The Problem of Kings

Michael Larsen*
University of Pennsylvania
Philadelphia, PA 19104
larsen@math.upenn.edu

Submitted: May 30, 1994; Accepted: September 23, 1995

Abstract

Let $f(n)$ denote the number of configurations of n^2 mutually non-attacking kings on a $2n \times 2n$ chessboard. We show that $\log f(n)$ grows like $2n \log n - 2n \log 2$, with an error term of $O(n^{4/5} \log n)$. The result depends on an estimate for the sum of the entries of a high power of a matrix with positive entries.

In chess, two kings can attack one another if their squares are horizontally, vertically, or diagonally adjacent. Consider the problem of placing mutually non-attacking kings on a chessboard with $2m$ rows and $2n$ columns. Partitioning the chessboard into 2×2 *cells*, we see that no cell can contain more than one king, so there can be no more than mn kings:

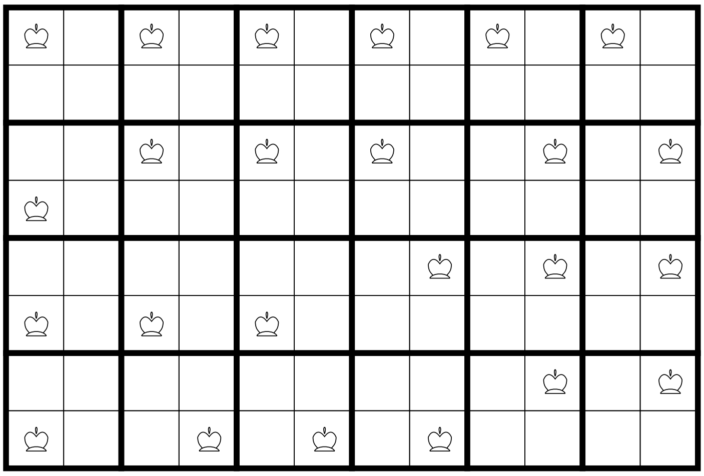

Figure 1

In this note, we estimate the number $K(m, n)$ of configurations of mn kings. H. Wilf [5] has obtained good estimates in the case that m is fixed and $n \gg m$. We consider the order of growth when both m and n tend to infinity, and especially the case $m = n$. Our main result is stated at the end of the paper.

Before proceeding with the problem of kings, it may be worth a brief look at the analogous problem for other chess pieces. No more than n mutually non-attacking rooks can fit on an $n \times n$ board; the legal configurations have one rook in each row and column and are therefore given by the $n!$ permutations. It is known that there exist configurations of n mutually non-attacking queens as long as $n \geq 4$, and the number of such configurations is conjecturally super-exponential in n [4]. For bishops, it is not too difficult to see that

* Partially supported by N.S.A. Grant No. MDA 904-92-H-3026

there are exactly 2^n ways of placing $2n-2$ pieces on an $n \times n$ board, when $n > 1$ [3]. The maximum number of knights on an $n \times n$ board is $\lfloor \frac{n^2+1}{2} \rfloor$, when $n > 2$. For $n > 4$, the number of such configurations is 1 or 2 according to whether n is odd or even; this follows from the existence of a knight's tour (resp. a closed knight's tour) [3].

All such problems can be reformulated in the language of statistical mechanics, but from this point of view the problem of kings is certainly the most natural. Assign spin at each vertex in a (region of a) square lattice according to whether there is or is not a king in the corresponding square of the chessboard. The generating function $\sum_n a_n z^n$ for the number of configurations of n kings is closely related to the $L = M = -\infty$ limit of partition function of Baxter's "hard square model" [1] §14.2. The only difference is that in the hard square model, a king on a corner (resp. edge) of the board counts for only one quarter (resp. one half). Of course, questions in statistical mechanics can be quite sensitive to boundary conditions; see, *e.g.*, [3]. In any case, the hard square model has been solved in certain regions of the (L, M) plane, but not in the "unphysical" third quadrant.

Our approach to the problem of the kings is completely elementary. It depends on estimates for the entries of powers of matrices. These depend, in turn, on eigenvalue estimates. It is clear that for fixed M, the entries of M^n grow like polynomial multiples of R^n, where R is the largest eigenvalue of n. The coefficients of this polynomial can be bounded if the entries of M are bounded. Theorem 2 below gives the estimates we need, which may be be of some independent interest.

