Hanani-Tutte for Radial Planarity II*

Radoslav Fulek†
Department of Mathematics
Stanford University
450 Jane Stanford Way
Stanford CA 94305-2125
radoslav.fulek@gmail.com

Michael J. Pelsmajer
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, Illinois 60616, USA
pelsmajer@iit.edu

Marcus Schaefer
School of Computing
DePaul University
Chicago, IL 60604, USA
mschaefer@cs.depaul.edu

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Abstract

A drawing of a graph G, possibly with multiple edges but without loops, is *radial* if all edges are drawn *radially*, that is, each edge intersects every circle centered at the origin at most once. G is *radial planar* if it has a radial embedding, that is, a crossing-free radial drawing. If the vertices of G are ordered or partitioned into ordered levels (as they are for leveled graphs), we require that the distances of the vertices from the origin respect the ordering or leveling.

A pair of edges e and f in a graph is *independent* if e and f do not share a vertex. We show that if a leveled graph G has a radial drawing in which every two independent edges cross an even number of times, then G is radial planar. In other words, we establish the *strong* Hanani-Tutte theorem for radial planarity. This characterization yields a very simple algorithm for radial planarity testing.

Mathematics Subject Classifications: 05C62,05C10,68R10

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1 Introduction

Radial planarity was first studied by Bachmaier, Brandenburg and Forster [3]. Radial and other layered layouts are a popular visualization tool (see [4, Section 11],[10]); early examples of radial graph layouts can be found in the literature on sociometry [21]. This paper continues work begun in "Hanani-Tutte for Radial Planarity" [17] by the same authors; some material will be revisited in order to make the current paper self-contained.

In a *leveled graph*, every vertex is assigned a level in $\{1, \ldots, k\}$. A curve is *radial* if it intersects each circle centered at the origin at most once. A *radial drawing* of a leveled graph visualizes the levels as concentric circles centered at the origin, with the level corresponding to the distance of the circles from the origin, and such that every edge is radial. A leveled graph is *radial planar* if it admits a radial embedding, that is, a radial drawing without crossings, that respects the leveling. *Radial planarity* generalizes *level planarity* [9] in which levels are visualized as vertical lines instead of concentric circles and radially-drawn edges are drawn as *x-monotone curves*. Figure 1 shows a radially embedded graph.

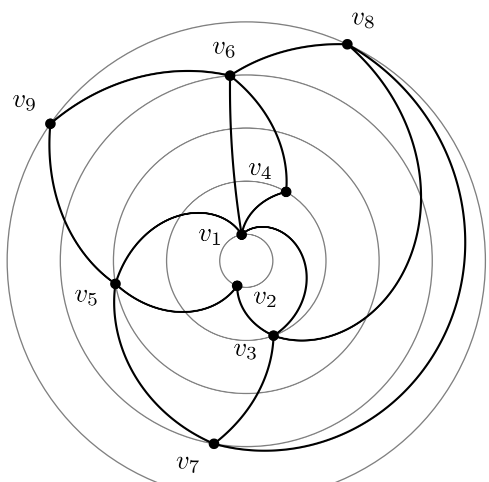

Figure 1: A radial embedding of a leveled graph with five levels.

We previously established the weak Hanani-Tutte theorem for radial planarity: a leveled graph G is radial planar if it has a radial drawing (respecting the leveling) in which every two edges cross an even number of times [17, Theorem 1]. Our main result is the following strengthening of the weak Hanani-Tutte theorem for radial planarity, also generalizing the strong Hanani-Tutte theorem for level-planarity [16]:

Theorem 1. *If a leveled graph, possibly with multiple edges but without loops, has a radial drawing in which every two independent edges cross an even number of times, then it has a radial embedding.*

In order to put the theorem in the context of previous developments, we remark that neither Theorem 1 nor the strong Hanani-Tutte theorem [8, 29] for ordinary planarity follow from each other, since the edge-drawing style is severely restricted in the radial version. The strong Hanani-Tutte theorem for level planarity [16] is a special case of Theorem 1, which we use as the stepping stone in its proof. The main challenge in extending the level version to the radial version comes from the presence of homologically non-trivial cycles over $\mathbb{Z}/2\mathbb{Z}$ in the drawing from the hypothesis of the theorem.

The weak variant of a Hanani-Tutte theorem makes the stronger assumption that *every* two edges cross an even number of times. Weak variants often lead to stronger conclusions. For example, it is known that if a graph can be drawn in a surface so that every two edges cross evenly, then the graph has an embedding on that surface with the same rotation system, i.e. the cyclic order of ends at each vertex remains the same [7, 23]. This, in a way, is a disadvantage, since it implies that the original drawing is combinatorially already an embedding, so that weak Hanani-Tutte theorems do not help in finding embeddings. On the other hand, strong Hanani-Tutte theorems are often algorithmic. Theorem 1 yields a very simple algorithm for radial planarity testing (described in Section 5) which is based on solving a system of linear equations over $\mathbb{Z}/2\mathbb{Z}$, see also [25, Section 1.4]. Our algorithm runs in time $O(n^{2\omega})$, where $n = |V(G)|$ and $O(n^{\omega})$ is the complexity of multiplication of two square $n \times n$ matrices. Since our linear system is sparse, it is also possible to use Wiedemann's randomized algorithm [30], with expected running time $O(n^4 \log^2 n)$ in our case.

Bachmaier, Brandenburg and Forster [3] showed that radial planarity can be tested, and an embedding can be found, in linear time. Their algorithm is based on a variant of PQ-trees [6] and is rather intricate. It generalizes an earlier linear-time algorithm for level-planarity testing by Jünger and Leipert [20]. Angelini et al. [2] devised a conceptually simpler algorithm for radial planarity testing with running time $O(n^4)$ (quadratic if the leveling is proper, that is, edges occur between consecutive levels only), by reducing the problem to a tractable case of Simultaneous PQ-ordering [5].

We prove Theorem 1 by ruling out the existence of a minimal counter-example. By the weak Hanani-Tutte theorem [17, Theorem 1] a minimal counter-example must contain a pair of adjacent edges crossing an odd number of times. Thus, [17, Theorem 1] serves as the base case in our argument (mirroring the development for level-planarity). In place of Theorem 1 we establish a stronger version, Theorem 6, which we discuss in Section 4.

We refer the reader to [28, 24, 25, 17, 27] for more background on the family of Hanani-Tutte theorems, but suffice it to say that strong variants are still rather rare, so we consider the current result as important evidence that Hanani-Tutte is a viable route to answer graph-drawing questions.

2 Terminology

For the purposes of this paper, graphs may have multiple edges, but no loops. For the definitions of graph theoretical notions such as *cycle*, *path*, *walk*, and *component* including those pertaining to graph embeddings such as a *face* and its *boundary walk* we refer the reader to [11]. An *ordered* graph $G = (V, E)$ is a graph whose vertices are equipped with a total order $v_1 < v_2 < \cdots < v_n$. We can think of an ordered graph as a special case of a *leveled* graph, in which every vertex of G is assigned a *level*, a number in $\{1, \ldots, k\}$ for some k. The leveling of the vertices induces a weak ordering of the vertices.

For convenience we represent radial drawings as drawings on a (standing) cylinder. Intuitively, imagine placing a cylindrically-shaped mirror in the center of a radial drawing as described in the introduction, e.g., the left illustration in Figure 2 shows the cylindrical drawing of the leveled graph in Figure 1.[1] In practice, we will work with flat representations of the cylinder, as shown on the right of the same figure.

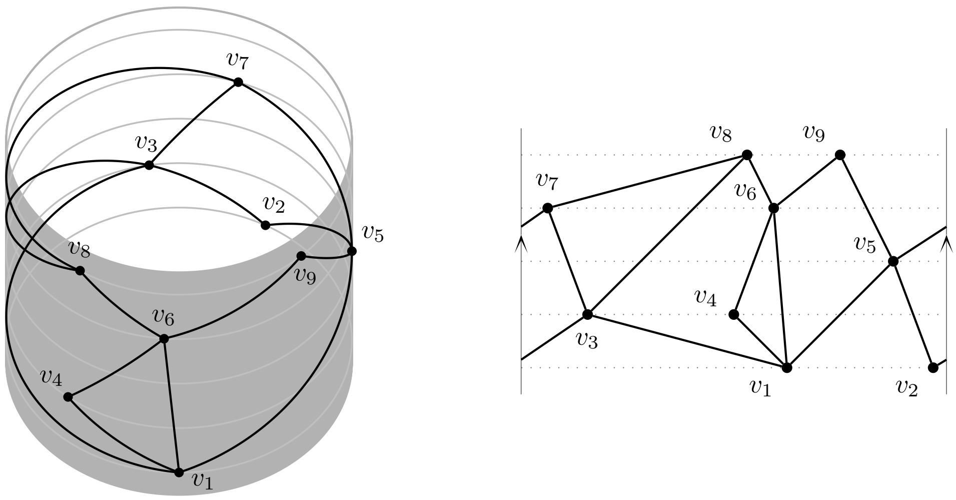

Figure 2: *(left)* A cylindrical embedding of the leveled graph Figure 1; *(right)* a flat representation of the same cylindrical embedding.

