\ &= \sum_i (\text{ multiplicity of } x_i \text{ in the set } Y) \\ &= \sum_j (\text{ multiplicity of } y_j \text{ in the set } X) \\ &= \sum_{i,j} \delta_{x_i,y_j} \end{aligned}$$

which also does not depend on σ. Thus we can write $\Delta(X, Y)$ instead of $\Delta(\zeta)$ too. ■

We will use both notations, depending whether we want to stress ζ or the sets (X, Y). Note that while Δ is completely determined by the sets (X, Y), w also depends on the poset P globally, i.e., it counts the sizes of intervals (x_i, y_i) not relative to the sets X and Y, but with respect to the whole poset P.

The simplicity of this formula is in the way the elements to which we are restricting the Laplacian, are obtained one from another, by simply transposing the labels. In general, this example shows that the Laplacian L can be broken down into diagonal blocks, which are generated by a pure wedge ζ, and all pure wedges obtained by permutations of the labels of ζ. Furthermore, since $a \wedge b = -b \wedge a$, we can always keep the x-labels in order, i.e, we will always put the element $z_{x_i,*}$ at the i^{th} position of the pure wedge.

4 The eigenvalues of the Laplacian

Let $\zeta = z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n}$ be an element of the exterior algebra of the Lie algebra of P. In the last section we saw that the Laplacian acts on pure wedges of Lie algebra elements $z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n}$ by summing the action of switching pairs of comparable x's, and pairs of comparable y's among themselves (plus a scalar).

That fact gives us the opportunity to divide our Laplacian into diagonal blocks where each block corresponds to all possible permutations of the x's and y's for a fixed choice of the element $z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n}$, i.e., for the fixed choice of the multisets $X = \{x_1, x_2, \ldots, x_n\}$, and $Y = \{y_1, y_2, \ldots, y_n\}$. In other words each block represents the "action" of the Laplacian on the subspace of the n^{th} exterior power of our Lie algebra spanned by the elements $\{z_{x_1,y_{\sigma(1)}} \wedge z_{x_2,y_{\sigma(2)}} \wedge \cdots \wedge z_{x_n,y_{\sigma(n)}} : \sigma \in S_n\}$. Here the element $z_{x_1,y_{\sigma(1)}} \wedge z_{x_2,y_{\sigma(2)}} \wedge \cdots \wedge z_{x_n,y_{\sigma(n)}}$ is defined if and only if $x_i <_P y_{\sigma(i)}$ for every $i = 1, 2, \ldots, n$. Thus each block is of size $n!$, if all the elements are defined, or less, if some of the elements are not defined which is the case in general. The size of the block depends on the structure of the poset, and in particular, it depends on the relations in the subposet of P spanned by the sets X and Y. More formally :

Definition 4.1 *The* **L-block** V **spanned by the (multi)-sets** $(X, Y)_P$*, subsets of a poset* P*, is the vector space with basis*

$$\{z_{x_1,y_{\sigma(1)}} \wedge z_{x_2,y_{\sigma(2)}} \wedge \cdots \wedge z_{x_n,y_{\sigma(n)}} : \sigma \in S_n\}$$

where $n = |X| = |Y|$, σ *is a permutation in* S_n*, and the element*

$$z_{x_1,y_{\sigma(1)}} \wedge z_{x_2,y_{\sigma(2)}} \wedge \cdots \wedge z_{x_n,y_{\sigma(n)}}$$

is zero unless $x_i <_P y_{\sigma(i)}$ *for all* $i = 1, \ldots, n$.

If we want to stress the dependence of the L-block V *of the sets* X *and* Y *and the poset* P*, we write* $V(X, Y)_P$.

The sets X and Y may be multisets since some of the x's or y's might appear more than once as a label. In that case the sizes $|X|$ and $|Y|$ are counting multiplicities as well.

Using this division of the chain space into L-blocks, we can use the results of the previous section, and state the theorem:

Theorem 11 *Let L_P be the Lie algebra corresponding to a linear poset P, and let $C_n(L_P)$ be the n^{th} chain space. Then*

$$C_n(L_P) = \bigoplus_{(X,Y)} V(X,Y)_P$$

where the direct sum is over all possible choices of (multi)-sets X and Y of equal cardinality, and each summand $V(X,Y)_P$ is invariant under the action of the Laplacian.

Thus we can now concentrate on the action of the Laplacian on each of these blocks.

4.1 Embedding of the L-block in $\mathbb{C}S_n$

Write the multisets X and Y as $X = \cup_{i \in A_1}\{x_i\} \cup \ldots \cup_{i \in A_l} \{x_i\}$, and $Y = \cup_{j \in B_1}\{y_j\} \cup \ldots \cup_{j \in B_m} \{y_j\}$, where the A_i's contain the sets of indices of equal x's, and B_i's contain the sets of indices of equal y's.

For example, if $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 = x_2, x_3 = x_4$, then $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$ and $A_3 = \{5\}$.

Switching two of the x's will displace x_i from its original position. To take into account the fact that we have to bring it back (by the choice of our basis) into the i^{th} place, we need a minus sign.

Let

$$\Pi_x = \sum_{\sigma_1 \in Sym(A_1), \sigma_2 \in Sym(A_2) \ldots} (\prod_i sgn(\sigma_i))\sigma_1\sigma_2 \cdots \sigma_l$$

and

$$\Pi_y = \sum_{\sigma_1 \in Sym(B_1), \sigma_2 \in Sym(B_2) \ldots} \sigma_1\sigma_2 \cdots \sigma_m.$$

Then the L-block V can be identified with a subspace of $\Pi_x \mathbb{C}S_n \Pi_y$. So Π_y symmetrizes over equal y's and Π_x anti-symmetrizes over equal x's. In other words, Π_x permutes the positions, while Π_y permutes indices.

4.2 The Laplacian L_Y

Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be two fixed (multi-)sets of vertices of the poset P, and consider the restriction of the Laplacian L to L-block $V(X,Y)$.

To simplify the notation, we will write the Laplacian L as:

$$L(\zeta) = \left((w(X,Y) + \Delta(X,Y))Id + \sum_{x_i <_P x_j} (x_i, x_j) + \sum_{y_i <_P y_j} (y_i, y_j) \right) \cdot \zeta,$$

where the "action" of (y_i, y_j) or (x_i, x_j) on $z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n}$ means switching the corresponding pairs of y's, or x's.

To simplify our examination, we will split it into these three parts:

$$L = L_D + L_X + L_Y,$$

where

- L_D is the scalar matrix, $L_D = w(X,Y) + \Delta(X,Y)$
- L_X is the "action" of the Laplacian on the set of the x's, i.e.

$$L_X = \sum_{x_i <_P x_j} (x_i, x_j)$$

- L_Y is the "action" of the Laplacian on the set of the y's

$$L_Y = \sum_{y_i <_P y_j} (y_i, y_j)$$

on our L-block $V(X,Y)$.

In the embedding of this L-block into $\mathbb{C}S_n$, the notation for the Laplacian L_Y would be $L_Y = \sum_{i<j: y_i <_P y_j} (i,j)$, where the actual multiplication is from the right. The proper notation for L_X in the $\Pi_x \mathbb{C}S_n \Pi_y$ is $L_X = \sum_{i<j: x_i <_P x_j} (i,j)$, but the multiplication in this case is from the left.

Lemma 12 *L_Y and Π_y commute, i.e.,*

$$L_Y \cdot \Pi_y = \Pi_y \cdot L_Y.$$

Proof: It is sufficient to prove that the Laplacian L_Y commutes with every transposition of the form (i,k), where $y_i = y_k$, because every permutation in Π_y can be written as a product of those permutations. So, let $y_i = y_k$. That means that Π_y has transposition (i,k) as one of its summands. Let $y_j \in Y$ be comparable to y_i (thus it is comparable to y_k). In that case, the Laplacian L_Y contains both transpositions, (i,j), and (k,j), i.e., $L_Y = \cdots + (i,j) + (k,j) + \cdots$.

But, $(i,k) \cdot (i,j) = (k,j) \cdot (i,k)$, which shows that $\Pi_y \cdot L_Y = L_Y \cdot \Pi_y$.■

Using exactly same argument we see that L_X and Π_x also commute.

We know from section 2 that L_D is a scalar matrix on each block, and thus it commutes with L_X and L_Y.

As for the L_X and L_Y, we have the following Lemma:

Lemma 13

$$L_X \cdot L_Y \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n}) = L_Y \cdot L_X \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n})$$

Proof:

The absence of certain relations in the poset may cause terms in the Laplacian to be missing. That is why this lemma is not obvious, and needs to be proved.

Let $\zeta = z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_n,y_n}$. Without loss of generality we can assume that all of the x's and all of the y's are distinct, because if they were not, we would just apply the same reasoning to each appearance of an observed element. Let (x_i, x_j) be a transposition of the operator L_X, and let (y_k, y_l) be a transposition of the operator L_Y. If all of the numbers i, j, k, l are distinct, we have nothing to prove since it would not make any difference which transposition was applied first. On the other hand, if $i = k$ and $j = l$, again there is nothing to prove, since their combined action would amount to multiplying with -1 no matter in which order they are applied.