I would like to thank Herb Wilf for introducing me to this problem and the referee for suggesting the simplified proof of Theorem 2 which appears below.

To each configuration of mn kings on a $2m \times 2n$ chessboard, we associate two diagrams. Each diagram consists of an $m \times n$ array of squares, with an arrow in each square. Squares are labelled by an ordered pair (i, j), $1 \le i \le n$, $1 \le j \le m$, with the top row is labelled m and the bottom row 1. A square in the *vertical* diagram contains a $\uparrow$ if the corresponding cell on the chessboard contains a king in one of its two upper squares; otherwise it contains a $\downarrow$. Thus figure 1 has the following vertical diagram:

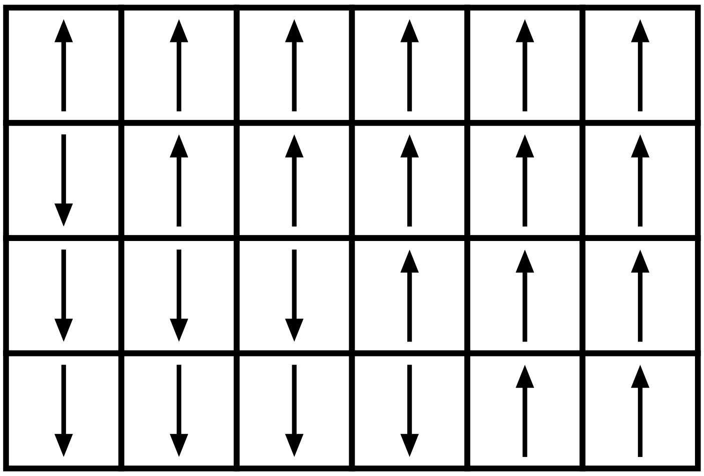

Figure 2

Likewise, each square in the horizontal diagram contains a $\leftarrow$ if the king in the corresponding cell is in one of the two left squares of the cell; otherwise it contains a $\rightarrow$. The horizontal diagram of figure 1 is

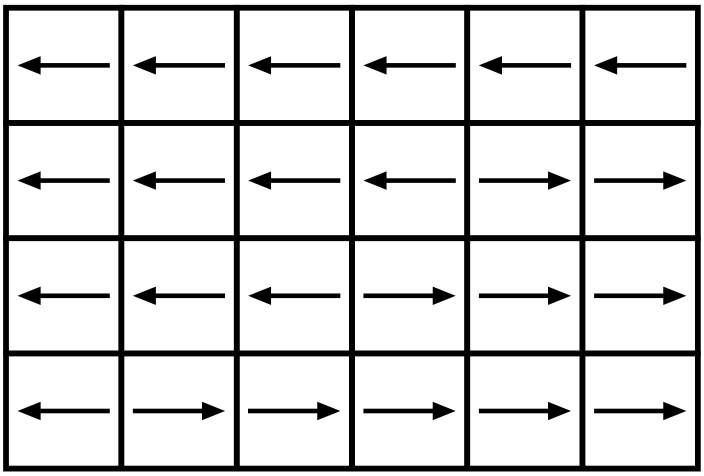

Figure 3

The condition that kings in vertically adjacent cells cannot attack each other amounts to the condition that every $\uparrow$ in a column of figure 2 must lie over every $\downarrow$ in that column; likewise, the condition that kings in horizontally adjacent cells cannot attack amounts to the condition that every $\leftarrow$ in a row in figure 3 lies to the left of every $\rightarrow$. Thus a vertical (resp. horizontal) diagram is specified uniquely by the number of $\downarrow$ (resp. $\leftarrow$) symbols in each column (resp. row). For a $2m \times 2n$ board, this is a an ordered n-tuple $(a_1, \ldots, a_n)$ (resp. m-tuple $(b_1, \ldots, b_m)$) of integers in $[0, m]$ (resp. $[0, n]$). Obviously a configuration of kings is uniquely specified by its two diagrams. The question that remains is which pairs of diagrams are compatible. The compatibility condition is dictated by the rule that diagonally adjacent squares in diagonally adjacent cells should not contain attacking kings. This actually amounts to two separate rules depending on whether the adjacency is NE$\leftrightarrow$SW or NW$\leftrightarrow$SE. The first gives rise to the rule that we cannot have

$$a_i < j < a_{i+1} \text{ and } b_j < i < b_{j+1} \tag{1}$$

for any i and j. The second implies that we cannot have

$$a_i > j > a_{i+1} \text{ and } b_j > i > b_{j+1}. \tag{2}$$

One way of visualizing the situation is to extend $j \mapsto a_j$ and $i \mapsto b_i$ to piecewise linear functions, $y = a(x)$ and $x = b(y)$ respectively, view their graphs as oriented curves, and ask that the curves not intersect with "positive" orientation. This is not quite right because the two diagrams are incompatible only if (1) or (2) is sharp for some (i, j). Nevertheless, this picture motivates the computations below.