A *drawing* of a graph is its representation on a surface in which vertices are represented as distinct points and edges as simple curves joining their respective end vertices. We assume that a drawing is non-degenerate, that is, edges do not overlap (they intersect at most finitely often) and edges do not pass through vertices in a drawing. Abusing notation, we denote by a vertex v and an edge e both the objects in the (abstract) graph as well as their representations in its drawing.

We work on the *cylinder* $\mathcal{C} = \mathbb{S}^1 \times (0, 1) = \{(\cos\theta, \sin\theta, \ell) : \theta \in \mathbb{R}, \ell \in (0, 1)\}$. A point can be specified as (θ, ℓ) for convenience. A curve on the cylinder is *radial* if it has the form $\theta = f(\ell)$ for some function f with interval domain. A drawing of a leveled graph on the cylinder is *radial* if the ℓ-coordinates of the vertices respect the leveling, and every edge is drawn radially.

Let $\ell(X)$ denote the projection of any $X \subseteq \mathcal{C}$ to its ℓ-coordinate on the interval $(0, 1)$. For an ordered graph H with a given drawing, we allow $\max H$ to be either $\max V(H)$, its vertex of maximum level, or its maximum ℓ-coordinate $\max \ell(H)$, and likewise for $\min H$; note that for a radial drawing of a graph, the maximum ℓ-coordinate occurs at its vertex of maximum level (and likewise for minimum).

[1] This transformation is well-known in the history of art as "cylindrical mirror anamorphosis".

Given a graph G with subgraph G' and a radial drawing $\mathcal{D}(G)$ of G, let $\mathcal{D}(G')$ be the drawing restricted to G' (i.e., treat $\mathcal{D}$ as a function from G as a topological space to the cylinder).

Given vertices u, v with $u \leqslant v$ in a graph G on the cylinder, let $G[u, v]$, $G[u, v)$, $G(u, v]$, and $G(u, v)$ denote the subgraphs with ℓ-coordinates between $\ell(u)$ and $\ell(v)$, including endpoints as indicated.

In a radial drawing of a graph, an *upper (lower) edge* at v is an edge incident to v for which $\min e = v$ ($\max e = v$). A *sink* (*source*) is a vertex with no upper (lower) edges.

Given a radial drawing of a graph G on the cylinder $\mathcal{C}$, if there is a simple curve γ from the bottom of the cylinder $\mathbb{S}^1 \times 0$ to its top $\mathbb{S}^1 \times 1$ such that C does not intersect the drawing of G, then we can cut $\mathcal{C}$ along γ, unroll $\mathcal{C} \setminus \gamma$ so that it is flat, and rotate the drawing so that the circle levels turn into vertical line segments. The resulting drawing of G is called an *x-monotone drawing.* We will always work with vertical levels whenever we discuss x-monotone drawings.

The *rotation* at a vertex in a drawing (on any surface) of a graph is the cyclic, clockwise order of the ends of edges incident to the vertex in the drawing. The *rotation system* is the set of rotations at all the vertices in the drawing. For radial drawings, we define the *upper* (*lower*) rotation at a vertex v to be the linear order of the ends of just the upper (lower) edges in the rotation at v, with the order corresponding to the clockwise orientation of $\mathbb{S}^1$ (left-to-right in the flattened cylinder). Thus the upper (lower) rotation is clockwise (counterclockwise) on the cylinder surface.

The rotation at a vertex in a radial drawing is completely determined by its upper and lower rotation. The *rotation system* of a radial drawing is the set of the upper and lower rotations at all the vertices in the drawing.

If G is connected, then the rotation system determines the plane embedding, in the sense that the faces are uniquely determined [19, Theorem 3.2.3]. For a leveled, connected graph, the rotation system determines the faces of the radial embedding, including the outer faces, since we know the vertices on the first and last levels.

The *winding number* of a closed curve on a cylinder is the number of times the projection of the curve to $\mathbb{S}^1$ winds around $\mathbb{S}^1$, i.e., the number of times the projection passes through an arbitrary point of $\mathbb{S}^1$ in the counterclockwise sense minus the number of times the projection passes through the point in the clockwise sense. A closed curve (or a closed walk in a graph) on a cylinder is *essential* if its winding number is odd.[2] A graph drawn on the cylinder is *essential* if it contains an essential cycle.

For any closed, possibly self-overlapping, curve in the plane (or cylinder), we can *two-color* the complement of the curve so that connected regions each get one color and crossing the curve switches colors; at self-overlaps, the color switches if the number of overlaps of the curve is odd at the point being crossed. (For example, a graph embedded

[2]In other words, a closed curve on a cylinder is essential if its homology class over $\mathbb{Z}/2\mathbb{Z}$ is non-trivial, and thus, uses the topology of the cylinder in an essential way. The reason to consider the essentialness over $\mathbb{Z}/2\mathbb{Z}$, rather than have it equivalent to the non-contractibility as is perhaps more common in topological graph theory, is due to the hypothesis of Theorem 1 being stated in terms of a parity relaxation of the embedding.

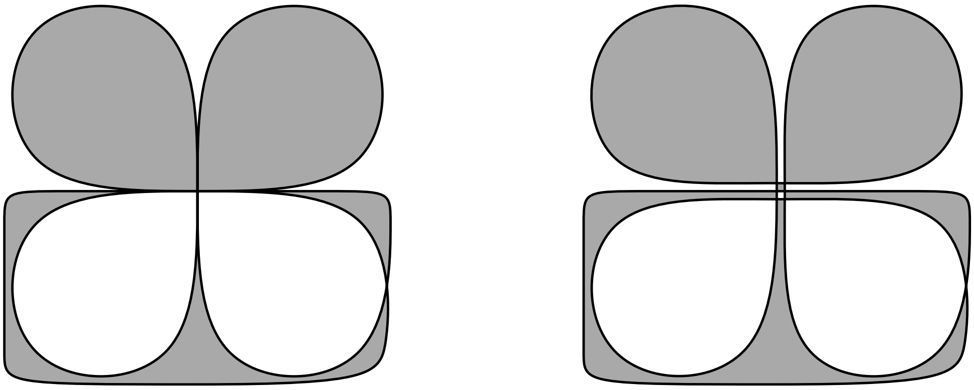

Figure 3: *(left)* A two-coloring of a curve with self-overlaps; *(right)* A two-coloring of the curve from the left after a small perturbation removing the overlaps. With self-overlaps removed, a two-coloring exists by a simple proof [26, Lemma A.2].

in the plane may have a face bounded by a closed curve that uses an edge e twice, once in each direction; for such a curve, the two-coloring will have the same color on both sides of e.) Refer to Figure 3 for an example. If the closed curve is non-essential, the region incident to $\mathbb{S}^1 \times 1$ and the region incident to $\mathbb{S}^1 \times 0$ will have the same color; all regions of that color form the *exterior* of the curve and the remaining regions form the *interior* of the curve.

Each pair of edges in a graph drawing crosses *evenly* or *oddly*, we call this the *crossing parity* of the pair of edges, and refer to *even* and *odd* pairs. A drawing of a graph is *even* if every pair of edges cross evenly. A drawing of a graph is *independently even* if every two independent edges in the drawing cross an even number of times, while two edges that share an endpoint may cross oddly or evenly.

For any (non-degenerate) continuous deformation of a drawing of G, the crossing parity of a pair of edges only changes when an edge passes through a vertex. We call this event an *edge-vertex switch*. When an edge e passes through a vertex v, the crossing parity changes between e and every edge incident to v.

3 Weak Hanani-Tutte for Radial Drawings

We first recall the weak variant of the result that we want to prove.

Theorem 2 (Fulek, Pelsmajer, Schaefer [17, Theorem 1])**.** *If a leveled graph has a radial drawing in which every two edges cross an even number of times, then it has a radial embedding with the same rotation system and leveling.*

We need a stronger version of this result that also keeps track of the parity of winding numbers.

Theorem 3 (Fulek, Pelsmajer, Schaefer [17, Theorem 2])**.** *If an ordered graph G has an even radial drawing, then it has a radial embedding with the same ordering and the same rotation system such that the winding number parity of every cycle remains the same.*

Theorem 2 easily follows from Theorem 3 using the pendant construction from [16, Section 4.2] that was used to reduce level-planarity to x-monotonicity: Given an even radial drawing of a leveled graph G, consider each level $\ell = c$ with more than one vertex; see the left illustration in Figure 4.

