Towards Lehel's Conjecture for 4-Uniform Tight Cycles

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Submitted: Jul 28, 2021; Accepted: Nov 28, 2022; Published: Jan 13, 2023

Abstract

A k-uniform tight cycle is a k-uniform hypergraph with a cyclic ordering of its vertices such that its edges are all the sets of size k formed by k consecutive vertices in the ordering. We prove that every red-blue edge-coloured $K_n^{(4)}$ contains a red and a blue tight cycle that are vertex-disjoint and together cover $n - o(n)$ vertices. Moreover, we prove that every red-blue edge-coloured $K_n^{(5)}$ contains four monochromatic tight cycles that are vertex-disjoint and together cover $n - o(n)$ vertices.

Mathematics Subject Classifications: 05C65, 05C35, 05C70

1 Introduction

An *r-edge-colouring* of a graph (or hypergraph) is a colouring of its edges with r colours. A *monochromatic subgraph* of an r-edge-coloured graph is one in which all the edges have the same colour.

Lehel conjectured that every 2-edge-colouring of the complete graph on n vertices admits a partition of the vertex set into two monochromatic cycles of distinct colours, where the empty set, a single vertex and a single edge are considered to be *degenerate cycles*. This conjecture was proved for large n by Łuczak, Rödl and Szemerédi [16] using Szemerédi's Regularity Lemma. Allen [1] improved the bound on n by giving a different proof. Finally Bessy and Thomassé [3] proved Lehel's conjecture for all $n \geqslant 1$.

Similar problems have also been considered for colourings with a general number of colours. In particular, a lot of attention has been given to the problem of determining the number of monochromatic cycles that are needed to partition an r-edge-coloured complete graph. Erdős, Gyárfás and Pyber [6] proved that every r-edge-coloured complete

graph can be partitioned into $O(r^2 \log r)$ monochromatic cycles and conjectured that r monochromatic cycles would suffice. Their result was improved by Gyárfás, Ruszinkó, Sárközy and Szemerédi [11] who showed that $O(r \log r)$ monochromatic cycles are enough. However, Pokrovskiy [17] disproved the conjecture and proposed a weaker version of the conjecture that each r-edge-coloured complete graph contains r monochromatic vertex-disjoint cycles that together cover all but at most c_r of the vertices, where c_r is a constant depending only on r. Pokrovskiy [18] subsequently proved that we can take $c_3 \leqslant 43000$ for large enough n.

Recently, generalisations of Lehel's conjecture to hypergraphs have also been considered. For any positive integer k, a *k-uniform hypergraph*, or *k-graph*, H is an ordered pair of sets $(V(H), E(H))$ such that $E(H) \subseteq \binom{V(H)}{k}$, where $\binom{S}{k}$ is the set of all subsets of S of size k. We abuse notation by identifying the k-graph H with its edge set $E(H)$. Hence by $|H|$ we mean the number of edges of H. Let $K_n^{(k)}$ be the complete k-graph on n vertices.

In k-graphs there are several notions of cycle. For integers $1 \leqslant \ell < k < n$, a k-graph C on n vertices is called an *ℓ-cycle* if there is an ordering of its vertices $V(C) = \{v_0, \dots, v_{n-1}\}$ such that $E(C) = \{\{v_{i(k-\ell)}, \dots, v_{i(k-\ell)+k-1}\} \colon 0 \leqslant i \leqslant n/(k-\ell) - 1\}$, where the indices are taken modulo n. That is, an ℓ-cycle is a k-graph with a cyclic ordering of its vertices such that its edges are sets of k consecutive vertices and consecutive edges share exactly ℓ vertices. (Note that $k - \ell$ divides n.) A single edge or any set of fewer than k vertices is considered to be a *degenerate ℓ-cycle*. Further, 1-cycles and $(k-1)$-cycles are called *loose cycles* and *tight cycles*, respectively.

For loose cycles, Gyárfás and Sárközy [9] showed that every r-edge-coloured complete k-graph on n vertices can be partitioned into $c(k, r)$ monochromatic loose cycles. Sárközy [19] showed that, for n sufficiently large, $50kr \log(kr)$ loose cycles are enough. For tight cycles, Bustamante, Corsten, Frankl, Pokrovskiy and Skokan [4] showed that every r-edge-coloured complete k-graph can be partitioned into $C(k, r)$ monochromatic tight cycles. See [10] for a survey on other results about monochromatic cycle partitions and related problems.

In this paper, we investigate monochromatic tight cycle partitions in 2-edge-coloured complete k-graphs on n vertices. When $k = 3$, Bustamante, Hàn and Stein [5] showed that there exist two vertex-disjoint monochromatic tight cycles of distinct colours covering all but at most $o(n)$ of the vertices. Recently, Garbe, Mycroft, Lang, Lo and Sanhueza-Matamala [7] proved that two monochromatic tight cycles are sufficient to cover all vertices. However, these cycles may not be of distinct colours. First we show that for all $k \geqslant 3$, there are arbitrarily large 2-edge-coloured complete k-graphs that cannot be partitioned into two monochromatic tight cycles of distinct colours.

Proposition 1. *For all $k \geqslant 3$ and $m \geqslant k+1$, there exists a 2-edge-colouring of $K^{(k)}_{k(m+1)+1}$ that does not admit a partition into two tight cycles of distinct colours.*

It is natural to ask whether we can cover almost all vertices of a 2-edge-coloured complete k-graph with two vertex-disjoint monochromatic tight cycles of distinct colours. The case when $k = 3$ is affirmed in [5]. Here, we show that this is true when $k = 4$.

Theorem 2. *For every $\varepsilon > 0$, there exists an integer n_1 such that, for all $n \geqslant n_1$, every 2-edge-coloured complete 4-graph on n vertices contains two vertex-disjoint monochromatic tight cycles of distinct colours covering all but at most εn of the vertices.*

When $k = 5$, we prove a weaker result that four monochromatic tight cycles are sufficient to cover almost all vertices.

Theorem 3. *For every $\varepsilon > 0$, there exists an integer n_1 such that, for all $n \geqslant n_1$, every 2-edge-coloured complete 5-graph on n vertices contains four vertex-disjoint monochromatic tight cycles covering all but at most εn of the vertices.*

To prove Theorems 2 and 3, we use the *connected matching method* that has often been credited to Łuczak [15]. We now present a sketch proof for Theorem 2. Consider a 2-edge-coloured complete 4-graph $K_n^{(4)}$ on n vertices. We start by applying the Hypergraph Regularity Lemma to the 2-edge-coloured complete 4-graph $K_n^{(4)}$. More precisely the Regular Slice Lemma of Allen, Böttcher, Cooley and Mycroft [2], see Lemma 11. We obtain a 2-edge-coloured reduced graph $\mathcal{R}$ that is almost complete. A monochromatic matching in a k-graph is a set of vertex-disjoint edges of the same colour. We say that it is tightly connected if, for any two edges f and f', there exists a sequence of edges $e_1, \dots, e_t$ of the same colour such that $e_1 = f$, $e_t = f'$ and $|e_i \cap e_{i+1}| = k - 1$ for all $i \in [t-1]$. Using Corollary 20, it suffices to find two vertex-disjoint monochromatic tightly connected matchings of distinct colours in the reduced graph $\mathcal{R}$. The main challenge is to identify the 'tightly connected components' (see Section 2 for the formal definition) in which we will find the matchings. To do so, we introduce the concept of 'blueprint', which is a 2-edge-coloured 2-graph with the same vertex set as $\mathcal{R}$. The key property is that connected components in the blueprint correspond to tightly connected components in $\mathcal{R}$.

We conclude the introduction by outlining the structure of the paper. In Section 2, we introduce some basic notation and definitions. In Section 3, we prove Proposition 1. In Section 4, we introduce the statements about hypergraph regularity and prove the crucial Corollary 20 that allows us to reduce our problem of finding cycles in the complete graph to one about finding tightly connected matchings in the reduced graph. In Section 5, we give the definition of blueprint and setup some useful results. In Sections 6 and 7, we prove Theorems 2 and 3, respectively. Finally, we make some concluding remarks in Section 8.

2 Preliminaries

If we say that a statement holds for $0 < a \ll b \leqslant 1$, then we mean that there exists a non-decreasing function $f\colon (0,1] \to (0,1]$ such that the statement holds for all $a, b \in (0,1]$ with $a \leqslant f(b)$. Similar expressions with more variables are defined analogously. If $1/n$ appears in one of these expressions, then we implicitly assume that n is a positive integer.

We often write $x_1 \dots x_j$ for the set $\{x_1, \dots, x_j\}$. Moreover, for each positive integer n, we let $[n] = \{1, \dots, n\}$.

Throughout this paper, any 2-edge-colouring uses the colours red and blue. Let H be a 2-edge-coloured k-graph. We denote by H^{red} (and H^{blue}) the subgraph of H on $V(H)$ induced by the red (and blue) edges of H. Two edges f and f' in H are *tightly connected* if there exists a sequence of edges $e_1, \dots, e_t$ such that $e_1 = f$, $e_t = f'$ and $|e_i \cap e_{i+1}| = k-1$ for all $i \in [t-1]$. A subgraph H' of H is *tightly connected* if every pair of edges in H' is tightly connected in H. A maximal tightly connected subgraph of H is called a *tight component* of H. Note that a tight component is a subgraph rather than a vertex subset as in the traditional graph case. A *red tight component* and a *red tightly connected matching* are a tight component and a tightly connected matching in H^{red}, respectively. We define these terms similarly for *blue*.

Let H be a k-graph and $S, W \subseteq V(H)$. We denote by $H - W$ the k-graph with $V(H-W) = V(H) \setminus W$ and $E(H-W) = \{e \in E(H) \colon e \cap W = \varnothing\}$. We call $H - W$ the *k-graph obtained from H by deleting W*. Further we let $H[W] = H - (V(H) \setminus W)$. Let F be a k-graph or a set of k-element sets. We denote by $H - F$ the subgraph of H obtained by deleting the edges in F. We define $N_H(S, W)$ to be the set $\{e \in \binom{W}{k-|S|} \colon e \cup S \in H\}$ and we define $d_H(S, W)$ to be its cardinality. Further we write $N_H(S)$ and $d_H(S)$ for $N_H(S, V(H))$ and $d_H(S, V(H))$, respectively. If H is 2-edge-coloured, then we write $N_H^{\mathrm{red}}(S, W)$, $d_H^{\mathrm{red}}(S, W)$, $N_H^{\mathrm{blue}}(S, W)$, $d_H^{\mathrm{blue}}(S, W)$ for $N_{H^{\mathrm{red}}}(S, W)$, $d_{H^{\mathrm{red}}}(S, W)$, $N_{H^{\mathrm{blue}}}(S, W)$, $d_{H^{\mathrm{blue}}}(S, W)$, respectively. The *link graph of H with respect to S*, denoted by H_S, is the $(k - |S|)$-graph satisfying $V(H_S) = V(H) \setminus S$ and $E(H_S) = N_H(S)$.

For $j \in [k-1]$, the *j-th shadow of H*, denoted by $\partial^j H$, is the $(k-j)$-graph with vertex set $V(\partial^j H) = V(H)$ and edge set

$$E(\partial^j H) = \left\{ e \in \binom{V(H)}{k-j} \colon e \subseteq f \text{ for some } f \in E(H) \right\}.$$

For the 1-st shadow of H, we also simply write ∂H instead of $\partial^1 H$.

For $\mu, \alpha > 0$, we say that a k-graph H on n vertices is *(μ, α)-dense* if, for each $i \in [k-1]$, we have $d_H(S) \geqslant \mu \binom{n}{k-i}$ for all but at most $\alpha \binom{n}{i}$ sets $S \in \binom{V(H)}{i}$ and $d_H(S) = 0$ for all other $S \in \binom{V(H)}{i}$.

Proposition 4. *Let $0 \leqslant \alpha, \mu \leqslant 1$ and let H be a (μ, α)-dense k-graph on n vertices. Then $|H| \geqslant (\mu - \alpha)\binom{n}{k}$. Moreover, if $\mu > 1/2$, then H is tightly connected.*

Proof. Note that

$$|H| = \frac{1}{k} \sum_{S \in \binom{V(H)}{k-1}} d_H(S) \geqslant \frac{1}{k}(1-\alpha)\binom{n}{k-1}\mu n \geqslant (\mu - \alpha)\binom{n}{k}.$$

Now suppose that $\mu > 1/2$. We show that H is tightly connected. Note that, for $S, S' \in \binom{V(H)}{k-1}$ with $d_H(S), d_H(S') > 0$, we have $d_H(S), d_H(S') \geqslant \mu n > n/2$ and thus

$$N_H(S) \cap N_H(S') \neq \varnothing.$$

Let $f = x_1 \dots x_k$ and $f' = y_1 \dots y_k$ be two edges of H. Iteratively choose vertices $z_1, \dots, z_{k-1} \in V(H)$ such that

$$z_i \in N_H(z_1 \dots z_{i-1} x_{i+1} \dots x_k) \cap N_H(z_1 \dots z_{i-1} y_{i+1} \dots y_k)$$

for all $i \in [k-1]$. It follows that f and f' are tightly connected. □

The following proposition shows that any k-graph that has all but a small fraction of the possible edges contains a $(1-\varepsilon, \alpha)$-dense subgraph. The proof was inspired by the proof of Lemma 8.8 in [12]. A different generalisation of this lemma can also be found as Lemma 2.3 in [14].

Proposition 5. *Let $1/n \ll \alpha \ll 1/k \leqslant 1/2$. Let H be a k-graph on n vertices with $|H| \geqslant (1-\alpha)\binom{n}{k}$. Then there exists a subgraph H' of H such that $V(H') = V(H)$ and H' is $(1 - 2\alpha^{1/4k^2}, 2\alpha^{1/4k})$-dense.*

Proof. We call a set $S \subseteq V(H)$ with $|S| \in [k-1]$ *bad* if $d_H(S) < (1-\alpha^{1/2})\binom{n}{k-|S|}$. For $i \in [k-1]$, let $\mathcal{B}_i$ be the set of all bad i-sets. For each $i \in [k-1]$, we have

$$(1-\alpha)\binom{k}{i}\binom{n}{k} \leqslant \binom{k}{i}|H| = \sum_{S \in \binom{V(H)}{i}} d_H(S) \leqslant \binom{n}{i}\binom{n}{k-i} - \alpha^{1/2}\binom{n}{k-i}|\mathcal{B}_i|.$$

This implies

$$|\mathcal{B}_i| \leqslant \frac{1}{\alpha^{1/2}}\left(\binom{n}{i} - \frac{(1-\alpha)\binom{k}{i}\binom{n}{k}}{\binom{n}{k-i}}\right) \leqslant 2\alpha^{1/2}\binom{n}{i}.$$

Let $\beta = \alpha^{1/2k}$. For all $j \in \{k-1, k-2, \dots, 1\}$ in turn, we construct $\mathcal{A}_j \subseteq \binom{V(H)}{j}$ inductively as follows. We set $\mathcal{A}_{k-1} = \mathcal{B}_{k-1}$. Given $2 \leqslant j \leqslant k-1$ and $\mathcal{A}_j$, we define $\mathcal{A}_{j-1} \subseteq \binom{V(H)}{j-1}$ to be the set of all $X \in \binom{V(H)}{j-1}$ such that $X \in \mathcal{B}_{j-1}$ or $d_{\mathcal{A}_j}(X) \geqslant \beta^{1/2}n$.

Claim 6. *For all $i \in [k-1]$, $|\mathcal{A}_i| \leqslant \beta^i\binom{n}{i}$. Moreover, if $1 \leqslant i < j \leqslant k-1$ and a set $S \in \binom{V(H)}{i}$ satisfies $d_{\mathcal{A}_j}(S) \geqslant \beta^{1/2(j-i)}\binom{n}{j-i}$, then $S \in \mathcal{A}_i$.*

Proof of Claim. We first prove the first part by induction on $k-i$. For $i = k-1$, we have $|\mathcal{A}_{k-1}| = |\mathcal{B}_{k-1}| \leqslant 2\alpha^{1/2}\binom{n}{k-1} \leqslant \beta^{k-1}\binom{n}{k-1}$.

Now suppose $2 \leqslant i \leqslant k-1$ and $|\mathcal{A}_i| \leqslant \beta^i\binom{n}{i}$. By double counting tuples (X, w) with $X \in \mathcal{A}_{i-1} \setminus \mathcal{B}_{i-1}$ and $X \cup w \in \mathcal{A}_i$, we have $(|\mathcal{A}_{i-1}| - |\mathcal{B}_{i-1}|)\,\beta^{1/2}n \leqslant i\,|\mathcal{A}_i|$. Hence

$$\begin{aligned}|\mathcal{A}_{i-1}| &\leqslant \frac{i}{\beta^{1/2}n}|\mathcal{A}_i| + |\mathcal{B}_{i-1}| \leqslant \frac{i}{\beta^{1/2}n}\beta^i\binom{n}{i} + 2\alpha^{1/2}\binom{n}{i-1} \\ &= \beta^{i-1/2}\binom{n-1}{i-1} + 2\alpha^{1/2}\binom{n}{i-1} \leqslant \beta^{i-1}\binom{n}{i-1}.\end{aligned}$$

This proves the first part of the claim.

We now prove the second part of the claim. Fix $i \in [k-1]$. We proceed by induction on $j-i$. For $j = i+1$, the statement holds by the definition of $\mathcal{A}_i$. Now let $S \in \binom{V(H)}{i}$ and $j \geqslant i+2$ be such that $d_{\mathcal{A}_j}(S) \geqslant \beta^{1/2(j-i)}\binom{n}{j-i}$. If $S \in \mathcal{B}_i$, then $S \in \mathcal{A}_i$. Recall that if $T \in \binom{V(H)}{j-1} \setminus \mathcal{A}_{j-1}$, then $d_{\mathcal{A}_j}(T) < \beta^{1/2}n$. We have

$$\begin{aligned}\beta^{1/2(j-i)}\binom{n}{j-i} &\leqslant d_{\mathcal{A}_j}(S) \leqslant \sum_{\substack{T\in\mathcal{A}_{j-1}\\ S\subseteq T}} d_{\mathcal{A}_j}(T) + \sum_{\substack{T\in\binom{V(H)}{j-1}\setminus\mathcal{A}_{j-1}\\ S\subseteq T}} d_{\mathcal{A}_j}(T)\\ &\leqslant n d_{\mathcal{A}_{j-1}}(S) + \beta^{1/2} n d_{\binom{V(H)}{j-1}\setminus\mathcal{A}_{j-1}}(S)\\ &\leqslant n d_{\mathcal{A}_{j-1}}(S) + \beta^{1/2} n\binom{n}{j-i-1},\end{aligned}$$

and thus

$$d_{\mathcal{A}_{j-1}}(S) \geqslant \beta^{1/2(j-i-1)}\binom{n}{j-i-1}.$$

Hence by the induction hypothesis we have $S \in \mathcal{A}_i$. □

For each $j \in [k-1]$, let F_j be the set of edges $e \in H$ for which there exists some $S \in \mathcal{A}_j$ with $S \subseteq e$. Let $F = \bigcup_{j\in[k-1]} F_j$ and let $H' = H - F$. We will show that H' is the desired k-graph. For $i \in [k-1]$, let $\mathcal{S}_i$ be the set of all $S \in \binom{V(H)}{i}$ such that $d_F(S) \geqslant \beta^{1/2k}\binom{n}{k-i}$.

