Smaller embeddings of partial k-star decompositions

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Abstract

A k-star is a complete bipartite graph $K_{1,k}$. For a graph G, a *k-star decomposition of G* is a set of k-stars in G whose edge sets partition the edge set of G. If we weaken this condition to only demand that each edge of G is in at most one k-star, then the resulting object is a *partial k-star decomposition of G*. An embedding of a partial k-star decomposition $\mathcal{A}$ of a graph G is a partial k-star decomposition $\mathcal{B}$ of another graph H such that $\mathcal{A} \subseteq \mathcal{B}$ and G is a subgraph of H. This paper considers the problem of when a partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for a given integer s. We improve a result of Noble and Richardson, itself an improvement of a result of Hoffman and Roberts, by showing that any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s < \frac{9}{4}k$ when k is odd and $s < (6-2\sqrt{2})k$ when k is even. For general k, these constants cannot be improved. We also obtain stronger results subject to placing a lower bound on n.

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1 Introduction

A *k-star decomposition* of a graph G is a collection of copies of $K_{1,k}$ in G such that each edge of G is in exactly one copy. If we weaken this condition to demand that each edge of G is in at most one copy, then the resulting object is a *partial k-star decomposition*. An *embedding* of a partial k-star decomposition $\mathcal{A}$ of a graph G is a partial k-star decomposition $\mathcal{B}$ of another graph H such that $\mathcal{A} \subseteq \mathcal{B}$ and G is a subgraph of H. The *leave* of a partial k-star decomposition of G is the graph L having vertex set $V(G)$ and edge set comprising all edges of G that are not in a k-star in the decomposition.

The problem of determining when a graph has a decomposition into k-stars has been thoroughly investigated. An obvious necessary condition for a graph to have a k-star decomposition is that its number of edges is divisible by k. Trivially, any graph has a decomposition into 1-stars. A simple inductive argument shows that any connected graph with an even number of edges has a 2-star decomposition (see [4, Theorem 1]). Tarsi [10] and Yamamoto et al. [13] independently proved that, for $n \geqslant 2$, a k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $\binom{n}{2} \equiv 0 \pmod{k}$. In fact, Tarsi gave necessary and sufficient conditions for the existence of a decomposition of a complete multigraph into k-stars while Yamamoto et al. also proved an analogous statement for complete bipartite graphs.

A result of Dor and Tarsi [5] implies that determining whether an arbitrary graph G has a k-star decomposition is NP-complete whenever $k \geqslant 3$. A result of Tarsi [11] gives a characterisation of when an arbitrary graph G has a k-star decomposition in which the number of k-stars that are centred on each vertex is specified. Other results in [11] imply various sufficient conditions for a graph to have a decomposition into k-stars. Hoffman and Roberts [8] exactly determined the maximum possible number of k-stars in a partial k-star decomposition of K_n and moreover characterised the possible leaves in some cases.

This paper is concerned with the problem of when a partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s}. In 2012, Hoffman and Roberts [7] proved that a partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some positive integer s such that $s \leqslant 7k-4$ when k is odd and $s \leqslant 8k-4$ when k is even. Furthermore, they conjectured that the smallest possible upper bound on s is around $2k$. In 2019, Noble and Richardson [9] improved the bounds on s to $s \leqslant 3k-2$ when k is odd and $s \leqslant 4k-2$ when k is even. As our first main result of the paper we further improve these bounds.

Theorem 1. *Let $k \geqslant 2$ and $n \geqslant 1$ be integers. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s < \frac{9}{4}k$ when k is odd and $s < (6-2\sqrt{2})k$ when k is even.*

If either of the constants $\frac{9}{4}$ or $6-2\sqrt{2} \approx 3.17$ in the above result were decreased then the result would fail to hold for infinitely many k (see Lemmas 18 and 21). Our next main result shows, however, that these constants can be improved if we impose a lower bound on n.

Theorem 2. *Let $k \geqslant 2$ and $n > \frac{k(k-1)}{\sqrt{8k-1}}$ be integers. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s \leqslant 2k-2$ when k is odd and $s \leqslant 3k-2$ when k is even.*

Neither of the upper bounds on s in this result can be decreased, no matter what lower bound we place on n (see Lemmas 9(c) and 16(b)). We prove Theorem 2 as a consequence of the following result which shows that, when $s \geqslant k$ and n is large enough, the obvious necessary condition is also sufficient for the existence of an embedding of a partial k-star decomposition of K_n in a k-star decomposition of K_{n+s}.

Theorem 3. *Let $k \geqslant 2$ and $n > \frac{k(k-1)}{\sqrt{8k-1}}$ be integers. Any nonempty partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for each $s \geqslant k$ such that $\binom{n+s}{2} \equiv 0 \pmod{k}$.*

The lower bound on s in this result cannot be decreased no matter what lower bound we place on n (see Lemma 9(b)). Moreover, the lower bound on n is asymptotically best possible as k becomes large (see Lemma 15).

2 Central functions and other preliminaries

We introduce some more notation that we use throughout the paper. Let G be a graph. Let $E(G)$, $V(G)$ and $\overline{G}$ denote the edge set, vertex set and complement of G respectively. For any $x \in V(G)$, $\deg_G(x)$ denotes the degree of x in G. The *neighbourhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices which are adjacent to x in G. For a subset U of $V(G)$ we use $G[U]$ to denote the subgraph of G induced by U.

For a set S of vertices we use K_S to denote the complete graph with vertex set S, and for disjoint sets S and T of vertices we use $K_{S,T}$ to denote the complete bipartite graph with parts S and T. For vertex-disjoint graphs G and H we use $G \vee H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup E(K_{V(G),V(H)})$. Our use of the notation $K_{S,T}$ will imply that S and T are disjoint and our use of the notation $G \vee H$ will imply that G and H are vertex-disjoint. As a special case, we take $G \vee K_\emptyset$ or $G \vee K_0$ to be simply the graph G. We can embed a partial k-star decomposition $\mathcal{D}$ of K_n in a k-star decomposition of K_{n+s} for some nonnegative integer s if and only if there is a k-star decomposition of $L \vee K_s$, where L is the leave of $\mathcal{D}$.