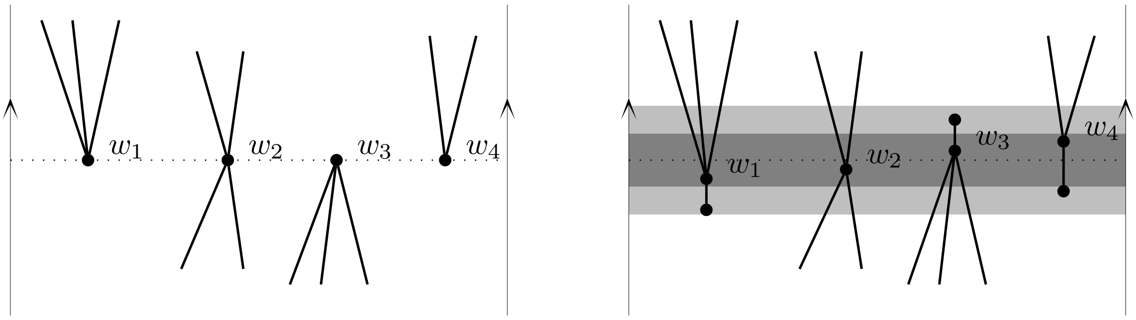

Figure 4: *(left)* A level with four vertices (including one sink and two sources); *(right)* the pendant vertices are placed in a small neighborhood (light gray) of the original level and the four original vertices perturbed in an even smaller neighborhood (dark gray) so that each vertex is at a unique level.

For each source or sink w on that level, add a short crossing-free edge incident to w on the empty side of that vertex, placing its other endpoint so that it doesn't share its level with any other vertex. (Since we work on an open cylinder, the pendant construction can be applied to the vertices on the bottom-most and top-most level.) We now slightly perturb all the vertices on the level $\ell = c$ so that no two vertices are at the same level, without moving them past any other level. All this can be done while keeping all edges radial, and without introducing any new crossings. We obtain an ordered graph, G'. By Theorem 3, G' has a radial embedding with the same rotation system, and the winding number of every cycle remains unchanged. The additional edges we added ensure that there is room to slide all of the perturbed vertices back to their original level without getting in the way of each other.

3.1 Working with Radial Embeddings and Even Drawings

Given a connected graph G with a rotation system, one can define a *facial walk* purely combinatorially by following the edges according to the rotation system (see, for example, [19, Section 3.2.6]), by traversing consecutive edges at each vertex, in clockwise order.

We need some terminology for embeddings of an ordered graph G with $v_1 < v_2 < \ldots < v_n$. The *maximum* (*minimum*) of a facial walk W in the radial drawing of G is the maximum (minimum) v that lies on W. A *local maximum* (*local minimum*) of a facial walk W is a vertex v on W that is larger (smaller) than both the previous and the next vertex on W. (A vertex v might appear more than once on W; the previous definition implicitly refers to one such appearance.)

A facial walk in an even radial drawing is an *upper (lower) facial walk* if it contains $\mathbb{S}^1 \times 1$ ($\mathbb{S}^1 \times 0$) in its interior. An *outer facial walk* is an upper or lower facial walk; other facial walks are *inner facial walks*. If a radial embedding of G has only one outer facial walk (and one outer face), then it also has an x-monotone embedding (by cutting the cylinder along a curve connecting the bottom and the top of the cylinder, as we described in Section 2).

Lemma 4. *In a radial embedding of a graph, there are two outer faces if and only if the graph contains an essential cycle.*

The lemma also holds for even radial drawings; since we don't need that stronger result, we do not prove it here.

Proof. If the radial embedding has one outer face, there is a curve γ lying in the outer face and connecting $\ell = 0$ to $\ell = 1$. Since the embedding is disjoint from γ, the winding number of all cycles of the graph in the embedding is zero.

On the other hand, if there are two outer faces, one incident to $\ell = 0$ and one incident to $\ell = 1$, then the lower face boundary of the radial embedding is homotopic to the circle $\ell = 0$ so it has odd winding number; by induction it must contain a cycle with odd winding number, too. ∎

Lemma 5. *A cycle C in an even radial drawing is essential if and only if the two paths connecting its extreme vertices do so in inverse order.*

Proof. Let e and f be the edges in C that are incident to the maximum v of C, and e' and f' be the two edges of C incident to the minimum u of C. Suppose that e, e', f', f appear in this order along C (where we allow $e = e'$ or $f = f'$). See Figure 5.

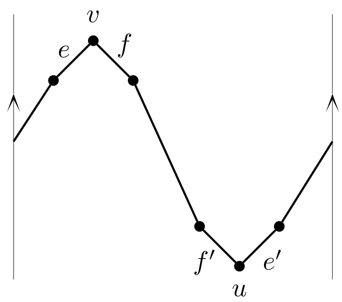

Figure 5: An essential cycle (ends of path are in inverse order at u and v).

Let $<_v$ be the lower rotation at v and let $<_u$ be the upper rotation at u. Suppose that $e <_v f$. We have to show that C is essential if and only if $f' <_u e'$.

Two-color the complement of C. Traverse the path in C which begins with v and e and ends with e' and u. At the beginning, the colored region to the right includes the empty space just above v. Since C is an even drawing, the color immediately to the right will be the same as we begin and end our path traversal. At the end, the colored region to the right includes the empty space just below u if and only if $e' <_u f'$. The space above v and the space below u have the same color if and only if the winding parity of C is even. ■

4 Strong Hanani-Tutte for Radial Drawings

Theorem 3 preserves the parity of the winding number of cycles in even radial drawings of ordered graphs and this property is used in the inductive proof of Theorem 3 in [15]. We cannot hope to do this when the drawings are only independently even; see Figure 6 for a counterexample.

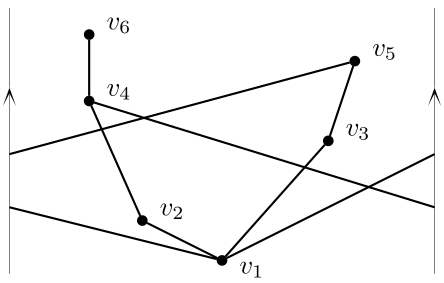

Figure 6: An independently even drawing with two essential cycles, $v_1v_2v_4$ and $v_1v_3v_5$; there is no radial embedding in which both these cycles remain essential.

Fortunately, we can make the proof of the main result work with a somewhat weaker property: Given an ordered graph G with radial drawing D_1, a radial redrawing D_2 *is supported by* D_1 if for every essential cycle C_2 in D_2, there is an essential cycle C_1 in D_1 such that $[\min C_1, \max C_1] \subseteq [\min C_2, \max C_2]$. In other words, while the redrawing D_2 may contain new essential cycles not in D_1, those cycles are propped up by essential cycles in D_1.

A radial drawing of an ordered connected graph is *weakly essential* if every essential cycle in the drawing passes through v_1 or v_n (the first or the last vertex). With this definition we can state the strengthened version of our main result which we need for the proof.

Theorem 6. *Let G be an ordered graph. Suppose that G has an independently even radial drawing. Then G has a radial embedding. Moreover,* (i) *if the given drawing of G is weakly essential, then G has an x-monotone embedding; and* (ii) *the new radial embedding is supported by the original drawing.*

Theorem 1 follows from Theorem 6 by the pendant construction from [16, Section 4.2] described in Section 3.

We will prove part (i) of Theorem 6 in Section 4.2 and complete the proof in Section 4.4. We develop some tools for these proofs in Section 4.1 and establish some properties of a minimal counterexample for part (ii) in Section 4.3.

4.1 Working with Independently Even Radial Drawings

An edge e drawn on $\mathcal{C}$ with endpoints $u < v$ is *bounded* if $\ell(u) < \ell(p) < \ell(v)$ for every point p in the interior of e. Call a drawing of G *bounded* if all edges are bounded.

Lemma 7. *If an ordered graph has an (independently) even bounded drawing, then it has an (independently) even radial drawing with the same rotation system which preserves whether cycles are essential or non-essential.*

The proof of the lemma uses an adaptation of the usual (e, v)-moves (e.g. [23]) to the radial setting: To perform a *radial* (e, v)*-move*, assuming $\min e < v < \max e$, we detour a portion of e within a narrow band $\mathbb{S}^1 \times [\ell(v) - \epsilon, \ell(v) + \epsilon]$ so that it passes over v; see Figure 7. The effect of an (e, v)-move is to change the crossing parities between e and any edge incident to v. All other crossing parities remain the same (since newly added crossings come in pairs except near v, assuming ϵ is sufficiently small).

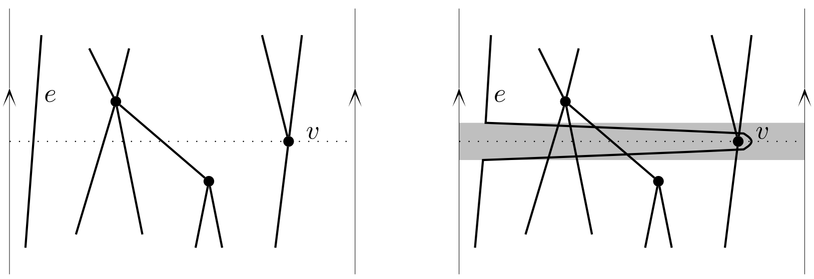

Figure 7: Performing a radial (e, v)-move in a small neighborhood $\mathbb{S}^1 \times [\ell(v) - \epsilon, \ell(v) + \epsilon]$ (gray).

Proof of Lemma 7. It is sufficient to show that we can redraw any particular edge $e = uv$ radially without changing the remainder of the drawing, so that the crossing parity between e and each other edge is unchanged, and so that winding number of each cycle is unchanged.