Claim 7. *For $i \in [k-1]$, $|\mathcal{S}_i| \leqslant \beta^{1/2}\binom{n}{i}$.*

Proof of Claim. For $j \in [k-1]$, we have

$$|F_j| \leqslant |\mathcal{A}_j|\binom{n-j}{k-j} \overset{\text{Claim 6}}{\leqslant} \beta^j\binom{n}{j}\binom{n-j}{k-j} = \beta^j\binom{k}{j}\binom{n}{k}.$$

Thus

$$|F| \leqslant \sum_{j\in[k-1]} |F_j| \leqslant \sum_{j\in[k-1]} \beta^j\binom{k}{j}\binom{n}{k} \leqslant 2^k\beta\binom{n}{k}.$$

Now, for $i \in [k-1]$, we have

$$\frac{|\mathcal{S}_i|\,\beta^{1/2k}\binom{n}{k-i}}{\binom{k}{i}} \leqslant |F| \leqslant 2^k\beta\binom{n}{k}$$

and thus $|\mathcal{S}_i| \leqslant \beta^{1/2}\binom{n}{i}$. □

Consider $i \in [k-1]$. Note that $|\mathcal{S}_i \cup \mathcal{B}_i| \leqslant 2\alpha^{1/4k}\binom{n}{i}$. Now let $S \in \binom{V(H)}{i} \setminus (\mathcal{S}_i \cup \mathcal{B}_i)$. As $S \notin \mathcal{B}_i$, we have $d_H(S) \geqslant (1-\alpha^{1/2})\binom{n}{k-i}$. As $S \notin \mathcal{S}_i$, we have

$$\begin{aligned}d_{H'}(S) = d_H(S) - d_F(S) &\geqslant d_H(S) - \beta^{1/2k}\binom{n}{k-i}\\ &\geqslant (1-\alpha^{1/2}-\beta^{1/2k})\binom{n}{k-i} \geqslant (1-2\alpha^{1/4k^2})\binom{n}{k-i}.\end{aligned}$$

Consider $X \in \binom{V(H)}{i}$ with $d_{H'}(X) \neq 0$. We want to show that $d_{H'}(X) \geqslant (1-2\alpha^{1/4k^2})\binom{n}{k-i}$. By the above, it suffices to show that $X \notin \mathcal{B}_i \cup \mathcal{S}_i$. Let $e \in H'$ with $X \subseteq e$. Since $e \notin F_i$, we have $X \notin \mathcal{A}_i$ and thus $X \notin \mathcal{B}_i$. It remains for us to show that $X \notin \mathcal{S}_i$. Assume the contrary that X is contained in more that $\beta^{1/2k}\binom{n}{k-i}$ edges of F. Let $\mathcal{Y} = N_F(X)$, so $|\mathcal{Y}| \geqslant \beta^{1/2k}\binom{n}{k-i}$. For each $Y \in \mathcal{Y}$, fix a set $A_Y \in \bigcup_{j\in[k-1]} \mathcal{A}_j$ such that $A_Y \subseteq X \cup Y$ and let $T_Y = X \cap A_Y$ and $S_Y = Y \setminus A_Y$. If $A_Y \subseteq X$, then $A_Y \subseteq e \in H'$, a contradiction. Hence $A_Y \setminus X \neq \varnothing$ for all $Y \in \mathcal{Y}$. Thus, for $Y \in \mathcal{Y}$, we have $|T_Y| \leqslant |A_Y| - 1 \leqslant k-2$. By an averaging argument, there exist $t \in \{0, 1, \ldots, k-2\}$, $T \in \binom{X}{t}$, $a \in [k-1]$, $S \in \binom{V(H)}{k-i-a+t}$ and $\widetilde{\mathcal{Y}} \subseteq \mathcal{Y}$ such that, for all $Y \in \widetilde{\mathcal{Y}}$, we have $T_Y = T$, $|A_Y| = a$, $S_Y = S$ and

$$\left|\widetilde{\mathcal{Y}}\right| \geqslant \frac{|\mathcal{Y}|}{2^i(k-1)\binom{n}{k-i-a+t}} \geqslant \beta^{1/2(k-1)}\binom{n}{a-t}.$$

Since $Y \setminus A_Y = S_Y = S$ and $|A_Y| = a$ for all $Y \in \widetilde{\mathcal{Y}}$, the A_Y are distinct for all $Y \in \widetilde{\mathcal{Y}}$. Recall that $T \subseteq A_Y \in \mathcal{A}_a$ for each $Y \in \widetilde{\mathcal{Y}}$. If $T = \varnothing$, then $t = 0$ and so $|\mathcal{A}_a| \geqslant \left|\widetilde{\mathcal{Y}}\right| > \beta^a\binom{n}{a}$ contradicting Claim 6. If $T \neq \varnothing$, then we have $d_{\mathcal{A}_a}(T) \geqslant \left|\widetilde{\mathcal{Y}}\right| \geqslant \beta^{1/2(k-1)}\binom{n}{a-t}$. Claim 6 implies that $T \in \mathcal{A}_t$. Since $T \subseteq X \subseteq e$, we have $e \in F_t$ contradicting the fact that $e \in H' = H - \bigcup_{j\in[k-1]} F_j$. □

3 Extremal example

In this section, we prove Proposition 1, that is, we prove that, for $k \geqslant 3$, there exist arbitrarily large 2-edge-coloured complete k-graphs that do not admit a partition into two tight cycles of distinct colours.

A *k-uniform tight path* is a k-graph obtained by deleting a vertex from a tight cycle. First we need the following proposition.

Proposition 8. *Let $k \geqslant 3$, let P and C be a k-uniform tight path and tight cycle, respectively. We have the following.*

(i) *If X and Y partition $V(P)$ such that $|e \cap Y| \geqslant 2$ for all $e \in P$, then $2(|X|-(k-1)) \leqslant (k-2)|Y|$.*

(ii) *If X and Y partition $V(C)$ such that $|e \cap Y| \geqslant 2$ for all $e \in C$, then $2|X| \leqslant (k-2)|Y|$.*

Proof. We first prove (i). Let M be a matching of maximum size in P. Since each edge of P contains at least 2 vertices of Y,

$$|X| \leqslant |X \cap V(M)| + |V(P) \setminus V(M)| \leqslant (k-2)|M| + k - 1 \leqslant \frac{(k-2)|Y|}{2} + k - 1.$$

Now we prove (ii). Since $|e \cap Y| \geqslant 2$ and $|e \cap X| \leqslant k-2$ for each edge $e \in C$, we have

$$|X| = \frac{1}{k}\sum_{e\in C} |e \cap X| = \frac{1}{k}\sum_{e\in C} \frac{|e \cap X|}{|e \cap Y|}\,|e \cap Y|$$

$$\leqslant \frac{1}{k}\sum_{e\in C}\frac{k-2}{2}\left|e\cap Y\right| = \frac{k-2}{2}\left|Y\right|. \qquad \square$$

We are now ready to give our extremal example. Note that the case $k = 3$ of the extremal example is already given in [7]. Recall that, in a k-graph, we consider a single edge and any set of fewer than k vertices to be degenerate cycles.

Proof of Proposition 1. Let $k \geqslant 3$, $m \geqslant k+1$ and $n = k(m+1)+1$. Let X, Y and $\{z\}$ be three disjoint vertex sets of $K_n^{(k)}$ of sizes $(k-1)m+k-2$, $m+2$ and 1, respectively. We colour an edge e in $K_n^{(k)}$ red if $z \in e$ and $|e\cap Y| \geqslant 2$ or $z \notin e$ and $|e\cap Y| = 1$. Otherwise we colour it blue. Note that $K_n^{(k)} - z$ has the following 3 monochromatic tight components:

$$B_1 = \binom{X}{k},\ B_2 = \left\{e \in \binom{X\cup Y}{k} :\ |e\cap Y| \geqslant 2\right\},\ R = \left\{e \in \binom{X\cup Y}{k} :\ |e\cap Y| = 1\right\}.$$

Note that B_1 and B_2 are blue and R is red. Suppose for a contradiction that $K_n^{(k)}$ can be partitioned into a red tight cycle C_R and a blue tight cycle C_B.

First assume $z \in V(C_R)$. Since all the red edges containing z are in a red tight component disjoint from R, we have $|V(C_R)| \leqslant k$. Hence $|V(C_B)| = n - |V(C_R)| \geqslant n-k \geqslant km > k$ and $|V(C_B)\cap Y| = |Y \setminus V(C_R)| \geqslant m+2-(k-1) \geqslant 1$. So C_B is not degenerate and $C_B \subseteq B_2$. Any edge $e \in C_B$ contains at least 2 vertices in Y. By Proposition 8(ii), $2\left|V(C_B)\cap X\right| \leqslant (k-2)\left|V(C_B)\cap Y\right|$. It follows that

$$\begin{aligned} 2(k-1)m-2 = 2(|X|-(k-1)) &\leqslant 2\left|V(C_B)\cap X\right| \\ &\leqslant (k-2)\left|V(C_B)\cap Y\right| \leqslant (k-2)\left|Y\right| = (k-2)(m+2). \end{aligned}$$

This implies that $m \leqslant 2$, a contradiction.

Hence, we may assume that $z \in V(C_B)$. This implies that $C_R \subseteq R$ or $|V(C_R)| \leqslant k-1$. Let $x_R = |V(C_R)\cap X|$, $y_R = |V(C_R)\cap Y|$, $x_B = |V(C_B)\cap X|$ and $y_B = |V(C_B)\cap Y|$. Let P_B be the tight path $C_B - z$. Clearly $|V(P_B)\cap X| = x_B$ and $|V(P_B)\cap Y| = y_B$. Since $C_R \subseteq R$ or $|V(C_R)| \leqslant k-1$,

$$y_R \leqslant \max\left\{\left\lfloor\frac{|X|}{k-1}\right\rfloor, k-1\right\} = m < \left|Y\right|. \tag{1}$$

Hence, $V(P_B)\cap Y \neq \varnothing$ and $|V(P_B)| \geqslant (n-1)-km \geqslant k$. We must have $P_B \subseteq B_2$. By Proposition 8(i), we have that

$$2(x_B-(k-1)) \leqslant (k-2)y_B. \tag{2}$$

Thus

$$\begin{aligned} |V(P_B)| = x_B + y_B &\leqslant \frac{k}{2}y_B + k - 1 \leqslant \frac{k}{2}\left|Y\right| + k - 1 = \frac{k}{2}(m+2) + k - 1 \\ &\leqslant mk = n-1-k. \end{aligned}$$

This implies that $|V(C_R)| \geqslant k$. Hence $C_R \subseteq R$ and thus

$$x_R = (k-1)y_R. \tag{3}$$

Since $x_R + x_B = |X| = (k-1)m + k - 2$ and $y_R + y_B = |Y| = m + 2$, (2) implies

$$\begin{aligned}(k-2)(m+2-y_R) &\geqslant 2(|X| - x_R - (k-1)) \\ &= 2((k-1)m + k - 2 - (k-1)y_R - (k-1)),\end{aligned}$$

which implies $y_R \geqslant m-1$. If $y_R = m-1$, then (3) implies that $x_R = (k-1)(m-1)$ and thus $x_B = 2k-3$ and $y_B = 3$. Let $P_B = v_1 \dots v_{2k}$. Either the edge $v_1 \dots v_k$ or the edge $v_{k+1} \dots v_{2k}$ contains at most one vertex of Y, a contradiction to $P_B \subseteq B_2$. Thus we may assume $y_R \geqslant m$ and since $y_R \leqslant m$ by (1), we have $y_R = m$. By (3), we have $x_R = (k-1)m$ and thus $x_B = k-2$ and $y_B = 2$. Hence, C_B is a copy of $K^{(k)}_{k+1}$ that has a blue edge containing z and at least two vertices of Y, a contradiction. □

4 Hypergraph regularity

In this section, we give the formulation of hypergraph regularity that we use, following closely the presentation of Allen, Böttcher, Cooley and Mycroft [2]. A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$. Again, we identify the hypergraph $\mathcal{H}$ with its edge set $E(\mathcal{H})$. A subgraph $\mathcal{H}'$ of $\mathcal{H}$ is a hypergraph with $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $E(\mathcal{H}') \subseteq E(\mathcal{H})$. It is *spanning* if $V(\mathcal{H}') = V(\mathcal{H})$. For $U \subseteq V(\mathcal{H})$, we define $\mathcal{H}[U]$ to be the subgraph of $\mathcal{H}$ with $V(\mathcal{H}[U]) = U$ and $E(\mathcal{H}[U]) = \{e \in E(\mathcal{H})\colon e \subseteq U\}$. We call $\mathcal{H}$ a *complex* if $\mathcal{H}$ is down-closed, that is if $e \in \mathcal{H}$ and $f \subseteq e$, then $f \in \mathcal{H}$. A *k-complex* is a complex with only edges of size at most k. We denote by $\mathcal{H}^{(i)}$ the spanning subgraph of $\mathcal{H}$ containing only the edges of size i. Let $\mathcal{P}$ be a partition of $V(\mathcal{H})$ into parts $V_1, \dots, V_s$. Then we say that a set $S \subseteq V(\mathcal{H})$ is *$\mathcal{P}$-partite* if $|S \cap V_i| \leqslant 1$ for all $i \in [s]$. For $\mathcal{P}' = \{V_{i_1}, \dots, V_{i_r}\} \subseteq \mathcal{P}$, we define the subgraph of $\mathcal{H}$ induced by $\mathcal{P}'$, denoted by $\mathcal{H}[\mathcal{P}']$ or $\mathcal{H}[V_{i_1}, \dots, V_{i_r}]$, to be the subgraph of $\mathcal{H}[\bigcup \mathcal{P}']$ containing only the edges that are $\mathcal{P}'$-partite. The hypergraph $\mathcal{H}$ is $\mathcal{P}$-partite if all of its edges are $\mathcal{P}$-partite. In this case we call the parts of $\mathcal{P}$ the *vertex classes* of $\mathcal{H}$. We say that $\mathcal{H}$ is *s-partite* if it is $\mathcal{P}$-partite for some partition $\mathcal{P}$ of $V(\mathcal{H})$ into s parts. Let $\mathcal{H}$ be a $\mathcal{P}$-partite hypergraph. If X is a k-set of vertex classes of $\mathcal{H}$, then we write $\mathcal{H}_X$ for the k-partite subgraph of $\mathcal{H}^{(k)}$ induced by $\bigcup X$, whose vertex classes are the elements of X. Moreover, we denote by $\mathcal{H}_{X^<}$ the k-partite hypergraph with $V(\mathcal{H}_{X^<}) = \bigcup X$ and $E(\mathcal{H}_{X^<}) = \bigcup_{X' \subsetneq X} \mathcal{H}_{X'}$. In particular, if $\mathcal{H}$ is a complex, then $\mathcal{H}_{X^<}$ is a $(k-1)$-complex because X is a set of size k.

Let $i \geqslant 2$, and let $\mathcal{P}_i$ be a partition of a vertex set V into i parts. Let H_i and H_{i-1} be a $\mathcal{P}_i$-partite i-graph and a $\mathcal{P}_i$-partite $(i-1)$-graph on a common vertex set V, respectively. We say that a $\mathcal{P}_i$-partite i-set in V is *supported on* H_{i-1} if it induces a copy of the complete $(i-1)$-graph $K_i^{(i-1)}$ on i vertices in H_{i-1}. We denote by $K_i(H_{i-1})$ the $\mathcal{P}_i$-partite i-graph on V whose edges are all $\mathcal{P}_i$-partite i-sets contained in V which are supported on H_{i-1}. Now we define the *density of H_i with respect to H_{i-1}* to be

$$d(H_i \mid H_{i-1}) = \frac{|K_i(H_{i-1}) \cap H_i|}{|K_i(H_{i-1})|}$$

if $|K_i(H_{i-1})| > 0$ and $d(H_i \mid H_{i-1}) = 0$ if $|K_i(H_{i-1})| = 0$. So $d(H_i \mid H_{i-1})$ is the proportion of $\mathcal{P}_i$-partite copies of K_i^{i-1} in H_{i-1} which are also edges of H_i. More generally, if $\mathbf{Q} = (Q_1, Q_2, \dots, Q_r)$ is a collection of r (not necessarily disjoint) subgraphs of H_{i-1}, we define $K_i(\mathbf{Q}) = \bigcup_{j=1}^{r} K_i(Q_j)$ and

$$d(H_i \mid \mathbf{Q}) = \frac{|K_i(\mathbf{Q}) \cap H_i|}{|K_i(\mathbf{Q})|}$$

if $|K_i(\mathbf{Q})| > 0$ and $d(H_i \mid \mathbf{Q}) = 0$ if $|K_i(\mathbf{Q})| = 0$. We say that H_i is (d_i, ε, r)-*regular with respect to* H_{i-1}, if we have $d(H_i \mid \mathbf{Q}) = d_i \pm \varepsilon$ for every r-set $\mathbf{Q}$ of subgraphs of H_{i-1} with $|K_i(\mathbf{Q})| > \varepsilon\,|K_i(H_{i-1})|$. We say that H_i is (ε, r)-*regular with respect to* H_{i-1} if there exists some d_i for which H_i is (d_i, ε, r)-regular with respect to H_{i-1}. Finally, given an i-graph G whose vertex set contains that of H_{i-1}, we say that G is (d_i, ε, r)-*regular with respect to* H_{i-1} if the i-partite subgraph of G induced by the vertex classes of H_{i-1} is (d_i, ε, r)-regular with respect to H_{i-1}. We refer to the density of this i-partite subgraph of G with respect to H_{i-1} as the *relative density of* G *with respect to* H_{i-1}.

Now let $s \geqslant k \geqslant 3$ and let $\mathcal{H}$ be an s-partite k-complex on vertex classes $V_1, \dots, V_s$. For any set $A \subseteq [s]$, we write V_A for $\bigcup_{i \in A} V_i$. Note that, if $e \in \mathcal{H}^{(i)}$ for some $2 \leqslant i \leqslant k$, then the vertices of e induce a copy of K_i^{i-1} in $\mathcal{H}^{(i-1)}$. Therefore, for any set $A \in \binom{[s]}{i}$, the density $d(\mathcal{H}^{(i)}[V_A] \mid \mathcal{H}^{(i-1)}[V_A])$ is the proportion of 'possible edges' of $\mathcal{H}^{(i)}[V_A]$, which are indeed edges. We say that $\mathcal{H}$ is $(d_k, \dots, d_2, \varepsilon_k, \varepsilon, r)$-*regular* if

(a) for any $2 \leqslant i \leqslant k-1$ and any $A \in \binom{[s]}{i}$, the induced subgraph $\mathcal{H}^{(i)}[V_A]$ is $(d_i, \varepsilon, 1)$-regular with respect to $\mathcal{H}^{(i-1)}[V_A]$, and

(b) for any $A \in \binom{[s]}{k}$, the induced subgraph $\mathcal{H}^{(k)}[V_A]$ is (d_k, ε_k, r)-regular with respect to $\mathcal{H}^{(k-1)}[V_A]$.

For $\mathbf{d} = (d_k, \dots, d_2)$, we write $(\mathbf{d}, \varepsilon_k, \varepsilon, r)$-regular to mean $(d_k, \dots, d_2, \varepsilon_k, \varepsilon, r)$-regular. We say that a $(k-1)$-complex $\mathcal{J}$ is (t_0, t_1, ε)-*equitable* if it has the following properties.

(a) $\mathcal{J}$ is $\mathcal{P}$-partite for some $\mathcal{P}$ which partitions $V(\mathcal{J})$ into t parts, where $t_0 \leqslant t \leqslant t_1$, of equal size. We refer to $\mathcal{P}$ as the *ground partition* of $\mathcal{J}$, and to the parts of $\mathcal{P}$ as the *clusters* of $\mathcal{J}$.

(b) There exists a *density vector* $\mathbf{d} = (d_{k-1}, \dots, d_2)$ such that, for each $2 \leqslant i \leqslant k-1$, we have $d_i \geqslant 1/t_1$ and $1/d_i \in \mathbb{N}$, and $\mathcal{J}$ is $(\mathbf{d}, \varepsilon, \varepsilon, 1)$-regular.