Therefore assume that $i \neq k$ but $j = l$, i.e., we have two transpositions, (x_i, x_j) and (y_j, y_k) in L_X and L_Y respectively, which overlap at one position. Without loss of generality assume that $n = 3$. There are only three elements of the pure wedge, call them $z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}$, i.e., $i = 1, j = 2, n = k = 3$.

Let

$$\begin{aligned} \mathcal{A} &= (x_1, x_2) \cdot (y_2, y_3) \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}) \\ &= (x_1, x_2) \cdot (z_{x_1,y_1} \wedge z_{x_2,y_3} \wedge z_{x_3,y_2}) \\ &= -(z_{x_1,y_3} \wedge z_{x_2,y_1} \wedge z_{x_3,y_2}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{B} &= (y_2, y_3) \cdot (x_1, x_2) \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}) \\ &= -(y_2, y_3) \cdot (z_{x_1,y_2} \wedge z_{x_2,y_1} \wedge z_{x_3,y_3}) \\ &= -(z_{x_1,y_3} \wedge z_{x_2,y_1} \wedge z_{x_3,y_2}). \end{aligned}$$

Thus

$$L_X \cdot L_Y \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}) = L_Y \cdot L_X \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}),$$

whenever all of the relations used above are present, i.e., whenever every x_i is beneath each y_j. That can be explained by the fact that L_X is acting on the x-indices and L_Y is acting on the y-indices.

The question remains whether the answer would be the same if some of the relations needed above were missing, and only one of the expressions above gets annulled. The final expressions in both $\mathcal{A}$ and $\mathcal{B}$ are 0 unless:

$$x_1 < y_3, \qquad x_2 < y_1, \qquad x_3 < y_2.$$

Suppose (without loss of generality) that $\mathcal{B}$ above survives the procedure, i.e., we have the relation $x_1 < y_2$. On the other hand if $\mathcal{A}$ is annulled in the middle step, the only possible conflict left is $x_2 \not< y_3$. We have that y_2 and y_3 are comparable, otherwise the transposition (y_2, y_3) wouldn't be a summand of L_Y. If $y_2 < y_3$, then $x_2 < y_2 < y_3$, which is contrary to the just stated assumption. Thus, we must have $y_3 < y_2$. Also $x_1 < y_3$ by our assumption above, and (x_1, x_2) is a transposition in L_X, so they must also be comparable. By the same argument as above, x_2 must be larger than x_1. Hence in the interval (x_1, y_2) there are two elements x_2 and y_3. Since the poset is linear – those two elements must be comparable, and since we assumed that $x_2 \not< y_3$, it must be that $x_2 > y_3$.

All together, the relations are:

$$\begin{matrix} y_2 \\ y_1 \end{matrix} > x_2 > y_3 > \begin{matrix} x_3 \\ x_1 \end{matrix}$$

So in this case we have $y_1 > y_3$, and $x_2 > x_3$. Let

$$\begin{aligned} \mathcal{C} &= (x_2, x_3) \cdot (y_1, y_3) \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}) \\ &= (x_2, x_3) \cdot (z_{x_1,y_3} \wedge z_{x_2,y_2} \wedge z_{x_3,y_1}) \\ &= -(z_{x_1,y_3} \wedge z_{x_2,y_1} \wedge z_{x_3,y_2}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{D} &= (y_1, y_3) \cdot (x_2, x_3) \cdot (z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge z_{x_3,y_3}) \\ &= -(y_1, y_3) \cdot (z_{x_1,y_1} \wedge z_{x_2,y_3} \wedge z_{x_3,y_2}) \\ &= 0 \end{aligned}$$

since $x_2 > y_3$.

The expressions $\mathcal{A}$ and $\mathcal{C}$ are two summands of the product $L_X L_Y$, while $\mathcal{B}$ and $\mathcal{D}$ are two summands of the product $L_Y L_X$. As we can see, $\mathcal{A}+\mathcal{C} = \mathcal{B}+\mathcal{D}$. Thus $L_X \cdot L_Y = L_Y \cdot L_X$.

■

In view of Lemma 13, L_X, L_Y and L_D are commuting linear transformations. So, to analyze the spectrum of their sum, we can compute the eigenvalues and eigenspaces of each separately. We will begin with L_Y.

4.3 A poset tableau of type $(X, Y)_P$

Definition 4.2 *The* **diagram of the L-block**, *$P[X, Y]$, spanned by the sets $(X, Y)_P$, is the Hasse diagram of the subposet $X \cup Y$ with order inherited from the poset P. Furthermore every vertex of P, which is in the intersection $X \cap Y$ is split into two nodes, with the x-node above the y-node.*

Definition 4.3 *Given a node v in $P[X, Y]$ define the* **repetition number of** *v , $k(v)$, to be the number of times that v appears in the multiset X if v is an x-node of $P[X, Y]$, or the multiset Y if v is a y-node of $P[X, Y]$.*

Let $C(v)$ be the set of covers of node v in $P[X, Y]$. If v is a maximal node, than $C(v) = \emptyset$.

Definition 4.4 *A* **poset tableau of type** $(X,Y)_P$ *(or just of type (X,Y)) is any labeling Λ of the diagram, $P[X,Y]$, of the L-block V spanned by (X,Y), where the labels are partitions $\Lambda(v)$, such that $\Lambda(v)$ is a partition of the number $\sum_{w\geq v}\epsilon(w)k(w)$, where*

$$\epsilon(w) = \begin{cases} +1 & \text{if } w \text{ is a } y\text{-node} \\ -1 & \text{if } w \text{ is an } x\text{-node.} \end{cases}$$

Given a poset tableau Λ we will define the **multiplicity of Λ,** $m(\Lambda)$, and the **eigenvalues of Λ,** $e(\Lambda)$**.**

Definition 4.5

- *Let v be a y-node of the diagram $P[X,Y]$, labeled with the partition $\Lambda(v)$ and with repetition number $k(v)$. Let $C(v)=\{v_1,v_2,\ldots,v_l\}$ be the set of covers of v. Let λ_i denote $\Lambda(v_i)$, and let k_i denote the repetition numbers, $k(v_i)$. The* **multiplicity of Λ at** *v is defined to be*

$$m_v(\Lambda) = c^{\Lambda(v)}_{\lambda_1,\ldots,\lambda_l,k(v)}$$

- *Let v be an x-node of the diagram $P[X,Y]$, labeled with the partition $\Lambda(v)$ and with repetition number $k(v)$. Let $C(v)=\{v_1,v_2,\ldots,v_l\}$ be the set of covers of v. Let λ_i denote $\Lambda(v_i)$, and let k_i denote the repetition numbers, $k(v_i)$. The* **multiplicity of Λ at** *v is defined to be*

$$m_v(\Lambda) = \sum_{\mu} c^{\mu}_{\lambda_1,\ldots,\lambda_l} c^{\mu}_{\Lambda(v),1^{k(v)}}.$$

If the multiplicity $m_v(\Lambda)=0$ then we know that that particular labeling is not valid.

Now, we will define the y-eigenvalues for each y-node v of the diagram $P[X,Y]$. We want to have as many y-eigenvalues as the value of multiplicity. From the representation theory of the symmetric group, we know that

$$c^{\Lambda(v)}_{\lambda_1,\ldots,\lambda_l,k(v)} = \sum_{\Lambda(v)/\mu \,=\, k(v)\text{–horizontal strip}} c^{\mu}_{\lambda_1,\ldots,\lambda_l}. \tag{14}$$

The **node–eigenvalue,** $e_v(\Lambda)$**,** for each node v, is the set of the sums of the content over all squares in $\Lambda(v)/\mu$ for all possible μ for which $\Lambda(v)/\mu$ is a $k(v)$–horizontal strip minus the binomial coefficient $\binom{k(v)}{2}$.

Recall that the content of a square is given by $c(i,k)=k-i$ if the square is at position (i,k) in a partition (i th row and k th column).

This gives $m_v(\Lambda)$ eigenvalues at each y-node v. We now define **y-eigenvalue of** Λ, $e_y(\Lambda)$, to be the set of numbers obtained by taking a sum of one element of $e_v(\Lambda)$ for each y-node v. So $|e_y(\Lambda)| = \prod_{y\text{-nodes } v} m_v(\Lambda)$.

4.4 Example

Let the poset $P=\{1,2,3,4,5\}$ with the relations $1<_P 2$, $2<_P 3$, $3<_P 4$ and $4<_P 5$. The Hasse diagram of this poset is given in figure 1.