While keeping $\ell(e) = [u, v]$ and the rotation system fixed, we can continuously deform e so that it is radial. As e is deformed, it passes through some vertices an odd number of times; call this set of vertices S. To reestablish the original crossing parities between e and all other edges, we perform a radial (e, w)-move for every vertex $w \in S$ (see Figure 7). ∎

The following result is a simple corollary of Lemma 7; see Figure 8 *(left)* for an illustration.

Corollary 8. *In a radial embedding of a connected ordered graph G we can subdivide any face f by adding an edge joining its maximum with its minimum while keeping the embedding radial. If f is an outer face, we can subdivide it by adding at most two edges so that the new outer face contains exactly one local minimum and maximum.*

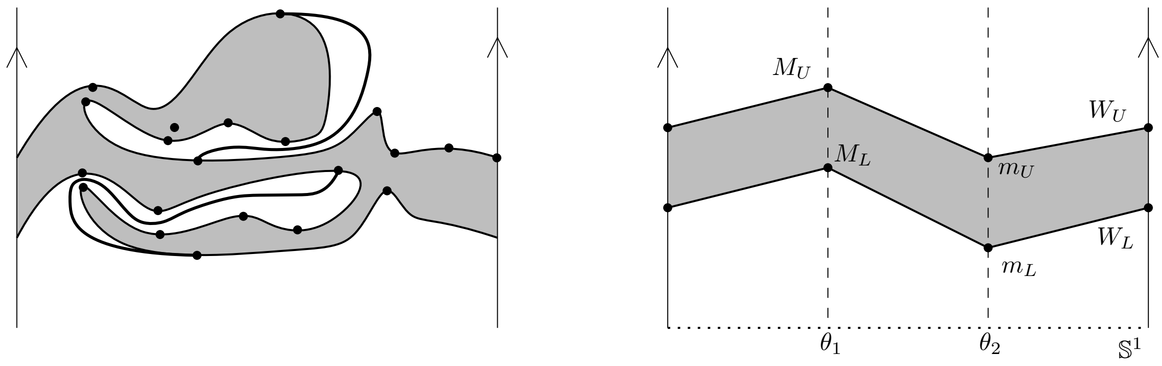

Figure 8: *(left)* Subdividing the lower and upper outer face in a radial embedding by bounded edges. *(right)* The lower and upper face G bounded by essential 2-cycles.

Proof. Given a face f we add a bounded edge e to the interior of f, drawn along the boundary of f, that joins the minimum and the maximum of f. An application of Lemma 7 and Theorem 2 then concludes the proof of the first part of the statement.

For the second part, if f is an outer face, then its boundary W is a facial walk which can be broken into two sub-walks between its minimum and maximum. We can add an edge in f drawn alongside each sub-walk, unless the sub-walk already consists of a single edge only, in which case we use that edge. These edges form a walk W' that bounds a 2-face f' which is now the outer face instead of f. We apply Lemma 7 and Theorem 3 to get an embedding that is radial. Since the rotation system is unchanged and the graph is connected, W' still bounds a face f'_* in the new embedding. The winding number of W' is the same, so by Lemma 5, f'_* is essential if and only if f' was essential. Then the rotation ensures that f'_* is an outer face, just as f' was. ■

We use Corollary 8 below to derive Lemma 11, which allows us to augment the embedding of essential components so that the outer faces are 2-cycles; see Figure 8 (right).

We often make use of the following fact.

Lemma 9. *Let P be a path and let C be an essential cycle, vertex-disjoint from P, in an independently even radial drawing of a graph. Then $\ell(P)$ does not contain $\ell(C)$.*

Proof. Suppose $\ell(P)$ contains $\ell(C)$. We can then find a vertex u on P above C and a vertex v on P below C. Thus, the sub-path of P between u and v, and hence, an edge of P on the sub-path between u and v, intersects an edge of C an odd number of times, which is a contradiction. ■

Finally, we present some re-embedding techniques that will help us combine embeddings of components. If a graph H has an independently even radial drawing with no essential cycles, then it has an x-monotone embedding (see Lemma 13, which actually shows something a bit stronger). The x-monotone embedding can be "squeezed" to have an arbitrarily narrow width. Then, given any radial curve e with $\ell(H) \subseteq \ell(e)$, we can deform the embedding of H so that it lies arbitrarily close to e (within ε for any $\varepsilon > 0$). We call this a "skinny" embedding.

Observation 10. *If an ordered graph H has an x-monotone embedding and e is a radial curve on the cylinder with $\ell(H) \subseteq \ell(e)$, then H has an embedding on the cylinder that lies arbitrarily close to e.*

A radially-embedded graph G that is not free of essential cycles also has a useful level-preserving deformation, whose existence is proved in the next lemma. Refer to Figure 8 (right).

For any distinct θ_1, θ_2 on $\mathbb{S}^1$ and m, M in $[0, 1]$, let $\gamma(\theta_1, m, \theta_2, M)$ be the essential curve consisting of the two helix segments from (θ_1, m) to (θ_2, M).

Lemma 11. *Suppose an ordered graph G is radially embedded. Let m_L and M_L be the minimum and maximum of the lower face boundary and let m_U and M_U be the minimum and maximum of the upper face boundary. Then for any distinct θ_1, θ_2 on $\mathbb{S}^1$, there is a radial embedding of G that lies between the curves $\gamma(\theta_1, m_L, \theta_2, M_L)$ and $\gamma(\theta_1, m_U, \theta_2, M_U)$.*

Proof. By Lemma 4, there are separate upper and lower outer faces. By Corollary 8, we can add edges to the upper and lower face so that each is bounded by a 2-cycle with radially-drawn edges. Then we can do a level-preserving deformation that maps the new edges to the curves $\gamma(\theta_1, m_L, \theta_2, M_L)$ and $\gamma(\theta_1, m_U, \theta_2, M_U)$. ∎

4.2 Weakly Essential Drawings

If two consecutive edges in the rotation at a vertex v cross oddly, we can make them cross evenly by a local redrawing: we "flip" the order of the two edges in the rotation at v, adding a crossing, which makes them cross evenly. (For any two crossing adjacent edges in a radial drawing, either both are in the upper rotation or both are in the lower rotation of their shared endpoint.) In the proof of the strong Hanani-Tutte theorem for x-monotone drawings [16, Theorem 3.1], if the edges incident to a vertex v cannot all be made to cross (pairwise) evenly using edge flips, then there must either be a component H of $G \setminus \{v, w\}$, for some $w \in V(G)$, satisfying $v \leqslant \min H < \max H \leqslant w$ or a multiple edge vw (for some vertex w). Both cases lead to a reduction; then an application of the weak Hanani-Tutte theorem for x-monotone drawings [22] (see also [16, Theorem 2.11]) completes the proof.

We want to use the same approach for radial drawings, and we already know that the weak Hanani-Tutte theorem holds for radial drawings. But in this case, it is possible that we have a vertex v whose incident edges cannot be made to cross pairwise evenly using edge flips and yet there may be no obstacle like H or a multiple edge. However, we will show (Lemma 13) that this can only occur when v is the first or last vertex of the ordered graph. The next lemma helps us deal with this case.

Given an ordered graph G with vertices $v_1 < \ldots < v_n$ without the edge $v_1 v_n$, let G' denote the ordered graph obtained by removing v_1 and v_n, and replacing each edge $v_i w$ with $i \in \{1, n\}$ by a "pendant edge" uw where u is a new degree one vertex placed in the order before v_2 if $i = 1$ and after v_{n-1} if $i = n$ (and otherwise ordered arbitrarily).

Lemma 12. *If G is a connected ordered graph with an (independently) even radial drawing $\mathcal{D}(G)$, then G' has an (independently) even radial drawing $\mathcal{D}'(G')$ such that $\mathcal{D}'(G \setminus \{v_1, v_n\}) = \mathcal{D}(G \setminus \{v_1, v_n\})$ and $\mathcal{D}'(G')$ is supported by $\mathcal{D}(G)$.*

Proof. In $\mathcal{D}(G)$, erase v_1 and a small portion of its incident edges to create new pendant edges with new endpoints $v'_1 < \ldots < v'_k$ with distinct ℓ-coordinates between $\ell = \ell(v_1)$ and $\ell = \ell(v_2)$. This does not change the crossing parity of any pair of edges, but each pair of edges incident to v_1 that crossed oddly in $\mathcal{D}(G)$ is now a pair of independent edges that cross oddly (unless they were multiple edges, which share their upper endpoint). We will redraw these pendant edges so that every pair of these pendant edges crosses evenly.

Let w_i be the upper endpoint of the pendant edge incident to v'_i for $1 \leqslant i \leqslant k$. If $v'_i w_i$ and $v'_j w_j$ cross oddly and $i < j$, then a radial $(v'_i w_i, v'_j)$-move (as in Figure 7) will make them cross evenly. For each fixed j, we perform radial $(v'_i w_i, v'_j)$-moves for every edge $v'_i w_i$ that crosses $v'_j w_j$ oddly: doing so for all j with $1 \leqslant j \leqslant k$ makes every pair of pendant edges cross evenly.