For any k-set X of clusters of $\mathcal{J}$, we denote by $\hat{\mathcal{J}}_X$ the k-partite $(k-1)$-graph $(\mathcal{J}_{X^<})^{(k-1)}$ and call $\hat{\mathcal{J}}_X$ a *polyad*. Given a (t_0, t_1, ε)-equitable $(k-1)$-complex $\mathcal{J}$ and a k-graph G on $V(\mathcal{J})$, we say that G is (ε_k, r)-*regular with respect to a* k-*set* X *of clusters of* $\mathcal{J}$ if there exists some d such that G is (d, ε_k, r)-regular with respect to the polyad $\hat{\mathcal{J}}_X$. Moreover, we write $d^*_{G,\mathcal{J}}(X)$ for the relative density of G with respect to $\hat{\mathcal{J}}_X$; we may drop either subscript if it is clear from context.

We can now give the crucial definition of a regular slice.

Definition 9 (Regular slice)**.** Given $\varepsilon, \varepsilon_k > 0, r, t_0, t_1 \in \mathbb{N}$, a graph G and a $(k-1)$-complex $\mathcal{J}$ on $V(G)$, we call $\mathcal{J}$ a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$*-regular slice* for G if $\mathcal{J}$ is (t_0, t_1, ε)-equitable and G is (ε_k, r)-regular with respect to all but at most $\varepsilon_k\binom{t}{k}$ of the k-sets of clusters of $\mathcal{J}$, where t is the number of clusters of $\mathcal{J}$.

If we specify the density vector $\mathbf{d}$ and the number of clusters t of an equitable complex or a regular slice, then it is not necessary to specify t_0 and t_1 (since the only role of these is to bound $\mathbf{d}$ and t). In this situation we write that $\mathcal{J}$ is $(\cdot, \cdot, \varepsilon)$-equitable, or is a $(\cdot, \cdot, \varepsilon, \varepsilon_k, r)$-regular slice for G.

Given a regular slice $\mathcal{J}$ for a k-graph G, we define the d-reduced k-graph $\mathcal{R}_d^{\mathcal{J}}(G)$ as follows.

Definition 10 (The $\boldsymbol{d}$-reduced $\boldsymbol{k}$-graph)**.** Let $k \geqslant 3$. Let G be a k-graph and let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G. Then, for $d > 0$, we define the *d-reduced k-graph* $\mathcal{R}_d^{\mathcal{J}}(G)$ to be the k-graph whose vertices are the clusters of $\mathcal{J}$ and whose edges are all k-sets X of clusters of $\mathcal{J}$ such that G is (ε_k, r)-regular with respect to X and $d^*(X) \geqslant d$.

We now state the version of the Regular Slice Lemma that we need, which is a special case of [2, Lemma 10].

Lemma 11 (Regular Slice Lemma [2, Lemma 10])**.** *Let $k \geqslant 3$. For all positive integers t_0 and s, positive ε_k and all functions $r\colon \mathbb{N} \to \mathbb{N}$ and $\varepsilon\colon \mathbb{N} \to (0,1]$, there are integers t_1 and n_0 such that the following holds for all $n \geqslant n_0$ which are divisible by $t_1!$. Let K be a 2-edge-coloured complete k-graph on n vertices. Then there exists a $(k-1)$-complex $\mathcal{J}$ on $V(K)$ which is a $(t_0, t_1, \varepsilon(t_1), \varepsilon_k, r(t_1))$-regular slice for both K^{red} and K^{blue}.*

Given a 2-edge-coloured complete k-graph H we want to apply the Regular Slice Lemma to H^{red} and H^{blue}. The following lemma shows that in this setting the union of the corresponding reduced graphs $\mathcal{R}_d^{\mathcal{J}}(H^{\mathrm{red}}) \cup \mathcal{R}_d^{\mathcal{J}}(H^{\mathrm{blue}})$ is almost complete.

Lemma 12 ([7, Lemma 8.5])**.** *Let $k \geqslant 3$. Let K be a 2-edge-coloured complete k-graph and let $\mathcal{J}$ be a $(\cdot, \cdot, \varepsilon, \varepsilon_k, r)$-regular slice for both K^{red} and K^{blue}. Let t be the number of clusters of $\mathcal{J}$. Then, provided that $d \leqslant 1/2$, we have $\left|\mathcal{R}_d^{\mathcal{J}}(K^{\mathrm{red}}) \cup \mathcal{R}_d^{\mathcal{J}}(K^{\mathrm{blue}})\right| \geqslant (1 - 2\varepsilon_k)\binom{t}{k}$.*

Proof. Since $\mathcal{J}$ is a $(\cdot, \cdot, \varepsilon, \varepsilon_k, r)$-regular slice for both K^{red} and K^{blue} there are at least $(1 - 2\varepsilon_k)\binom{t}{k}$ k-sets X of clusters of $\mathcal{J}$ such that both K^{red} and K^{blue} are (ε_k, r)-regular with respect to X. Let X be such a k-set. Since K^{red} and K^{blue} are complements of each other, we have $d^*_{K^{\mathrm{red}}}(X) + d^*_{K^{\mathrm{blue}}}(X) = 1$. Hence $d^*_{K^{\mathrm{red}}}(X) \geqslant 1/2$ or $d^*_{K^{\mathrm{blue}}}(X) \geqslant 1/2$ and thus, since $d \leqslant 1/2$, we have $X \in \mathcal{R}_d^{\mathcal{J}}(K^{\mathrm{red}}) \cup \mathcal{R}_d^{\mathcal{J}}(K^{\mathrm{blue}})$. □

Let H be a k-graph. A *fractional matching* in H is a function $\omega : E(H) \to [0,1]$ such that for all $v \in V(H)$, $\omega(v) := \sum_{e \in H : v \in e} \omega(e) \leqslant 1$. The *weight* of the fractional matching is defined to be $\sum_{e \in H} \omega(e)$. A fractional matching is *tightly connected* if the subgraph induced by the edges e with $\omega(e) > 0$ is tightly connected in H. The following result from [2] converts a tightly connected fractional matching in the reduced graph into a tight cycle in the original graph.

Lemma 13 ([2, Lemma 13])**.** *Let k, r, n_0, t be positive integers, and let $\psi, \varepsilon, \varepsilon_k, d_k, \dots, d_2$ be positive constants such that $1/d_i \in \mathbb{N}$ for each $2 \leqslant i \leqslant k-1$, and such that $1/n_0 \ll 1/t$,*

$$\frac{1}{n_0} \ll \frac{1}{r}, \varepsilon \ll \varepsilon_k, d_{k-1}, \dots, d_2 \quad \text{and} \quad \varepsilon_k \ll \psi, d_k, \frac{1}{k}.$$

Then the following holds for all integers $n \geqslant n_0$. Let G be a k-graph on n vertices, and $\mathcal{J}$ be a $(\cdot, \cdot, \varepsilon, \varepsilon_k, r)$-regular slice for G with t clusters and density vector $(d_{k-1}, \dots, d_2)$. Suppose that $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$ contains a tightly connected fractional matching with weight μ. Then G contains a tight cycle of length ℓ for every $\ell \leqslant (1-\psi)k\mu n/t$ that is divisible by k.

We use the following fact, lemma and proposition to prove Lemma 18 which is a stronger version of Lemma 13 that allows us to control the location of the tight cycle.

Fact 14 ([2, Fact 7])**.** *Suppose that $1/m_0 \ll \varepsilon \ll 1/t_1, 1/t_0, \beta, 1/k \leqslant 1/3$ and that $\mathcal{J}$ is a (t_0, t_1, ε)-equitable $(k-1)$-complex with density vector $(d_{k-1}, \dots, d_2)$ whose clusters each have size $m \geqslant m_0$. Let X be a set of k clusters of $\mathcal{J}$. Then*

$$\left|K_k((\mathcal{J}_{X^<})^{(k-1)})\right| = (1 \pm \beta)m^k \prod_{i=2}^{k-1} d_i^{\binom{k}{i}}.$$

Lemma 15 (Regular Restriction Lemma [2, Lemma 28])**.** *Suppose integers k, m and reals $\alpha, \varepsilon, \varepsilon_k, d_k, \dots, d_2 > 0$ are such that*

$$\frac{1}{m} \ll \varepsilon \ll \varepsilon_k, d_{k-1}, \dots, d_2 \quad \text{and} \quad \varepsilon_k \ll \alpha, \frac{1}{k}.$$

For any $r, s \in \mathbb{N}$ and $d_k > 0$, set $\mathbf{d} = (d_k, \dots, d_2)$, and let $\mathcal{G}$ be an s-partite k-complex whose vertex classes $V_1, \dots, V_s$ each have size m and which is $(\mathbf{d}, \varepsilon_k, \varepsilon, r)$-regular. Choose any $V_i' \subseteq V_i$ with $|V_i'| \geqslant \alpha m$ for each $i \in [s]$. Then the induced subcomplex $\mathcal{G}[V_1' \cup \dots \cup V_s']$ is $(\mathbf{d}, \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular.

The following proposition shows that a refinement of a regular slice is also a regular slice.

Proposition 16. *Let $1/m \ll \varepsilon \ll 1/N, 1/t_0, 1/t_1, 1/k \leqslant 1/3$. Let $\mathcal{J}$ be a (t_0, t_1, ε)-equitable $(k-1)$-complex with density vector $(d_{k-1}, \dots, d_2)$ and clusters $V_1, \dots, V_t$ each of size m. Let $V_{i,1}, \dots, V_{i,N}$ be an equipartition of V_i for each $i \in [t]$. Then there exists a $(Nt_0, Nt_1, \sqrt{\varepsilon})$-equitable $(k-1)$-complex $\widetilde{\mathcal{J}}$ with density vector $(d_{k-1}, \dots, d_2)$, ground partition $\{V_{i,j} \colon i \in [t], j \in [N]\}$ and $\widetilde{\mathcal{J}}[V_1, \dots, V_t] = \mathcal{J}$.*

Proof. We construct $\widetilde{\mathcal{J}}$ from $\mathcal{J}$ as follows. Let the ground partition of $\widetilde{\mathcal{J}}$ be $\{V_{i,j} \colon i \in [t], j \in [N]\}$. Starting with the edges of $\mathcal{J}$ we iteratively add additional edges at random as follows. For each $2 \leqslant i \leqslant k-1$, beginning with $i = 2$, we add each i-edge that contains two vertices that are in vertex classes with the same first index and is supported on the $(i-1)$-edges independently with probability d_i.

We now show that with high probability $\widetilde{\mathcal{J}}$ is the desired $(k-1)$-complex. Note that it suffices to show that with high probability $\widetilde{\mathcal{J}}$ is $(\mathbf{d}, \sqrt{\varepsilon}, \sqrt{\varepsilon}, 1)$-regular.

Let $\widetilde{\mathcal{J}}^{\leqslant i} = \bigcup_{j\in[i]} \widetilde{\mathcal{J}}^{(j)}$ and $\mathbf{d}^{\leqslant i} = (d_i, \dots, d_2)$. For $i \in [k-1]$, let B_i be the event that $\widetilde{\mathcal{J}}^{\leqslant i}$ is not $(\mathbf{d}^{\leqslant i}, \sqrt{\varepsilon}, \sqrt{\varepsilon}, 1)$-regular. Note that $B_1 = \varnothing$. Consider $2 \leqslant i \leqslant k-1$ and $A \in \binom{[t]\times[N]}{i}$. Let $B_{i,A}$ be the event that $\widetilde{\mathcal{J}}^{(i)}[V_A]$ is not $(d_i, \sqrt{\varepsilon}, 1)$-regular with respect to $\widetilde{\mathcal{J}}^{(i-1)}[V_A]$.

Claim 17. *For $i \in [k-1]$ and $A \in \binom{[t]\times[N]}{i}$, we have $\mathbb{P}\left[B_{i,A} \mid \overline{B_{i-1}}\right] = e^{-\Omega(m^i)}$ as $m \to \infty$.*

Proof of Claim. Assume $\overline{B_{i-1}}$ holds. Let $A = \{(r_j, s_j) \colon j \in [i]\}$. Define $\widetilde{A} = \{r_j \colon j \in [i]\}$. If the r_j are distinct, then the claim holds by Lemma 15 with $\mathcal{G} = \mathcal{J}[V_{\widetilde{A}}]$ and $\alpha = 1/N$. If not all the r_j are distinct, then $\left|K_i(\widetilde{\mathcal{J}}^{(i-1)}[V_A])\right| \geqslant \frac{1}{2}\left(\prod_{j=2}^{i-1} d_j^{\binom{i}{j}}\right)(m/N)^i$, by Fact 14. Thus for each subgraph Q of $\widetilde{\mathcal{J}}^{(i-1)}[V_A]$ such that $|K_i(Q)| > \sqrt{\varepsilon}\left|K_i(\widetilde{\mathcal{J}}^{(i-1)}[V_A])\right|$, a Chernoff bound implies that

$$\begin{aligned}
&\mathbb{P}\left[d(\widetilde{\mathcal{J}}^{(i)}[V_A] \mid Q) \neq d_i \pm \sqrt{\varepsilon} \,\middle|\, \overline{B_{i-1}}\right] \\
=&\mathbb{P}\left[\left|\left|\widetilde{\mathcal{J}}^{(i)}[V_A] \cap K_i(Q)\right| - d_i\left|K_i(Q)\right|\right| > \frac{\sqrt{\varepsilon}}{d_i} d_i \left|K_i(Q)\right| \,\middle|\, \overline{B_{i-1}}\right] \\
\leqslant& 2\exp\left(-\frac{1}{3}\left(\frac{\sqrt{\varepsilon}}{d_i}\right)^2 d_i\left|K_i(Q)\right|\right) \leqslant 2\exp\left(-\frac{1}{3}\frac{\varepsilon^{3/2}}{d_i}\left|K_i(\widetilde{\mathcal{J}}^{(i-1)}[V_A]\right|\right) \\
\leqslant& 2\exp\left(-\frac{1}{6}\frac{\varepsilon^{3/2}}{d_i}\left(\prod_{j=2}^{i-1} d_j^{\binom{i}{j}}\right)\left(\frac{m}{N}\right)^i\right) \leqslant e^{-\Omega(m^i)}.
\end{aligned}$$

Since there are at most $2^{(im)^{i-1}}$ choices for Q, the claim follows by a union bound. □

Note that if $\widetilde{\mathcal{J}}$ is not $(\mathbf{d}, \sqrt{\varepsilon}, \sqrt{\varepsilon}, 1)$-regular, then there exists some $i \in [k-1]$ and $A \in \binom{[t]\times[N]}{i}$ such that $B_{i,A}$ holds. Further by choosing i minimal we can ensure that $\overline{B_{i-1}}$ holds. Thus, by a union bound and Claim 17, we have

$$\begin{aligned}
\mathbb{P}\left[\widetilde{\mathcal{J}} \text{ is not } (\mathbf{d}, \sqrt{\varepsilon}, \sqrt{\varepsilon}, 1)\text{-regular}\right] &\leqslant \sum_{i=1}^{k-1} \sum_{A\in\binom{[t]\times[N]}{i}} \mathbb{P}\left[B_{i,A} \cap \overline{B_{i-1}}\right] \\
&\leqslant \sum_{i=1}^{k-1} \sum_{A\in\binom{[t]\times[N]}{i}} \mathbb{P}\left[B_{i,A} \mid \overline{B_{i-1}}\right] = o(1). \qquad \square
\end{aligned}$$

The following lemma is a strengthening of Lemma 13. We believe the constant β and the corresponding condition could be removed if one were to go through the proof of Lemma 13 to prove a stronger result.

Lemma 18. *Let $1/n \ll 1/r, \varepsilon \ll \varepsilon_k, d_{k-1}, \dots, d_2$ and $\varepsilon_k \ll \varepsilon' \ll \psi, d_k, \beta, 1/k \leqslant 1/3$ and $1/n \ll 1/t$ such that t divides n and $1/d_i \in \mathbb{N}$ for all $2 \leqslant i \leqslant k-1$. Let G be a k-graph on n vertices and $\mathcal{J}$ be a $(\cdot, \cdot, \varepsilon, \varepsilon_k, r)$-regular slice for G. Further, let $\mathcal{J}$ have t clusters $V_1, \dots, V_t$ all of size n/t and density vector $\mathbf{d} = (d_{k-1}, \dots, d_2)$. Suppose that the reduced graph $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$ contains a tightly connected fractional matching φ with weight μ. Assume that all edges with non-zero weight have weight at least β. For each $i \in [t]$, let $W_i \subseteq V_i$ be such that $|W_i| \geqslant ((1-3\varepsilon')\varphi(V_i) + \varepsilon')n/t$. Then $G\left[\bigcup_{i\in[t]} W_i\right]$ contains a tight cycle of length ℓ for each $\ell \leqslant (1-\psi)k\mu n/t$ that is divisible by k.*

We first explain the main ideas of the proof. We would like to find a regular slice for $G' = G[\bigcup_{i\in[t]} W_i]$ so that we can then apply Lemma 13 to G'. The issue is that not all vertex classes in G' have the same size. To get around this we take a refinement of the original partition and use Proposition 16 to find a new regular slice with that ground partition. The reduced graph for this new regular slice will be a blow up of the original reduced graph. We can find a corresponding tightly connected matching in this new reduced graph. Then we simply apply Lemma 13.

Proof of Lemma 18. Let $m = n/t$ and $\widetilde{m} = \lfloor \varepsilon' m/2 \rfloor$. For each $i \in [t]$, let $\widetilde{V}_i \subseteq V_i$ such that $\widetilde{m} \mid \left|\widetilde{V}_i\right|$ and $\left|V_i \setminus \widetilde{V}_i\right| \leqslant \varepsilon' m/2$. By Lemma 15, $\mathcal{J}[\widetilde{V}_1, \dots, \widetilde{V}_t]$ is $(\cdot, \cdot, \sqrt{\varepsilon})$-equitable with density vector $(d_{k-1}, \dots, d_2)$. Let $N = \lfloor m/\widetilde{m} \rfloor$ and, for each $i \in [t]$, let $N_i = \lfloor ((1-3\varepsilon')\varphi(V_i) + \varepsilon')N \rfloor \leqslant \lfloor |W_i|/\widetilde{m} \rfloor$. For each $i \in [t]$, let $V_{i,1}, \dots, V_{i,N}$ be an equipartition of $\widetilde{V}_i$ such that $V_{i,1}, \dots, V_{i,N_i} \subseteq W_i$. Let $\widetilde{W} = \{V_{i,j} \colon i \in [t], j \in [N_i]\}$ and $\widetilde{t} = \left|\widetilde{W}\right|$. By Proposition 16, there exists a $(\cdot, \cdot, \varepsilon^{1/4})$-equitable $(k-1)$-complex $\mathcal{J}^*$ with density vector $(d_{k-1}, \dots, d_2)$ and ground partition $\{V_{i,j} \colon i \in [t], j \in [N]\}$ such that $\mathcal{J}[\widetilde{V}_1, \dots, \widetilde{V}_t] = \mathcal{J}^*[\widetilde{V}_1, \dots, \widetilde{V}_t]$. Let $\widetilde{\mathcal{J}} = \mathcal{J}^*_{\widetilde{W}}$, that is $\widetilde{\mathcal{J}}$ is the $(k-1)$-complex contained in $\mathcal{J}^*$ induced by the vertex classes in $\widetilde{W}$.

Let $\widetilde{G}$ be the subgraph of $G[\bigcup \widetilde{W}]$ obtained by removing all edges contained in k-tuples of density less than d_k and in irregular k-tuples. We show that $\widetilde{\mathcal{J}}$ is a regular slice for $\widetilde{G}$. Let X be a set of k clusters of $\widetilde{\mathcal{J}}$. If the k clusters in X are all contained in distinct clusters of $\mathcal{J}$ that form a regular k-tuple of density at least d_k, then let Y denote the k-set of these clusters. Note that $(G \cup \mathcal{J})[Y]$ is $((d, d_{k-1}, \dots, d_2), \varepsilon_k, \varepsilon, r)$-regular, for some $d \geqslant d_k - \varepsilon_k$, and thus, by Lemma 15, $(\widetilde{G} \cup \widetilde{\mathcal{J}})[X]$ is $((d, d_{k-1}, \dots, d_2), \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular. Hence $\widetilde{G}$ is $(d, \sqrt{\varepsilon_k}, r)$-regular with respect to $(\widetilde{\mathcal{J}}_{X^<})^{(k-1)}$. Note that, for all other k-sets of clusters X, the k-partite subgraph of $\widetilde{G}$ induced by the clusters in X is empty. For these k-sets of clusters, $\widetilde{G}$ is $(0, \sqrt{\varepsilon_k}, r)$-regular with respect to the polyad $(\widetilde{\mathcal{J}}_{X^<})^{(k-1)}$. Thus $\widetilde{\mathcal{J}}$ is a $(\cdot, \cdot, \sqrt{\varepsilon_k}, \varepsilon^{1/4}, r)$-regular slice for $\widetilde{G}$.