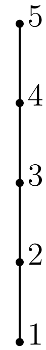


Figure 1: Example: poset P

Let X and Y be the sets $X = \{1, 2, 3\}$ and $Y = \{4, 4, 5\}$. So the node 4, is a node with non-trivial repetition number $k(4) = 2$. The L-block V is spanned by the following pure wedges:

$$\begin{aligned} \zeta &= z_{1,4} \wedge z_{2,4} \wedge z_{3,5} \\ \tau &= z_{1,4} \wedge z_{2,5} \wedge z_{3,4} \\ \eta &= z_{1,5} \wedge z_{2,4} \wedge z_{3,4}. \end{aligned}$$

Thus the L-block V is 3-dimensional. We calculate the Laplacian L_Y on these three elements. Note that the Laplacian L_Y is in fact $L_Y = (4, 5)$, since those are the only two comparable y's.

$$\begin{aligned} L(\zeta) &= \tau + \eta \\ L(\tau) &= \zeta + \eta \\ L(\eta) &= \zeta + \tau \end{aligned}$$

The matrix representation of L_Y with respect to the basis $< \zeta, \tau, \eta >$ is thus

$$L_Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

So the eigenvalues of the Laplacian are $-1, -1, +2$.

Now we will evaluate the y–eigenvalue for each of the poset tableaux for this L-block. The only nontrivial node is node 4. Thus, the y–eigenvalue, $e_y(\Lambda)$, is the node-eigenvalue, $e_4(\Lambda) = c(1, 2) + c(1, 3) - \binom{2}{2}$. The result is given in figure 2.

Note that the y–eigenvalues of this labeling give exactly the same numbers as the eigenvalues of the Laplacian L_Y. In the next section we will show that this is not coincidental.

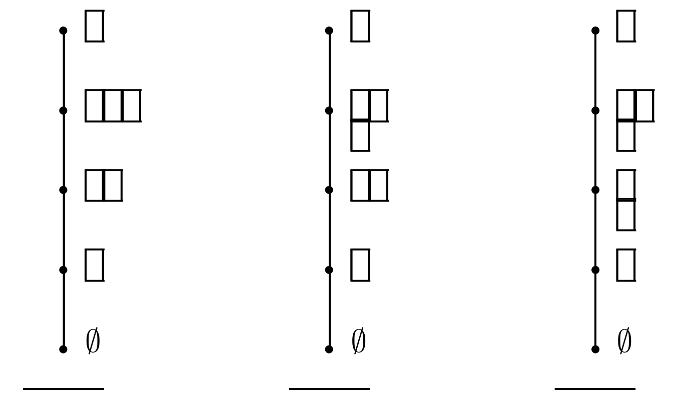


Figure 2: Example: the y-eigenvalues

5 Centerpiece Theorem for L_Y

Theorem 14 (L_Y-Centerpiece) *Let P be a linear poset with a minimum element, $\hat{0}$. Let X and Y be two (multi-)sets, subsets of P. For every labeling Λ of positive multiplicity, each element in $e_y(\Lambda)$ is an eigenvalue of L_Y with multiplicity $\prod_{x\text{-nodes } v} m_v(\Lambda)$.*

Proof:

The proof of this theorem will be by induction on the sizes of the (multi-)sets X and Y. So let $n = |X| = |Y|$ (counting multiplicities).

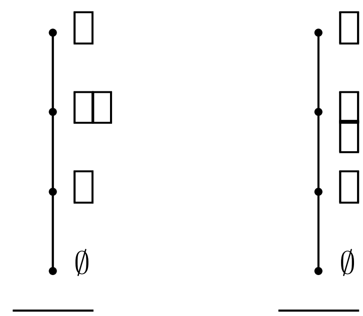
If $n = 1$ — there is nothing to prove as the Laplacian L_Y has no pairs to switch, and the only y-node is the maximal element for the diagram of the L-block. The Laplacian L_Y is the one-by-one zero matrix and the eigenvalue of this unique pair is zero.

Suppose $n = 2$. There are several different possible combinations of relations between sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$.

- The most obvious one is $x_1 < x_2 < y_1 < y_2$. In that case the Laplacian $L_Y = (y_1, y_2)$, and the two possible elements are $\zeta_1 = z_{x_1,y_1} \wedge z_{x_2,y_2}$, $\zeta_2 = z_{x_1,y_2} \wedge z_{x_2,y_1}$. The Laplacian L_Y has the following matrix representation with respect to the basis $\langle \zeta_1, \zeta_2 \rangle$:

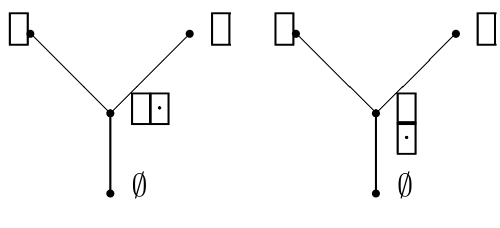
$$L_Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  The eigenvalues of L_Y are $+1$ and -1. The eigenvalue of the poset tableau of type $(X, Y)_P$ is given in figure 3. It also gives values $+1$ and -1, so the claim of the theorem holds.



$e_y(\Lambda) = +1$ $\quad$ $e_y(\Lambda) = -1$

Figure 3: poset tableaux



$e_y(\Lambda) = 0$ $\quad$ $e_y(\Lambda) = 0$

Figure 4: poset tableaux

- The second case is when

$$x_1 < x_2 < \begin{matrix} y_1 \\ y_2 \end{matrix}$$

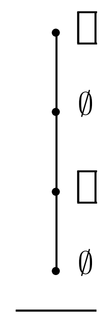
In that case the Laplacian L_Y does nothing (since y_1 and y_2 are not comparable), thus the eigenvalues of L_Y are 0. The dimension of the L-block spanned by (X, Y) is two. The eigenvalues of the poset tableaux give the same values (figure 4), where the "$\cdot$" in a box denotes which square was deleted in that step.

Thus in this case the theorem checks too.

- $x_1 < y_1 < x_2 < y_2$ or equivalently (for our purpose) $x_1 < y_1 = x_2 < y_2$.
There is only one poset tableau spanned by these sets X and Y, namely the one shown on the figure 5.

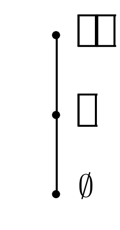
The y-eigenvalue for the poset tableau is zero in both cases.

The Laplacian L_Y can not switch the y's, since that would produce the element z_{x_2,y_1} which doesn't exist. So the Laplacian L_Y also acts as zero.



$e_y(\Lambda) = 0$

Figure 5: poset tableau



$e_y(\Lambda) = 0$

Figure 6: poset tableau

- $x_1 < x_2 < y_1 = y_2$.
  There is only one poset tableau spanned by these sets X and Y, namely the one shown on the figure 6.

  The y-eigenvalue is again zero (contents of the partition (2) minus the binomial coefficient $\binom{2}{2}$). The Laplacian L_Y doesn't have two distinct y's to switch, thus, it is zero.

- x's are the same.

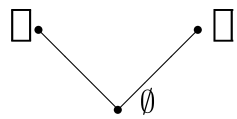
$$x_1 = x_2 < \begin{matrix} y_1 \\ y_2 \end{matrix}$$

  There is only one poset tableau spanned by these sets X and Y, namely the one shown on the figure 7.

  The y-eigenvalue is zero. The Laplacian L_Y has no comparable y's to switch - thus $L_Y = 0$.

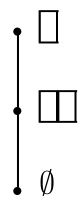
- $x_1 = x_2 < y_1 < y_2$. There is only one poset tableau spanned by these sets X and Y, namely the one shown on the figure 8.

  The y-eigenvalue is equal to 1. The Laplacian L_Y can switch y_1 and y_2 but the result would be the same element, since the x's are indistinguishable. Thus the Laplacian



$e_y(\Lambda) = 0$

Figure 7: poset tableau



$e_y(\Lambda) = 1$

Figure 8: poset tableau

$L_Y = Id$, with the eigenvalue 1.

- In the trivial case when the x's are not comparable the y's are not comparable because of linearity and the existence of a minimal element. So we have

$$\begin{array}{ccc} x_1 & < & y_1 \\ x_2 & < & y_2 \end{array}.$$

The poset tableau again gives zero as the y-eigenvalue, and since the y's are not comparable, the Laplacian L_Y is also zero.

So the theorem holds for the case $n = 2$.

Now, we will treat the general case $n > 2$.

- Label the y-nodes of the diagram of the L-block using the depth-first algorithm:

  1. Start with a leftmost minimal y-element v.
  2. If v is not the maximal unlabeled y-node go to the leftmost unlabeled cover of v, and repeat this step. Otherwise label v with next available number from the set $\{1, 2, \ldots, |Y|\}$.

  From this labeling we see that, $y_i > y_j \Rightarrow i < j$.