We begin by emphasising the necessary and sufficient conditions for the existence of a k-star decomposition of K_n that we mentioned in the introduction and highlighting their effects in the special case where k is a prime power.

Theorem 4. [10, 13] *Let $k \geqslant 2$ and $n \geqslant 2$ be positive integers.*

(a) *A k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $\binom{n}{2} \equiv 0 \pmod{k}$.*

(b) *If k is a power of 2 then a k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $n \equiv 0 \pmod{2k}$ or $n \equiv 1 \pmod{2k}$.*

(c) *If k is a power of an odd prime then a k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $n \equiv 0 \pmod{k}$ or $n \equiv 1 \pmod{k}$.*

Parts (b) and (c) of Theorem 4 follow immediately from part (a) because $\binom{n}{2} \equiv 0 \pmod{k}$ is equivalent to $n \equiv 0 \pmod{2k}$ or $n \equiv 1 \pmod{2k}$ when k is a power of 2 and is equivalent to $n \equiv 0 \pmod{k}$ or $n \equiv 1 \pmod{k}$ when k is a power of an odd prime. We often exploit this limitation of the possible values of n when k is a prime power in our constructions of partial k-star decompositions without small embeddings.

As mentioned in the introduction, a simple inductive argument shows that any connected graph with an even number of edges has a 2-star decomposition (see [4, Theorem 1]). This immediately implies the following characterisation of when a graph $L \vee K_s$ has a 2-star decomposition.

Lemma 5. *Let L be a graph. There is a 2-star decomposition of $L \vee K_s$ if and only if*

- *$s = 0$ and each connected component of L has an even number of edges; or*
- *$s \geqslant 1$ and $|E(L \vee K_s)| \equiv 0 \pmod{2}$.*

Let $k \geqslant 2$ be an integer. In a k-star, the vertex of degree k is called the *centre*. For a given k-star decomposition $\mathcal{D}$ of G, we can define a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$ called the *central function*, where $\gamma(x)$ is the number of k-stars of $\mathcal{D}$ whose centre is x for each $x \in V(G)$. It will be helpful to bear in mind the three following properties that must hold for any central function γ of a k-star decomposition of a graph G.

- $k \sum_{x \in V(G)} \gamma(x) = |E(G)|$.
- For each edge x_1x_2 of G, $\gamma(x_1) + \gamma(x_2) \geqslant 1$.
- For each vertex x of G, $k\gamma(x) \leqslant \deg_G(x)$ and if $k\gamma(x) = \deg_G(x)$ then each edge of G incident with x is in a k-star of $\mathcal{D}$ centred at x.

We call a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$ such that $k \sum_{x \in V(G)} \gamma(x) = |E(G)|$ a *k-precentral function* for G. Crucial to our approach in this paper is Lemma 6 below, which characterises when a k-star decomposition of a graph G with a specified central function exists. Lemma 6 is a simple consequence of a result of Tarsi [11, Theorem 2]. Because we will use Lemma 6 so extensively, we first introduce some notation that simplifies its statement and use.

Let $\mathcal{G}$ be a graph G equipped with a k-precentral function γ (note that G and γ determine the value of k). We call a k-star decomposition of G in which there are $\gamma(x)$ stars centred at x for each $x \in V(G)$ a *star $\mathcal{G}$-decomposition*. The notation we now define is implicitly dependent on $\mathcal{G}$, which will always be obvious from context. For any subset T of $V(G)$, let $\Delta_T = \Delta_T^+ - \Delta_T^-$ where $\Delta_T^- = k \sum_{x \in T} \gamma(x)$, $\Delta_T^+ = |E_T|$, and E_T is the set of edges of G that are incident to at least one vertex in T. Let Δ be the minimum of Δ_T over all subsets T of $V(G)$ and note that taking $T = \emptyset$ implies that $\Delta \leqslant 0$. Let $\mathcal{T}$ be the collection of subsets T of $V(G)$ for which $\Delta_T = \Delta$ and which, subject to this, have minimum cardinality.

Lemma 6. *Let $k \geqslant 2$ be an integer and let $\mathcal{G}$ be a graph G equipped with a k-precentral function γ.*

(i) *There exists a star $\mathcal{G}$-decomposition if and only if $\Delta = 0$.*

(ii) *For each $T \in \mathcal{T}$, $T \subseteq \{x \in V(G) : \gamma(x) \geqslant 1\}$.*

Proof. We first prove (i). It is clear that a star $\mathcal{G}$-decomposition exists if and only if there is an orientation of the edges of G such that exactly $k\gamma(x)$ edges are oriented out from x for each $x \in V(G)$. Remember that $k\sum_{x\in V(G)}\gamma(x) = |E(G)|$ because γ is a k-precentral function. Thus, by [11, Theorem 2] such an orientation exists if and only if $k\sum_{x\in S}\gamma(x) \geqslant |E(G[S])|$ for each subset S of $V(G)$. For a given subset S of $V(G)$, $k\sum_{x\in S}\gamma(x) = |E(G)| - \Delta_T^-$ and $E(G[S]) = E(G) \setminus E_T$, where $T = V(G) \setminus S$. Thus, such an orientation exists if and only if

$$\Delta_T \geqslant 0 \qquad \text{for each subset } T \text{ of } V(G). \tag{1}$$

Because $\Delta_\emptyset = 0$ and hence $\Delta \leqslant 0$, (1) is equivalent to $\Delta = 0$.