We apply a similar procedure to v_n: erase it, creating pendant edges with new endpoints $v''_1 < \ldots < v''_\ell$, then redraw the pendant edges so that every pair crosses evenly. We have obtained a drawing $\mathcal{D}'(G')$ of G' in which every pair of oddly-crossing edges is also a pair of oddly-crossing edges, as desired. The second part of the claim follows because every essential cycle of $\mathcal{D}'(G')$ is an essential cycle in $\mathcal{D}(G)$. ∎

Using Lemma 12 we can establish part (i) of Theorem 6.

Lemma 13. *Suppose that G has an independently even radial drawing that is weakly essential. Then G has an x-monotone embedding.*

Proof. Let $\mathcal{D}(G)$ be the independently even radial drawing of G that is weakly essential. By Lemma 12 there is an independently even radial drawing $\mathcal{D}'(G')$ of G' (as defined before the lemma) such that $\mathcal{D}'(G')$ is supported by $\mathcal{D}(G)$. Since $\mathcal{D}(G)$ is weakly essential, $\mathcal{D}'(G')$ contains no essential cycles.

Let v, v' be new vertices such that $v < \min G'$ and $\max G' < v'$. Draw a new edge $e = vv'$ radially and so that its interior does not pass through any vertex of G'. Let E' denote the set of edges in G' crossed oddly by e. Since G' contains no essential cycle, each cycle C of G' crosses e evenly, and hence, the number of edges of $E(C)$ crossed oddly by e is even. In a graph, the cycle space is orthogonal to the cut space over $\mathbb{Z}/2\mathbb{Z}$ [11, Section 1.9]. It follows that E', if it is non-empty, is an edge-cut of G'. In that case, we perform radial (e, w)-moves with all vertices w on one side of the cut E'; then e crosses every edge of G' evenly (if E' is empty, that is automatically true).

The edge e can then be cleaned of crossings by [15, Lemma 8] (where the graph is $(V(G') \cup \{v, v'\}, E(G') \cup \{e\})$). By cutting the cylinder along e, we can conformally deform $\mathcal{C} \setminus e$ to a subset of the plane so that levels become parallel line segments and our radial drawing becomes an x-monotone drawing as discussed in Section 2. Then the strong variant of the Hanani-Tutte theorem for x-monotone drawings [16] applies, giving us an x-monotone embedding of G'. Finally, we can extend the pendant edges in the drawing to reach two (new) shared endpoints v_1, v_n, giving us an x-monotone embedding of G. ∎

4.3 Components of a Minimal Counterexample

In the remainder of Section 4, G will denote a minimum counterexample to Theorem 6, by which we mean a counterexample with the fewest possible number of vertices, and, subject to that, the fewest edges possible. Let $\mathcal{D}(G)$ be the independently even radial drawing of G. Then G has no radial embedding supported by $\mathcal{D}(G)$ and, by Lemma 13, $\mathcal{D}(G)$ is not weakly essential.

Lemma 14. *A minimal counterexample does not have multiple edges.*

Proof. Suppose that e and e' both have endpoints u, v in a minimal counterexample G which is drawn on the cylinder. Apply induction to $G - e$ to get a drawing $\mathcal{D}(G - e)$, then draw e alongside e' without any crossings, producing an embedding of G. The new embedding is supported by the original drawing of G, because for any essential cycle C through e, the cycle $C - e + e'$ is essential and in $G - e$, and $\ell(C') = \ell(C)$. ∎

Lemma 15. *G is connected.*

Proof. Suppose that G is not connected. If G has a non-essential component H, embed the rest of the graph G' by induction. Let $m_H = \min V(H)$ and $M_H = \max V(H)$. It suffices to add an edge $m_H M_H$ to the embedding of G', by Observation 10, which we do next.

Let E' be the set of edges e such that $\min e < M_H < \max e$ and let G^- and G^+ be the (embedded) subgraphs induced by $\{v \in V(G') : v < M_H\}$ and $\{v \in V(G') : v > M_H\}$, respectively; see Figure 9. Since the upper face of G^- contains $\ell = M_H$, it must also intersect $\ell = m_H$, since otherwise the upper boundary of G^- would contain an essential cycle between m_H and M_H, contradicting part (ii) of Theorem 6 with Lemma 9. Let x be any point in the upper face of G^- with $\ell(x) = m_H$; moreover, choose x to avoid intersecting E'. The face of G' that contains x must also intersect $\ell = M_H$, so it will contain a radial curve from $\ell = m_H$ to $\ell = M_H$ (using Corollary 8 to make it radial), along which we can embed the edge $m_H M_H$ or H itself, by Observation 10. This yields an embedding of G, which satisfies part (ii) since G' contains the same essential cycles as G.

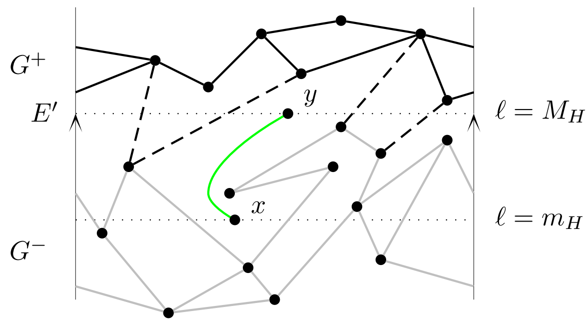

Figure 9: Induced graphs G^+ (black edges) and G^- (gray edges) and the set of edges E' (dashed) ; we will draw $m_H M_H$ (and H) along the radial curve xy (green).

Now instead suppose that G is not connected and every component is essential. Let H be the component of G with $\max H = \max G$. Let G' be the rest of G, embedded by induction. Since the upper face of G' contains $\ell = \max H$, it must also intersect $\ell = \min H$, since otherwise the upper boundary of G' would contain an essential cycle between $\min H$ and $\max H$, contradicting part (ii) of Theorem 6 with Lemma 9. So the minimum m_U of the upper boundary of G' satisfies $m_U < \min H$. Note that $\max G'$ is the maximum of the upper boundary of G'. By Lemma 11, G' can be embedded on the cylinder so that it lies below the curve $\gamma(0, m_U, \pi, \max G')$ as described just prior to Lemma 11.

Similarly, H has an embedding by induction, the maximum M_L of the lower boundary of H satisfies $\max G' < M_L$, and by Lemma 11 it can be embedded so that it lies strictly above the curve $\gamma(0, \min H, \pi, M_L)$. Since $m_U < \min H$ and $\max G' < M_L$, the embeddings of H and G' do not interesect. Thus we have obtained an embedding of G and it satisfies part (ii) of Theorem 6 since every essential cycle is in a component. ■

Let v be a vertex and suppose that B is a component of $G \setminus v$ with $\min B > v$. By Lemma 15, there exists at least one edge from v to a vertex in B.

Lemma 16. *Let v be a vertex and B be a component of $G \setminus v$ with $\min B > v$. Then either $|V(B)| = 1$ (and the vertex of B has just one neighbor, v) or B is essential in $\mathcal{D}(G)$.*

Proof. Suppose that B contains no essential cycle and $|V(B)| \neq 1$. Let B' be the subgraph induced by $V(B) \cup \{v\}$; i.e., B' contains B, v, and all edges from v to B. By Lemma 13 we obtain an x-monotone embedding $\mathcal{E}(B')$ of B'. Let vPw be a path in B' from v to $\max B = w$. Let G' be a graph obtained from G by replacing B' with a single edge e from v to w. Let $\mathcal{D}(G')$ denote the drawing of G' inherited from $\mathcal{D}(G)$ such that $\mathcal{D}(P) = \mathcal{D}(e)$. Thus, the drawing of e in $\mathcal{D}(G')$ is obtained by suppressing the interior vertices of P. The drawing $\mathcal{D}(G')$ may not be radial due to e, but it is still bounded and independently even. By Lemma 7, we obtain an independently even radial drawing $\mathcal{D}'(G')$ of G'. By the minimality of G we get a radial embedding $\mathcal{E}'(G')$ of G'. Finally, using Observation 10 we replace e in $\mathcal{E}'(G')$ by a "skinny" copy of $\mathcal{E}(B')$ intersecting $\mathcal{E}'(G')$ in v, thereby obtaining a radial embedding of G. This embedding of G is supported by $\mathcal{D}(G)$. ■

Lemma 17. *Let v be a vertex and B be a component of $G \setminus v$ with $\min B > v$. If B is essential in $\mathcal{D}(G)$, then $v = v_1$.*

Proof. For the sake of contradiction we assume the contrary. Thus, B is essential and $v \neq v_1$. Let G_1 denote the union of all components H of $G \setminus v$ for which $\min H > v$. Let G_2 denote the union of components of $G \setminus v$ not included in G_1. Since $v \neq v_1$, G_2 is non-empty, and G_1 is also non-empty due to the existence of B. Let G_1' denote the subgraph of G induced by $V(G_1) \cup \{v\}$. Let G_2' denote the union of the subgraph of G induced by $V(G_2) \cup \{v\}$ and an edge e between v and $w := \max G_1' = \max G$. Since there is a path vPw connecting v and w in G_1, we can argue as in the proof of Lemma 16 that there is an independently even radial drawing of G_2'. G_2' is not the same as G since G_1' must contain a cycle (for B to be essential). See Figure 10.