Note that $\widetilde{\mathcal{R}} = \mathcal{R}^{\widetilde{\mathcal{J}}}_{d_k - 2\sqrt{\varepsilon_k}}(\widetilde{G})$ is a blow-up of $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$. Consider the tightly connected fractional matching φ on $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$ with weight μ. We construct a tightly connected matching on $\widetilde{\mathcal{R}}$ as follows. For each $e \in \mathcal{R}^{\mathcal{J}}_{d_k}(G)$, we will pick a matching M_e in $\widetilde{\mathcal{R}}$ of size

$\widetilde{\varphi}(e) = \lfloor (1-3\varepsilon')\varphi(e)N \rfloor$. Note that, for each $i \in [t]$,

$$\sum_{e \ni V_i} \widetilde{\varphi}(e) \leqslant \lfloor ((1-3\varepsilon')\varphi(V_i) + \varepsilon')N \rfloor = N_i. \tag{4}$$

For each vertex V_i in $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$ and each edge $e \in \mathcal{R}^{\mathcal{J}}_{d_k}(G)$ that contains V_i, we choose disjoint sets $I_{i,e} \subseteq [N_i]$ such that $|I_{i,e}| = \widetilde{\varphi}(e)$. This is possible by (4). Recall that $\widetilde{\mathcal{R}}$ is a blow-up of $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$. For each edge $e = \{V_{i_1}, V_{i_2}, \dots, V_{i_k}\} \in \mathcal{R}^{\mathcal{J}}_{d_k}(G)$, the subgraph $\widetilde{\mathcal{R}}_e$ of $\widetilde{\mathcal{R}}$ induced by the set of edges $\{\{V_{i_1,j_1}, \dots, V_{i_k,j_k}\} \colon j_1 \in I_{i_1,e}, \dots, j_k \in I_{i_k,e}\}$ is a balanced complete k-partite k-graph. Pick a perfect matching M_e in $\widetilde{\mathcal{R}}_e$. Let $M = \bigcup_{e \in \mathcal{R}^{\mathcal{J}}_{d_k}(G)} M_e$. Note that M is a matching of size

$$\begin{aligned}
\sum_{e \in \mathcal{R}^{\mathcal{J}}_{d_k}(G)} \widetilde{\varphi}(e) = \sum_{e \in \mathcal{R}^{\mathcal{J}}_{d_k}(G)} \lfloor (1-3\varepsilon')\varphi(e)N \rfloor &\geqslant \sum_{\substack{e \in \mathcal{R}^{\mathcal{J}}_{d_k}(G) \\ \varphi(e)>0}} ((1-3\varepsilon')\varphi(e)N - 1) \\
&\geqslant (1-3\varepsilon')\mu N - \mu/\beta = \left(1 - 3\varepsilon' - \frac{1}{N\beta}\right)\mu N \\
&\geqslant (1 - 3\varepsilon' - \varepsilon'/\beta)\mu N \geqslant (1-\sqrt{\varepsilon'})\mu N \geqslant (1 - 2\sqrt{\varepsilon'})\mu \frac{m}{\widetilde{m}}.
\end{aligned}$$

In the second inequality above we used the fact that since φ is a fractional matching with weight μ and all edges have weight at least β, there are at most μ/β edges of positive weight. Since $\widetilde{\mathcal{R}}$ is a blow-up of $\mathcal{R}^{\mathcal{J}}_{d_k}(G)$, M is tightly connected. We conclude by applying Lemma 13 with $k, r, n, \widetilde{t}, \psi^2, \varepsilon^{1/4}, \sqrt{\varepsilon_k}, d_k - 2\sqrt{\varepsilon_k}, d_{k-1}, \dots, d_2, \widetilde{\mathcal{J}}, \widetilde{G}, \ell$ playing the roles of $k, r, n_0, t, \psi, \varepsilon, \varepsilon_k, d_k, \dots, d_2, \mathcal{J}, G, \ell$. □

For the next result, we need the following definition.

Definition 19. Let $\mu^s_k(\beta, \varepsilon, n)$ be the largest μ such that every 2-edge-coloured $(1-\varepsilon, \varepsilon)$-dense k-graph on n vertices contains a fractional matching with weight μ such that all edges with non-zero weight have weight at least β and lie in s monochromatic tight components. Let $\mu^s_k(\beta) = \liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \mu^s_k(\beta, \varepsilon, n)/n$. Similarly, let $\mu^*_k(\beta, \varepsilon, n)$ be the largest μ such that every 2-edge-coloured $(1-\varepsilon, \varepsilon)$-dense k-graph on n vertices contains a fractional matching with weight μ such that all edges with non-zero weight have weight at least β and lie in one red and one blue tight component. Let $\mu^*_k(\beta) = \liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \mu^*_k(\beta, \varepsilon, n)/n$.

The following is the crucial result that reduces finding cycles in the original graph to finding tightly connected matchings in the reduced graph.

Corollary 20. *Let $1/n \ll \eta, \beta, 1/k, 1/s$ with $k \geqslant 3$. Let K be a 2-edge-coloured complete k-graph on n vertices. Then the following hold.*

(i) *K contains s vertex-disjoint monochromatic tight cycles covering at least $(\mu^s_k(\beta) - \eta)kn$ vertices,*

(ii) *K contains two vertex-disjoint monochromatic tight cycles of distinct colours covering at least $(\mu_k^*(\beta)-\eta)kn$ vertices, and*

(iii) *K contains a monochromatic tight cycle of length ℓ for any $\ell \leqslant (\mu_k^1(\beta)-\eta)kn$ divisible by k.*

Proof. We prove the first statement. The other two statements can be proved similarly (where for the third statement we additionally make use of the fact that Lemma 18 also allows us to control the length of the resulting cycle). Without loss of generality assume that $\eta \leqslant 1/3$. Let $d_k = 1/2$ and $1/t_0 \ll \varepsilon_k \ll \varepsilon' \ll \varepsilon \ll \eta, \beta, 1/k, 1/s$. Note that $\mu_k^s(\beta,\varepsilon,t) \geqslant (\mu_k^s(\beta)-\eta^2)t$ for all $t \geqslant t_0$. We choose functions $\widetilde{\varepsilon}(\cdot)$ and $r(\cdot)$ where $\widetilde{\varepsilon}(\cdot)$ approaches zero sufficiently quickly and $r(\cdot)$ increases sufficiently quickly such that for any integer $t^* \geqslant t_0$ and $d_2,\dots,d_{k-1} \geqslant 1/t^*$ we may apply Lemma 18 with $\widetilde{\varepsilon}(t^*)$ and $r(t^*)$ playing the roles of ε and r, respectively. We apply Lemma 11 to obtain n_0 and t_1. Let $\widetilde{\varepsilon} = \widetilde{\varepsilon}(t_1)$ and $r = r(t_1)$. Let $n_1 \geqslant n_0$ be large enough such that for all $n \geqslant n_1$ and $d_2,\dots,d_{k-1} \geqslant 1/t_1$ we may apply Lemma 18. Let $n_2 = n_1 + t_1!$. We show that the theorem holds for all $n \geqslant n_2$. Let K be a 2-edge-coloured complete k-graph on n vertices. Let $\widetilde{n} \leqslant n$ be the largest integer such that $t_1!$ divides $\widetilde{n}$. Let $\widetilde{K}$ be a complete subgraph of K on $\widetilde{n}$ vertices. Note that $\widetilde{n} \geqslant n_1$. By Lemma 11, there exists a $(t_0, t_1, \widetilde{\varepsilon}, \varepsilon_k, r)$-regular slice $\mathcal{J}$ for both $\widetilde{K}^{\mathrm{red}}$ and $\widetilde{K}^{\mathrm{blue}}$. Let t be the number of clusters of $\mathcal{J}$ and let $(d_{k-1},\dots,d_2)$ be the density vector of $\mathcal{J}$. Let $\widetilde{H} = \mathcal{R}^{\mathcal{J}}_{d_k}(\widetilde{K}^{\mathrm{red}}) \cup \mathcal{R}^{\mathcal{J}}_{d_k}(\widetilde{K}^{\mathrm{blue}})$ be a 2-edge-coloured k-graph such that $\mathcal{R}^{\mathcal{J}}_{d_k}(\widetilde{K}^{\mathrm{red}}) \backslash \mathcal{R}^{\mathcal{J}}_{d_k}(\widetilde{K}^{\mathrm{blue}}) \subseteq \widetilde{H}^{\mathrm{red}}$ and $\mathcal{R}^{\mathcal{J}}_{d_k}(\widetilde{K}^{\mathrm{blue}}) \backslash \mathcal{R}^{\mathcal{J}}_{d_k}(\widetilde{K}^{\mathrm{red}}) \subseteq \widetilde{H}^{\mathrm{blue}}$. By Lemma 12, we have $\left|\widetilde{H}\right| \geqslant (1-2\varepsilon_k)\binom{t}{k}$. By Proposition 5, there exists a $(1-(2\varepsilon_k)^{1/(4k^2+1)}, (2\varepsilon_k)^{1/(4k^2+1)})$-dense subgraph $H \subseteq \widetilde{H}$ with $V(H) = V(\widetilde{H})$. Since $\varepsilon_k \ll \varepsilon$, H is $(1-\varepsilon,\varepsilon)$-dense. Let φ be a fractional matching in H of weight $\mu = \mu_k^s(\beta,\varepsilon,t) \geqslant (\mu_k^s(\beta)-2\eta^2)t$ such that all edges with non-zero weight have weight at least β and lie in s monochromatic tight components $K_1,\dots,K_s$ of H. For each $j \in [s]$, we define a fractional matching φ_j in H by setting $\varphi_j(e) = \varphi(e)$ if $e \in K_i$ and $\varphi(e) = 0$ otherwise. For each $j \in [s]$, let μ_j be the weight of φ_j. It follows that $\sum_{j\in[s]} \mu_j = \mu$.

Let $V_1,\dots,V_t$ be the clusters of $\mathcal{J}$. For each $i \in [t]$ and $j \in [s]$, we define

$$w_{i,j} = \max\{\sum_{\substack{e\in H\\ V_i\in e}} \varphi_j(e) - s\varepsilon', \varepsilon'\}.$$

For each $i \in [t]$, let $V_{i,1},\dots,V_{i,s}$ be disjoint subsets of V_i such that $|V_{i,j}| = \lceil w_{i,j}n/t \rceil$. By Lemma 18, there exist tight cycles $C_1,\dots,C_s$ in K such that, for all $j \in [s]$, $|C_j| = (1-\eta^2)\mu_j k\widetilde{n}/t$, $C_j \subseteq K\left[\bigcup_{i\in[t]} V_{i,j}\right]$ and C_j has the same colour as K_j. Hence $C_1,\dots,C_s$ are vertex-disjoint and together cover

$$(1-\eta^2)\mu k\widetilde{n}/t \geqslant (1-\eta^2)(\mu_k^s(\beta)-\eta^2)k\widetilde{n} \geqslant (\mu_k^s(\beta)-\eta)kn$$

vertices of K. □

5 Blueprints

Let H be a 2-edge-coloured k-graph. We define what we call a *blueprint for* H which is an auxiliary graph that can be used as a guide when finding connected matchings in H. A form of the notion of a blueprint for $k = 3$ already appeared in [13].

Definition 21. Let $\varepsilon > 0$, $k \geqslant 3$ and let H be a 2-edge-coloured k-graph on n vertices. We say that a 2-edge-coloured $(k-2)$-graph G with $V(G) \subseteq V(H)$ is an *ε-blueprint for H*, if

(BP1) for every edge $e \in G$, there exists a monochromatic tight component $H(e)$ in H such that $H(e)$ has the same colour as e and $d_{\partial H(e)}(e) \geqslant (1-\varepsilon)n$ and

(BP2) for $e, e' \in G$ of the same colour with $|e \cap e'| = k-3$, we have $H(e) = H(e')$.

We say that e *induces* $H(e)$ and write $R(e)$ or $B(e)$ instead of $H(e)$ if e is red or blue, respectively. We simply say that G is a *blueprint*, when H and ε are clear from context. For $S \in \binom{V(H)}{k-3}$, all the red (blue) edges of a blueprint containing S induce the same red (blue) tight component, so we call that component the red (blue) tight component induced by S. Note that any subgraph of a blueprint is also a blueprint.

Example 22. Let $k \geqslant 3$ and let n be a positive integer. Let A and B be disjoint vertex sets with $|A \cup B| = n$. Let $K^{(k)}(A, B)$ be the 2-edge-coloured complete k-graph with vertex set $A \cup B$ where an edge e is red if and only if $|e \cap A|$ is even (and blue otherwise). Let H be $K^{(k)}(A, B)$ and let G be $K^{(k-2)}(A, B)$ with colours reversed. If $\varepsilon \geqslant \frac{k-2}{n}$, then G is an ε-blueprint for H. Indeed, for an edge $e \in G$ we can set $H(e) = \{f \in H \colon |f \cap A| = |e \cap A| + 1\}$.

The main aim of this section is to prove the following lemma that establishes the existence of blueprints for 2-edge-coloured $(1-\varepsilon, \alpha)$-dense graphs.

Lemma 23. *Let $1/n \ll \varepsilon \leqslant \alpha \ll 1/k \leqslant 1/3$. Let H be a 2-edge-coloured $(1-\varepsilon, \alpha)$-dense k-graph on n vertices. Then there exists a $3\sqrt{\varepsilon}$-blueprint G_* for H with $V(G_*) = V(H)$ and $|G_*| \geqslant (1 - \alpha - 24k\sqrt{\varepsilon})\binom{n}{k-2}$. Moreover, if $k \geqslant 4$ and $\varepsilon \ll \alpha$, there exists a $(1-\alpha^{1/(4(k-2)^2+1)}, \alpha^{1/(4(k-2)^2+1)})$-dense spanning subgraph G of G_*.*

We need a few simple preliminary results to prove Lemma 23. For a 2-graph G, we denote by $\delta(G)$ the minimum degree of G. First we show that any 2-edge-coloured 2-graph with large minimum degree contains a large monochromatic connected subgraph. This proposition is implied by [8, Lemma 1.5] but we include a proof for completeness.

Proposition 24. *Let $0 < \beta \leqslant 1/6$ and let F be a 2-edge-coloured 2-graph with $|V(F)| \leqslant n$ and $\delta(F) \geqslant (1-\beta)n$. Then there exists a subgraph F' of F of order at least $(1-\beta)n$ that contains a spanning monochromatic component and $\delta(F') \geqslant (1-2\beta)n$.*

Proof. Let F' be an induced subgraph of F of maximum order that contains a spanning monochromatic component. Assume without loss of generality that F' contains a spanning red component. Let $S = V(F')$ and $\overline{S} = V(F) \setminus V(F')$. Since $\delta(F) \geqslant (1-\beta)n$, we have that $|S| \geqslant (1-\beta)n/2$. Suppose, for a contradiction, that $|S| < (1-\beta)n$. Note that all edges between S and $\overline{S}$ are blue. If $\delta(F) - |S| + 1 > \left|\overline{S}\right|/2$, then each pair of vertices in S has a common neighbour in $\overline{S}$ and so there is a blue component strictly containing S which contradicts the maximality of F'. Therefore

$$\delta(F) - |S| + 1 \leqslant \left|\overline{S}\right|/2 = (|V(F)| - |S|)/2 \leqslant (n - |S|)/2.$$

Hence

$$|S| \geqslant 2\delta(F) - n + 2 \geqslant 2(1-\beta)n - n + 2 = (1-2\beta)n + 2.$$

But now every pair of vertices in $\overline{S}$ has a common neighbour in S, since $\left|\overline{S}\right| = |V(F)| - |S| \leqslant 2\beta n$ and so

$$\delta(F) - \left|\overline{S}\right| + 1 \geqslant (1-\beta)n - 2\beta n + 1 = (1-3\beta)n + 1 > n/2.$$

Thus $\overline{S} \cup N_F(\overline{S})$ is spanned by a blue component. But since

$$\left|\overline{S} \cup N_F(\overline{S})\right| \geqslant \delta(F) \geqslant (1-\beta)n,$$

we have a contradiction. It is easy to see that $\delta(F') \geqslant (1-2\beta)n$. □

Proposition 25. *Let $1/n \ll \gamma \leqslant 1/9$. Let F be a 2-graph with $|V(F)| \leqslant n$ and $|E(F)| \geqslant (1-\gamma)\binom{n}{2}$. Then there exists a subgraph of F with minimum degree at least $(1-3\sqrt{\gamma})n$.*

Proof. Let $W = \{v \in V(F) \colon d(v) < (1-2\sqrt{\gamma})n\}$. We have that

$$(1-2\gamma)n^2 \leqslant 2\,|E(F)| = \sum_{v \in V(F)} d(v) \leqslant n^2 - 2\sqrt{\gamma}n\,|W|\,.$$

This implies that $|W| \leqslant \sqrt{\gamma}n$. Let $F^* = F - W$. It follows that $\delta(F^*) \geqslant (1-2\sqrt{\gamma})n - |W| \geqslant (1-3\sqrt{\gamma})n$. □

Corollary 26. *Let $1/n \ll \varepsilon \leqslant 1/324$. Let F be a 2-edge-coloured 2-graph with $|V(F)| \leqslant n$ and $|E(F)| \geqslant (1-\varepsilon)\binom{n}{2}$. Then there exists a subgraph F' of F of order at least $(1-3\sqrt{\varepsilon})n$ that contains a spanning monochromatic component and $\delta(F') \geqslant (1-6\sqrt{\varepsilon})n$.*

Proof. By Proposition 25, there exists a subgraph F^* of F with $\delta(F^*) \geqslant (1-3\sqrt{\varepsilon})n$. We conclude by applying Proposition 24 with $F = F^*$ and $\beta = 3\sqrt{\varepsilon}$. □

5.1 Proof of Lemma 23

Now we show that for any $(1-\varepsilon, \alpha)$-dense 2-edge-coloured graph we can find a dense blueprint.

Proof of Lemma 23. Let $F = \partial^2 H$. Since H is $(1-\varepsilon, \alpha)$-dense,

$$E(F) = \left\{ e \in \binom{V(H)}{k-2} \colon d_H(e) > 0 \right\} = \left\{ e \in \binom{V(H)}{k-2} \colon d_H(e) \geqslant (1-\varepsilon)\binom{n}{2} \right\}$$

and

$$|E(F)| \geqslant (1-\alpha)\binom{n}{k-2}. \tag{5}$$

We now colour each edge e of F as follows. Note that the link graph H_e is a 2-graph. We induce a 2-edge-colouring on H_e by colouring the 2-edge $f \in H_e$ with the colour of the k-edge $e \cup f \in H$. By Corollary 26, there exists a monochromatic component in H_e of order at least $(1-3\sqrt{\varepsilon})n$. Let K_e be such a component chosen arbitrarily. We colour the edge e according to the colour of K_e. If e is red in F, then we define $R(e) \subseteq H$ to be the red tight component containing all the edges $e \cup f$ where $f \in K_e$. If e is blue in F, then we define $B(e)$ analogously.

In the next claim we show that, for each $S \in \binom{V(H)}{k-3}$, almost all edges in F of the same colour containing S induce the same monochromatic tight component in H.