We now prove (ii). Let $T \in \mathcal{T}$ and suppose for a contradiction that $\gamma(x) = 0$ for some $x \in T$. We have that $\Delta_{T\setminus\{x\}} \leqslant \Delta_T$ because $\Delta_{T\setminus\{x\}}^- = \Delta_T^-$ and $\Delta_{T\setminus\{x\}}^+ \leqslant \Delta_T^+$ since $E_{T\setminus\{x\}} \subseteq E_T$. So, because $|T \setminus \{x\}| < |T|$, we have a contradiction to the definition of $\mathcal{T}$. □

Lemma 6 can also be obtained by specialising results in [6] or [1] concerning star decompositions of multigraphs. Through our notation Δ_T^+ and Δ_T^-, the condition of Lemma 6(i) is stated in the complement when compared to [11, Theorem 2], but this makes it consistent with the statements in [1, 6], which generalise more naturally to star packings of graphs.

We call a set U of vertices of a graph G *pairwise twin*, if $N_G(x) \setminus \{y\} = N_G(y) \setminus \{x\}$ for all $x, y \in U$. The next lemma aids us when applying Lemma 6 to graphs containing sets of pairwise twin vertices. Note that in a graph $G = L \vee K_S$, the vertices in S are pairwise twin and so we can apply the lemma with U chosen to be S.

Lemma 7. *Let $k \geqslant 2$ be an integer, let G be a graph and let U be a pairwise twin subset of $V(G)$. Let $\mathcal{G}$ be the graph G equipped with some k-precentral function γ and let $T \in \mathcal{T}$. For any $x_1 \in U \setminus T$ and $x_2 \in T \cap U$ we have $\gamma(x_1) < \gamma(x_2)$. In particular, if $\gamma(x) = \gamma(x')$ for all $x, x' \in U$ then, for each $T \in \mathcal{T}$, either $U \subseteq T$ or $T \cap U = \emptyset$.*

Proof. Suppose that $T \in \mathcal{T}$, $x_1 \in U \setminus T$ and $x_2 \in U \cap T$. Let $A = N_G(x_1) \setminus T$, and note that $A = N_G(x_2) \setminus (T \cup \{x_1\})$ because x_1 and x_2 are twin. Let $a = |A|$, $T_1 = T \cup \{x_1\}$ and $T_2 = T \setminus \{x_2\}$. Because $T \in \mathcal{T}$ and $|T_2| < |T|$ we have $\Delta_{T_1} \geqslant \Delta_T$ and $\Delta_{T_2} > \Delta_T$.

Observe that $\Delta_{T_1}^- = \Delta_T^- + k\gamma(x_1)$ and $\Delta_{T_1}^+ = \Delta_T^+ + a$ since $E_{T_1} = E_T \cup \{x_1 z : z \in A\}$. Therefore, $\Delta_{T_1} = \Delta_T + a - k\gamma(x_1)$ and so, because $\Delta_{T_1} \geqslant \Delta_T$, $k\gamma(x_1) \leqslant a$. Now,

$\Delta^-_{T_2} = \Delta^-_T - k\gamma(x_2)$ and $\Delta^+_{T_2} \leqslant \Delta^+_T - a$ since $E_{T_2} = E_T \setminus (\{x_2z : z \in A\} \cup X)$, where $X = \{x_1\}$ if $x_1x_2 \in E(G)$ and $X = \emptyset$ if $x_1x_2 \notin E(G)$. Therefore, $\Delta_{T_2} \leqslant \Delta_T - a + k\gamma(x_2)$ and so, because $\Delta_{T_2} > \Delta_T$, $a < k\gamma(x_2)$. Combining $k\gamma(x_1) \leqslant a$ and $a < k\gamma(x_2)$, we see we must have $\gamma(x_1) < \gamma(x_2)$.

Now suppose $\gamma(x) = \gamma(x')$ for all $x, x' \in U$. By what we have just proved, either $U \setminus T = \emptyset$ and hence $U \subseteq T$, or $T \cap U = \emptyset$. □

Many of the results in this paper (including Theorem 3) effectively concern k-star decompositions of $L \vee K_s$ for some specified graph L and integer $s \geqslant k$. Lemma 9 below illustrates why we usually impose the condition that s be at least k in these results. First we state a special case of a result of Tarsi [11, Theorem 4] that we will often use to show that a certain graph is the leave of a partial k-star decomposition.

Theorem 8 ([11]). *Let G be a graph of order n such that $\deg_G(x) \geqslant \frac{1}{2}n + k - 1$ for each $x \in V(G)$. Then G has a k-star decomposition if $|E(G)| \equiv 0 \pmod{k}$.*

Lemma 9. *Let $k \geqslant 2$ and $n \geqslant 2$ be integers such that k is odd and $n \equiv 2 \pmod{2k}$. Let L be a graph of order n that has exactly one edge.*

(a) *There is a partial k-star decomposition of K_n whose leave is L.*

(b) *There is no k-star decomposition $L \vee K_{k-1}$, even though $|E(L \vee K_{k-1})| \equiv 0 \pmod{k}$.*

(c) *If k is a power of an odd prime, there is no k-star decomposition $L \vee K_s$ for any $s < 2k - 2$.*

Proof. We first prove (a) by showing that a k-star decomposition of $\overline{L}$ exists. This is trivial if $n = 2$. If $n \geqslant 2k + 2$, then $\deg_{\overline{L}}(y) \geqslant n - 2 \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ and $|E(\overline{L})| = \binom{n}{2} - 1 \equiv 0 \pmod{k}$ since $n \equiv 2 \pmod{2k}$. Therefore, by Theorem 8, a k-star decomposition of $\overline{L}$ exists.