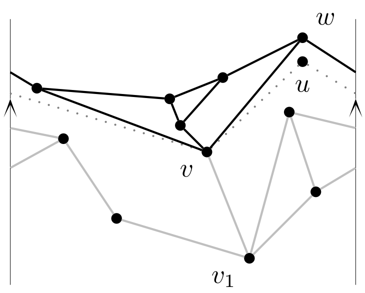

Figure 10: G_1' in black, and $G_2' \setminus w$ in gray. Also, vertex u of G_2'', just below $\max W_1$ (which happens to be w), and the upper outer 2-face of G_2'' (dotted).

By the minimality of G there are radial embeddings $\mathcal{E}_1(G_1')$ and $\mathcal{E}_2(G_2')$ of G_1' and G_2', each supported by the drawings we had of G_1' and G_2'.

If $\mathcal{E}_1(G_1')$ is non-essential, we can insert a "skinny" embedding of $\mathcal{E}_1(G_1')$ alongside the embedding of e in $\mathcal{E}_2(G_2')$, thereby obtaining a radial embedding of G. Otherwise, let W_1 be the lower facial walk of G_1' in $\mathcal{E}_1(G_1')$; then W_1 is essential.

We claim that $\max W_1 > \max(G_2' \setminus w)$. If the claim were false, there would be a y in $G_2' \setminus w$ with $y > \max W_1$. Since every component H of G_2 has $\min H < v$, there must also be an $x < v$ and a path xPy from x to y in $G_2' \setminus w$. But then W_1 and P contradict Lemma 9. This shows that $\max W_1 > \max(G_2' \setminus w)$.

From G_2' we define G_2'' by (i) subdividing e in G_2' by adding a vertex u just below $\max W_1$ (and, therefore, above $\max(G_2' \setminus w)$), (ii) remove w (and uw), and (iii) adding a second edge e'' between the v and u so that $ve'ue''v$ is an essential 2-cycle; e'' can be added to the embedding by Corollary 8 since w and hence e is on the upper boundary walk of G_2'. So in the resulting embedding $\mathcal{E}_2(G_2'')$ of G_2'' the upper facial walk is essential and it is the 2-face $e'e''$ with maximum u and minimum v; see Figure 10. Using Corollary 8, we can add edges to G_1' so that the lower outer face of $\mathcal{E}_1(G_1')$ is a 2-face with endpoints v and $\max W_1$. The embeddings of $\mathcal{E}_1(G_1')$ and $\mathcal{E}_2(G_2'')$ can be combined using Lemma 11 into an embedding which contains G. By deleting all edges we added during the construction, we obtain a radial embedding of G that is supported by $\mathcal{D}(G)$. This contradicts the choice of G. ■

Lemma 18. *Suppose that $v, w \in V$ and B is a component of $G \setminus \{v, w\}$ with $v < \min B$ and $\max B < w$, and there is at least one edge from B to v and at least one edge from B to w. Then B is essential.*

Proof. We proceed similarly as in the proof of Lemma 16. Suppose that B contains no essential cycle. Let B' be the subgraph induced by $V(B) \cup \{v, w\}$. By Lemma 13, we obtain an x-monotone embedding $\mathcal{E}(B')$ of B'. Let vPw be a path in B' from v to w. Let G' be obtained from G by replacing B' with a single new edge e from v to w. Let $\mathcal{D}(G')$ denote the drawing of G' inherited from $\mathcal{D}(G)$ by letting $\mathcal{D}(e) = \mathcal{D}(P)$, i.e., the drawing of e in $\mathcal{D}(G')$ is obtained by suppressing the interior vertices of P. The drawing $\mathcal{D}(G')$

may not be radial due to e, but it is still bounded and independently even. By Lemma 7, we obtain an independently even radial drawing $\mathcal{D}'(G')$ of G'. By the minimality of G we get a radial embedding $\mathcal{E}'(G')$ of G'. Finally, we replace e in $\mathcal{E}'(G')$ by a "skinny" copy of $\mathcal{E}(B')$ intersecting $\mathcal{E}'(G')$ in v and w, thereby obtaining a radial embedding of G.

It remains to show that the obtained radial embedding of G is supported by $\mathcal{D}(G)$. Suppose that C is an essential cycle in our embedding of G. Then C is not contained in B, so either $C \cap B'$ is a path between v and w, or C does not intersect B at all. In the former case, replace that path by the edge e to get an essential cycle C' in the embedding $\mathcal{E}'(G')$ of G'; otherwise C is an essential cycle in G' so just let $C' = C$. The embedding $\mathcal{E}'(G')$ is supported by $\mathcal{D}(G')$, so there is an essential cycle C'' in $\mathcal{D}(G')$ such that $\ell(C'') \subseteq \ell(C')$. If C'' contains e, then replace e by P to get a new cycle C''' in G. Since P can be smoothly deformed to e within the cylinder, C''' is essential in the original drawing of G. If C'' does not contain e, then just let $C''' = C''$. Then we have $\ell(C''') = \ell(C'') \subseteq \ell(C') = \ell(C)$, which proves that the obtained radial embedding of G is supported by $\mathcal{D}(G)$. ∎

4.4 Completing the Proof of Theorem 6

If $\mathcal{D}(G)$ is even, then Theorem 3 gives us a radial embedding of G and cycles preserve whether they are essential or non-essential because their winding number parities are unchanged, so part (ii) of the theorem is satisfied. Any two edge that cross oddly and are consecutive in the upper (or lower) rotation at their common endpoint can be redrawn flipped so that they cross evenly; we will repeatedly do that until no such pairs remain. We are done unless there is still a pair of edges that cross oddly; without loss of generality we may assume that two such edges e and f lie in the upper rotation of a common endpoint v. Choose e and f to be such a pair at minimum distance within the upper rotation at v; then there must be an edge g between them in the upper rotation at v, and g must cross e and f each evenly.

The distance in the upper rotation won't actually matter; only their order in the rotation and parity of crossings matter. Thus, let us define an *unflippable triple* to be three edges a, b, c in that order in the upper or lower rotation of a vertex v such that the outer pair a, c crosses oddly and the other pairs a, b and b, c each cross evenly. The following lemma will be useful later.

Lemma 19. *If a, b, c is an unflippable triple with common vertex v, we can redraw the ends of a,b, and c at v so their order is b, c, a, and we can redraw the ends at v so their order is c, a, b. In either case, the edges form an unflippable triple after the redrawing.*

Proof. If we flip a with every edge to its right until it flips with c, then a will be on the right and b on the left with c in the middle, and a will now cross c evenly and b oddly, while b and c still cross evenly. Thus we still have the outer pair crossing evenly and the other two pairs crossing oddly, only now the order in the upper rotation has shifted cyclically by one: b, c, a. Similarly we can move c to the far left of the rotation and obtain the same configuration with order c, a, b in the upper rotation. ∎

The next lemma lies at the heart of the remaining part of the proof. Figure 11 illustrates the set-up in the lemma.

Lemma 20. *Let e_1, e_2, e_3 be three edges (in that order) in the upper rotation of a vertex v such that e_1 crosses e_3 oddly and e_2 crosses both e_1 and e_3 evenly. Suppose that P, Q, Q' are paths that begin at v and their first edges are e_1, e_2, e_3, not necessarily respectively in that order, such that $V(P) \cap V(Q) = \{v\} = V(P) \cap V(Q')$ and $v = \min Q = \min Q' > \min P$. (See Figure 11.) Then it cannot be that both* $\max Q > \max P$ *and* $\max Q' > \max P$.

Proof. For the sake of contradiction we assume the contrary. Without loss of generality, we can choose the three paths to be minimal; then all their vertices except for their last vertices are in the region $\mathbb{S}^1 \times [\ell(v), \max \ell(P)]$. We can add a simple curve γ^* in the region $\mathbb{S}^1 \times (\max \ell(P), 1)$ joining the endpoints of Q and Q' so that altogether, γ^* with Q and Q' form a non-essential curve, which we'll call γ.

The end of P in $\mathbb{S}^1 \times (0, \ell(v))$ is in the exterior of γ so P crosses γ oddly if and only if it reaches v from the interior of γ, which occurs if the first edge of P is e_2; if the first edge of P is e_1 or e_3, then P crosses γ evenly.

P does not cross γ^* since γ^* is in the region $\mathbb{S}^1 \times (\ell(\max P), 1)$ and the only edges of P and γ that could cross oddly are in $\{e_1, e_2, e_3\}$ since independent edges in G cross evenly. Thus, if e_2 is the first edge of P, then since e_2 crosses e_1 and e_3 evenly, P crosses γ evenly. And if e_1 (or e_3) is the first edge of P, then since it crosses e_2 oddly and e_3 (or e_1) evenly, P crosses γ oddly. In both cases, this contradicts what we observed in the previous paragraph. ■

Figure 11: A configuration of three paths P, Q, and Q' starting at v that cannot occur in an independently even radial drawing if the triple of edges e_1, e_2 and e_3 incident to v from left to right is unflippable by Lemma 20, *(left)* P starts with e_1, *(right)* P starts with e_2.