Claim 27. *For each $S \in \binom{V(H)}{k-3}$, there exist $\Gamma^{\mathrm{red}}(S) \subseteq N_F^{\mathrm{red}}(S)$ and $\Gamma^{\mathrm{blue}}(S) \subseteq N_F^{\mathrm{blue}}(S)$ with $\left|\Gamma^{\mathrm{red}}(S)\right| \geqslant \left|N_F^{\mathrm{red}}(S)\right| - 6\sqrt{\varepsilon}n$ and $\left|\Gamma^{\mathrm{blue}}(S)\right| \geqslant \left|N_F^{\mathrm{blue}}(S)\right| - 6\sqrt{\varepsilon}n$ such that, for all $y_1, y_2 \in \Gamma^{\mathrm{red}}(S)$, $R(S \cup y_1) = R(S \cup y_2)$ and, for all $y_1', y_2' \in \Gamma^{\mathrm{blue}}(S)$, $B(S \cup y_1') = B(S \cup y_2')$.*

Proof of Claim. We only prove the statement for $N_F^{\mathrm{red}}(S)$ as the proof of the statement for $N_F^{\mathrm{blue}}(S)$ is analogous. Assume $\left|N_F^{\mathrm{red}}(S)\right| > 6\sqrt{\varepsilon}n$ (or else we simply set $\Gamma^{\mathrm{red}}(S) = \varnothing$). Let D be the directed graph with vertex set $N_F^{\mathrm{red}}(S)$ and edge set

$$E(D) = \left\{ y_1 y_2 \colon y_1 \in V(K_{S \cup y_2}) \right\}.$$

Note that, for $y_1 y_2 \in E(D)$, there exists an edge in $R(S \cup y_2)$ containing $S \cup y_1 y_2$. So if $y_1 y_2$ is a double edge (that is, $y_1 y_2, y_2 y_1 \in E(D)$), then $R(S \cup y_1) = R(S \cup y_2)$. For $y \in N_F^{\mathrm{red}}(S)$,

$$d_D^-(y) \geqslant \left|N_F^{\mathrm{red}}(S) \cap V(K_{S \cup y})\right| \geqslant \left|N_F^{\mathrm{red}}(S)\right| - 3\sqrt{\varepsilon}n,$$

since $|V(K_{S \cup y})| \geqslant (1-3\sqrt{\varepsilon})n$. Hence the number of double edges in D is at least

$$\left|N_F^{\mathrm{red}}(S)\right| \left(\left|N_F^{\mathrm{red}}(S)\right| - 3\sqrt{\varepsilon}n\right) - \frac{1}{2}\left|N_F^{\mathrm{red}}(S)\right|^2 = \frac{1}{2}\left|N_F^{\mathrm{red}}(S)\right| \left(\left|N_F^{\mathrm{red}}(S)\right| - 6\sqrt{\varepsilon}n\right).$$

Thus there exists a vertex $y_0 \in N_F^{\mathrm{red}}(S)$ that is incident to at least $\left|N_F^{\mathrm{red}}(S)\right| - 6\sqrt{\varepsilon}n$ double edges. Let $\Gamma^{\mathrm{red}}(S) = \{y_0\} \cup \{y \in N_F^{\mathrm{red}}(S) \colon yy_0, y_0 y \in E(D)\}$. Note that $\left|\Gamma^{\mathrm{red}}(S)\right| \geqslant \left|N_F^{\mathrm{red}}(S)\right| - 6\sqrt{\varepsilon}n$ and $R(S \cup y) = R(S \cup y_0)$ for all $y \in \Gamma^{\mathrm{red}}(S)$. □

Consider the multi-$(k-2)$-graph D^* with

$$E(D^*) = \left\{ S \cup y \colon S \in \binom{V(H)}{k-3}, y \in \Gamma^{\mathrm{red}}(S) \cup \Gamma^{\mathrm{blue}}(S) \right\}.$$

Note that

$$|E(D^*)| = \sum_{S\in\binom{V(H)}{k-3}} \left|\Gamma^{\mathrm{red}}(S)\cup\Gamma^{\mathrm{blue}}(S)\right| \geqslant \sum_{S\in\binom{V(H)}{k-3}} (d_F(S) - 12\sqrt{\varepsilon}n)$$
$$\geqslant (k-2)\,|F| - 24k\sqrt{\varepsilon}\binom{n}{k-2}.$$

Observe that every edge of D^* is an edge of F. Every edge in D^* has multiplicity at most $k-2$. So at least $|F| - 24k\sqrt{\varepsilon}\binom{n}{k-2}$ edges $e \in \binom{V(H)}{k-2}$ have multiplicity $k-2$ in D^*. Let G_* be the $(k-2)$-graph on $V(H)$ such that $e \in G_*$ if and only if e has multiplicity $k-2$ in D^*. So, by (5), $|G_*| \geqslant |F| - 24k\sqrt{\varepsilon}\binom{n}{k-2} \geqslant (1-\alpha-24k\sqrt{\varepsilon})\binom{n}{k-2}$.

We now show that G_* is a $3\sqrt{\varepsilon}$-blueprint for H. Consider any $e, e' \in G_*^{\mathrm{red}}$ with $|e\cap e'| = k-3$. Let $S = e\cap e'$, $y = e'\setminus S$ and $y' = e\setminus S$. Since $e, e' \in G_*^{\mathrm{red}}$, we have $y, y' \in \Gamma^{\mathrm{red}}(S)$ and so $R(e) = R(S\cup y) = R(S\cup y') = R(e')$. Further, for $e \in G_*^{\mathrm{red}}$, we have $d_{\partial R(e)}(e) \geqslant |V(K_e)| \geqslant (1-3\sqrt{\varepsilon})n$. Analogous statements hold for edges of G_*^{blue}.

If $k \geqslant 4$ and $\varepsilon \ll \alpha$, then $|G_*| \geqslant (1-2\alpha)\binom{n}{k-2}$ and thus by Proposition 5 there exists a subgraph $G \subseteq G_*$ such that G is $(1-\alpha^{1/(4(k-2)^2+1)}, \alpha^{1/(4(k-2)^2+1)})$-dense and $V(G) = V(G_*) = V(H)$. □

5.2 Some lemmas about blueprints

Let H be a k-graph and G be a blueprint for H. We write $H(G)$ for $\bigcup_{e\in G} H(e)$. We write G^+ for the subgraph of $H(G)$ with edge set

$$E(G^+) = \{e \in H(G)\colon f \subseteq e \text{ for some } f \in G\},$$

that is, the subgraph of $H(G)$ obtained by deleting all edges that do not contain an edge of G. Note that this also defines $(G')^+$ for any subgraph G' of G as a subgraph of a blueprint for H is also a blueprint for H. Moreover, note that G^+ is a subgraph of H, not of G. For a red tight component R_* and a blue tight component B_* in H, we denote by R_*^{k-2} and B_*^{k-2} the edges of G that induce R_* and B_*, respectively.

We prove some lemmas that we will use several times later on. Roughly speaking, the following lemma says that if S is a set of $k-4$ vertices of H contained in many edges of both R_*^{k-2} and B_*^{k-2}, then S is contained in an edge of R_* or B_*.

Lemma 28. *Let $1/n \ll \varepsilon \ll \alpha \ll 1$. Let H be a 2-edge-coloured $(1-\varepsilon,\alpha)$-dense k-graph on n vertices and G a $3\sqrt{\varepsilon}$-blueprint for H. Let R_* and B_* be a red and a blue tight component of H, respectively. Let $U \subseteq V(G)$ and $S \in \binom{U}{k-4}$ such that*

$$d_{R_*^{k-2}}(S,U), d_{B_*^{k-2}}(S,U) \geqslant \varepsilon^{1/4}n^2.$$

Then there exist $x, x', y, y' \in U$ such that $S\cup xx' \in R_*^{k-2}$, $S\cup yy' \in B_*^{k-2}$, $S\cup xx'y \in \partial R_*$, $S\cup yy'x \in \partial B_*$ and $S\cup xx'yy' \in H$. In particular, $(R_*^{k-2})^+[U]\cup(B_*^{k-2})^+[U] \neq \varnothing$.

Proof. Let $X_{R_*} = \{x \in U \colon d_{R_*^{k-2}}(S \cup x, U) \geqslant \varepsilon^{1/2} n\}$ and $X_{B_*} = \{x \in U \colon d_{B_*^{k-2}}(S \cup x, U) \geqslant \varepsilon^{1/2} n\}$. Note that

$$\varepsilon^{1/4} n^2 \leqslant d_{R_*^{k-2}}(S, U) = \frac{1}{2} \sum_{x \in U} d_{R_*^{k-2}}(S \cup x, U) \leqslant n \left|X_{R_*}\right| + \varepsilon^{1/2} n^2.$$

Thus $|X_{R_*}| \geqslant (\varepsilon^{1/4} - \varepsilon^{1/2}) n \geqslant \frac{1}{2} \varepsilon^{1/4} n$. Similarly, $|X_{B_*}| \geqslant \frac{1}{2} \varepsilon^{1/4} n$.

For each $x \in X_{R_*}$, let

$$\begin{aligned} Y_x &= \{y \in X_{B_*} \colon S \cup yy' \in B_*^{k-2} \text{ and } S \cup xyy' \in \partial B_* \text{ for some } y' \in U\} \\ &= \bigcup_{y' \in U} N_{B_*^{k-2}}(S \cup y') \cap N_{\partial B_*}(S \cup xy'). \end{aligned}$$

For each $y \in X_{B_*}$, there exists $y' \in U$ with $S \cup yy' \in B_*^{k-2}$. By (BP1), $d_{\partial B_*}(S \cup yy', X_{R_*}) \geqslant |X_{R_*}| - 3\sqrt{\varepsilon} n$. Hence each $y \in X_{B_*}$ is contained in at least $|X_{R_*}| - 3\sqrt{\varepsilon} n$ of the sets Y_x. By averaging, there exists an $x \in X_{R_*}$ such that

$$|Y_x| \geqslant \frac{(|X_{R_*}| - 3\sqrt{\varepsilon} n) \left|X_{B_*}\right|}{2 \left|X_{R_*}\right|} \geqslant \frac{1}{4} \left|X_{B_*}\right| \geqslant \frac{1}{8} \varepsilon^{1/4} n.$$

Fix such an $x \in X_{R_*}$. For each $y \in Y_x$, choose a vertex $y' \in U$ such that $S \cup yy' \in B_*^{k-2}$ and $S \cup xyy' \in \partial B_*$. Let $X = N_{R_*^{k-2}}(S \cup x, U)$, so $|X| \geqslant \varepsilon^{1/2} n$, since $x \in X_{R_*}$. For each $y \in Y_x$, since H is $(1 - \varepsilon, \alpha)$-dense, there are at least $|X| - \varepsilon n$ vertices $x' \in X$ such that $S \cup xx'yy' \in H$. Thus, by averaging, there exists a vertex $x' \in X$ and a set $\widetilde{Y}_x \subseteq Y_x$ with

$$\left|\widetilde{Y}_x\right| \geqslant \frac{(|X| - \varepsilon n) \left|Y_x\right|}{2 \left|X\right|} \geqslant \frac{1}{4} \left|Y_x\right| \geqslant \frac{1}{32} \varepsilon^{1/4} n$$

such that $S \cup xx'yy' \in H$ for all $y \in \widetilde{Y}_x$. Fix such an $x' \in X$. Since $S \cup xx' \in R_*^{k-2}$, we have that

$$\left|N_{\partial R_*}(S \cup xx') \cap \widetilde{Y}_x\right| \geqslant \left|\widetilde{Y}_x\right| - 3\sqrt{\varepsilon} n \geqslant \left(\frac{1}{32} \varepsilon^{1/4} - 3\sqrt{\varepsilon}\right) n > 0.$$

Choose $y \in N_{\partial R_*}(S \cup xx') \cap \widetilde{Y}_x$. We have $S \cup xx' \in R_*^{k-2}$, $S \cup yy' \in B_*^{k-2}$, $S \cup xx'y \in \partial R_*$, $S \cup xyy' \in \partial B_*$ and $S \cup xx'yy' \in H$ as required. □

The following lemma shows that if we have a vertex set $T \in \binom{V(G)}{k-3}$ such that $d_G^{\text{red}}(T)$ and $d_G^{\text{blue}}(T)$ are both large, then T is contained a lot of sets in $\partial R \cap \partial B$, where R and B are the red and blue tight components induced by the red and blue edges incident to T, respectively.

Lemma 29. *Let $1/n \ll \varepsilon \ll 1$, $k \geqslant 3$ and $\delta > 5\sqrt{\varepsilon}$. Let H be a 2-edge-coloured k-graph on n vertices and G a $3\sqrt{\varepsilon}$-blueprint for H. Let $T \in \binom{V(H)}{k-3}$. Let $S^{\text{blue}} \subseteq N_G^{\text{blue}}(T)$ and*

$S^{\mathrm{red}} \subseteq N_G^{\mathrm{red}}(T)$ *be such that* $|S^{\mathrm{blue}}|, |S^{\mathrm{red}}| \geqslant \delta n$*. Then there exists a vertex* $y \in S^{\mathrm{blue}}$ *such that, for*

$$\Gamma_y^{\mathrm{red}} = \{x \in S^{\mathrm{red}} \colon T \cup xy \in \partial R(T \cup x) \cap \partial B(T \cup y)\},$$

we have $\left|\Gamma_y^{\mathrm{red}}\right| \geqslant (\delta - 6\sqrt{\varepsilon})n$*. Moreover, if* $\delta \geqslant \varepsilon^{1/9}$*, then* $\left|\Gamma_y^{\mathrm{red}}\right| \geqslant (1-\varepsilon^{1/4})\left|S^{\mathrm{red}}\right|$*. The same statements hold when the colours are reversed.*

Proof. Let $m_{\mathrm{blue}} = |S^{\mathrm{blue}}|$ and $m_{\mathrm{red}} = |S^{\mathrm{red}}|$. If $\delta < \varepsilon^{1/9}$, then we may assume that $m_{\mathrm{blue}} = m_{\mathrm{red}} = \lceil \delta n \rceil$ by deleting vertices in S^{blue} and S^{red} if necessary. Let D be the bipartite directed graph with vertex classes S^{blue} and S^{red} such that, for each $y \in S^{\mathrm{blue}}$ and $x \in S^{\mathrm{red}}$, we have $N_D^+(y) = N_{\partial B}(T \cup y) \cap S^{\mathrm{red}}$ and $N_D^+(x) = N_{\partial R}(T \cup x) \cap S^{\mathrm{blue}}$. Since G is a $3\sqrt{\varepsilon}$-blueprint for H, we have that

$$\begin{aligned} |E(D)| &\geqslant m_{\mathrm{blue}}(m_{\mathrm{red}} - 3\sqrt{\varepsilon}n) + m_{\mathrm{red}}(m_{\mathrm{blue}} - 3\sqrt{\varepsilon}n) \\ &= 2m_{\mathrm{blue}}m_{\mathrm{red}} - 3\sqrt{\varepsilon}n(m_{\mathrm{blue}} + m_{\mathrm{red}}). \end{aligned}$$

Thus the number of double edges in D is at least $m_{\mathrm{blue}}m_{\mathrm{red}} - 3\sqrt{\varepsilon}n(m_{\mathrm{blue}} + m_{\mathrm{red}})$. For each $y \in S^{\mathrm{blue}}$, let $\Gamma_y = \{x \in S^{\mathrm{red}} \colon xy, yx \in D\}$. Hence there is some vertex $y \in S^{\mathrm{blue}}$ such that

$$|\Gamma_y| \geqslant m_{\mathrm{red}} - 3\sqrt{\varepsilon}n\left(\frac{m_{\mathrm{blue}} + m_{\mathrm{red}}}{m_{\mathrm{blue}}}\right) \geqslant \begin{cases} (\delta - 6\sqrt{\varepsilon})n, & \text{if } \delta < \varepsilon^{1/9}; \\ m_{\mathrm{red}}(1-\varepsilon^{1/4}), & \text{otherwise.} \end{cases}$$

Note that if $xy, yx \in D$ with $x \in S^{\mathrm{red}}$ and $y \in S^{\mathrm{blue}}$, then $T \cup xy \in \partial R(T \cup x) \cap \partial B(T \cup y)$. Hence $\Gamma_y \subseteq \Gamma_y^{\mathrm{red}}$ and thus the lemma follows. □

Roughly speaking, in the next lemma we consider the following situation. Let R be a red tight component in H, G be a blueprint for H and $R_G \subseteq G^{\mathrm{red}}$ be such that $H(R_G) \subseteq R$. We pick a maximal matching in R_G^+ and let U be the remaining vertices of H not in this matching, so $R_G^+[U]$ is empty. Then the lemma implies that the number of monochromatic tight components in U is less than what we would expect. In particular, if $k = 4$, then the edges in $G[U]$ induce only two monochromatic tight components in H.

Lemma 30. *Let* $k \geqslant 4$ *and* $1/n \ll \varepsilon \ll \alpha, \delta \ll \eta \ll 1$*. Let* H *be a* $(1-\varepsilon, \alpha)$*-dense* k*-graph and* G *a* $3\sqrt{\varepsilon}$*-blueprint for* H*. Let* R *be a red tight component in* H*. Let* $R_G \subseteq G^{\mathrm{red}}$ *be such that* $H(R_G) \subseteq R$*. Let* $U \subseteq V(H)$ *be such that* $|U| \geqslant \eta n/2$ *and* $R_G^+[U] = \varnothing$*. Let* $S \in \binom{U}{k-4}$ *be such that the link graph* G_S *of* G *satisfies* $G_S^{\mathrm{red}}[U] \subseteq (R_G)_S$ *and* $\delta(G_S[U]) \geqslant |U| - \delta n$*. Then there exists a subgraph* J_S *of* $G_S[U]$ *such that* $|J_S| \geqslant |G_S[U]| - 7\delta^{1/4}n^2$ *and* $H(S \cup e) = H(S \cup e')$ *for all* $e, e' \in J_S$ *of the same colour. In particular, if* $k = 4$*, then the edges in* J *induce only one red and one blue tight component in* H*. The same statement holds when the colours are reversed.*

Proof. Set $J_S^{\mathrm{red}} = G_S^{\mathrm{red}}[U]$. Note that for $e, e' \in J_S^{\mathrm{red}}$, we have $e, e' \in (R_G)_S$ and thus $H(S \cup e) = H(S \cup e') = R$ since $H(R_G) \subseteq R$. Therefore to prove the lemma, it suffices

to prove that there exists $J_S^{\text{blue}} \subseteq G_S^{\text{blue}}[U]$ such that $\left|J_S^{\text{red}}\right| + \left|J_S^{\text{blue}}\right| \geqslant |G_S[U]| - 7\delta^{1/4}n^2$ and $H(S \cup e) = H(S \cup e')$ for all $e, e' \in J_S^{\text{blue}}$.

For simplicity we assume $k = 4$ and $S = \varnothing$. It is easy to see that an analogous argument works in the general case. Thus for the rest of the proof, we omit the subscript S.