We now prove (b). Note that $|E(L \vee K_{k-1})| = 1 + n(k-1) + \binom{k-1}{2} \equiv 0 \pmod{k}$ because $n \equiv 2 \pmod{2k}$ and k is odd. Let r be the nonnegative integer such that $n = 2kr + 2$. Suppose for a contradiction that there is a k-star decomposition $\mathcal{D}$ of $L \vee K_S$, where $|S| = k - 1$, and let γ be the central function of $\mathcal{D}$. Now $|E(L \vee K_S)| = 1 + n(k-1) + \binom{k-1}{2}$ and so $\sum_{x \in V(L) \cup S} \gamma(x) = (2r + \frac{1}{2})(k-1) + 1$. Observe that $\deg_{L \vee K_S}(y_1) = \deg_{L \vee K_S}(y_2) = k$, where y_1y_2 is the only edge in L, and $\deg_{L \vee K_S}(y) = k - 1$ for each $y \in V(L) \setminus \{y_1, y_2\}$. So, without loss of generality, $\gamma(y_1) = 1$, every edge of $L \vee K_S$ incident with y_1 is in the star in $\mathcal{D}$ centred at y_1, and $\gamma(y) = 0$ for each $y \in V(L) \setminus \{y_1\}$. Thus $\sum_{z \in S} \gamma(z) = (2r + \frac{1}{2})(k-1)$. By the pigeonhole principle, it follows that $\gamma(z_1) = 2r + 1$ for some $z_1 \in S$ because $|S| = k - 1$. Now $\deg_{L \vee K_S}(z_1) = n + k - 2 = k(2r + 1)$ noting that $n = 2kr + 2$. So every edge incident with z_1 is in a star in $\mathcal{D}$ centred at z_1. But this contradicts the fact that the edge y_1z_1 is in the star in $\mathcal{D}$ centred at y_1.

We now prove (c). Suppose that k is a power of an odd prime. Assume for a contradiction that $\mathcal{D}$ is a k-star decomposition of $L \vee K_S$ where $|S| = s$ for some nonnegative integer $s < 2k-2$. By part (a) of this lemma and Theorem 4(c), we have that $n+s \equiv 0 \pmod{k}$ or $n+s \equiv 1 \pmod{k}$ and hence, because $n \equiv 2 \pmod{2k}$, that $s \equiv k-2 \pmod{k}$ or $s \equiv k-1 \pmod{k}$. So $s \in \{k-2, k-1\}$ because $s < 2k-2$. So then $s = k-1$ because a k-star in $\mathcal{D}$ must be centred at an end vertex of the edge in L and these vertices have degree $s+1$ in $L \vee K_S$. However, a k-star decomposition of $L \vee K_{k-1}$ does not exist by (b). □

3 Embedding maximal partial k-star decompositions

A partial k-star decomposition of a graph G is *maximal* if there is no star that can be added to it to produce a partial k-star decomposition of G containing more stars. Thus, a partial k-star decomposition of a graph G is maximal if and only if its leave has maximum degree at most $k-1$. In this section we prove results about embedding maximal partial k-star decompositions of K_n in k-star decompositions of K_{n+s} where $s \geqslant k$. These results will be crucial in proving the main theorems.

An *independent set* in a graph is a set of its vertices that are pairwise non-adjacent. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality of an independent set in G. In [2, Corollary 2], Caro and Roditty note that if a graph G has a decomposition into k-stars then $\alpha(G) \geqslant |V(G)| - \frac{1}{k}|E(G)|$. This can be seen by observing that any edge in G must have a star of the decomposition centred on at least one of its end-vertices. For the cases we are interested in, we formalise this observation in the following lemma.

Lemma 10. *Let $k \geqslant 2$, $n \geqslant 1$ and $s \geqslant 0$ be integers, and let L be a graph of order n. If there is k-star decomposition of $L \vee K_s$, then $\alpha(L) \geqslant n + s - \frac{1}{k}|E(L \vee K_s)|$.*

Proof. If there is a k-star decomposition of $L \vee K_s$, then $\alpha(L \vee K_s) \geqslant n+s-\frac{1}{k}|E(L \vee K_s)|$ by [2, Corollary 2]. Furthermore, it is easy to see that $\alpha(L \vee K_s) = \alpha(L)$. □

In this section we show that, for a maximal partial k-star decomposition $\mathcal{D}$ of K_n and an integer $s \geqslant k$ such that $\binom{n+s}{2} \equiv 0 \pmod{k}$, the obstacle described by Lemma 10 is the only thing that can prevent the existence of an embedding of $\mathcal{D}$ in a k-star decomposition of K_{n+s}. We do this in two lemmas: Lemma 11 deals with the case where the number of stars to be added is small and the obstacle may arise whereas Lemma 12 deals with the case where the number of stars to be added is large and the obstacle cannot arise.

Lemma 11. *Let k, n and s be integers with $s \geqslant k \geqslant 2$, and let L be a graph of order n with maximum degree at most $k-1$ and $|E(L \vee K_s)| \leqslant k(n+s)$. Then there is a k-star decomposition of $L \vee K_s$ if and only if $|E(L \vee K_s)| \equiv 0 \pmod{k}$ and $\alpha(L) \geqslant n + s - \frac{1}{k}|E(L \vee K_s)|$.*

Proof. The 'only if' direction follows from Lemma 10, so we only need to prove the 'if' direction.

Suppose that $|E(L \vee K_S)| \equiv 0 \pmod{k}$, where S is a set with $|S| = s$. Let $V = V(L \vee K_S)$ and $b = \frac{1}{k}|E(L \vee K_S)|$, and suppose that L has an independent set A containing $n + s - b$ vertices. Note that $n + s - b \geqslant 0$ because $|E(L \vee K_S)| \leqslant k(n + s)$ by our hypotheses. Define a k-precentral function γ for $L \vee K_S$ by $\gamma(x) = 0$ for each $x \in A$ and $\gamma(x) = 1$ for each $x \in V \setminus A$. This is indeed a k-precentral function for $L \vee K_S$ because $\sum_{x \in V} \gamma(x) = n + s - |A| = b$. Let $\mathcal{G}$ be the graph $L \vee K_S$ equipped with γ. We complete the proof by showing that $\Delta = 0$ and hence a star $\mathcal{G}$-decomposition exists by Lemma 6. Let $T \in \mathcal{T}$ and suppose for a contradiction that $\Delta_T < 0$. Since $\gamma(z) = 1$ for all $z \in S$, we can apply Lemma 7 with $U = S$ to conclude that either $T \cap S = \emptyset$ or $S \subseteq T$. We consider these cases separately, with the latter splitting into two subcases.