The proof of Theorem 6 splits into two cases. Recall that $v_1 = \min V(G)$.

Case 1: Assume that $v \neq v_1$.

First we will show that there must be a path P from v through e, f, or g, which ends in the region $I < v$. If not, then consider any component of $G \setminus \{v\}$ containing the upper endpoint of e, f or g: it lies in the region $\mathcal{C}(v, 1)$, so by Lemmas 16 and 17, it must be a single vertex. Then the upper endpoints v_e, v_f, v_g of e, f, g (respectively) each have degree 1 in G. Without loss of generality suppose that v_e is the one with smallest i-coordinate: remove v_e (and e) from the graph, apply induction to embed $G - v_e$, then embed e alongside f to obtain the required embedding of G.

Let P be a path from v through e, f, or g, which ends in the region $\mathcal{C}(0, v)$, chosen so as to minimize $\max P$. Let w_P be its maximum vertex. Choose a minimal such P, so that every vertex except its last is in the region $\mathcal{C}[v, 1)$. Let P_1 be the subpath of P from v to w_P and let P_2 be the subpath of P from w_P to the region $\mathcal{C}(0, v)$.

Let H be the subgraph induced by $\{u \in V : v < u < w_P\}$, that is, $G(v, w_P)$. Let H_2 be the component of H that intersects P_2 (let $H_2 = \emptyset$ if P_2 has just one edge) and let H_e, H_f, H_g be the (not necessarily distinct) components of H incident to e, f, g, respectively (or $\emptyset$ if the upper endpoint of that edge is in the region $\mathcal{C}(w_P, 1)$); see Figure 12.

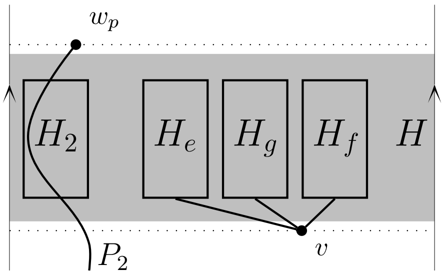

Figure 12: Parts of the subgraph H which lies in the gray region between w_p and v.

By the choice of P, the subgraph H_2 is disjoint (and distinct) from each of H_e, H_f, H_g and there is no edge from $H_e \cup H_f \cup H_g$ to the region $\mathcal{C}(0, v)$.

Claim 21. *H_e is non-essential and adjacent to a vertex in the region $\mathcal{C}[w_P, 1)$, unless H_e is empty, and likewise for H_f and H_g. (H_e is empty if and only if $\max e \geqslant w_P$, and likewise for H_f and H_g.)*

Proof of Claim. H_e cannot be essential since then H_e and P_2 would contradict Lemma 9. H_e cannot be adjacent to a vertex in $G(0, v)$, by the choice of P. If H_e is not adjacent to any vertex in the region $\mathcal{C}[w_P, 1)$, then v is the only vertex adjacent to H_e. Then by Lemma 16 and Lemma 17, H_e is a single vertex, which is the upper endpoint of e, which we call v_e.

Remove v_e (and e): by the choice of G (minimal, aka induction), there is a radial embedding $\mathcal{E}(G \setminus v_e)$ of $G \setminus v_e$ that is supported by $\mathcal{D}(G \setminus v_e)$ (the original drawing restricted to $G \setminus v_e$). If there is an edge vw' in $G \setminus v_e$ with $w' \geqslant w_P$, simply embed e alongside vw to obtain the embedding of G that we need. So let's assume that there is no such edge. Then P_1 contains at least one vertex besides its endpoints, which is in the region $\mathcal{C}(v, w_P)$.

Let H_1 be the (non-empty) component of H that intersects P_1. (Since P begins with e, f, or g, H_1 equals H_e, H_f, or H_g.) Let H_1' be the subgraph induced by $V(H_1) \cup \{v\}$. By the choice of P, H_1 is not incident to an edge intersecting the region $\mathcal{C}[0, v)$.

If the radial embedding $\mathcal{E}(H_1')$ is essential, then the original drawing contains an essential cycle in $\mathcal{C}[\min H_1', \max H_1']$, but this contradicts Lemma 9 since $\ell(H_1') \subseteq \ell(P_2)$. Hence, the embedding of H_1' must be non-essential.

Let H_1'' denote the union of H_1' with all its incident edges (if any) intersecting the region $\mathcal{C}(w_P, 1)$. We can draw e alongside the boundary of the lower outer face of H_1'' in $\mathcal{E}(H_1'')$ so that it is bounded. Hence, we can apply Lemma 7 to re-embed e without crossings (contradiction).

Thus, if H_e is non-empty, then it must be adjacent to vertices other than v, which means vertices in either region $\mathcal{C}(0, v)$ or $\mathcal{C}[w_P, 1)$, where the former is ruled out due to the choice of P. By similar arguments, H_f and H_g have neighbors in $I \geqslant w_P$ unless they are empty. Thus, Claim 21 is proved. ■

Claim 22. *There exists a cycle C in $G[v, w_P]$ that contains v and (exactly) two edges in $\{e, f, g\}$.*

Proof of Claim. By Lemma 19, we can suppose that P goes through e, i.e., if not, then we can redraw near v to flip the relative order of the ends of e, f, g at v so that P ends at the leftmost edge, renaming it e, renaming the middle one g, and the right one f.

By Claim 21, there must be a path which begins with v then f and ends in $G[w_p, 1)$, with interior vertices (if any) in H_f; let P_f be a minimal such path. Define P_g similarly. If neither P_f nor P_g intersects $P_1 \setminus v$, then neither intersects P and both end in $\mathcal{C}(w_P, 1)$, which contradicts Lemma 20. Thus, we may assume that there exists a path from v through f or g to P_1 which lies in the region $\mathcal{C}[v, w_P]$. Hence, there exists a cycle C through e, v, f or e, v, g which lies in the region $\mathcal{C}[v, w_P]$. This proves the claim. ■

Let C be a cycle in $G[v, w_P]$ that contains (exactly) two edges of $\{e, f, g\}$, choosing C to minimize $\max C$. Let w be the vertex of C with $w = \max C$. (Note that $w \leqslant w_P$.)

We will need the following in the proof of the next claim about C. Let B_e be the component of $G(v, w)$ that contains v_e (the upper endpoint of e) if v_e is in $G(v, w)$, then let B'_e be the union of B_e and all incident edges (including e) and their endpoints. Otherwise, $v_e \geqslant w$; then let $B_e = \emptyset$ and let B'_e be the graph with just e and its endpoints v and v_e. Define B_f, B'_f, B_g, B'_g similarly. (Since $w \leqslant w_P$, $B_e \subseteq H_e$, $B_f \subseteq H_f$, and $B_g \subseteq H_g$.) By the choice of C, we have $B_e \cap B_f = B_e \cap B_g = B_f \cap B_g = \emptyset$. By the choice of P, none of B_e, B_f and B_g is adjacent to a vertex in $G(0, v)$.

Claim 23. *If C is non-essential, then w is the upper endpoint of the edge in $\{e, f, g\} \setminus E(C)$.*

Proof of Claim. First consider the case that C contains e and f. Since g crosses every edge of C evenly and g is between e and f near v—which is in the interior of C—the other endpoint of g must be in the interior of C or on C. If it is in the interior of C, then every vertex of B_g must be in the interior of C, because C and B_g are disjoint so their edges cross evenly. For the same reason, B_g cannot be adjacent to any vertices in $G(w, 1)$, so $V(B'_g) \setminus V(B_g) \subseteq \{v, w\}$. If B_g has no neighbors in $G[w, 1)$, then $B_g = H_g$, but H_g has a neighbor in $G[w_P, 1)$ by Claim 21, a contradiction since $w_P \geqslant w$. If B_g is adjacent to w, then B_g is essential by Lemma 18, a contradiction since $B_g \subseteq H_g$ and H_g is non-essential by Claim 21. Therefore B_g must be empty, so the upper endpoint of g is in C, and by the choice of C, it must be w; i.e., $g = vw$.

Next, consider the case that C contains e and g. Since g is between e and f near v, the edge f near v is in the exterior of C. Since f crosses e oddly and g evenly, the upper end of g is in the interior of C, ending at a vertex in the interior or at a vertex on C. The rest of the argument is the same as the previous case but with f and g switched, concluding that $f = vw$. The case that C contains f and g is similar to this case, with the conclusion that $e = vw$. ■

Claim 24. *We may assume that C is essential and $w_P = w$, so $B_e = H_e$, $B_f = H_f$, and $B_g = H_g$.*

Proof of Claim. The cycle C consists of two v, w-paths. If C is non-essential, then each v, w-path forms a cycle with the edge in $\{e, f, g\} \setminus E(C)$ as its maximum is w. If neither of those cycles is essential, Claim 23 applies to each one, but then $e = f = g = vw$, contradicting minimality by Lemma 14. Hence, there is a cycle C as defined prior to Claim 23 which is essential. Thus, we can choose C to be essential.