Lemma 1. For all m and n,

$$K(m, n) \geq \lfloor n/2 \rfloor^m \lfloor m/2 \rfloor^n.$$

Proof. Consider all vertical diagrams satisfying

$$a_i \in \begin{cases} [0, m/2] & \text{if } i \leq n/2, \\ [m/2, m] & \text{if } i > n/2, \end{cases}$$

and all horizontal diagrams satisfying

$$b_j \in \begin{cases} [n/2, n] & \text{if } j \leq m/2, \\ [0, n/2] & \text{if } j > m/2. \end{cases}$$

There cannot be an incompatibility among two such diagrams. Indeed, if (1) holds for some (i, j), we have

$$i \leq \frac{n}{2} \Rightarrow b_j < \frac{n}{2} \Rightarrow j > \frac{m}{2} \Rightarrow a_{i+1} > \frac{m}{2} \Rightarrow i+1 > \frac{n}{2} \Rightarrow b_{j+1} \geq i+1 > \frac{n}{2} \Rightarrow j+1 \leq \frac{m}{2}$$

and

$$i > n/2 \Rightarrow j > a_i \geq m/2 \Rightarrow i < b_{j+1} \leq n/2,$$

a contradiction either way. If (2) holds for some (i, j),

$$j \leq m/2 \Rightarrow a_i > m/2 \Rightarrow i > n/2 \Rightarrow j > a_{i+1} > m/2,$$

and

$$j > m/2 \Rightarrow i < b_j \leq n/2 \Rightarrow m/2 \geq a_i > j,$$

again a contradiction. The lemma follows. □

From this and the trivial upper bound $K(m, n) \leq (m+1)^n (n+1)^m$ we obtain

Theorem 1. For all positive integers m and n,

$$\log K(m, n) = m \log n + n \log m + O(m + n).$$

□

In order to improve the error term, we need a better upper bound for K. To get it, we need to bound the size of entries of powers of matrices. Let $\Sigma(M)$ denote the sum of the entries of a matrix M.

Theorem 2. Let M be a $k \times k$ matrix with entries in $[0, 1]$. For all $n > k$,

$$\Sigma(M^n) < 35(15k)^k n^k \sup(r(M)^n, 1)$$

where $r(M)$ denotes the largest absolute value of an eigenvalue of M.

Proof. Note first that if M is nilpotent, $\Sigma(M^n) = 0$ for all $n \geq k$. We are therefore justified in assuming henceforth that $r(M) > 0$.

If v denotes the column vector of length k with all entries 1, the sum of the entries of M^n is

$$S_n := {}^t v M^n v.$$

By the Cayley-Hamilton theorem, the sequence S satisfies the linear recurrence

$$\sum_{i=0}^{k} c_i S_{n+i} = 0 \quad \forall n \geq 0, \tag{3}$$

where $\sum_i c_i x^i$ is the characteristic polynomial of M. Moreover, if $I_{p,q}$ denotes the $p \times q$ matrix with every entry equal to 1, then each entry of M^n is dominated by the corresponding entry of $I^n_{k,k}$, so $S_n \leq k^{n+1}$ for all n. Setting

$$P(x) = \sum_{i=0}^{k} c_i x^{k-i}, \quad S(x) = \sum_{i=0}^{\infty} S_i x^i,$$

we conclude that $P(x)S(x)$ is a polynomial $Q(x) = \sum_i q_i x^i$ of degree less than k. As the absolute value of the determinant of a $j \times j$ matrix with entries in $[0, 1]$ is bounded by $j!$, we have