Case 1: Suppose that $T \cap S = \emptyset$. This implies $T \subseteq V(L)$. Then $\Delta_T^+ \geqslant s|T|$, because $E(K_{S,T}) \subseteq E_T$ and $\Delta_T^- = k|T|$ by the definition of γ and Lemma 6(ii). Therefore, we have $\Delta_T^- \leqslant \Delta_T^+$ as $s \geqslant k$. This contradicts $\Delta_T < 0$.

Case 2a: Suppose that $S \subseteq T$ but $T \neq V \setminus A$. Then there is a vertex $y \in V(L) \setminus (A \cup T)$ and, by the definition of γ, $\gamma(y) = 1$. Let $T_1 = T \cup \{y\}$. Then $\Delta_{T_1}^+ \leqslant \Delta_T^+ + k - 1$, noting that $\deg_L(y) \leqslant k - 1$ and $\Delta_{T_1}^- = \Delta_T^- + k$. Therefore, $\Delta_{T_1} \leqslant \Delta_T - 1$ contradicting $T \in \mathcal{T}$.

Case 2b: Suppose that $T = V \setminus A$. Then $\Delta_T^+ = |E(L \vee K_S)|$ because $E_T = E(L \vee K_S)$ since A is independent. Moreover, $\Delta_T^- = |E(L \vee K_S)|$ because γ is a k-precentral function for $L \vee K_S$. So $\Delta_T^+ = \Delta_T^-$ contradicting $\Delta_T < 0$. $\square$

Note that the condition $n \geqslant k$ in the following lemma will certainly hold whenever L is the leave of a nontrivial k-star decomposition.

Lemma 12. *Let k, n and s be positive integers with $s \geqslant k \geqslant 2$ and $n \geqslant k$, and let L be a graph of order n with maximum degree at most $k - 1$ and $|E(L \vee K_s)| \geqslant k(n + s)$. Then there is a k-star decomposition of $L \vee K_s$ if and only if $|E(L \vee K_s)| \equiv 0 \pmod{k}$.*

Proof. If $L \vee K_s$ has a k-star decomposition, then obviously $|E(L \vee K_s)| \equiv 0 \pmod{k}$. So it suffices to prove the 'if' direction.

Assume that $|E(L \vee K_S)| \equiv 0 \pmod{k}$, where S is a set with $|S| = s$, let $b = \frac{1}{k}|E(L \vee K_S)|$ and note $b \geqslant n + s$ by the hypotheses of the lemma. Thus, we can define a k-precentral function γ on $L \vee K_S$ such that $\gamma(y) = 1$ for each $y \in V(L)$ and $\gamma(z) \in \{d, d + 1\}$ for each $z \in S$, where $d = \lfloor \frac{b-n}{s} \rfloor$. Note that $d \geqslant 1$ since $b \geqslant n + s$ and let $S_0 = \{z \in S : \gamma(z) = d\}$. We will show there is a star $\mathcal{G}$-decomposition where $\mathcal{G}$ is $L \vee K_S$ equipped with γ.

Let $T \in \mathcal{T}$, $H = L[V(L) \setminus T]$, $h = |V(H)|$, and $e = |E(H)|$. By Lemma 6, it suffices to show that $\Delta_T \geqslant 0$. By Lemma 7 with $U = S$, we have that $T \cap S \in \{\emptyset, S \setminus S_0, S\}$. We separate the proof into three cases accordingly.

Case 1: Suppose that $T \cap S = \emptyset$. Then $T = V(L) \setminus V(H)$. Noting that $E_T = E(K_{S,V(L)\setminus V(H)}) \cup (E(L) \setminus E(H))$ and $\Delta_T^- = k(n-h)$, we have

$$\Delta_T = \big((n-h)s + |E(L)| - e\big) - k(n-h) = (n-h)(s-k) + |E(L)| - e.$$

This last expression is nonnegative because $n \geqslant h, s \geqslant k$ and $|E(L)| \geqslant e$.

Case 2: Suppose that $T \cap S = S$. Noting that $E_T = E(L \vee K_S) \setminus E(H)$, that $|E(L \vee K_S)| = bk$, and that $\Delta_T^- = k(b-h)$, we see that

$$\Delta_T = (bk - e) - k(b-h) = kh - e.$$

This last expression is nonnegative because $e \leqslant \frac{1}{2}h(k-1)$ since H has maximum degree at most $k-1$.

Case 3: Suppose that $T \cap S = S \setminus S_0$. Let $s_0 = |S_0|$. Noting that

$$E_T = E(L \vee K_S) \setminus \big(E(K_{S_0}) \cup E(K_{S_0,V(H)}) \cup E(H)\big),$$

that $|E(L \vee K_S)| = bk$, and that $\Delta_T^- = k(b - ds_0 - h)$, we see that

$$\Delta_T = \left(bk - \binom{s_0}{2} - hs_0 - e\right) - k(b - ds_0 - h) = \frac{s_0}{2}(2dk + 1 - s_0) + h(k - s_0) - e. \quad (2)$$

The remainder of the proof is a somewhat tedious verification that this last expression is nonnegative. We first observe the following three useful facts.