- Let $P[X_1, Y_1], P[X_2, Y_2], \ldots, P[X_c, Y_c]$ be the connected components of $P[X, Y]$. In that case the L-block V is the tensor product of the L-blocks of the $P[X_i, Y_i]$. The

Laplacian L_Y switches only comparable y's, and two y's from different connected components are not comparable. Thus

$$L_Y(v_1 \otimes \cdots \otimes v_c) = \sum_{i=1}^{c} v_1 \otimes \cdots \otimes (L_{Y_i} v_i) \otimes \cdots \otimes v_c.$$

So if $v_1, \ldots, v_c$ are eigenvectors of L_Y with the eigenvalues $e_1, \ldots, e_c$, then $v_1 \otimes \cdots \otimes v_c$ is an eigenvector with eigenvalue $e_1 + \cdots + e_c$. Thus, by induction, we can label each of the components of $P[X,Y]$ to get the eigenvalues of L_Y on the total L-block V.

- Suppose that $P[X,Y]$ is connected. In that case, there must be a minimal element in $P[X,Y]$, which must be an x-node.

  Call the minimal element x_n. Then define $x_{n-1}, x_{n-2}, \ldots, x_a$ by:

  1. $x_a > x_{a+1} > \cdots > x_n$, where all $>$ are covering relations.
  2. Either

  Case 1: There is more than one element covering x_a.

  Case 2: x_a has unique cover in $P[X,Y]$ but it is a y-element.

  Let $B = \{x_a, \ldots, x_n\}$, and let $G =$Sym(B).

Lemma 15 *Let $\sigma \in G$, and let $\zeta = z_{x_{i_1},y_1} \wedge z_{x_{i_2},y_2} \wedge \cdots \wedge z_{x_{i_n},y_n}$ be non-zero. Then*

$$\zeta^\sigma = z_{\sigma(x_{i_1}),y_1} \wedge z_{\sigma(x_{i_2}),y_2} \wedge \cdots \wedge z_{\sigma(x_{i_n}),y_n}$$

is also non-zero.

Proof: It is sufficient to prove the lemma for the transposition $(x_{i_k}, x_{i_l}) \in G$.

$$\zeta^{(x_{i_k},x_{i_l})} = z_{x_{i_1},y_1} \wedge \cdots \wedge z_{x_{i_l},y_l} \wedge \cdots \wedge z_{x_{i_k},y_k} \wedge \cdots \wedge z_{x_{i_n},y_n}.$$

Now, since $(x_{i_k}, x_{i_l}) \in B$, i.e., x_{i_k}, x_{i_l} are both less or equal to x_a, which is below all of the y's – the lemma is clear.■

Lemma 16 *This action of G commutes with L_Y.*

Proof: Let $\zeta = z_{x_{i_1},y_1} \wedge z_{x_{i_2},y_2} \wedge \cdots \wedge z_{x_{i_n},y_n}$ be in our L-block, V, let $y_k <_P y_l$ and let $\sigma \in G$. Then

$$\sigma \cdot (y_k, y_l) \cdot \zeta = z_{\sigma(x_{i_1}),y_1} \wedge \cdots \wedge z_{\sigma(x_{i_k}),y_l} \wedge \cdots \wedge z_{\sigma(x_{i_l}),y_k} \wedge \cdots \wedge z_{\sigma(x_{i_n}),y_n}$$

and

$$(y_k, y_l) \cdot \sigma \cdot \zeta = z_{\sigma(x_{i_1}),y_1} \wedge \cdots \wedge z_{\sigma(x_{i_k}),y_l} \wedge \cdots \wedge z_{\sigma(x_{i_l}),y_k} \wedge \cdots \wedge z_{\sigma(x_{i_n}),y_n}.$$

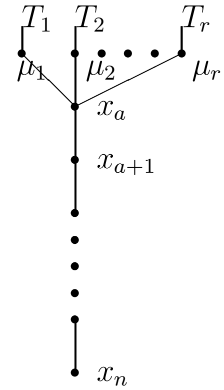


Figure 9: Case 1

So they are equal unless one of the expressions above is zero, and the other is not. The only way for that to happen is in the middle step, i.e., either $\sigma\cdot\zeta = 0$ or $(y_k, y_l)\cdot\zeta = 0$. But we assumed the $\zeta \neq 0$, and by our lemma above $\sigma \cdot \zeta \neq 0$. So the only possible conflict is $(y_k, y_l) \cdot \zeta = 0$, and $(y_k, y_l) \cdot \sigma \cdot \zeta \neq 0$. But since $\sigma \in G$, it only moves elements of B which are allowed to be paired with any y_k.■

Let $C(x_a) = \{v_1, v_2, \ldots, v_l\}$ be the set of covers of x_a. We will prove the following generalization of the Centerpiece Theorem:

Theorem 17 *Let Λ be a poset tableau of positive multiplicity, and let λ_i be the label of v_i in Λ. Let λ be a partition such that*

$$c^{\lambda}_{\lambda_1,\ldots,\lambda_l} c^{\lambda}_{\Lambda(x_a),1^{k(x_a)}} \neq 0.$$

Then the occurrences of G-irreducibles S^λ in V can be indexed by such poset tableaux Λ, and the Laplacian L_Y acts on S^λ as one of the scalars in $e_Y(\Lambda)$.

5.0.1 Case 1

Suppose there are two or more subtrees above the node x_a, in our poset P (as in figure 9). Label the subtrees above the x_a by $T_1, T_2, \ldots, T_r$. Let $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$ ($Y_i \subset T_i$), where $Y_i \cap Y_j = \emptyset$. Let $k_i = |Y_i|$, and let

$$Y_i = \{y_{k_1+k_2+\cdots+k_{i-1}+1}, \ldots, y_{k_1+\cdots+k_{i-1}+k_i}\}.$$

Because of our labeling, we know that all relations between y's are contained within the sets Y_i, i.e., $y_i <_P y_j$ implies that both y_i, y_j are in the same Y_k.

Let b_i be the number of y_j's in T_i minus the number of x_j's in T_i (note that in general T_i will have more y_j's than x_j's). In other words, $b_i = |Y_i| - |X \cap T_i|$.

Consider element $\zeta = z_{x_{i_1},y_1} \wedge z_{x_{i_2},y_2} \wedge \cdots \wedge z_{x_{i_n},y_n}$. If we want ζ to be non-zero, there will be exactly b_i x_j's from B in ζ paired up with the y_j's of T_i.

Split the L-block

$$V = \oplus_{(S_1,S_2,\ldots,S_r)} V[S_1, S_2, \ldots, S_r], \tag{15}$$

where $S_1 \cup S_2 \cup \cdots \cup S_r = B$, $|S_i| = b_i$, and $V[S_1, \ldots, S_r]$ is the span of all ζ with exactly the elements of S_i paired with the y's from T_i.

Lemma 18 *1. As a vector space*

$$V[S_1, \ldots, S_r] \cong V(X_1 \cup S_1, Y_1) \otimes V(X_2 \cup S_2, Y_2) \otimes \cdots \otimes V(X_r \cup S_r, Y_r),$$

where X_i and Y_i are the multisets of the x and y elements of the subtree T_i.

2. With respect to the decomposition in 1., the Laplacian L_Y acts as:

$$L_Y(v_1 \otimes \cdots \otimes v_r) = \sum_i v_1 \otimes \cdots \otimes (L_{Y_i} v_i) \otimes \cdots \otimes v_r.$$

Proof: Statement 1. is clear by the definition of $V[S_1, \ldots, S_r]$. To prove statement 2., we only need to recall that the Laplacian L_Y switches comparable y's, and that y's in different subtrees can not be comparable because of linearity of the poset. L_Y can switch only y's in the same subtree T_i.∎

Let $G_i = \mathrm{Sym}(S_i)$. Note that $G_1 \times G_2 \times \ldots \times G_r$ acts on $V[S_1, S_2, \ldots, S_r]$. Let s_i be the minimal node of subtree T_i.

Now apply the induction hypothesis to L-blocks, $V_i = V(X_i \cup S_i, Y_i)$. According to our theorem this gives the decomposition of the L-block V_i as G_i-module, and the eigenvalues of L_{Y_i} are indexed by poset tableau of shape $\mu_i \vdash |S_i|$. Moreover each poset tableau of shape μ_i with eigenvalue e_i represents a copy of the irreducible S^{μ_i} in the e_i-eigenspace.

Now, as a $G_1 \times \ldots \times G_r$ module we know the eigenspaces of L_Y are given by our labeling up to the points s_i where the partitions μ_i come together at x_a.

Lemma 19 *As a G =Sym(B)-module, the space V is*

$$V \cong \mathrm{ind}^{\mathrm{Sym}(B)}_{(\mathrm{Sym}(S_1^0)\,\times\cdots\times\,\mathrm{Sym}(S_r^0)\,)} (V[S_1^0, \ldots, S_r^0]),$$

where $(S_1^0, \ldots, S_r^0)$ is any fixed ordered partition of B.