$$c_i \leq k(k-1)\cdots(i+1) \leq k^{k-i}$$

and therefore

$$q_i \leq (i+1)k^{i+1}.$$

By the residue theorem,

$$S_n = \oint \frac{Q(z)}{z^{n+1}P(z)} dz,$$

as long as the contour of integration is contained in an open disk about the origin of radius $1/r(M)$. We choose a counter-clockwise circle of radius $\frac{n}{(n+k+1)r(M)}$. For every point z on this contour,

$$|P(z)| \geq \left(\frac{k+1}{(n+k+1)r(M)}\right)^k$$

and

$$|Q(z)| \leq (k^{k+1} + (k-1)k^{k-1} + (k-2)k^{k-2} + \cdots) \sup(r(M)^{1-k}, 1) \leq k^{k+2} \sup(r(M)^{1-k}, 1).$$

Therefore, the integral is bounded above by

$$2\pi r(M)^{-1} k^{k+2} \sup(r(M)^{1-k}, 1) r(M)^k (n+k+1)^k (k+1)^{-k} r(M)^{n+1} (n+k+1)^{n+1} n^{-n-1}$$

and thus by

$$2\pi k^2 \sup(r(M)^{n+k}, 1)(n+k+1)^k \frac{n+k+1}{n} e^{k+1}.$$

As $n \geq k+1$, $\sup(r(M)) \leq k$, $e^k \geq k^e > k^2$, $4\pi e < 35$, and $2e^2 < 15$, we obtain the desired upper bound of

$$4\pi e k^2 (2ek)^k n^k \sup(r(M)^n, 1) < 35(15k)^k n^k \sup(r(M)^n, 1). \qquad \square$$

Fix $k \geq 3$ and assume that $m' = m/k$ and $n' = n/k$ are integers. We divide each diagram, horizontal and vertical, into a $k \times k$ array of $m' \times n'$ *blocks*, indexed by ordered pairs (p, q), $1 \leq p, q \leq k$, and consisting of the cells with integral coordinates in the rectangle $[(p-1)n' + 1, pn'] \times [(q-1)m' + 1, qm']$. We say that the (p, q) block is of type $\uparrow$ if there exists $i \in [(p-1)n' + 1, pn' - 2]$ such that $a_i \leq (q-1)m' < qm' \leq a_{i+1}$. In effect, the graph of $i \mapsto a_i$ cuts entirely through the block from below. We define blocks of type ↓, ←, and → analogously. For example, the figure below shows a vertical diagram, its a-function, and its ↑ and ↓ blocks:

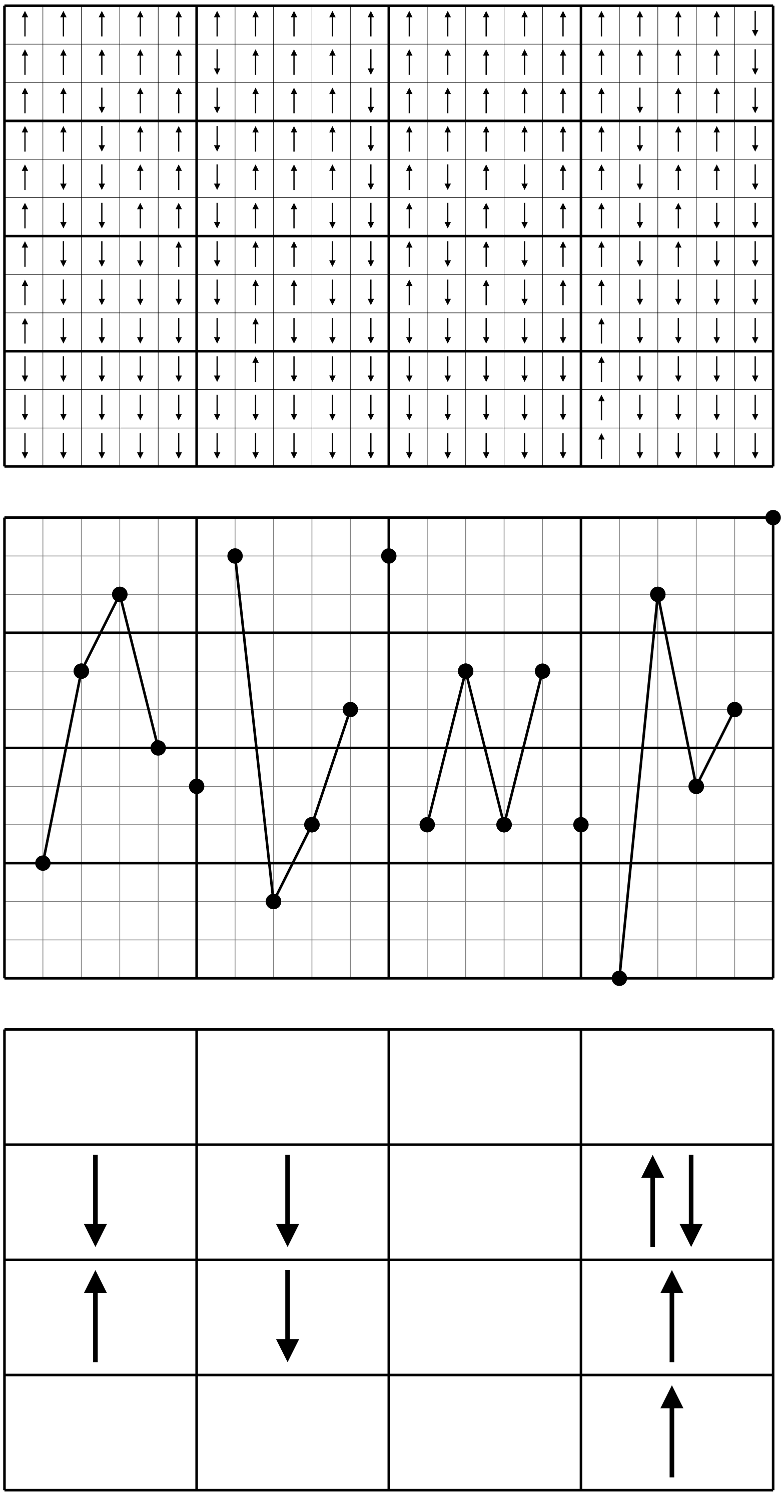

Figure 4

Thanks to condition (1), a block cannot be both of type ↑ and of type →, and thanks to condition (2), it cannot be both of type $\downarrow$ and of type $\leftarrow$. It turns out that we need to use only the first of these conditions.

We consider the single column X^p of blocks of the form $\{(p,i) \mid 1 \leq i \leq k\}$. Given a subset $U = U^p$ of X^p, we consider the number of sequences a_j, $j \in [(p-1)n'+1, pn'-2]$, for which the set of blocks of type $\uparrow$ is contained in U. Such sequences are characterized by the rule

$$a_j \leq (i-1)m' < im' \leq a_{j+1} \Rightarrow \ (p,i) \in U. \tag{4}$$

The number of such sequences is $\Sigma(M_U^{n'-2})$, where M_U is the incidence matrix defined by condition (4). More precisely, M_U is a $\{0,1\}$-matrix whose $(u+1, v+1)$ entry is 1 if and only if

$$v \leq (i-1)m' < im' \leq u \Rightarrow \ (p,i) \in U, \quad 0 \leq u, v \leq m$$

We extend the $m+1 \times m+1$ matrix M_U to a $(k+1)m' \times (k+1)m'$ matrix $\tilde{M_U}$ by adding $m'-1$ zero-rows and $m'-1$ zero-columns, on the top and left respectively. For example, if $m' = 2$, $k = 3$, and U is empty, we obtain

$$\tilde{M_U} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{5}$$

Let N_U denote the $(k+1)m' \times (k+1)m'$ $\{0,1\}$-matrix whose (u,v) entry is 1 if and only if

$$\lfloor v/m' \rfloor \leq i-1 < i+1 \leq \lfloor u/m' \rfloor \Rightarrow \ (p,i) \in U, \quad 0 \leq u, v \leq m+m'-1.$$

For example, under the same conditions as (5), we obtain

$$N_U = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

As $i+1 \leq \lfloor u/m' \rfloor$ implies $im' \leq u-(m'-1)$, each term in N_U is at least as great as the corresponding term in $\tilde{M_U}$, and

$$\Sigma(N_U^{n'-2}) \geq \Sigma(\tilde{M}_U^{n'-2}) \geq \Sigma(M_U^{n'-2}) \tag{6}$$

On the other hand,

$$N_U = I_{m',m'} \otimes P_U,$$

where P_U is the $\{0,1\}$-matrix with $(u+1, v+1)$ entry 1 if and only if

$$v \le i-1 < i+1 \le u \Rightarrow \ (p,i) \in U.$$

Thus

$$\Sigma(N_U^{n'-2}) = \Sigma(I_{m',m'}^{n'-2})\Sigma(P_U^{n'-2}) = (m')^{n'-1}\Sigma(P_U^{n'-2}). \tag{7}$$

By Theorem 2,

$$\Sigma(P_U^{n'-2}) < 35(15k)^k (n')^k r(P_U)^{n'}, \tag{8}$$

so we would like to estimate $r(P_U)$.