(F1) $2e \leqslant h(k-1)$

(F2) $e \leqslant k(n + s(d+1) - s_0) - ns - \binom{s}{2}$

(F3) $d \leqslant \frac{1}{2ks}(n(2s-k-1) + s(s-1) + 2ks_0) - 1$

Note that (F1) holds because H is a subgraph of L and thus has maximum degree at most $k-1$. Also, (F2) holds because $e \leqslant |E(L)| = bk - ns - \binom{s}{2}$ and $b = n + s(d+1) - s_0$ from the definition of γ. Further, (F3) holds because $b = n + s(d+1) - s_0$, $b = \frac{1}{k}(|E(L)| + ns + \binom{s}{2})$ and $|E(L)| \leqslant \frac{1}{2}n(k-1)$ since L has maximum degree at most $k-1$. We divide this case into subcases depending on the value of s_0.

Case 3a: Suppose that $s_0 \geqslant k$. Then substituting $h \leqslant n$ and (F2) into (2) we obtain

$$\Delta_T \geqslant \frac{s - s_0}{2}\big(s + s_0 + 2(n-k) - 2dk - 1\big). \quad (3)$$

Substituting (F3) into (3) and rearranging, we obtain

$$\Delta_T \geqslant \frac{s - s_0}{2s}\big((s_0 - k)(s-k) + k(s + n - s_0 - k) + n\big).$$

This last expression is nonnegative because $n \geqslant k$ and $s \geqslant s_0 \geqslant k$ using the conditions of this case.

Case 3b: Suppose that $s_0 \leqslant \frac{k+1}{2}$. Then substituting $e \leqslant \frac{1}{2}h(k-1)$ from (F1) into (2) we obtain

$$\Delta_T \geqslant \frac{s_0}{2}(2dk+1-s_0) + h(\tfrac{k+1}{2} - s_0).$$

This last expression can be seen to be nonnegative using $d \geqslant 1$ and $1 \leqslant s_0 \leqslant \frac{k+1}{2}$ from the conditions of this case.

Case 3c: Suppose that $\frac{k+2}{2} \leqslant s_0 \leqslant k-1$. Then substituting $h \geqslant \frac{2e}{k-1}$ from (F1) into (2) we obtain

$$\Delta_T \geqslant \frac{s_0}{2}(2dk+1-s_0) - \frac{e}{k-1}(2s_0-k-1). \tag{4}$$

Observing that $2s_0 - k - 1 > 0$ by the conditions of this case, substituting (F2) and rearranging, we obtain

$$\Delta_T \geqslant \frac{2s_0-k-1}{k-1}\left(\binom{s}{2} + n(s-k) - k(s-s_0)\right) - \binom{s_0}{2} + \frac{dk}{k-1}\big(s(k+1) - s_0(2s-k+1)\big). \tag{5}$$

We further divide this subcase according to the sign of the coefficient of d in (5).

Case 3c(i): Suppose that $s(k+1) < s_0(2s-k+1)$. Substituting (F3) into (5) and simplifying, we obtain

$$\Delta_T \geqslant \frac{s-s_0}{2s}\big(n + k(n-s_0) + s_0(s-k)\big). \tag{6}$$

We can easily see that Δ_T is nonnegative since $s \geqslant k$, $n \geqslant k$ and $s_0 \leqslant k-1$ by the conditions of Case 3c.

Case 3c(ii): Suppose that $s(k+1) \geqslant s_0(2s-k+1)$. Substituting $d \geqslant 1$ and $n \geqslant k$ in (5) and rearranging yields

$$\Delta_T \geqslant \frac{2s_0-k-1}{2(k-1)}\left(s^2 - (2k+1)s - 2k^2\right) + \frac{3k+1}{k-1}\binom{s_0}{2}. \tag{7}$$

Recall that $2s_0 > k-1$ by the conditions of Case 3c. Since $s \geqslant k$ is an integer, either $s = k$ or $s \geqslant k+1$, and hence $s^2 - (2k+1)s \geqslant -k(k+1)$. Substituting this into (7) and rearranging, we obtain

$$\Delta_T \geqslant \frac{3k+1}{k-1}\binom{k-s_0+1}{2}. \tag{8}$$

This last expression is clearly nonnegative since $s_0 \leqslant k-1$ by the conditions of Case 3c. $\square$

4 Proof of Theorems 2 and 3

Caro [3] and Wei [12] independently established the following lower bounds on the independence number of a graph.

Theorem 13 ([3], [12])**.** *For any graph G, the following hold.*

(a) $\alpha(G) \geqslant \sum_{x \in V(G)} \frac{1}{\deg_G(x)+1}$

(b) $\alpha(G) \geqslant \frac{|V(G)|^2}{2|E(G)|+|V(G)|}$

Part (b) of Theorem 13 follows immediately from part (a) because, by convexity,

$$\sum_{x \in V(G)} \frac{1}{\deg_G(x)+1} \geqslant \frac{|V(G)|}{d+1} \qquad \text{where} \qquad d = \frac{2|E(G)|}{|V(G)|}.$$

In Lemma 14 below we combine Theorem 13(b) with Lemmas 11 and 12 to show that, for any graph L, a k-star decomposition of $L \vee K_s$ must exist if $|E(L \vee K_s)| \equiv 0 \pmod{k}$ and s is greater than a certain function of k and $|V(L)|$. Theorem 3 then follows from Lemma 14 and, in turn, Theorem 2 follows from Theorem 3. For technical reasons we restrict Lemma 14 to $k \geqslant 3$. Lemma 5 covers the case when $k = 2$.

Lemma 14. *Let k, n and s be positive integers with $s \geqslant k \geqslant 3$ and $n \geqslant k$, and let L be a graph of order n such that $|E(L \vee K_s)| \equiv 0 \pmod{k}$. Then there is a k-star decomposition of $L \vee K_s$ if*

$$s > k - n + \frac{1}{2} + \sqrt{\left(n - \sqrt{2k}\right)^2 + k(k-3) + \tfrac{1}{4}}. \tag{9}$$

In particular, such a decomposition exists if $n > \frac{k(k-1)}{\sqrt{8k-1}}$.