Let $K = G[U]$. If $\left|K^{\text{blue}}\right| < 2\delta^{1/2}n^2$, then we are done by setting $J^{\text{blue}} = \varnothing$ as

$$\left|J^{\text{red}}\right| = \left|K^{\text{red}}\right| = |K| - \left|K^{\text{blue}}\right| \geqslant |K| - 2\delta^{1/2}n^2 \geqslant |K| - 7\delta^{1/4}n^2.$$

Now assume $\left|K^{\text{blue}}\right| \geqslant 2\delta^{1/2}n^2$. Let $X = \{x \in V(K) \colon d_K^{\text{blue}}(x) \geqslant \delta n\}$. We have that

$$2\delta^{1/2}n^2 \leqslant \left|K^{\text{blue}}\right| \leqslant \sum_{x \in U} d_K^{\text{blue}}(x) \leqslant n\,|X| + \delta n^2.$$

Thus $|X| \geqslant \delta^{1/2}n$. Let D be the digraph with vertex set X such that, for each $x \in X$,

$$N_D^+(x) = N_K^{\text{blue}}(x, X) \cup \{x' \in N_K^{\text{red}}(x, X) \colon xx'y \in \partial R \cap \partial B(xy) \text{ for some } y \in N_K^{\text{blue}}(x)\}.$$

We now bound $\delta^+(D)$ as follows. If $d_K^{\text{red}}(x, X) \geqslant \delta n$, then by applying Lemma 29 (with $x, N_G^{\text{blue}}(x, U), N_G^{\text{red}}(x, X), \delta$ playing the roles of $T, S^{\text{blue}}, S^{\text{red}}, \delta$), we deduce that

$$\begin{aligned}&\left|\{x' \in N_K^{\text{red}}(x, X) \colon xx'y \in \partial R(xx') \cap \partial B(xy) \text{ for some } y \in N_K^{\text{blue}}(x)\}\right| \\ &\geqslant (1 - \varepsilon^{1/4}) d_K^{\text{red}}(x, X).\end{aligned}$$

Recall that $R = R(xx')$ for all $x' \in N_K^{\text{red}}(x, X)$, $|X| \geqslant \delta^{1/2}n$ and $\varepsilon \ll \delta$. Hence

$$\begin{aligned}d_D^+(x) &\geqslant d_K^{\text{blue}}(x, X) + (1 - \varepsilon^{1/4}) d_K^{\text{red}}(x, X) \geqslant (1 - \varepsilon^{1/4})(d_K^{\text{blue}}(x, X) + d_K^{\text{red}}(x, X)) \\ &= (1 - \varepsilon^{1/4}) d_K(x, X) \geqslant (1 - \varepsilon^{1/4})(|X| - \delta n) \geqslant (1 - 2\delta^{1/2})\,|X|\,.\end{aligned}$$

On the other hand, if $d_K^{\text{red}}(x, X) < \delta n$, then

$$d_D^+(x) \geqslant d_K^{\text{blue}}(x, X) \geqslant |X| - \delta n - d_K^{\text{red}}(x, X) \geqslant |X| - 2\delta n \geqslant (1 - 2\delta^{1/2})\,|X|\,.$$

Therefore, we have $\delta^+(D) \geqslant (1 - 2\delta^{1/2})\,|X|$ and so $|E(D)| \geqslant (1 - 2\delta^{1/2})\,|X|^2 \geqslant 2(1 - 2\delta^{1/2})\binom{|X|}{2}$. Let F be the graph with vertex set X in which xx' forms an edge if and only if it forms a double edge in D. Note that $|F| \geqslant (1 - 4\delta^{1/2})\binom{|X|}{2}$. By Proposition 25, there exists a subgraph F^* of F with $\delta(F^*) \geqslant (1 - 6\delta^{1/4})\,|X|$. Clearly, F^* is connected.

Let $J^{\text{blue}} = \{xx' \in K^{\text{blue}} \colon x \in V(F^*)\}$. We have

$$\begin{aligned}\left|J^{\text{red}} \cup J^{\text{blue}}\right| &\geqslant |K| - \sum_{x' \in U \setminus X} d_K^{\text{blue}}(x') - |X \setminus V(F^*)|\,n \\ &\geqslant |K| - \delta n^2 - 6\delta^{1/4}n^2 \geqslant |G[U]| - 7\delta^{1/4}n^2.\end{aligned}$$

We now show that $B(x_1z_1) = B(x_2z_2)$ for all $x_1z_1, x_2z_2 \in J^{\text{blue}}$. Since F^* is connected and $d_{J^{\text{blue}}}(x) > 0$ for all $x \in V(F^*)$, it suffices to consider the case when $x_1x_2 \in F^*$. If $x_1x_2 \in K^{\text{blue}}$, then $x_1z_1, x_1x_2, x_2z_2 \in G^{\text{blue}}$ and so $B(x_1z_1) = B(x_1x_2) = B(x_2z_2)$, since G is a blueprint. Now assume that $x_1x_2 \in K^{\text{red}}$. Since $x_1x_2 \in F^* \subseteq F$, there are $y_1 \in N_K^{\text{blue}}(x_1)$ and $y_2 \in N_K^{\text{blue}}(x_2)$ such that $x_1x_2y_1 \in \partial R \cap \partial B(x_1y_1)$ and $x_1x_2y_2 \in \partial R \cap \partial B(x_2y_2)$. Let $u \in N_H(x_1x_2y_1) \cap N_H(x_1x_2y_2) \cap U$. Since $R_G^+[U] = \varnothing$, we have $x_1x_2y_1u, x_1x_2y_2u \in H^{\text{blue}}$. Hence, $B(x_1y_1) = B(x_2y_2)$. Moreover, since $x_1y_1, x_1z_1, x_2y_2, x_2z_2 \in G^{\text{blue}}$, we have $B(x_1z_1) = B(x_1y_1) = B(x_2y_2) = B(x_2z_2)$ as required. □

6 Monochromatic connected matchings in $K_n^{(4)}$

In this section, we prove that every almost complete red-blue edge-coloured 4-graph H contains a red and a blue tightly connected matching that are vertex-disjoint and together cover almost all vertices of H.

Lemma 31. *Let $1/n \ll \varepsilon \ll \alpha \ll \eta < 1$. Let H be a 2-edge-coloured $(1-\varepsilon, \alpha)$-dense 4-graph on n vertices. Then H contains two vertex-disjoint monochromatic tightly connected matchings of distinct colours such that their union covers all but at most $3\eta n$ of the vertices of H.*

Note that this implies $\mu_4^*(1, \varepsilon, n) \geqslant (1-3\eta)n/4$ for $1/n \ll \varepsilon \ll \eta < 1$. Hence $\mu_4^*(1) \geqslant 1/4$. Therefore, together with Corollary 20, Lemma 31 implies Theorem 2.

To prove Lemma 31 we first need the following lemma which chooses the initial tight components in H in which we find our tightly connected matchings.

Lemma 32. *Let $1/n \ll \varepsilon \ll \alpha \ll \eta < 1$. Let H be a 2-edge-coloured $(1-\varepsilon, \alpha)$-dense 4-graph on n vertices. Suppose that H does not contain two vertex-disjoint monochromatic tightly connected matchings of distinct colours such that their union covers all but at most $3\eta n$ of the vertices of H. Then, there exists a red tight component R in H, a blue tight component B in H, a $3\sqrt{\varepsilon}$-blueprint G for H with $\delta(G) \geqslant (1-\alpha^{1/30})n$ and a matching M_0 in $R \cup B$ such that the following holds, where $W_0 = V(G) \setminus V(M_0)$.*

(i) *$R(e) = R$ and $B(e') = B$ for all edges $e \in G^{\mathrm{red}}[V(M_0^{\mathrm{red}}) \cup W_0]$ and all edges $e' \in G^{\mathrm{blue}}[V(M_0^{\mathrm{blue}}) \cup W_0]$,*

(ii) *$M_0 \subseteq (G^{\mathrm{red}})^+ \cup (G^{\mathrm{blue}})^+$,*

(iii) *$(G^{\mathrm{red}})^+[W_0] \cup (G^{\mathrm{blue}})^+[W_0]$ is empty.*

Proof. By Lemma 23, there exists a $3\sqrt{\varepsilon}$-blueprint G_0 for H with $V(G_0) = V(H)$ and $|G_0| \geqslant (1-\alpha-96\sqrt{\varepsilon})\binom{n}{2} \geqslant (1-4\alpha)\binom{n}{2}$. By Corollary 26, there exists a subgraph G_1 of G_0 of order at least $(1-6\sqrt{\alpha})n$ that contains a spanning monochromatic component and $\delta(G_1) \geqslant (1-12\sqrt{\alpha})n$. Note that that G_1 is also a $3\sqrt{\varepsilon}$-blueprint for H.

We assume without loss of generality that G_1 contains a spanning red component. Since G_1 is a blueprint, all the red edges in G_1 induce the same red tight component R in H. Let $R^+ = (G_1^{\mathrm{red}})^+ \subseteq R$. Let M be a matching in R^+ of maximum size. Let $U = V(G_1) \setminus V(M)$.

Thus $|U| \geqslant \eta n$ (or else $|V(M)| \geqslant |V(G_1)| - |U| \geqslant (1-2\eta)n$, a contradiction). Moreover, $R^+[U] = \varnothing$. Since $\delta(G_1) \geqslant (1-12\sqrt{\alpha})n$, we have $\delta(G_1[U]) \geqslant |U| - \alpha^{1/3}n$. Hence, by Lemma 30 (with $4, U, \varnothing, \alpha^{1/3}$ playing the roles of k, U, S, δ), there exists a subgraph J of $G_1[U]$ such that $|J| \geqslant |G_1[U]| - 2\alpha^{1/13}n^2$, such that $H(e) = H(e')$ for all $e, e' \in J$ of the same colour. Let $G_2 = (G_1 - G_1^{\mathrm{blue}}[U]) \cup J$ and $B = B(e)$ for $e \in J^{\mathrm{blue}}$. Note that $|G_2| \geqslant (1-\alpha^{1/14})\binom{n}{2}$. By Proposition 25, there exists a subgraph G of G_2 such that $\delta(G) \geqslant (1-\alpha^{1/30})n$.

Let $W = V(G) \setminus V(M)$. Next, we show that (i) and (ii) hold but with M, W instead of M_0, W_0. Note that $M^{\text{blue}} = \varnothing$, so (ii) holds by our construction. Since $G^{\text{red}} \subseteq G_1^{\text{red}}$ and G_1^{red} is connected and a blueprint, $R(e) = R$ for all $e \in G^{\text{red}}$. Note that $G^{\text{blue}}[V(M^{\text{blue}}) \cup W] = G^{\text{blue}} - V(M) \subseteq G_2^{\text{blue}}[U] = J^{\text{blue}}$, so $B(e) = B$ for all $e \in G^{\text{blue}}[V(M^{\text{blue}}) \cup W]$. Hence (i) holds. We now add vertex-disjoint edges of $(G^{\text{red}})^+[W] \cup (G^{\text{blue}})^+[W]$ to M and call the resulting matching M_0. We deduce that M_0 satisfies (i)–(iii). □

We now prove Lemma 31.

Proof of Lemma 31. Suppose the contrary that H does not contain two vertex-disjoint monochromatic tightly connected matchings of distinct colours such that their union covers all but at most $3\eta n$ of the vertices of H. We call this the initial assumption. Apply Lemma 32 and obtain a red tight component R, a blue tight component B in H, a $3\sqrt{\varepsilon}$-blueprint G for H with $\delta(G) \geqslant (1 - \alpha^{1/30})n$ and a matching M_0 in $R \cup B$ satisfying Lemma 32(i)–(iii).

We now fix G, R and B. We use the following notation for the rest of the proof. For a matching M in $R \cup B$, we set

$$\begin{aligned} W &= W(M) = V(G) \setminus V(M), \\ W_{\text{red}} &= W_{\text{red}}(M) = \{w \in W \colon d^{\text{blue}}_{G[W]}(w) \leqslant 8\sqrt{\varepsilon}n\}, \\ W_{\text{blue}} &= W_{\text{blue}}(M) = \{w \in W \colon d^{\text{red}}_{G[W]}(w) \leqslant 8\sqrt{\varepsilon}n\}. \end{aligned}$$

Note that $|W| \geqslant \eta n$ by the initial assumption. Without loss of generality, $|W_{\text{blue}}(M_0)| \leqslant |W_{\text{red}}(M_0)|$.

We define $\mathcal{M}$ be the set of matchings M in $R \cup B$ such that

(i′) $R(e) = R$ and $B(e') = B$ for all edges $e \in G^{\text{red}}[W]$ and $e' \in G^{\text{blue}}[V(M^{\text{blue}}) \cup W]$,

(ii′) $M^{\text{blue}} \subseteq (G^{\text{blue}})^+$,

(iii′) $(G^{\text{red}})^+[W] \cup (G^{\text{blue}})^+[W]$ is empty.

Note that (i′) and (ii′) are weaker statements of those in Lemma 32(i) and (ii), so $M_0 \in \mathcal{M}$. Let $\mathcal{M}'$ be the set of $M \in \mathcal{M}$ also satisfying

(iv′) $|W_{\text{blue}}| \leqslant |W_{\text{red}}|$.

Observe that $M_0 \in \mathcal{M}'$, so $\mathcal{M}'$ is nonempty.

Let $\gamma = 10\alpha^{1/30}$. We now show that, for all $M \in \mathcal{M}$, W_{red} and W_{blue} partition W, and moreover one of them is small.

Claim 33. *Let $M \in \mathcal{M}$. The following holds:*

(a) *for all $w \in W$, either $d^{\text{red}}_{G[W]}(w) \leqslant 7\sqrt{\varepsilon}n$ or $d^{\text{blue}}_{G[W]}(w) \leqslant 7\sqrt{\varepsilon}n$,*

(b) *W_{red} and W_{blue} partition W,*

(c) *either* $|W_{\text{blue}}| \leqslant \gamma n$ *or* $|W_{\text{red}}| \leqslant \gamma n$.

In particular, if $M \in \mathcal{M}'$*, then* $|W_{\text{blue}}| \leqslant \gamma n$.

Proof of Claim. Suppose that $w \in W$ satisfies $d^{\text{red}}_{G[W]}(w), d^{\text{blue}}_{G[W]}(w) > 7\sqrt{\varepsilon}n$. By Lemma 29 (with $7\sqrt{\varepsilon}, w, N^{\text{red}}_{G[W]}(w), N^{\text{blue}}_{G[W]}(w)$ playing the roles of $\delta, T, S^{\text{red}}, S^{\text{blue}}$), there exist $x \in N^{\text{red}}_{G[W]}(w)$ and $y \in N^{\text{blue}}_{G[W]}(w)$ such that $wxy \in \partial R \cap \partial B$. In particular, $d_H(wxy) \neq 0$ and thus $d_H(wxy) \geqslant (1-\varepsilon)n$, which implies that there exists a vertex $w' \in W$ such that $ww'xy \in H$. Note that $ww'xy \in (G^{\text{red}})^+[W] \cup (G^{\text{blue}})^+[W]$ contradicting (iii′). Hence, $\min\{d^{\text{red}}_{G[W]}(w), d^{\text{blue}}_{G[W]}(w)\} \leqslant 7\sqrt{\varepsilon}n$. Since $\delta(G) \geqslant (1-\alpha^{1/30})n$ implies that $\delta(G[W]) \geqslant |W| - \alpha^{1/30}n > 16\sqrt{\varepsilon}n$, we deduce that (a) and (b) hold.

Recall that $|W| \geqslant \eta n > 2\gamma n$. So one of W_{red} and W_{blue} has size greater than γn. Suppose both are (that is, (c) is false). Since $\delta(G) \geqslant (1-\alpha^{1/30})n = (1-\gamma/10)n$, we have that there are at least

$$|W_{\text{blue}}|\left(|W_{\text{red}}| - \gamma n/10 - 8\sqrt{\varepsilon}n\right) \geqslant |W_{\text{blue}}|\left(|W_{\text{red}}| - \gamma n/5\right) > 3\,|W_{\text{red}}|\,|W_{\text{blue}}|\,/4$$

blue edges between W_{blue} and W_{red} and similarly there are at least $3\,|W_{\text{red}}|\,|W_{\text{blue}}|\,/4$ red edges between W_{blue} and W_{red}. Thus $e(W_{\text{red}}, W_{\text{blue}}) > |W_{\text{red}}|\,|W_{\text{blue}}|$, a contradiction. □

Let $M_* \in \mathcal{M}'$ be such that $(|M_*|, |M^{\text{red}}_*|)$ is lexicographically maximum. We write $W^*, W^*_{\text{red}}, W^*_{\text{blue}}$ for $W(M_*), W_{\text{red}}(M_*), W_{\text{blue}}(M_*)$, respectively.

The next claim shows that almost all 4-edges in $H[W^*]$ are blue and they form a tight component. Indeed, this follows from the fact that almost all edges in $G[W^*]$ are red and thus almost all triples in W^* are in ∂R.

Claim 34. *There exists a blue tight component* B' *in* H *such that the number of triples* $xyz \in \binom{W^*_{\text{red}}}{3} \cap \partial B'$ *with* $d_{B'}(xyz, W^*_{\text{red}}) \geqslant |W^*_{\text{red}}| - \varepsilon n$ *is at least* $(1-\alpha^{1/31})\left|\binom{W^*_{\text{red}}}{3}\right|$.

Proof of Claim. Let $\mathcal{T}$ be the set of triples $xyz \in \binom{W^*_{\text{red}}}{3} \cap \partial R$ such that $xy \in G^{\text{red}}$. Note that, for any $x \in W^*_{\text{red}}$, $y \in N^{\text{red}}_G(x, W^*_{\text{red}})$ and $z \in N_{\partial R}(xy, W^*_{\text{red}})$, we have $xyz \in \mathcal{T}$. Thus

$$\begin{aligned}|\mathcal{T}| &\geqslant \frac{1}{3!}\,|W^*_{\text{red}}|\left(|W^*_{\text{red}}| - \alpha^{1/30}n - 8\sqrt{\varepsilon}n\right)\left(|W^*_{\text{red}}| - 3\sqrt{\varepsilon}n\right)\\ &\geqslant \frac{|W^*_{\text{red}}|^3}{3!}\left(1 - \frac{2\alpha^{1/30}n}{|W^*_{\text{red}}|}\right) \geqslant \left(1-\alpha^{1/31}\right)\left|\binom{W^*_{\text{red}}}{3}\right|,\end{aligned}$$

as $|W^*_{\text{red}}| \geqslant \eta n/2$. By (iii′), we have that if $xyz \in \mathcal{T}$ and $w \in N_H(xyz, W^*_{\text{red}})$, then $wxyz \in H^{\text{blue}}$. For $xyz \in \mathcal{T}$, let $B(xyz)$ be the maximal blue tight component containing all the edges $xyzw$, where $w \in N_H(xyz, W^*_{\text{red}})$. We say that xyz generates the blue tight component $B(xyz)$. It suffices to show that all $xyz \in \mathcal{T}$ generate the same blue tight component. First we show that triples that share two vertices generate the same blue tight component. Note that, for $xyz_1, xyz_2 \in \mathcal{T}$, we have $d_H(xyz_1, W^*_{\text{red}}), d_H(xyz_2, W^*_{\text{red}}) \geqslant |W^*_{\text{red}}| - \varepsilon n > |W^*_{\text{red}}|\,/2$ and thus there exists $w \in N_H(xyz_1) \cap N_H(xyz_2) \cap W^*_{\text{red}}$. Since the edges $wxyz_1$ and $wxyz_2$ are blue, it follows that $B(xyz_1) = B(xyz_2)$.

Now let $x_1y_1z_1, x_2y_2z_2 \in \mathcal{T}$, where $x_1y_1, x_2y_2 \in G^{\text{red}}$. Let $w_1 \in N_{\partial R}(x_1y_1) \cap N_{\partial R}(x_2y_2) \cap N_{G^{\text{red}}}(x_1) \cap N_{G^{\text{red}}}(x_2) \cap W^*_{\text{red}}$ and $w_2 \in N_{\partial R}(x_1w_1) \cap N_{\partial R}(x_2w_1) \cap W^*_{\text{red}}$. It follows that $x_1y_1w_1, x_1w_1w_2, x_2w_1w_2, x_2y_2w_1 \in \mathcal{T}$. Hence $B(x_1y_1z_1) = B(x_1y_1w_1) = B(x_1w_1w_2) = B(x_2w_1w_2) = B(x_2y_2w_1) = B(x_2y_2z_2)$. Let B' be the unique blue tight component generated by all triples $xyz \in \mathcal{T}$. □

The previous claim and a greedy argument imply that there is a matching $M^{B'}_*$ in $B'[W^*]$ that covers all but ηn of the vertices in W^*. Thus we may assume that $\left|M^{\text{blue}}_*\right| \geqslant \eta n/4$, otherwise $\left|V(M^{\text{red}}_* \cup M^{B'}_*)\right| \geqslant n - 3\eta n$, which is a contradiction to the initial assumption. To complete the proof, we will show that in fact $B' = B$, implying M^{red}_* and $M^{\text{blue}}_* \cup M^{B'}_*$ are tightly connected matchings, a contradiction to the initial assumption.