Consider the equivalence relation on $\{1,2,\ldots,k+1\}$ generated by the relation $i \sim i+1$ if $(p,i) \in U$. Let c_i be the cardinality of the i^{th} equivalence class, where the equivalence classes are arranged by the increasing size of their elements. Thus

$$P_U = \begin{pmatrix} I_{c_1,c_1} & I_{c_1,c_2} & I_{c_1,c_3} & \cdots & I_{c_1,c_r} \\ R_{c_2,c_1} & I_{c_2,c_2} & I_{c_2,c_3} & \cdots & I_{c_2,c_r} \\ 0 & R_{c_2,c_3} & I_{c_3,c_3} & \cdots & I_{c_3,c_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{c_r,c_r} \end{pmatrix},$$

where $R_{p,q}$ is the $p \times q$ matrix whose first row consists of entries 1 but whose other entries are zero. Letting Q_ℓ denote the $\ell \times \ell$ matrix whose (j,k) entry is

$$e^{\frac{2\pi i(j-1)(k-1)}{\ell}},$$

and conjugating by

$$\begin{pmatrix} Q_{c_1} & 0 & \cdots & 0 \\ 0 & Q_{c_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{c_r} \end{pmatrix},$$

we obtain

$$P_U' = \begin{pmatrix} c_1 E_{c_1,c_1} & \sqrt{c_1 c_2} E_{c_1,c_2} & \sqrt{c_1 c_3} E_{c_1,c_3} & \cdots & \sqrt{c_1 c_r} E_{c_1,c_r} \\ \sqrt{c_1/c_2}\, C_{c_2,c_1} & c_2 E_{c_2,c_2} & \sqrt{c_2 c_3} E_{c_2,c_3} & \cdots & \sqrt{c_2 c_r} E_{c_2,c_r} \\ 0 & \sqrt{c_2/c_3}\, C_{c_2,c_3} & c_3 E_{c_3,c_3} & \cdots & \sqrt{c_3 c_r} E_{c_3,c_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_r E_{c_r,c_r} \end{pmatrix},$$

where $E_{i,j}$ denotes the $i \times j$ matrix with zero entries except for a one in the upper left corner, and $C_{i,j}$ denotes the $i \times j$ matrix with zero entries except for ones in the first column. It follows that the non-zero eigenvalues of P_U coincide with the non-zero eigenvalues of

$$\begin{pmatrix} c_1 & \sqrt{c_1 c_2} & \sqrt{c_1 c_3} & \cdots & \sqrt{c_1 c_r} \\ \sqrt{c_1/c_2} & c_2 & \sqrt{c_2 c_3} & \cdots & \sqrt{c_2 c_r} \\ 0 & \sqrt{c_2/c_3} & c_3 & \cdots & \sqrt{c_3 c_r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_r \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\left\{1-\left(\frac{c_1}{(c_1-\lambda)(c_2-\lambda)}+\frac{c_2}{(c_2-\lambda)(c_3-\lambda)}+\cdots+\frac{c_{r-1}}{(c_{r-1}-\lambda)(c_r-\lambda)}\right)\right.$$
$$\left.+\left(\frac{c_1}{(c_1-\lambda)(c_2-\lambda)(c_3-\lambda)}+\cdots+\frac{c_{r-2}}{(c_{r-2}-\lambda)(c_{r-1}-\lambda)(c_r-\lambda)}\right)-\cdots\right\}\prod_{i=1}^{r}(c_i-\lambda).$$

As $\sum_i c_i = k+1$,

$$r(P_U) \leq 2\sqrt{k+1} + \sup_i c_i.$$

On the other hand,

$$(\sup_i c_i) - 1 \leq \sum_i (c_i - 1) = |U| + 1.$$

We deduce that

$$r(P_U) \leq 2\sqrt{k+1} + 1 + |U|.$$

Combining this with $r \leq k+1$, and the inequalities (6), (7), and (8), we conclude that

$$\Sigma(M_{U^p}^{n'-2}) \leq 35(m')^{n'}(15n'(k+1))^{k+1}\left(|U^p|+1+2\sqrt{k+1}\right)^{n'}.$$