Proof. Observe that the right hand side of (9) is real because $k \geqslant 3$. We first prove the first part of the lemma. Suppose that (9) holds. We may assume that L has maximum degree at most $k-1$ because otherwise we can greedily delete k-stars from L until this is the case, apply the proof, and finally add the deleted k-stars to the decomposition produced. Let $b = \frac{1}{k}|E(L \vee K_s)|$, note that b is an integer because $|E(L \vee K_s)| \equiv 0 \pmod{k}$, and let $e = |E(L)|$. If $b \geqslant n + s$, then a k-star decomposition of $L \vee K_s$ exists by Lemma 12, so we may assume that $b < n + s$. By Lemma 11 it suffices to show that $\alpha(L) \geqslant n + s - b$.

By Theorem 13 we have $\alpha(L) \geqslant \frac{n^2}{2e+n}$. So, because $\alpha(L)$ and $n+s-b$ are both integers, it is enough to show that $\frac{n^2}{2e+n} > n + s - b - 1$. Using $b = \frac{1}{k}(e + ns + \binom{s}{2})$ and multiplying through by $2k$, this is equivalent to showing that

$$s^2 + (2n - 2k - 1)s - 2kn + 2k + 2e + \frac{2kn^2}{(2e+n)} \tag{10}$$

is positive. Considered as a function of a real variable $e \geqslant 0$, (10) is minimised when $e = \frac{n}{2}(\sqrt{2k} - 1)$. Substituting this value for e and rearranging, we see that (10) is at least

$$s^2 + (2n - 2k - 1)s + 2k - \left(2k - 2\sqrt{2k} + 1\right) n.$$

Considering this last expression as a quadratic in s, it can be seen that it is positive when (9) holds. Thus, (10) is positive and $\alpha(L) \geqslant n + s - b$, as required.

We now prove the second part of the lemma. Suppose that $n > \frac{k(k-1)}{\sqrt{8k-1}}$. Since $s \geqslant k$, substituting $s = k$ into (9) and rearranging shows that (9) will hold if

$$n - \frac{1}{2} > \sqrt{\left(n - \sqrt{2k}\right)^2 + k(k-3) + \frac{1}{4}}.$$

By squaring both sides of this expression and rearranging, we see that it is equivalent to $n > \frac{k(k-1)}{\sqrt{8k-1}}$. Therefore, by the first part of the lemma, a k-star decomposition of $L \vee K_s$ exists. $\square$

We can now prove Theorem 3 directly from Lemma 14.

Proof of Theorem 3. Let L be the leave of a nonempty partial k-star decomposition of K_n and note that this implies that $n > k$. Let s be an integer such that $s \geqslant k$ and $\binom{n+s}{2} \equiv 0 \pmod{k}$. Since L is the leave of a partial k-star decomposition and $\binom{n+s}{2} \equiv 0 \pmod{k}$, it follows that $|E(L \vee K_s)| \equiv 0 \pmod{k}$. So, by Lemma 14 if $k \geqslant 3$ and by Lemma 5 if $k = 2$, there is a k-star decomposition of $L \vee K_s$. $\square$

Lemma 9(b) demonstrates that the lower bound on s in Theorem 3 cannot be decreased no matter what lower bound we place on n. Next, in Lemma 15, we show that in the case $s = k$ the lower bound on n in Theorem 3 is asymptotically best possible. To see that Lemma 15 implies this, note that $\frac{k(k-1)}{\sqrt{8k-1}} = (\frac{k}{2})^{3/2} + O(k)$ as k becomes large.

Lemma 15. *Let $k = 2^t$ for some odd integer $t \geqslant 7$, let $m = \sqrt{2k}$, and let $n = \frac{1}{4}km - k = (\frac{k}{2})^{3/2} - k$. Let L be a graph of order n that is a vertex disjoint union of $\frac{n}{m}$ copies of K_m. Then a partial k-star decomposition of K_n whose leave is L exists and furthermore it cannot be embedded in a k-star decomposition of K_{n+k}, even though $\binom{n+k}{2} \equiv 0 \pmod{k}$.*

Proof. Note that $m = 2^{(t+1)/2}$ is an integer divisible by 8 because t is odd and $t \geqslant 7$. Thus $n \equiv k \pmod{2k}$, $\frac{n}{m}$ is an integer and $\binom{n+k}{2} \equiv 0 \pmod{k}$. Note that $|E(L)| = \frac{n}{m}\binom{m}{2} = \frac{n}{2}(m-1)$. We first show that L is the leave of a partial k-star decomposition of K_n. Note that $\deg_{\overline{L}}(y) = n - m \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ because $n = \frac{1}{4}km - k$ and $k \geqslant 128$. Furthermore, $E(\overline{L}) = \binom{n}{2} - \frac{n}{2}(m-1) = \frac{n}{2}(n-m) \equiv 0 \pmod{k}$ because $n \equiv 0 \pmod{k}$ and $n - m$ is even. Therefore, by Theorem 8, there is a k-star decomposition of $\overline{L}$.

We complete the proof by using Lemma 10 to show that there is no k-star decomposition of $L \vee K_k$. Observe that

$$n + k - \frac{1}{k}|E(L \vee K_k)| = n + k - \frac{1}{k}\left(\frac{n}{2}(m-1) + kn + \binom{k}{2}\right) = \frac{k}{4} + \frac{5\sqrt{2k}}{8}$$

where the first equality follows using $|E(L)| = \frac{n}{2}(m-1)$ and the second follows using $n = \frac{1}{4}km - k$ and $m = \sqrt{2k}$. On the other hand, $\alpha(L) = \frac{n}{m} = \frac{k}{4} - \frac{k}{m}$ because an independent set in L can contain at most one vertex from each copy of K_m. So we have $\alpha(L) < n + k - \frac{1}{k}|E(L \vee K_k)|$ and hence there is no k-star decomposition of $L \vee K_k$ by Lemma 10. $\square$

Theorem 2 follows readily from Theorem 3.