If $w_P > w$, then C and P_2 would contradict Lemma 9. Therefore $w_P = w$, so $B_e = H_e$, $B_f = H_f$, and $B_g = H_g$. ■

Using Claim 24, we derive the following.

Claim 25. *If B'_e (or B'_f or B'_g) does not intersect $G(w, 1)$, then B'_e (or B'_f or B'_g) has only one edge, $vw = e$ (or f or g).*

Proof. Suppose that B'_e does not intersect $G(w, 1)$. By Claim 24, $w_P = w$ and $B_e = H_e$, so H_e has no neighbors in $G(w_P, 1)$.

If H_e is not empty, then by Claim 21, H_e must be non-essential, with at least one neighbor in $G[w_P, 1)$; so w_P is the only neighbor of H_e in $G[w_P, 1)$. But also H_e has no neighbors in $G(0, v)$ by the choice of P, which contradicts Lemma 18.

If $H_e = \emptyset$, then B'_e is just the edge e and its endpoints v and v_e, where v_e is in $G[w_P, 1)$. Then $v_e = w$, since B'_e does not intersect $G(w, 1)$, so $e = vv_e = vw$. ■

We now assume without loss of generality that C passes through e and f. In the remainder of Case 1, we show that B'_e cannot intersect $G(w, 1)$, and, by symmetry, that B'_f cannot intersect $G(w, 1)$. Then it follows that $e = vw = f$, which is a contradiction because e and f are distinct and there are no multiple edges by Lemma 14, which will complete the proof of this case.

Claim 26. *B'_g intersects $G(w, 1)$.*

Proof. If not, then $B'_g = vw$ by Claim 25. By Claim 24, C is essential. The two v, w-paths of C form cycles with g and at least one must be non-essential; otherwise C would be obtainable as the symmetric difference of two essential cycles and therefore C would be non-essential, a contradiction. Claim 23 applies to this non-essential cycle (according to the definition of C just prior to Claim 23), which implies that $e = vw$ or $f = vw$. But then there are multiple edges with endpoints v and w, a contradiction by Lemma 14. ■

Claim 27. *B'_e and B'_f do not intersect $G(w, 1)$.*

Proof. For the sake of contradiction, by symmetry, assume that B'_e intersects $G(w, 1)$. Then there is a path Q in B'_e that starts with v, e which reaches $G(w, 1)$ at its other endpoint. By Claim 26, there is likewise a path Q' in B'_g starting at v with g which reaches $G(w, 1)$ at its other endpoint. Let P' be the concatenation of a v, w-path that in $C \cap B'_f$ and P_2; then $P' \cap Q = v$ and $P' \cap Q' = v$. Thus, we can apply Lemma 20 to P', Q, and Q' through f, e and g, respectively (contradiction). ■

By Claim 27 and 25, $e = vw = f$, which contradicts Lemma 14. Thus, we have shown that if there are edges crossing oddly in the upper rotation of a vertex v, then $v = v_1$. By symmetry, no two edges in the lower rotation of a vertex v cross oddly unless $v = v_n$.

Case 2: Only pairs of edges incident to v_1 or to v_n may cross oddly. We can assume that G does not contain edge v_1v_n, since otherwise $\mathcal{D}(G)$ is weakly essential by Lemma 9, and we are done by Lemma 13.

Modify G to create the graph G' with pendant edges as described in the paragraph preceding Lemma 12; then Lemma 12 implies that G' has an even drawing, so by Theorem 3 it has a radial embedding. We can redraw pendant edges and identify their endpoints to obtain a radial embedding of G, but we need to do this carefully to satisfy part (*ii*) of Theorem 6.

When we redraw the edges of G incident to v_1, we will do so such that the maximum vertex x on the lower face boundary walk W of G' is also on the outer face boundary of G: we can assume W begins and ends at x, then order the edges in the upper rotation of v_1 so that the upper endpoints form a subsequence of the vertices on W. See Figure 13. Likewise, we can redraw the edges of G incident to v_n so that the minimum vertex on the upper boundary walk of G' is also on the outer boundary of the embedding of G.

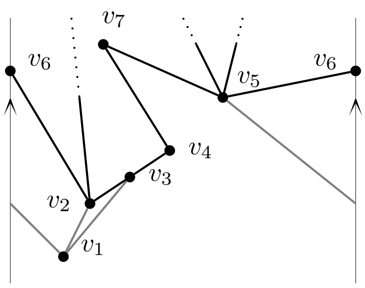

Figure 13: Lower parts of the graph G' (in black). The edges incident to v_1 have been added (in gray) so that v_7, the maximum vertex of the lower face boundary of G', remains on the lower outer face of G.

Any essential cycle C that is in G but not in G' must pass through v_1 or v_n. In order to satisfy part (ii) of Theorem 6, we need an essential cycle C' in the embedding of G' for which $[\min C', \max C'] \subseteq [\min C, \max C]$. A lower or upper facial walk of G' contains such a cycle.

5 Algorithm

Theorem 1 allows us to reduce the algorithmic problem of radial planarity testing to a system of linear equations over $\mathbb{Z}/2\mathbb{Z}$. For planarity testing, systems like this were first constructed by Wu and Tutte [25, Section 1.4.2].

Fix an arbitrary radial drawing of G and let $\mathrm{cr}(e, f)$ denote the parity of the number of crossings between edges e and f in that drawing. We need only check whether there is a radial redrawing of G such that the number of crossings is even between every pair of independent pair of edges e, f for which $\ell(e) \cap \ell(f) \neq \emptyset$.

Any two radial drawings of an edge with fixed endpoints can be obtained from one another by a continuous deformation during which the edges remain radial, as well as a certain number of Dehn twists. (This differs from the x-monotone case which needs no twists.) A *(Dehn) twist* is carried out by removing a small portion γ_e of e such that there is no vertex w with $w \in \ell(\gamma_e)$, and reconnecting the severed pieces of e by a curve γ'_e for which $\gamma_e \cup \gamma'_e$ has winding number 1.

For determining crossing parities, we only need to consider deforming an edge e via radial (e, w)-moves for vertices $w \in \ell(e)$, and we need to allow just one twist per edge e which we can assume occurs near its upper endpoint. The twist changes the crossing parity between e and e' for every edge e' with $\ell(\gamma_e) \subseteq \ell(e')$.

A linear system for testing radial planarity can then be constructed as follows. The system has a variable $x_{e,v}$, modeling an (e, v)-move, for every $v \in \ell(e)$, and a variable x_e, modeling an edge twist, for every e. For edges $e = uv$ and $f = wz$ with $u < w < z < v$, we require $\mathrm{cr}(e, f) \equiv x_{e,w} + x_{e,z} + x_f \pmod 2$. For edges $e = uv$ and $f = wz$ with $u < w < v < z$, we require $\mathrm{cr}(e, f) \equiv x_{e,w} + x_{f,v} + x_e \pmod 2$.

Then G is radial planar if and only if this linear system has a solution.

6 Open Questions

The cylinder model is generalized by a planarity notion introduced as *torus level planarity* in [2]. Unfortunately, already the weak Hanani-Tutte theorem fails for this variant for the leveled graph shown in Figure 14. The graph is based on the construction of the counterexample in the setting of the approximating maps of graphs [12, Figure 2]. To extend that example to the torus, we add a $C_3 \square P_3$, part of a toroidal grid, to block the original counterexample from using the full torus. The resulting instance is not toroidal level planar, yet it admits a toroidal level planar drawing that is even.

Nevertheless, there is a Hanani-Tutte theorem for torus level planarity if the underlying graph is a tree; see [12, Section 11], where a more general problem akin to strip planarity [1] is discussed. This more general problem can be seen as a special case of c-planarity. The computational complexity of c-planarity was recently settled by Fulek and Tóth [18]. The proof works with *atomic embeddability* which generalizes both c-planarity and thickenability of 3-dimensional manifolds. This suggests the question whether a variant of the Hanani-Tutte theorem holds in the setting of atomic embeddability if the underlying abstract graph does not contain a cycle.

Finally, if cycles are allowed in the underlying abstract graph, can we extend the Hanani-Tutte variant for approximating maps of graphs from [13, Theorem 1] to atomic embeddability?

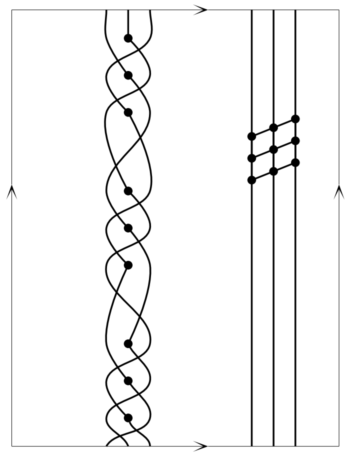

Figure 14: A leveled graph that has an even drawing on the torus, but is not toroidal level planar.

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