We fix an array of subsets $\vec{U} = (U^1, \ldots, U^k)$, $U^i \subset X^i$, with a total of

$$h = |U^1| + \cdots + |U^p|$$

blocks. The total number of vertical diagrams whose blocks of type $\uparrow$ all lie in $U^1 \cup \cdots \cup U^k$, is

$$\prod_{p=1}^{k} 35(m')^{n'}(15n'(k+1))^{k+1}\left(|U^p|+1+2\sqrt{k+1}\right)^{n'}(m+1)$$
$$\leq 35^k(m')^n(15n'(k+1))^{k^2+k}\left(\frac{h+k+2\sqrt{2}k\sqrt{k}}{k}(m+1)^k\right)^n$$
$$= (35m+35)^k m^n (15n'(k+1))^{k^2+k}(hk^{-2}+k^{-1}+2\sqrt{2}k^{-1/2})^n.$$

Likewise, fixing an array $\vec{L} = (L^1, \ldots, L^k)$, each L^i a subset of the i^{th} row of blocks, the number of horizontal diagrams whose blocks of type $\rightarrow$ are contained in $L^1 \cup \cdots \cup L^k$ is bounded by

$$(35n+35)^k n^m (15m'(k+1))^{k^2+k}(h'k^{-2}+k^{-1}+2\sqrt{2}k^{-1/2})^n.$$

As there are only 2^{k^2} ways of choosing $\vec{U}$ and 2^{k^2} ways of choosing $\vec{L}$, the number of pairs of compatible diagrams is less than

$$(35m+35)^k(35n+35)^k m^n n^m (900m'n'(k+1)^2)^{k^2+k} S,$$

where

$$S = \sup_{0 \le h \le k^2} ((2\sqrt{2}k^{-1/2} + k^{-1}) + hk^{-2})^n ((2\sqrt{2}k^{-1/2} + k^{-1}) + (1 - hk^{-2}))^m.$$

If n is a perfect 5^{th} power, we set $k = n^{2/5}$, and obtain

$$\begin{aligned}\log K(m,n) &\le n^{2/5}(\log(35m+35) + \log(35n+35)) + m\log n + n\log m \\ &\quad + n^{4/5}\log(1800mn) + n^{2/5}\log(1800mn) + \log S.\end{aligned}$$

When $m = n$, by the arithmetic-geometric mean inequality,

$$\begin{aligned}\log K(n,n) &\le 2n^{2/5}\log(35n+35) + 2n\log n + n^{4/5}\log(1800n^2) \\ &\quad + n^{2/5}\log(1800n^2) + 2n\log(1/2 + 2\sqrt{2}k^{-1/2} + 2k^{-1}) \\ &= 2n\log n - 2n\log 2 + (4\sqrt{2}+2)n^{4/5}\log(n) + O(n^{4/5}).\end{aligned}$$

This result applies even when n is not a perfect 5^{th} power. Indeed, $K(m,n)$ is monotonically increasing in each variable, so setting $N = \lceil n^{\frac{1}{5}} \rceil^n$,

$$\begin{aligned}K(n,n) \le K(N,N) &\le 2N\log N - 2N\log 2 + O(N^{4/5}\log N) \\ &= 2n\log n - 2n\log 2 + O(n^{4/5}\log n).\end{aligned}$$

Combining this upper bound with Lemma 1, we deduce

Theorem 3. For all positive integers n,

$$\log K(n,n) = 2n\log n - 2n\log 2 + O(n^{4/5}\log n).$$

$\square$

REFERENCES

[1] R. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

[2] N. Elkies, G. Kuperberg, M. Larsen, J. Propp, Domino tilings of Aztec diamonds I, II *J. of Algebraic Comb.* **1** (1992) 111–132, 219–234.

[3] N. Elkies, R. Stanley, Mathematical Chess, in preparation.

[4] I. Rivin, I. Vardi, P. Zimmermann, The n-queens problem, *American Mathematical Monthly* (1994) 629–638.

[5] H. Wilf, The problem of the kings, *Electronic Journal of Combinatorics* **2** (1995) R3.