We now pick a special edge $e^* \in M^{\text{blue}}_*$. Its special property that we desire is stated in Claim 36.

Claim 35. *There exist an edge $e^* = v^*_1v^*_2v^*_3v^*_4 \in M^{\text{blue}}_*$ and distinct vertices $w_1, \dots, w_4$, $w'_1, \dots, w'_4 \in W^*_{\text{red}}$ such that, for each $j \in [4]$,*

(a) *all the red edges of G incident to v^*_j induce R, or*

(b) *$v^*_jw_j \in G^{\text{blue}}$ and $v^*_jw_jw'_j \in \partial R \cap \partial B$.*

Proof of Claim. For each edge $e \in M^{\text{blue}}_*$, let $v^e_1, v^e_2, v^e_3, v^e_4$ be an enumeration of its vertices. It is easy to see that there exists $M^{\text{blue}}_1 \subseteq M^{\text{blue}}_*$ with $\left|M^{\text{blue}}_1\right| = \left|M^{\text{blue}}_*\right|/16$ such that for each $j \in [4]$ we have that either

(a′) for all $e \in M^{\text{blue}}_1$, there is a red edge in G between v^e_j and W^*_{red}, or

(b′) for all $e \in M^{\text{blue}}_1$, all edges in G between v^e_j and W^*_{red} are blue.

Let J_1 be the set of $j \in [4]$ such that (a′) holds and $J_2 = [4] \setminus J_1$. Since each vertex in W^*_{red} is incident to a red edge of G that induces R and G is a blueprint for H, we have that, for all $e \in M^{\text{blue}}_1$ and all $j \in J_1$, all the red edges incident to v^e_j induce R. For every $j \in J_2$, we have that

$$\left|G^{\text{blue}}[\left\{v^e_j \colon e \in M^{\text{blue}}_1\right\}, W^*_{\text{red}}]\right| \geqslant \left|M^{\text{blue}}_1\right| \left(|W^*_{\text{red}}| - \alpha^{1/30}n\right) \geqslant \left(1 - \alpha^{1/31}\right) \left|M^{\text{blue}}_1\right| |W^*_{\text{red}}|.$$

Thus there exists $w_j \in W^*_{\text{red}}$ such that $w_jv^e_j$ is blue for at least $\left|M^{\text{blue}}_1\right|(1 - \alpha^{1/32})$ of the vertices v^e_j, with $e \in M^{\text{blue}}_1$. It is easy to see that we can choose the w_j to be distinct. Hence there exist distinct vertices $w_1, w_2, w_3, w_4 \in W^*_{\text{red}}$ and $M^{\text{blue}}_2 \subseteq M^{\text{blue}}_1$ with $\left|M^{\text{blue}}_2\right| = \left|M^{\text{blue}}_1\right|/2 \geqslant \eta n/128$ such that for all $j \in J_2$ and all $e \in M^{\text{blue}}_2$ we have that $w_jv^e_j \in G^{\text{blue}}$.

For $j \in J_2$, let $V_j = \{v^e_j \colon e \in M^{\text{blue}}_2\}$ and note that $d^{\text{blue}}_G(w_j, V_j) = \left|M^{\text{blue}}_2\right| \geqslant \eta n/128$ and $d^{\text{red}}_G(w_j, W^*_{\text{red}}) \geqslant \eta n/2$. For each $j \in J_2$, we apply Lemma 29 with colours reversed and $w_j, V_j, \widetilde{W}^*_{\text{red}}$ playing the roles of $T, S^{\text{blue}}, S^{\text{red}}$ where $\widetilde{W}^*_{\text{red}}$ denotes W^*_{red} with all previously chosen vertices removed. Thus, we find distinct $w'_j \in W^*_{\text{red}} \setminus \{w_1, w_2, w_3, w_4\}$ and $M^{\text{blue}}_3 \subseteq$

M_2^{blue} with $\left|M_3^{\text{blue}}\right| = \left|M_2^{\text{blue}}\right|/2$ such that, for all $j \in J_2$ and all $e \in M_3^{\text{blue}}$, we have that $v_j^e w_j \in G^{\text{blue}}$ and $v_j^e w_j w_j' \in \partial R \cap \partial B$. We complete the proof by choosing $e^* = v_1^* v_2^* v_3^* v_4^* \in M_3^{\text{blue}}$ and a distinct vertex $w_j' \in W_{\text{red}}^*$ for each $j \in J_1$. □

Let $W' = W_{\text{red}}^* \setminus \{w_1, \dots, w_4, w_1', \dots, w_4'\}$.

Claim 36. *The 4-graph $R[e^* \cup W']$ is empty and $B[e^* \cup W']$ does not contain two vertex-disjoint edges each containing an edge of G^{blue}. In particular, there do not exist two vertex-disjoint edges f_1 and f_2 in $(R \cup B)[e^* \cup W']$ each containing an edge of G^{blue}.*

Proof of Claim. First suppose there exist two vertex-disjoint edges $f_1, f_2 \in B[e^* \cup W']$ each of which contains an edge of G^{blue}. By the maximality of $|M_*|$, both f_1 and f_2 must intersect e^*. For simplicity, we only consider the case that $e^* \setminus (f_1 \cup f_2) = \{v_1^*\}$ (the other cases can be proved similarly). By Claim 35, we have that all red edges of G incident to v_1^* induce R or $v_1^* w_1 \in G^{\text{blue}}$ and $v_1^* w_1 w_1' \in \partial R \cap \partial B$.

First suppose that $v_1^* w_1 \in G^{\text{blue}}$ and $v_1^* w_1 w_1' \in \partial R \cap \partial B$. Let $w_1'' \in N_H(v_1^* w_1 w_1', W^* \setminus (f_1 \cup f_2))$ and $f_3 = v_1^* w_1 w_1' w_1''$. Let $M' = (M_* \setminus \{e^*\}) \cup \{f_1, f_2, f_3\}$. Note that $W(M') \subseteq W^*$. Since $|W| \geqslant \eta n \geqslant 3\gamma n$ and $|W_{\text{blue}}^*| \leqslant \gamma n$ by Claim 33, we deduce that M' satisfies (iv′). Hence $M' \in \mathcal{M}'$ contradicting the maximality of $|M_*|$.

Now assume that all the red edges of G incident to v_1^* induce R. Let M be a matching in $R \cup B$ containing $(M_* \setminus \{e^*\}) \cup \{f_1, f_2\}$ satisfying (ii′) and (iii′). We now show that $M \in \mathcal{M}'$, which then contradicts the maximality of $|M_*|$. Recall that $v_1^* \in e^* \in M_*^{\text{blue}}$, so

$$W \subseteq (W^* \setminus (f_1 \cup f_2)) \cup \{v_1^*\} \text{ and } V(M^{\text{blue}}) \cup W \subseteq V(M_*^{\text{blue}}) \cup W^*. \tag{6}$$

Together with our assumption on v_1^*, M satisfies (i′). Hence $M \in \mathcal{M}$. For all $w \in W \cap W_{\text{red}}^*$,

$$d_{G[W]}^{\text{blue}}(w) \overset{(6)}{\leqslant} d_{G[W^*]}^{\text{blue}}(w) + |v_1^*| \overset{\text{Claim 33(a)}}{\leqslant} 7\sqrt{\varepsilon} n + 1 \leqslant 8\sqrt{\varepsilon} n.$$

and a similar inequality holds for all $w \in W \cap W_{\text{blue}}^*$. This implies that $W_{\text{blue}} \subseteq W_{\text{blue}}^* \cup \{v^*\}$. Since $|W| \geqslant \eta n \geqslant 3\gamma n$ and $|W_{\text{blue}}^*| \leqslant \gamma n$ by Claim 33, we deduce that M satisfies (iv′). Hence, $M \in \mathcal{M}'$ as required, a contradiction.

Therefore, $B[e^* \cup W']$ does not contain two vertex-disjoint edges each of which contains an edge of G^{blue}. If $R[e^* \cup W']$ contains an edge f, then a similar argument holds with f replacing $\{f_1, f_2\}$. Note that if $|M| = |M_*|$, then we obtain a contradiction by showing that $\left|M_*^{\text{red}}\right| < \left|M^{\text{red}}\right|$. □

Since $e^* \in M_*^{\text{blue}} \subseteq (G^{\text{blue}})^+$, we may assume without loss of generality that $v_1^* v_2^* \in G^{\text{blue}}$. The following claim shows that one of the vertices v_1^* and v_2^* has small blue degree in G to W' (and thus it has large red degree to W').

Claim 37. *We have $d_G^{\text{blue}}(v_1^*, W') \leqslant 3\gamma n$ or $d_G^{\text{blue}}(v_2^*, W') \leqslant 3\gamma n$.*

Proof of Claim. Suppose to the contrary that we have $d_G^{\text{blue}}(v_1^*, W'), d_G^{\text{blue}}(v_2^*, W') > 3\gamma n$. By Claim 36, it suffices to show that we can find two vertex-disjoint edges f_1 and f_2 in $(R \cup B)[e^* \cup W']$ each containing an edge of G^{blue}. It is easy to see that we can

greedily choose vertices $x \in N_G^{\text{blue}}(v_1^*, W')$, $x' \in N_G^{\text{red}}(x, W') \cap N_{\partial B}(v_1^* x, W')$ and $x'' \in N_{\partial R}(xx', W') \cap N_H(v_1^* xx', W')$. Set $f_1 = v_1^* xx'x''$. By our construction, $v_1^* xx' \in \partial B$ and $xx'x'' \in \partial R$ implying $f_1 \in (R \cup B)[e^* \cup W']$. Similarly there exists an edge $f_2 = v_2^* yy'y'' \in (R \cup B)[e^* \cup W']$ disjoint from f_1 with $y, y', y'' \in W'$. □

Without loss of generality assume $d_G^{\text{blue}}(v_1^*, W') \leqslant 3\gamma n$ and so $d_G^{\text{red}}(v_1^*, W') \geqslant |W'| - \alpha^{1/31} n$. Let $w \in N_{\partial B}(v_1^* v_2^*) \cap N_G^{\text{red}}(v_1^*) \cap W'$, $w' \in N_G^{\text{red}}(w) \cap N_{\partial R}(v_1^* w) \cap N_H(v_1^* v_2^* w) \cap W'$ and $w'' \in N_H(v_1^* ww', W')$. (We can find these vertices greedily one by one.) By Claim 34, we may further assume that $ww'w'' \in \partial B'$. By construction, we have that $v_1^* ww' \in \partial R$ and thus Claim 36 implies that both $v_1^* v_2^* ww'$ and $v_1^* ww'w''$ are blue. Since $v_1^* v_2^* w \in \partial B$, we deduce that $v_1^* v_2^* ww', v_1^* ww'w'' \in B$ and so $ww'w'' \in \partial B$ implying that $\partial B \cap \partial B' \neq \varnothing$. Therefore $B = B'$ as required. □

7 Monochromatic connected matchings in $K_n^{(5)}$

The aim of this section is to prove the following lemma which shows that 2-edge-coloured dense 5-graphs can be almost partitioned into four monochromatic tightly connected matchings.

Lemma 38. *Let $1/n \ll \varepsilon \ll \alpha \ll \eta < 1$. Let H be a 2-edge-coloured $(1-\varepsilon, \alpha)$-dense 5-graph on n vertices. Then H contains four vertex-disjoint monochromatic tightly connected matchings such that their union covers all but at most $3\eta n$ of the vertices of H.*

Note that this implies $\mu_5^4(1, \varepsilon, n) \geqslant (1 - 3\eta)n/5$ for $1/n \ll \varepsilon \ll \eta < 1$. Hence $\mu_5^4(1) \geqslant 1/5$. Together with Corollary 20, Lemma 38 implies Theorem 3.

We use the following notation throughout this section. Let H be a 2-edge-coloured 5-graph and let G be a blueprint for H. Given a red tight component $R \subseteq H$, we write R^3 for the edges of G that induce R. We use analogous notation for blue tight components.

Let H be a 2-edge-coloured dense 5-graph. We first apply Lemma 23 to H to get a blueprint G for H. Since G is 2-edge-coloured dense 3-graph, we can apply Lemma 23 again to G to obtain a blueprint for G, which is a 2-coloured 1-graph. The following lemma summarises the structural information about H that we obtain in this way.

Lemma 39. *Let $1/n \ll \varepsilon \ll \alpha \ll 1$. Let H be a 2-edge-coloured $(1-\varepsilon, \alpha)$-dense 5-graph on n vertices. Then there exists a 3-graph G with $V(G) = V(H)$, two disjoint subsets V^{red} and V^{blue} of $V(H)$, a red tight component $R \subseteq H$ and a blue tight component $B \subseteq H$ such that the following properties hold.*

(i) *G is a $(1 - \alpha^{1/37}, \alpha^{1/37})$-dense $3\sqrt{\varepsilon}$-blueprint for H.*

(ii) *$\big|V(H) \setminus (V^{\text{red}} \cup V^{\text{blue}})\big| \leqslant \alpha^{1/75} n$.*

(iii) *$d_{\partial R^3}(v) \geqslant (1 - \alpha^{1/75})n$ for all $v \in V^{\text{red}}$.*

(iv) *$d_{\partial B^3}(v) \geqslant (1 - \alpha^{1/75})n$ for all $v \in V^{\text{blue}}$.*

Proof. By Lemma 23, there exists a $(1-\alpha^{1/37}, \alpha^{1/37})$-dense $3\sqrt{\varepsilon}$-blueprint G for H with $V(G) = V(H)$. We apply Lemma 23 to G and obtain a $\alpha^{1/75}$-blueprint J for G with $|J| \geqslant (1-\alpha^{1/75})n$. Note that, as a blueprint for a 3-graph, J is a 1-graph. Hence each edge of J contains precisely one vertex. By the definition of a blueprint all the red edges of J induce the same red tight component R_G of G. Let $V^{\text{red}} = \bigcup J^{\text{red}}$. Since R_G is a red tight component of G all its edges induce the same red tight component R of H. Define V^{blue} and B analogously. □

Two edges f and f' in H are *loosely connected* if there exists a sequence of edges $e_1, \dots, e_t$ such that $e_1 = f$, $e_t = f'$ and $|e_i \cap e_{i+1}| \geqslant 1$ for all $i \in [t-1]$. A subgraph H' of H is *loosely connected* if every pair of edges in H' is loosely connected. A maximal loosely connected subgraph of H is called a *loose component* of H.

We now prove Lemma 38. The proof works by first finding a maximal matching in $R \cup B$, where R and B are the components given by Lemma 39, and then finding maximal connected matchings in the remaining vertices.

Proof of Lemma 38. Assume, for a contradiction, that such matchings do not exist. We call this the initial assumption. Apply Lemma 39 and obtain $V^{\text{red}}, V^{\text{blue}}, G, R^3, R, B^3, B$ and let $V^* = V^{\text{red}} \cup V^{\text{blue}}$. Since there are only few vertices in $V(H) \setminus V^*$ we ignore these vertices from the start and construct our matchings in $H[V^*]$.

We begin by choosing a matching $M \subseteq (R \cup B)[V^*]$ of maximum size. Let $U = V^* \setminus V(M)$. Note that we have $R[U] = B[U] = \varnothing$ and $|U| \geqslant \eta n$ by the initial assumption. Let $U^{\text{red}} = U \cap V^{\text{red}}$ and $U^{\text{blue}} = U \cap V^{\text{blue}}$. The following claim shows that if U^{red} and U^{blue} are both large, then $G[U]$ must contain many edges in R^3 or many edges in B^3.

Claim 40. *If* $\left|U^{\text{red}}\right|, \left|U^{\text{blue}}\right| \geqslant \alpha^{1/309}n$, *then*

$$\max\{\left|R^3[U]\right|, \left|B^3[U]\right|\} \geqslant \frac{1}{2}\left|U^{\text{red}}\right|\left|U^{\text{blue}}\right||U| - 3\alpha^{1/155}n^3.$$

Proof of Claim. Define a bipartite graph K_0 with vertex classes U^{red} and U^{blue} such that $x \in U^{\text{red}}$ and $y \in U^{\text{blue}}$ are joined by an edge if and only if $xy \in \partial R^3 \cap \partial B^3$. Recall that $d_{\partial R^3}(x) \geqslant (1-\alpha^{1/75})n$ and $d_{\partial B^3}(y) \geqslant (1-\alpha^{1/75})n$ for all $x \in U^{\text{red}}$ and $y \in U^{\text{blue}}$. Hence

$$|K_0| \geqslant \left|U^{\text{blue}}\right|\left|U^{\text{red}}\right| - \alpha^{1/75}n^2.$$

Since G is $(1-\alpha^{1/37}, \alpha^{1/37})$-dense, we have $d_G(xy, U) \geqslant |U| - \alpha^{1/37}n$ for $xy \in K_0$. We now colour the edges of K_0 such that $xy \in K_0$ is red if $d_{R^3}(xy, U) \geqslant |U| - 2\alpha^{1/76}n$ and blue if $d_{B^3}(xy, U) \geqslant |U| - 2\alpha^{1/76}n$. Since $K_0 \subseteq \partial R^3 \cap \partial B^3$, if $xyz \in G$ with $xy \in K_0$, then $xyz \in R^3 \cup B^3$. Hence it suffices to show that almost all edges of K_0 are of the same colour. Indeed, if we have that at least $\left|U^{\text{red}}\right|\left|U^{\text{blue}}\right| - 3\alpha^{1/154}n^2$ edges of K_0 are red, then we have

$$\left|R^3[U]\right| \geqslant \frac{1}{2}(\left|U^{\text{red}}\right|\left|U^{\text{blue}}\right| - 3\alpha^{1/154}n^2)(|U| - 2\alpha^{1/76}n) \geqslant \frac{1}{2}\left|U^{\text{red}}\right|\left|U^{\text{blue}}\right||U| - 3\alpha^{1/155}n^3.$$

We show that each edge $xy \in K_0$ is coloured either red or blue. It suffices to show that either $d_{R^3}(xy, U) < \alpha^{1/76}n$ or $d_{B^3}(xy, U) < \alpha^{1/76}n$. Indeed if $d_{R^3}(xy, U), d_{B^3}(xy, U) \geqslant$

$\alpha^{1/76}n$, then by Lemma 29, there exists $u, u' \in U$ such that $xyu \in R^3$, $xyu' \in B^3$ and $xyuu' \in \partial R \cap \partial B$. For any $u'' \in N_H(xyuu', U)$, we would have $xyuu'u'' \in R[U] \cup B[U]$, a contradiction to the maximality of M. Moreover, by Lemma 28, we have that $\min\{d^{\mathrm{red}}_{K_0}(u), d^{\mathrm{blue}}_{K_0}(u)\} \leqslant \alpha^{1/76}n$ for all $u \in U$.

Let K_1 be the graph obtained from K_0 by, for each $u \in U$, deleting all red edges incident to u if $d^{\mathrm{red}}_K(u) \leqslant \alpha^{1/76}n$ and all blue edges incident to u if $d^{\mathrm{blue}}_K(u) \leqslant \alpha^{1/76}n$. Note that $|K_1| \geqslant \left|U^{\mathrm{red}}\right| \left|U^{\mathrm{blue}}\right| - \alpha^{1/77}n^2$ and that, in K_1, each vertex is incident to only edges of one colour. It is not too hard to see that by deleting at most $2\alpha^{1/154}n^2$ additional edges, we can obtain a subgraph K_2 of K_1 for which each vertex has degree 0 or large degree. More precisely, for all $u \in U^{\mathrm{red}}$,

$$d_{K_2}(u) \geqslant \left|U^{\mathrm{blue}}\right| - 3\alpha^{1/308}n \text{ or } d_{K_2}(u) = 0$$

and, for all $u \in U^{\mathrm{blue}}$,

$$d_{K_2}(u) \geqslant \left|U^{\mathrm{red}}\right| - 3\alpha^{1/308}n \text{ or } d_{K_2}(u) = 0.$$

Since each vertex is incident to only edges of one colour and any two vertices in U^{red} that have non-zero degree have a common neighbour this implies that all edges in K_2 are of the same colour. Since $|K_2| \geqslant \left|U^{\mathrm{red}}\right| \left|U^{\mathrm{blue}}\right| - 3\alpha^{1/154}n^2$, this concludes the proof. □

The following claim shows that there is a red tight component R_* and a blue tight component B_* of H such that almost all the edges in $G[U]$ induce one of these components.