Proof: Choose $S_i^0 = \{x_{a+b_1+\cdots+b_{i-1}}, x_{a+b_1+\cdots+b_{i-1}+1}, \ldots, x_{a+b_1+\cdots+b_i-1}\}$. Let $\mathcal{S}h$ denote the set of permutations $\sigma \in \mathrm{Sym}(B)$ such that $\sigma(u) < \sigma(v)$ whenever u, v are in the same set S_i^0 for some i. There is 1–1 correspondence between the $\sigma \in \mathcal{S}h$ and the sequences indexing the summands in the 15, namely

$$\sigma \leftrightarrow (S_1^\sigma, \ldots, S_r^\sigma)$$

where $S_i^\sigma = \{\sigma(u) : u \in S_i^0\}$. Also $\mathcal{S}h$ is a collection of coset representatives for $\mathrm{Sym}(S_1^0)\times\cdots\times\mathrm{Sym}(S_r^0)$ in $\mathrm{Sym}(B)$. Thus we have a natural vector space isomorphism between V and

$$V[S_1^0,\ldots,S_r^0]\otimes_{(\mathrm{Sym}(S_1^0)\ \times\cdots\times\ \mathrm{Sym}(S_r^0))}\mathrm{Sym}(B).$$

It is straightforward to check that this isomorphism commutes with the action of $\mathrm{Sym}(B)$.■

Let Λ_i be a poset tableau of type $(X_i \cup S_i, Y_i)$, where $\mu_i \vdash b_i$ is the label of the vertex s_i. By our inductive hypothesis the Laplacian L_Y acts as a scalar on the irreducible S^{μ_i}, i.e., $L_Y|_{S^{\mu_i}} = e_Y(\Lambda_i)$. Applying Lemma 18 part 2., we have

$$\begin{aligned} L_Y(v_1\otimes\cdots\otimes v_r) &= \sum_i v_1\otimes\cdots\otimes(L_{Y_i}v_i)\otimes\cdots\otimes v_r \\ &= \sum_i v_1\otimes\cdots\otimes e_Y(\Lambda_i)v_i\otimes\cdots\otimes v_r \\ &= (\sum_i e_Y(\Lambda_i))(v_1\otimes\cdots\otimes v_r). \end{aligned}$$

Now, we will use the fact ([16, 17, 24]) that

$$(S^{\mu_1}\otimes\cdots\otimes S^{\mu_r})\uparrow^G_{G_1\times\cdots\times G_r} = \oplus_{\lambda\vdash|B|}c^\lambda_{\mu_1,\mu_2,\ldots,\mu_r}S^\lambda.$$

Thus we have $c^\lambda_{\mu_1,\mu_2,\ldots,\mu_r}$ copies of the G-module S^λ which explains why this is the multiplicity of the label λ on node x_a in our labeling.

- Now we have to decide what is the dimension of each eigenspace. But that is something we will have to do in the second case too - so we will do it for both cases at the end.

5.0.2 Case 2

Let $A = \{y_{n-k+1},\ldots,y_n\}$ be the largest possible set so that

$$x_a \leq_P y_n \leq_P y_{n-1} \leq_P \cdots \leq_P y_{n-k+1}$$

and there are no x_i's with $y_n \leq_P x_i < y_{n-k+1}$. Call A the "terminal Y-set of V" (figure 10). Note that $|B| \geq |A|$. Split the L-block

$$V = \oplus_{(a_1,\ldots,a_k)}V(a_1,\ldots,a_k)$$

for $(a_1,\ldots a_k)$, a sequence of distinct elements of length k from B, and the vector space $V(a_1,a_2,\ldots,a_k)$ represents the span of all ζ which are of the form

$$\zeta = z_{x_1,y_1}\wedge\cdots\wedge z_{x_{n-k},y_{n-k}}\wedge z_{a_1,y_{n-k+1}}\wedge\cdots\wedge z_{a_k,y_n}$$

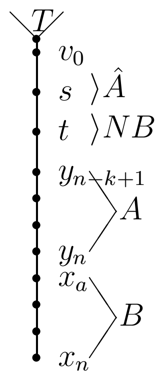


Figure 10: Case 2

For the moment assume that all of the elements $x_a, \ldots, x_n$ are distinct and all of $y_{n-k+1}, \ldots, y_n$ are distinct. Then we will look back at how we must modify the argument when some of the x_i's and y_j's are equal.

Fix the sequence $(a_1, \ldots, a_k)$, let $B' = B \backslash \{a_1, \ldots, a_k\}$ and let G' be the subgroup $G' = \mathrm{Sym}(B') \le G$. Note that G' acts on $V(a_1, \ldots, a_k)$.

Lemma 20 *$V(a_1, \ldots, a_k)$ is isomorphic to the L-block V_0 given by the following sets: $X' = X \backslash \{a_1, a_2, \ldots, a_k\}$ and $Y' = Y \backslash \{y_{n-k+1}, \ldots, y_n\}$.*

Proof: The isomorphism $\psi : V(a_1, \ldots, a_k) \to V_0$ is an obvious one

$$\psi(z_{x_1,y_1} \wedge \cdots \wedge z_{x_{n-k},y_{n-k}} \wedge z_{a_1,y_{n-k+1}} \wedge \cdots \wedge z_{a_k,y_n}) = z_{x_1,y_1} \wedge \cdots \wedge z_{x_{n-k},y_{n-k}}.$$

It is clearly a bijective linear map.■

- Now let's examine the L-block given by X' and Y'. Let $\hat{A}$ be the terminal Y-set for X', Y'. Let $\hat{B} = B' \cup \{\text{new } x_i\text{'s}\}$ be the x_i's below $\hat{A}$. Let NB denote the set of those new x_i's. Let $\hat{G} = \mathrm{Sym}(\hat{B})$. Note that $G' \le \hat{G}$.

  If $\hat{A} = \emptyset$ then there are no y's in the interval $(y_{n-k+1}, v_0]$. In that case the group $\hat{G}$ is the group described in the case 1., i.e., $\hat{G} = G_1 \times G_2 \times \cdots \times G_r$, where G_i is acting on the subtree T_i above x_a.

  Now apply the induction hypothesis to X', Y'. This gives the decomposition of the L-block as a $\hat{G}$-module. The theorem says that the irreducible summands are indexed by the (X', Y') poset tableaux $\hat{\Lambda}$ of shape λ' (where $\lambda' \vdash |\hat{B}|$), and "shape" means that the minimal element s of $\hat{A}$ is labeled with λ'. Also the theorem tells us that the Y-Laplacian L_Y for (X', Y') acts like the scalar $e_Y(\hat{\Lambda})$ on this copy of the irreducible $S^{\lambda'}$. From this we can deduce by restriction from $\hat{G}$ to G' the following

Lemma 21 *As a G'-module, the space $V(a_1, \ldots, a_k)$ decomposes as a sum over all (X', Y') tableaux $\hat{\Lambda}$ of shape λ' of the module*

$$S^{\lambda'} \downarrow_{G'}^{\hat{G}}$$

Moreover the Laplacian L_Y for (X', Y') acts like the scalar $e_Y(\hat{\Lambda})$ on this entire restriction.

Let $\mathcal{C}h(\lambda', \mu')$ be the set of chains $\lambda' \geq \lambda_1 \geq \cdots \geq \mu'$ where the steps in the chain of the partitions are all of size 1.

Now using the fact [16] that

$$(S^{\lambda'}) \downarrow_{G'}^{\hat{G}} = \oplus_{\mu' \vdash |B'|} S^{\mu'} |\mathcal{C}h(\lambda', \mu')|,$$

we can rewrite this lemma to say that the sum is over all extensions of $\hat{\Lambda}$ to a labeling of the new x_i's of $S^{\mu'}$ where the extension gives the label μ' to t, the minimal element of the set NB.

- Now we need one last lemma.

Lemma 22 *As a G-module $V \cong \mathrm{ind}_{G'}^{G}(V(x_{n-k+1}, \ldots, x_n))$.*

Proof: For each sequence $\underline{\alpha} = (a_1, \ldots, a_k)$ let $\pi_{\underline{\alpha}}$ be the permutation in G which maps a_i to x_{n-k+i} and which leaves the elements of $B \backslash \{a_1, \ldots, a_k\}$ in increasing order. Then $\pi_{\underline{\alpha}}$ is a set of coset representatives for $G \backslash G'$. Since there is one for every $\underline{\alpha}$ this shows that as vector spaces

$$V \cong V(x_{n-k+1}, \ldots, x_n) \otimes_{G'} \mathbb{C}(G).$$

Let $g \in G$, let $(a_1, \ldots, a_k) = \underline{\alpha}$ be a sequence, and let $b_i = g(a_i), \underline{\beta} = (b_1, \ldots, b_k)$. Then

$$\begin{aligned} g(z_{x_1,y_1} \wedge \cdots \wedge z_{x_{n-k},y_{n-k}} \wedge z_{a_1,y_{n-k+1}} \wedge \cdots \wedge z_{a_k,y_n}) = \\ = \; & z_{g(x_1),y_1} \wedge \cdots \wedge z_{g(x_{n-k}),y_{n-k}} \wedge z_{b_1,y_{n-k+1}} \wedge \cdots \wedge z_{b_k,y_n} \\ = \; & (\pi_{\underline{\beta}}^{-1} g \pi_{\underline{\alpha}}) z_{x'_1,y_1} \wedge \cdots \wedge z_{x'_{n-k},y_{n-k}} \wedge z_{x_{n-k+1},y_{n-k+1}} \wedge \cdots \wedge z_{x_n,y_n}, \end{aligned}$$

where x'_i is obtained by replacing the elements of $\{x_a, \ldots, x_n\} \backslash \{a_1, \ldots, a_k\}$ by the elements of the set $\{x_a, \ldots, x_{n-k}\}$ in order. This computation shows that the vector space isomorphism above is a G-module isomorphism.■

Now putting all the claims together with the fact([16]) that

$$\mathrm{ind}_{\mathrm{Sym}(B')}^{\mathrm{Sym}(B)}(S^{\mu'}) = \oplus_{\mathcal{C}h(\lambda,\mu')} S^{\lambda}, \tag{16}$$

shows that as a module for G, V decomposes as a sum, over all (X, Y) poset tableaux Λ of shape λ, of a copy of S^{λ}. In the expression 16 we know by induction that the Laplacian $L_{Y'}$ involving all switches which do not involve the terminal set, i.e., $L_{Y'} = L_Y - L_0$ (where L_0 is the sum of the switches involving the terminal Y-set), acts like the scalar $e_Y(\Lambda')$, where Λ' is the labeling as far as the point t (the minimal x above A). The switches involving the terminal Y-set A must be studied. But those y_i in A can either be switched with each other or with y_j that are above an $x_i \in B$. By the choice of the terminal set A, y_i is comparable to y_j for all $y_i \in A$. It follows that L_0 acts on $V = \oplus V(a_1, \ldots, a_k)$ by the sum of all transpositions (x_i, x_j) for $x_i \in B, x_j \in \{a_1, \ldots, a_k\}$. In terms of our induced module, L_0 acts on

$$\mathrm{ind}_{G'}^{G}(S^{\mu'}) = S^{\mu'} \otimes_{\mathbb{C}G'} \mathbb{C}G$$

like left multiplication on $\mathbb{C}G$ by $\sum_{x_i,x_j \in B}(x_i, x_j) - \sum_{x_i,x_j \in B'}(x_i, x_j)$.

But $\sum_{x_i,x_j \in B'}(x_i, x_j)$ passes through $\otimes_{\mathbb{C}G'}$ to act on $S^{\mu'}$ like the scalar $\sum_{x \in \mu'} c_x$ [24]. And $\sum_{x_i,x_j \in B}(x_i, x_j)$ acts on each G-irreducible S^λ in $S^{\mu'} \otimes_{\mathbb{C}G'} \mathbb{C}G$ like $\sum_{y \in \lambda} c_y$. The result is that L_0 acts on each copy of S^λ in $\mathrm{ind}_{G'}^{G}(S^{\mu'})$ like $\sum_{y \in \lambda/\mu'} c_y$. This explains the scalars $e_Y(\Lambda)$ and their multiplicity.

In order to be able to add the eigenvalues of $L_{Y'}$ and L_0, we need the following lemma.

Lemma 23

$$L_0 \cdot L_{Y'} = L_{Y'} \cdot L_0$$

Proof: Let (x_i, x_j) be a transposition in $\mathrm{Sym}(B)$, where $x_i \in B'$, and let (x_i, x_k) be a transposition of $L_{Y'}$. By the choice of B we know that (x_j, x_k) is also transposition in $L_{Y'}$, and since

$$(x_i, x_j)(x_i, x_k) = (x_j, x_k)(x_i, x_j)$$

the lemma is clear.■

- Now we want to consider the case where some of the elements of the sets A, B, and $NB = \{\text{new } x\text{'s}\}$ are equal. So let's write

$$\begin{aligned} A &= \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_l, \\ B &= \beta_1 \cup \beta_2 \cup \ldots \cup \beta_m, \\ NB &= \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_n, \end{aligned}$$

where $|\alpha_i| = a_i$, $|\beta_i| = b_i$ and $|\gamma_i| = c_i$, and the y's in each of the α_i are equal, the x's in each of the β_i are equal, and the x's in each γ_i are equal.

Let

$$\begin{aligned} \Pi_A &= \sum_{\sigma_i \in S_{\alpha_i}} \sigma_1 \times \sigma_2 \times \ldots \times \sigma_l \\ \Pi_B &= \sum_{\sigma_i \in S_{\beta_i}} (\prod \mathrm{sgn}(\sigma_i))\sigma_1 \times \ldots \times \sigma_m \\ \Pi_{NB} &= \sum_{\sigma_i \in S_{\gamma_i}} (\prod \mathrm{sgn}(\sigma_i))\sigma_1 \times \ldots \times \sigma_n. \end{aligned}$$

These projections commute with the Laplacian L_Y according to lemma 12.

Let Π be the projection $\Pi = \Pi_A \times \Pi_B \times \Pi_{NB}$, where the Π_A acts on the y's while the other two projections act on the set of the x's. Then the projection Π projects our original space (with all of the x's and y's distinct) examined in the previous paragraph to the space where y_i's in each α_i, x's in each β_i and γ_i are equal. Each of the components of Π commutes with L_Y so we know that Π and L_Y commute.

Lemma 24 *The projection Π maps the L-block V with all x's and y's distinct to the L-block ΠV, where α_i, β_i and γ_i are sets of equal elements.*

Proof: Let $\zeta = z_{x_1,y_1} \wedge \cdots \wedge z_{a_1,y_{i_1}} \wedge \cdots \wedge z_{a_k,y_{i_k}} \wedge \cdots \wedge z_{x_n,y_n}$. Note that if $y_i = y_j$, the element $\zeta^{(i,j)}$ and ζ are the same. In general, if $\sigma_i \in S_{\alpha_i}$, then $\zeta^{\sigma_i} = \zeta$. So we need to identify all equal elements. This is accomplished by projecting with a symmetrizer, i.e., we identify the class of elements $\cup_{\sigma \in S_{\alpha_i}} \zeta^\sigma$ with the sum $\sum_{\sigma \in S_{\alpha_i}} \zeta^\sigma$. But Π_A does exactly this identification, $\Pi_A = \sum_{\sigma \in S_{\alpha_i}} \zeta^\sigma$. The same is true for Π_B and Π_{NB}.$\blacksquare$

NOTE: When we deal with the projection Π_A there is one important point we have to make. The Laplacian L_Y switches all comparable pairs of y's. If two of the y's are the same – they would not get switched.

Therefore, when we observed the Laplacian L_0, we have to subtract all switches involving two y's from the same α_k. Each of these transpositions doesn't move any of the y's (or x's), and there are exactly $\binom{a_k}{2}$ of them. Thus, we have to subtract $\binom{a_k}{2}$ from the eigenvalue $\sum_{y \in \lambda/\mu'} c_y$ of L_0.

Thus we can write our space V as a direct sum of the eigenspaces

$$V = \oplus_w V_w,$$

where the sum is over all eigenvalues w of L_Y.

We want to know the eigenvalues of L_Y on the image ΠV. We will use that fact that the multiplicity of w as an eigenvalue on ΠV is the dimension of ΠV_w.

So we need to compute the dimension of ΠV_w. At present we have V_w written in terms of labellings of poset tableaux. So pick such a labeling which at the end has a λ' at vertex s coming down to a μ' at vertex t then back up to a λ at vertex y_n (using the notation of this proof). This represents a piece of the eigenspace of the corresponding eigenvalue w where $\hat{G} = \mathrm{Sym}(\hat{B})$ acts like $S^{\lambda'}$ and $G' = \mathrm{Sym}(B')$ acts like $S^{\mu'}$. We need the following lemma.

Lemma 25 *Let $B' \subset \hat{B}$, $NB = \hat{B} \backslash B'$. Then the multiplicity of $S^{\mu'}$ in $\Pi_{NB} S^{\lambda'}$ is equal to the number of ways to get μ' from λ' by successfully removing vertical strips of lengths $c_1, c_2, \ldots$.*

Note: As a consequence of this lemma, when we came to an x-node v of the repetition number $k(v)$ we have to remove a vertical strip of length $k(v)$.

Proof of lemma: Recall that $NB = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_n$. The projection Π_{NB} projects onto the $\mathrm{sgn}_{\gamma_1} \otimes \mathrm{sgn}_{\gamma_2} \otimes \cdots \otimes \mathrm{sgn}_{\gamma_n}$ isotypic component of $S^{\lambda'}$ considered as a $S_{\gamma_1} \times \ldots S_{\gamma_n}$ module. Thus, for μ' a partition of $|B'|$,

$$\begin{aligned} \langle \Pi_{NB} S^{\lambda'}, S^{\mu'} \rangle_{S_{B'}} &= \langle S^{\lambda'}, S^{\mu'} \otimes \mathrm{sgn}_{\gamma_1} \otimes \cdots \otimes \mathrm{sgn}_{\gamma_n} \rangle_{S_{B'} \times S_{\gamma_1} \times \ldots \times S_{\gamma_n}} \\ &= (\#\text{ of ways to get } \mu' \text{ from } \lambda' \text{ by successively removing} \\ &\qquad \text{vertical strips of lengths } c_1, c_2, \ldots) \end{aligned}$$

Now consider the next step of going up from μ' to λ. At this point we have a piece of the w-eigenspace on which B' acts like $S^{\mu'}$.