Claim 41. *Let $\gamma = \alpha^{1/1110}$. There exists a red tight component R_* and a blue tight component B_* of H such that*

(i) $|R^3_*[U]| \geqslant \left|G^{\mathrm{red}}[U]\right| - 8\gamma^{1/5}n^3$ *and* $|B^3_*[U]| \geqslant \left|G^{\mathrm{blue}}[U]\right| - 8\gamma^{1/5}n^3$,

(ii) $|(R^3_* \cup B^3_*)[U]| \geqslant (1-\gamma^{1/6})\binom{|U|}{3}$ *and*

(iii) $R_* = R$ *or* $B_* = B$.

Proof of Claim. First we show that, for each $u \in U$, there exists $J_u \subseteq G_u[U]$, where G_u is the link graph of G at u, such that $|J_u| \geqslant |G_u[U]| - \alpha^{1/14}n^2$ and $R(e \cup u) = R(e' \cup u)$ for $e, e' \in J^{\mathrm{red}}_u$ and $B(e \cup u) = B(e' \cup u)$ for $e, e' \in J^{\mathrm{blue}}_u$.

To show this fix $u \in U$. Without loss of generality assume that $u \in U^{\mathrm{red}}$. By Lemma 39, $d_{\partial R^3}(u, U) \geqslant |U| - \alpha^{1/75}n$. Let $U_* = N_{\partial R^3}(u, U)$. Clearly, $|U_*| \geqslant \eta n/2$ and $G^{\mathrm{red}}_u[U_*] \subseteq R^3_u$. Moreover, for all $x \in U_*$, we have $d_G(ux) > 0$ and thus, since G is $(1-\alpha^{1/37}, \alpha^{1/37})$-dense, $d_G(ux) \geqslant (1-\alpha^{1/37})n$. It follows that $\delta(G_u[U_*]) \geqslant |U_*| - \alpha^{1/37}n$. Thus by applying Lemma 30 with $R^3, u, U_*, \alpha^{1/37}$ playing the roles of R_G, S, U, δ, there exists $J_u \subseteq G_u[U_*] \subseteq G_u[U]$ such that

$$|J_u| \geqslant |G_u[U_*]| - 7\alpha^{1/148}n^2 \geqslant |G_u[U]| - \alpha^{1/75}n^2 - 7\alpha^{1/148}n^2 \geqslant |G_u[U]| - \alpha^{1/149}n^2$$

and $H(u \cup e) = H(u \cup e')$ for $e, e' \in J_u$ of the same colour.

Now consider the auxiliary multi-3-graph $D = \bigcup_{u\in U}\{e \cup u \colon e \in J_u\}$. Note that

$$|D| = \sum_{u\in U} |J_u| \geqslant \sum_{u\in U} \left(|G_u[U]| - \alpha^{1/149}n^2\right) \geqslant 3\,|G[U]| - \alpha^{1/149}n^3.$$

Let F be the subgraph of $G[U]$ for which $e \in F$ if and only if e is an edge of multiplicity 3 in D. Since G is $(1-\alpha^{1/37}, \alpha^{1/37})$-dense, Proposition 4 implies that $|G| \geqslant (1-2\alpha^{1/37})\binom{n}{3}$. Hence

$$\begin{aligned}|G[U]| &\geqslant \binom{|U|}{3} - 2\alpha^{1/37}\binom{n}{3} \geqslant \binom{|U|}{3} - 2\alpha^{1/37}\binom{|U|/\eta}{3}\\ &\geqslant \binom{|U|}{3} - \frac{4\alpha^{1/37}}{\eta^3}\binom{|U|}{3} \geqslant (1-\alpha^{1/38})\binom{|U|}{3}.\end{aligned}$$

Therefore $|F| \geqslant |G[U]| - \alpha^{1/149}n^3 \geqslant (1-\alpha^{1/150})\binom{|U|}{3}$. Recall that $\gamma = \alpha^{1/1110}$. By Propositions 4 and 5, there exists a $(1-\gamma^{1/5}, \gamma^{1/5})$-dense subgraph $\widetilde{F} \subseteq F$ with $V(\widetilde{F}) = V(F) = U$ and, by Proposition 4, $\left|\widetilde{F}\right| \geqslant (1-2\gamma^{1/5})\binom{|U|}{3}$. Hence $\left|\widetilde{F}^{\mathrm{red}}\right| \geqslant \left|G^{\mathrm{red}}[U]\right| - 2\gamma^{1/5}n^3$. Let $S^{\mathrm{red}} = \{x \in U \colon d_{\widetilde{F}^{\mathrm{red}}}(x) \geqslant 6\gamma^{1/5}n^2\}$. Let F_0^{red} be the subgraph of $\widetilde{F}^{\mathrm{red}}$ consisting of all edges that contain a vertex in S^{red}. Note that $\left|F_0^{\mathrm{red}}\right| \geqslant \left|\widetilde{F}^{\mathrm{red}}\right| - 6\gamma^{1/5}n^3 \geqslant \left|G^{\mathrm{red}}[U]\right| - 8\gamma^{1/5}n^3$.

We claim that all the edges in F_0^{red} induce the same red tight component R_* in H. Let $e, e' \in \widetilde{F}^{\mathrm{red}}$ with $u \in e \cap e'$. Note that $e \setminus u, e' \setminus u \in J_u^{\mathrm{red}}$ and so $R(e) = R(e')$. Hence edges in the same loose component of $\widetilde{F}^{\mathrm{red}}$ induce the same red tight component in H. In particular, since $F_0^{\mathrm{red}} \subseteq \widetilde{F}^{\mathrm{red}}$, for $u \in S^{\mathrm{red}}$, all the edges in $N_{F_0^{\mathrm{red}}}(u)$ induce the same red tight component $R(u)$ of H.

Let $u, v \in S^{\mathrm{red}}$. We want to show that $R(u) = R(v)$. We may assume that u and v are in distinct loose components L and L' of $\widetilde{F}^{\mathrm{red}}$, respectively. In particular, any edge of $\widetilde{F}$ that intersects both $V(L)$ and $V(L')$ is in $\widetilde{F}^{\mathrm{blue}}$. If $u, v \in V^{\mathrm{red}}$, then $d_{\partial R^3}(u), d_{\partial R^3}(v) \geqslant (1-\alpha^{1/75})n$ implying $R(u) = R = R(v)$. Thus we may assume that one of u and v is in V^{blue}, say $v \in V^{\mathrm{blue}}$. Let $\Gamma_L(u) = \{u' \in V(L) \colon d_L(uu') \geqslant \gamma^{1/5}n\}$ and $\Gamma_{L'}(v) = \{v' \in V(L') \colon d_{L'}(vv') \geqslant \gamma^{1/5}n\}$. It is easy to see that $|\Gamma_L(u)|, |\Gamma_{L'}(v)| \geqslant 5\gamma^{1/5}n$. Let D' be the bipartite directed graph with parts $\Gamma_L(u)$ and $\Gamma_{L'}(v)$ such that, for $u' \in \Gamma_L(u)$,

$$\begin{aligned}N^+_{D'}(u') = \{v' \in \Gamma_{L'}(v) \colon\ & uu'v' \in \widetilde{F}^{\mathrm{blue}} \text{ and } uu'u''v' \in \partial R(uu'u'') \cap \partial B(uu'v')\\ & \text{and } uu'u''vv' \in H \text{ for some } u'' \in N_L(uu')\},\end{aligned}$$

and, for $v' \in \Gamma_{L'}(v)$,

$$\begin{aligned}N^+_{D'}(v') = \{u' \in \Gamma_L(u) \colon\ & vv'u' \in \widetilde{F}^{\mathrm{blue}} \text{ and } u'v \in \partial B^3 \text{ and } vv'v''u' \in \partial B \cap \partial R(vv'v'')\\ & \text{and } vv'v''uu' \in H \text{ for some } v'' \in N_{L'}(vv')\}.\end{aligned}$$

By Lemma 29, the fact that $\widetilde{F}$ is $(1-\gamma^{1/5}, \gamma^{1/5})$-dense and the fact that H is $(1-\varepsilon, \alpha)$-dense, we have, for $u' \in \Gamma_L(u)$,

$$d^+_{D'}(u') \geqslant |\Gamma_{L'}(v)| - \gamma^{1/5}n - \varepsilon^{1/4}n - \varepsilon n > |\Gamma_{L'}(v)|/2.$$

Similarly, also using the fact that $d_{\partial B^3}(v) \geqslant (1-\alpha^{1/75})n$, we have, for $v' \in \Gamma_{L'}(v)$,

$$d^+_{D'}(v') \geqslant |\Gamma_L(u)| - \gamma^{1/5}n - \alpha^{1/75}n - \varepsilon^{1/4}n - \varepsilon n > |\Gamma_L(u)|/2.$$

It follows that D' contains a double edge $u'v'$, where $u' \in \Gamma_L(u)$ and $v' \in \Gamma_{L'}(v)$. Let $u'' \in N_L(uu')$ and $v'' \in N_{L'}(vv')$ be the vertices that are guaranteed to exist by the definition of D'. Since $u'v \in \partial B^3$, we have that $vv'u \in B^3$ and thus also $uu'v' \in B^3$. As $B[U] = \varnothing$, we have $vv'v''uu', uu'u''vv' \in H^{\mathrm{red}}$ and thus $R(uu'u'') = R(vv'v'')$. Hence $R(u) = R(v)$. We define F_0^{blue} and B_* in an analogous way. This proves (i).

Note that (ii) follows from (i) using the facts $|U| \geqslant \eta n$ and $|G[U]| \geqslant (1-\alpha^{1/38})\binom{|U|}{3}$, which were noted earlier in this proof.

We will now prove (iii). We distinguish between two cases.

Case 1: $\left|U^{\mathbf{red}}\right|, \left|U^{\mathbf{blue}}\right| \geqslant \gamma^{1/13}n$.
By Claim 40, we have $\max\{|R^3[U]|, |B^3[U]|\} \geqslant \frac{1}{2}\left|U^{\mathrm{red}}\right|\left|U^{\mathrm{blue}}\right||U| - 3\alpha^{1/155}n^3$. Since $\frac{1}{2}\left|U^{\mathrm{red}}\right|\left|U^{\mathrm{blue}}\right||U| - 3\alpha^{1/155}n^3 \geqslant \frac{1}{2}\gamma^{2/13}\eta n^3 - 3\alpha^{1/155}n^3 \geqslant 2\gamma^{1/6}n^3$, we have $R^3_* \cap R^3 \neq \varnothing$ or $B^3_* \cap B^3 \neq \varnothing$ and thus $R_* = R$ or $B_* = B$.

Case 2: $\left|U^{\mathbf{blue}}\right| \leqslant \gamma^{1/13}n$ or $\left|U^{\mathbf{red}}\right| \leqslant \gamma^{1/13}n$.
Say $\left|U^{\mathrm{blue}}\right| \leqslant \gamma^{1/13}n$. Then $\left|U^{\mathrm{red}}\right| = |U| - \left|U^{\mathrm{blue}}\right| \geqslant |U| - \gamma^{1/13}n$. Let $Q^3 = \{T \in \binom{U}{3} : \binom{T}{2} \cap \partial R^3 \neq \varnothing\}$. Since $d_{\partial R}(u, U) \geqslant |U| - \alpha^{1/75}n$ for $u \in U^{\mathrm{red}}$, there can be at most $\left|U^{\mathrm{red}}\right|\alpha^{2/75}n^2$ triples that intersect U^{red} and are not in Q^3. Hence

$$\begin{aligned}\left|Q^3\right| &\geqslant \binom{|U|}{3} - \left|U^{\mathrm{blue}}\right|^3 - \left|U^{\mathrm{red}}\right|\alpha^{2/75}n^2 \\ &\geqslant \binom{|U|}{3} - \gamma^{3/13}n^3 - \alpha^{2/75}n^3 \geqslant \binom{|U|}{3} - 2\gamma^{1/5}n^3.\end{aligned}$$

Note that $|R^3[U]| \geqslant \left|Q \cap G^{\mathrm{red}}[U]\right| \geqslant \left|G^{\mathrm{red}}[U]\right| - 2\gamma^{1/5}n^3$. Therefore, we have $R_* = R$. □

We define $R_\diamond = R \cup R_*$ and $B_\diamond = B \cup B_*$. Note that, by Claim 41(iii), $R_\diamond \cup B_\diamond$ is the union of at most three monochromatic tight components. Let $M_\diamond$ be a maximal matching in $(R_\diamond \cup B_\diamond)[V^*]$ containing M. Let $W = V^* \setminus V(M_\diamond)$. Since $M \subseteq M_\diamond$, we have $W \subseteq U$. By the initial assumption, we have $|W| \geqslant \eta n$. Note that $(R_* \cup B_*)[W] = \varnothing$ and, since $W \subseteq U$, $(R_* \cup B_*)[W] \geqslant \binom{|W|}{3} - \gamma^{1/6}n^3$. The following claim shows that almost all the edges in $G[W]$ are of the same colour.

Claim 42. *We have $|R^3_*[W]| \geqslant \binom{|W|}{3} - \gamma^{1/9}n^3$ or $|B^3_*[W]| \geqslant \binom{|W|}{3} - \gamma^{1/9}n^3$.*

Proof of Claim. Let $G_* = R^3_* \cup B^3_*$. We define

$$\begin{aligned}W_{\mathrm{red}} &= \{u \in W : d_{G_*}(u, W) \geqslant 2\alpha n^2 \text{ and } d_{B^3_*}(u, W) < \alpha n^2\},\\ W_{\mathrm{blue}} &= \{u \in W : d_{G_*}(u, W) \geqslant 2\alpha n^2 \text{ and } d_{R^3_*}(u, W) < \alpha n^2\},\\ W_0 &= \{u \in W : d_{G_*}(u, W) < 2\alpha n^2\}.\end{aligned}$$

Since $(R_* \cup B_*)[W] = \varnothing$, by Lemma 28, $W_{\mathrm{red}}, W_{\mathrm{blue}}$ and W_0 partition W. Let J be the subgraph of $G_*[W]$ obtained by deleting all red edges containing a vertex in $W_{\mathrm{blue}} \cup W_0$

and all blue edges containing a vertex in $W_{\mathrm{red}} \cup W_0$. Note that $|J| \geqslant |G_*[W]| - 2\alpha n^3 \geqslant (1-\gamma^{1/7})\binom{|W|}{3}$ and $J \subseteq \binom{W_{\mathrm{red}}}{3} \dot\cup \binom{W_{\mathrm{blue}}}{3}$. Hence

$$(1-\gamma^{1/7})\binom{|W|}{3} \leqslant \binom{|W_{\mathrm{red}}|}{3} + \binom{|W_{\mathrm{blue}}|}{3}. \tag{7}$$

Suppose that $|W_{\mathrm{red}}|, |W_{\mathrm{blue}}| \leqslant (1-\alpha^{1/8})|W|$. By (7), we may assume without loss of generality assume that $|W_{\mathrm{red}}| \geqslant |W|/2$. Noting that $x \mapsto x^3 + (|W|-x)^3$ is an increasing function for $x \geqslant |W|/2$ we have

$$\begin{aligned}\binom{|W_{\mathrm{red}}|}{3} + \binom{|W_{\mathrm{blue}}|}{3} &\leqslant \frac{1}{6}\left(|W_{\mathrm{red}}|^3 + |W_{\mathrm{blue}}|^3\right) \leqslant \frac{1}{6}\left(|W_{\mathrm{red}}|^3 + (|W| - |W_{\mathrm{red}}|)^3\right) \\ &\leqslant ((1-\gamma^{1/8})^3 + \gamma^{3/8})\frac{|W|^3}{6} < (1-\gamma^{1/7})\binom{|W|}{3},\end{aligned}$$

a contradiction to (7).

Hence at least one of W_{red} and W_{blue} has size at least $(1-\gamma^{1/8})|W|$. Without loss of generality assume $|W_{\mathrm{red}}| \geqslant (1-\gamma^{1/8})|W|$. Note that any edge of J contained in W_{red} is in R^3_*, hence

$$\left|R^3_*[W]\right| \geqslant |J| - \left|W \setminus W^{\mathrm{red}}\right| n^2 \geqslant \binom{|W|}{3} - \gamma^{1/9}n^3.$$

This proves the claim. □

Now assume without loss of generality that $|R^3_*[W]| \geqslant \binom{|W|}{3} - \gamma^{1/9}n^3$. Note that almost all edges in $H[W]$ are blue (otherwise there would have to be an edge in $R_*[W]$, which would contradict the maximality of M). More precisely, we have

$$\left|H^{\mathrm{blue}}[W]\right| \geqslant \frac{3!}{5!}|R_*[W]|\,(|W| - 3\sqrt{\varepsilon}n)(|W| - \varepsilon n) \geqslant (1-\gamma^{1/10})\binom{|W|}{5}.$$

By Propositions 4 and 5, there exists a $(1-\gamma^{1/1010}, \gamma^{1/1010})$-dense tightly connected subgraph $\widetilde{H}^{\mathrm{blue}}$ of $H^{\mathrm{blue}}[W]$ with $V(\widetilde{H}^{\mathrm{blue}}) = W$ and $\left|\widetilde{H}^{\mathrm{blue}}\right| \geqslant (1-2\gamma^{1/1010})\binom{|W|}{5}$. By an easy greedy argument, there exists a matching M' in $\widetilde{H}^{\mathrm{blue}}$ that covers all but at most ηn of the vertices in W. The matching $M' \cup M_\diamond$ covers all but at most $3\eta n$ of the vertices of H. This contradicts the initial assumption. □

8 Concluding Remarks

For $k \geqslant 3$, let $f(k)$ be the minimum integer m such that, for all large 2-edge-coloured complete k-graphs, there exists m vertex-disjoint monochromatic tight cycles covering almost all vertices. Note that $f(k)$ is well defined by [4] but the bound is very large. It is easy to see that $f(k) \geqslant 2$ for all $k \geqslant 3$. Indeed, consider the k-graph $H = K^{(k)}(A, B)$ given

in Example 22 with $|A| = \frac{3k-1}{3k}n$. Note that $H[A]$ is a red tight component. Moreover, note that any tight cycle contained in a monochromatic tight component other than $H[A]$ covers at most about a third of the vertices of H and any tight cycle in $H[A]$ leaves all $\frac{n}{3k}$ vertices in B uncovered. Hence no monochromatic tight cycle covers almost all vertices in H. We have $f(3) = 2$ by [5]. Theorems 2 and 3 imply $f(4) = 2$ and $f(5) \leqslant 4$, respectively. In general, we believe that $f(k) = 2$ for all k. However, we believe that new ideas may be needed as indicated by again considering the k-graph $H = K^{(k)}(A, B)$ with $|A| = \frac{3k-1}{3k}n$ (as above). If H contains two vertex-disjoint monochromatic tight cycles of distinct colour covering almost all vertices, then one of the two cycles must lie entirely in the red tight component $H[A]$. However, this tight component is not induced by any edge in the blueprint of H (which is $K^{(k-2)}(A, B)$ with colours swapped). Thus we ask the weaker question of whether one can bound $f(k)$ by some suitable function of k.

Acknowledgements

We thank Richard Lang, Nicolás Sanhueza-Matamala, and two anonymous referees for their helpful comments.

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