The multiplicity coming from this $S^{\mu'}$ is $\dim(\Pi_A \mathrm{ind}_{G'}^{G}(S^{\mu'})\Pi_B)$. We need to check how Π_A acts on the induction $\mathrm{ind}_{G'}^{G}(S^{\mu'}) = S^{\mu'} \otimes_{\mathbb{C}G'} \mathbb{C}G$. The $\sigma \in \Pi_A$ permutes the y's. We identified the induction by identifying the sequence of the x_i's that are paired with the set A. So switching y_i's has the effect in $\mathrm{Sym}(B)$ of switching the positions corresponding to $B \backslash B'$. In other words, $\Pi_A - \Pi_B$ has the effect of projecting onto the (trivial $\otimes$ sgn) characters of $(S_{\alpha_1} \times \ldots \times S_{\alpha_l}) \times (S_{\beta_1} \times \ldots \times S_{\beta_m}) \subset S_{B\backslash B'} \times S_B$ where $S_{B\backslash B'} \times S_B$ is acting on $S^{\mu'} \otimes_{\mathbb{C}G'} \mathbb{C}G$ via left multiplication on $\mathbb{C}G$ by $S_{B\backslash B'}$ and right multiplication on $\mathbb{C}G$ by S_B. So to determine $\dim(\Pi_A \mathrm{ind}_{G'}^{G}(S^{\mu'})\Pi_B)$ it will be helpful to know the decomposition of the induction $\mathrm{ind}_{G'}^{G}(S^{\mu'})$ as a $S_{B\backslash B'} \times S_B$-module.

Lemma 26 ([12]) *Let $\mu' \vdash m$, $G = S_r$, $G' = S_m$ and $H = S_{r-m}$ (acting on $\{m+1, \ldots, r\}$). Then as a $H \times G$ module, the induced representation $S^{\mu'} \otimes_{\mathbb{C}G'} \mathbb{C}G$ decomposes as*

$$ind_{G'}^{G}(S^{\mu'}) = \oplus_{\lambda \vdash r, \mu \subset \lambda} S^{\lambda/\mu} \otimes S^{\lambda}.$$

Now armed with that lemma, let us return to the dimension count.

$$\begin{aligned} &\dim(\Pi_A \mathrm{ind}_{G'}^{G}(S^{\mu'})\Pi_B) = \\ &= \langle (\mathrm{ind}_{G'}^{G}(S^{\mu'})) \downarrow, (\epsilon_{\alpha_1} \otimes \cdots \otimes \epsilon_{\alpha_l}) \otimes (\mathrm{sgn}_{\beta_1} \otimes \cdots \otimes \mathrm{sgn}_{\beta_m}) \rangle \\ &= \sum_{\lambda \vdash n, \mu' \subset \lambda} \langle (S^{\lambda/\mu'}) \downarrow, (\epsilon_{\alpha_1} \otimes \cdots \otimes \epsilon_{\alpha_l}) \rangle \langle S^{\lambda} \downarrow, \mathrm{sgn}_{\beta_1} \otimes \cdots \otimes \mathrm{sgn}_{\beta_m} \rangle \\ &= \sum_{\mu' \subset \lambda \vdash n} \langle (S^{\lambda/\mu'} \otimes S^{\lambda}) \downarrow, (\epsilon_{\alpha_1} \otimes \cdots \otimes \epsilon_{\alpha_l}) \otimes (\mathrm{sgn}_{\beta_1} \otimes \cdots \otimes \mathrm{sgn}_{\beta_m}) \rangle \\ &= \sum_{\lambda \vdash n} (\#\text{ of ways to get } \lambda \text{ from } \mu' \text{ by adding horizontal strips of lengths} \\ &\qquad \alpha_1, \alpha_2, \ldots) \cdot (\#\text{ of ways to get } \lambda \text{ from } \emptyset \text{ by removing a vertical} \\ &\qquad \text{strips of lengths } \beta_1, \beta_2, \ldots) \end{aligned}$$

$=$ (# of poset tableaux labellings from μ' up to λ then down to $\emptyset$, which add a horizontal strip of length k for every y-vertex of repetition number k and subtract a vertical strip of length k for every x-vertex of repetition number k.)

This completes the proof of the theorem.■

5.1 Adding the L_X

Consider the Laplacian L_X. Since we have identified the L-block V with a subspace of the symmetric group algebra $\mathbb{C}S_n$, by fixing the order on the x's, every time the Laplacian L_X switches a pair of x's, it is actually putting a minus sign in front of the corresponding basis element, with the x's ordered. Since L_X acts as a sum of transpositions, every eigenvalue we obtain from the L_X, will have a minus sign.

Recall the multiplicity of the x-node. Let v be an x-node of the diagram $P[X,Y]$, labeled with the partition $\Lambda(v)$ and with repetition number $k(v)$. Let $C(v) = \{v_1, v_2, \ldots, v_l\}$ be the set of covers of v. Let λ_i denote $\Lambda(v_i)$, and let k_i denote the repetition numbers, $k(v_i)$. The **multiplicity of Λ at** v is defined to be

$$m_v(\Lambda) = \sum_{\mu} c^{\mu}_{\lambda_1,\ldots,\lambda_l} c^{\mu}_{\Lambda(v),1^{k(v)}}.$$

The **node–eigenvalue, $e_v(\Lambda)$,** for each node v, is the set of sums of the content over all squares in $\mu/\Lambda(v)$ above for a given μ minus the binomial coefficient $\binom{k(v)}{2}$.

This gives $m_v(\Lambda)$ eigenvalues at each x-node v. We now define the x**-eigenvalue of** Λ, $e_x(\Lambda)$, to be the set of numbers obtained by taking a sum of one element of $e_v(\Lambda)$ for each x-node v. So $|e_x(\Lambda)| = \prod_{x\text{-nodes } v} m_v(\Lambda)$.

Theorem 27 (L_X-Centerpiece) *Let P be a linear poset with a minimum element, $\hat{0}$. Let X and Y be two (multi-)sets, subsets of P. For every labeling Λ of positive multiplicity, each element in $e_x(\Lambda)$ is an eigenvalue of L_X.*

Proof:

The proof of this theorem is similar to the proof of the L_Y-Centerpiece Theorem. We will omit the details here.■

Since L_X and L_Y commute (as established in Lemma 13), the eigenvalues of $L_X + L_Y$ will be the sum of the eigenvalues on the corresponding irreducibles of the eigenspaces.

Recall that the complete Laplacian L is the sum of three things (from the beginning of this section):

$$L = L_D + L_X + L_Y$$

The L_D component is the diagonal matrix, which on the L-block V spanned by the sets (X,Y), has value:

$$e_D(X,Y) \quad = \quad w(X,Y) + \Delta(X,Y)$$

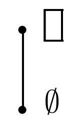


Figure 11: A diagram of the L-block

$$
\begin{aligned}
w(X,Y) &= \sum_{m=1}^{k} |(x_m, y_m)| \\
\Delta(X,Y) &= \sum_{i,j} \delta_{x_i, y_j}
\end{aligned}
$$

The computer evidence strongly supports the following conjecture:

6 Complete Centerpiece Conjecture

Conjecture 28 *Let P be a linear poset with a minimum element, $\hat{0}$. Let X and Y be two multisets of vertices of P. Let $e_D(X,Y)$ be defined as above. For every poset tableau Λ of positive multiplicity, let $\tau_Y(\Lambda) \in e_Y(\Lambda)$ and let $\tau_X(\Lambda) \in e_X(\Lambda)$. Then the scalars $e(\Lambda) = \tau_Y(\Lambda) - \tau_X(\Lambda) + e_D(X,Y)$ are the complete set of eigenvalues of the Laplacian L.*

This conjecture claims that the same poset tableau will work simultaneously for both Laplacians (L_X and L_Y),i.e., that the eigenvalues of the Laplacian L are the sum of the eigenvalues of L_Y and the eigenvalues of L_X evaluated simultaneously with the same poset tableau.

7 Homology

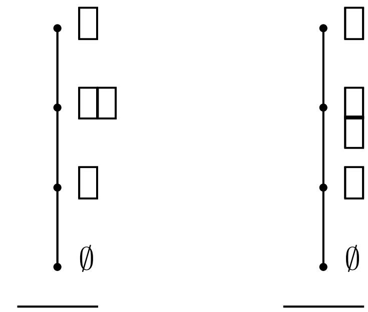
The object of the paper is to get a step closer to evaluating the homology of any Lie algebra corresponding to a linear poset, using only combinatorial properties of the poset. In these two small cases (n=1 and n=2) we had no difficulty. For larger n, we need some extra results.

7.1 H_1

For example, if we want to evaluate the homology $H_1(L_P)$ of a Lie algebra L_P corresponding to a linear poset P, with $\hat{0}$, our construction gives an immediate answer.

An L-block V of size 1, is determined by the sets (X,Y), $X = \{x\}$, and $Y = \{y\}$. Obviously, if we want V to be non-zero, $x <_P y$. So the corresponding diagram of this L-block is given in figure 11.

Both indicators e_Y and e_X are zero, so the eigenvalues are given by e_D. But $\Delta(X,Y) = 0$ too, since $X \cap Y = \emptyset$. Thus $L(z_{x,y}) = w(X,Y) z_{x,y} = |(x,y)| z_{x,y}$. In other words, the



$e(\Lambda) = 0$ $e(\Lambda) = 0$

Figure 12: poset tableaux

eigenvectors of the Laplacian L_1 are the basis vectors $z_{x,y}$, and the corresponding eigenvalues are $|(x, y)|$, i.e., the number of the vertices in the poset P, between x and y.

The dimension of the homology is the number of zero eigenvalues, i.e., the number of the intervals $z_{x,y}$, such that y covers x.

Thus

$$\dim(H_1(L_P)) = \ (\#\text{ of covering relations in } P).$$

7.2 H_2

In this case the L-block V in question is spanned by the (multi–)sets (X, Y), each of size 2, i.e., $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. There are several possibilities for the L-block.

1. All four elements are comparable, and x's are below the y's.

$$x_1 < x_2 < y_1 < y_2.$$

   All possible poset tableaux are shown in figure 12.

   As we can see, both e_Y and e_X eigenvalues are zero. So we don't have to worry how to add them up - we will always get zero. Δ is also zero, since the sets X and Y are disjoint. Thus again, the Laplacian is $L(z_{x_1,y_1} \wedge z_{x_2,y_2}) = w(X, Y)(z_{x_1,y_1} \wedge z_{x_2,y_2})$. But in this case, both intervals contain at least one element, so $w(X, Y) > 0$. Thus in this case we never get a zero eigenvalue, which might contribute to the homology H_2.

2. y's are not comparable.

$$x_1 < x_2 < \begin{matrix} y_1 \\ y_2 \end{matrix}.$$

   All possible poset tableaux are shown in figure 13.

   The value of e_Y is zero, while the values of e_X are +1 and -1. Since e_D is always non-negative, the value +1 cannot contribute to homology H_2. The other value can, but only if e_D is +1. That means that all of the relations indicated: $x_1 < x_2$, $x_2 < y_1$ and $x_2 < y_2$ are covering relations in the poset P. Whenever we have a four-element

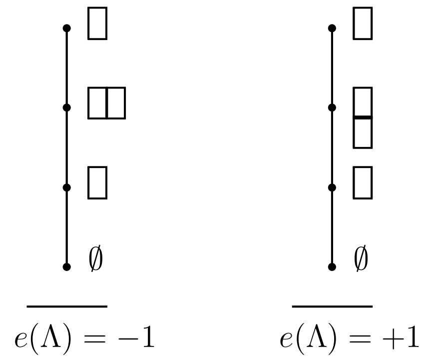


Figure 13: poset tableaux

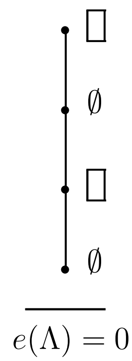


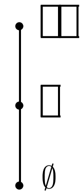
Figure 14: poset tableau

subset of the poset P, with covering relations $x_1 < x_2$, $x_2 < y_1$ and $x_2 < y_2$, we will call that a "Y"-configuration. Thus in this case every occurrence of "Y"-configuration (described as above) in the Hasse diagram of the poset contributes to the homology H_2.

3. $x_1 < y_1 < x_2 < y_2$ or equivalently (for our purpose) $x_1 < y_1 = x_2 < y_2$.
   There is only one poset tableau spanned by these sets X and Y, namely the one shown in figure 14.

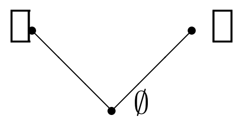
   Since the space is one dimensional, we will add the eigenvalues in the only possible way. In the first case, the eigenvalue will be zero, if and only if both relations, $x_1 < y_1$ and $x_2 < y_2$, are covering relations, i.e., every occurrence of a distinct (all vertices are distinct) pair of covering relations contributes to the dimension of the homology H_2. If the second case occurs, i.e., if $y_1 = x_2$ then Δ will contribute to the eigenvalue, and it won't be zero anymore.

4. $x_1 < x_2 < y_1 = y_2$.
   There is only one poset tableau spanned by these sets X and Y, namely the one shown on the figure 15.



$$e(\Lambda) = -1$$

Figure 15: poset tableau



$$e(\Lambda) = 0$$

Figure 16: poset tableau

Since the space is one dimensional, we will add the eigenvalues in the only possible way. The eigenvalue will be zero, if and only if x_2 covers x_1, and both y_1 and y_2 cover x_2. Note that this is a degenerate letter "Y", with a joint top vertex. Every such occurrence contributes to the homology H_2.

5. x's are the same.

$$x_1 = x_2 < \begin{array}{c} y_1 \\ y_2 \end{array}.$$

There is only one poset tableau spanned by these sets X and Y, namely the one shown in figure 16.

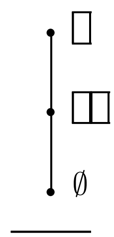
Since the space is one dimensional, we will add the eigenvalues in the only possible way. The eigenvalue will be zero, if and only if both y_1 and y_2 cover $x_1 = x_2$. Again, it is a pair of distinct covering relations with joint vertex, this time the x-vertex. Every such occurrence contributes to the homology H_2.

6. $x_1 = x_2 < y_1 < y_2$. There is only one poset tableau spanned by these sets X and Y, namely the one shown on the figure 17.

Since the space is one dimensional, we will add the eigenvalues in the only possible way. Since e_D is always non-negative, the value $+1$ cannot contribute to homology H_2.

7. The trivial case when x's are not comparable (so because of linearity and the minimum element neither are y's).

$$\begin{array}{ccc} x_1 & < & y_1 \\ x_2 & < & y_2 \end{array}.$$



$e(\Lambda) = 1$

Figure 17: poset tableau

This is equivalent to case 3. We have two distinct covering relations, so it contributes to the homology H_2.

All together, the dimension of the homology H_2 is in fact the number of distinct pairs of covering relations + the number of occurrences of letter "Y" in the poset P. In other words, if $\mathcal{H}$ is a Hasse diagram of a poset P, with e edges, and γ letter "Y"'s (degenerate or not), we have $H_1(L_P) = \mathbb{C}^e$ and $H_2 = \mathbb{C}^{\binom{e}{2}+\gamma}$.

7.3 Examples

- For example, suppose that we are dealing with the chain poset on n vertices $(1 < 2 < \cdots < n)$, T_n. The number of non-degenerate Y's is zero. The number of degenerate letters Y is (n-2). The number of edges in the Hasse diagram is (n-1). Hence

  $$H_1(T_n) = \mathbb{C}^{n-1},$$

  and

  $$H_2(T_n) = \mathbb{C}^{n-2+\binom{n-1}{2}} = \mathbb{C}^{\frac{(n+1)(n-2)}{2}}$$

- Let the poset P be given in figure 18, where the length of the chains are m and n. This is in fact equivalent to having two disjoint chains, of length m and n. Thus the corresponding homologies will be $H_1 = \mathbb{C}^{n-1+m-1}$, and $H_2(T_n) = \mathbb{C}^{\frac{(n+1)(n-2)}{2}+\frac{(m+1)(m-2)}{2}}$

8 Conclusion

These results have several interesting corollaries that are of a combinatorial nature. We will state one. Let P be a rooted tree on n nodes and let Σ be the sum in the group algebra of S_n of all transpositions (i, j) such that i is on the unique path from j to the root in P. Then Σ acting on $\mathbb{C}S_n$ by left multiplication has non-negative integer eigenvalues and the corresponding eigenspaces can be identified in representation-theoretic terms.

There is still a lot to do in this area. Although my work provides partial answers and a conjecture for all rooted trees, the question is still open for other posets. What

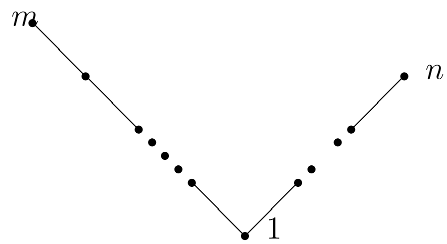


Figure 18: poset P

happens in those cases is very difficult to control, since the expression for the Laplacian becomes more complicated. In the tree case I wish to examine a twisting of the Laplacian by a parameter α which, my advisor has shown, is related to Jack polynomials and the Krawtchouk polynomials in certain special cases. Lastly I would like to see if the more algebraic consequences of Kostant's theorem have sensible analogues in my case.

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