Proof of Theorem 2. Let $\mathcal{D}$ be a partial k-star decomposition of K_n. If $\mathcal{D}$ is empty and n=1, then $\mathcal{D}$ is trivially its own embedding. If $\mathcal{D}$ is empty and $n \geqslant 2$, then there is an embedding of $\mathcal{D}$ in a k-star decomposition of K_{2k} by Theorem 4(a). So in either case the result holds, and hence we may assume that $\mathcal{D}$ is nonempty.

If k is even, let s be an element of $\{k, \ldots, 3k-2\}$ such that $n + s \equiv 0 \pmod{2k}$ or $n + s \equiv 1 \pmod{2k}$. If k is odd, let s be an element of $\{k, \ldots, 2k-2\}$ such that $n + s \equiv 0 \pmod{k}$ or $n + s \equiv 1 \pmod{k}$. In either case such an s exists because $\{k, \ldots, 3k-2\}$ contains $2k-1$ consecutive integers and $\{k, \ldots, 2k-2\}$ contains $k-1$ consecutive integers. Then $\binom{n+s}{2} \equiv 0 \pmod{k}$ by our definition of s. So by Theorem 3 there is an embedding of $\mathcal{D}$ in a k-star decomposition of K_{n+s} and hence the result is proved. $\square$

Lemma 9(c) shows that the upper bound of $2k-2$ on s in the k odd case of Theorem 2 cannot be improved for any k that is a power of an odd prime. Next, in Lemma 16, we show that the upper bound of $3k-2$ on s in the k even case of Theorem 2 cannot be improved for any $k \geqslant 16$ that is a power of 4.

Lemma 16. *Let $k = 2^t$ for some even $t \geqslant 4$, and let $n \geqslant 3k+2$ be an integer such that $n \equiv k + 2 \pmod{2k}$. Let L be a graph of order n that is a vertex disjoint union of one copy of $K_{\sqrt{k}}$, $\frac{1}{2}\sqrt{k} + 1$ copies of K_2 and $n - 2\sqrt{k} - 2$ copies of K_1. A partial k-star decomposition of K_n whose leave is L exists and furthermore it cannot be embedded in a k-star decomposition of K_{n+s} for any $s < 3k-2$.*

Proof. A simple calculation shows that $|E(L)| = \frac{1}{2}(k+2)$. We first show that L is the leave of a partial k-star decomposition of K_n. Note that $\deg_{\overline{L}}(y) \geqslant n - \sqrt{k} \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ since $n \geqslant 3k+2$ and $k \geqslant 16$. Furthermore, $|E(\overline{L})| = \binom{n}{2} - \frac{1}{2}(k+2) \equiv 0 \pmod{k}$ since $n \equiv k+2 \pmod{2k}$. Therefore, a k-star decomposition of $\overline{L}$ exists by Theorem 8.

Now assume for a contradiction that $\mathcal{D}$ is a k-star decomposition of $L \vee K_S$ where $|S| = s$ for some nonnegative integer $s < 3k-2$ and let γ be the central function of $\mathcal{D}$.

We must have that $n+s \equiv 0 \pmod{2k}$ or $n+s \equiv 1 \pmod{2k}$ by Theorem 4(b) and hence, because $n \equiv k+2 \pmod{2k}$, that $s \equiv k-2 \pmod{2k}$ or $s \equiv k-1 \pmod{2k}$. Therefore, $s \in \{k-2, k-1\}$ since $s < 3k-2$.

Let V_1 be the vertex set of the copy of $K_{\sqrt{k}}$ in L and let V_2 be the set of vertices in the $\frac{1}{2}\sqrt{k}+1$ copies of K_2 in L. If $s = k-2$, then $\deg_{L \vee K_S}(y) = k-1$ and hence $\gamma(y) = 0$ for each $y \in V_2$ which contradicts the fact that each edge in $L[V_2]$ is in a star in $\mathcal{D}$. Thus it must be that $s = k-1$ and $\mathcal{D}$ is a k-star decomposition of $L \vee K_{k-1}$. Let r be the positive integer such that $n = 2kr + k + 2$. Observe the following.

- $\sum_{x \in V(L) \cup S} \gamma(x) = (2r+1)(k-1) + \frac{1}{2}k + 1$ because $|E(L \vee K_{k-1})| = \frac{1}{2}(k+2) + n(k-1) + \binom{k-1}{2}$.
- $\sum_{y \in V_1} \gamma(y) \leqslant \sqrt{k}$ because $\deg_{L \vee K_S}(y) = k + \sqrt{k} - 2 < 2k$ for each $y \in V_1$ and hence $\gamma(y) \leqslant 1$ for all $y \in V_1$.
- $\sum_{y \in V_2} \gamma(y) = \frac{1}{2}\sqrt{k} + 1$ because $\deg_{L \vee K_S}(y) = k$ for each $y \in V_2$ and hence $\gamma(y_1) + \gamma(y_2) = 1$ for each edge y_1y_2 in $L[V_2]$.
- $\sum_{y \in V(L) \setminus (V_1 \cup V_2)} \gamma(y) = 0$ because $\deg_{L \vee K_S}(y) = k-1$ for each $y \in V(L) \setminus (V_1 \cup V_2)$.

Using these four facts and simplifying we have

$$\sum_{z \in S} \gamma(z) = \sum_{x \in V(L) \cup S} \gamma(x) - \sum_{y \in V(L)} \gamma(y) \geqslant (2r+1)(k-1) + \tfrac{1}{2}k - \tfrac{3}{2}\sqrt{k} > (2r+1)(k-1)$$

where the last inequality follows because $k \geqslant 16$. So, by the pigeonhole principle, $\gamma(z_1) \geqslant 2r+2$ for some $z_1 \in S$ because $s = k-1$. Now $\deg_{L \vee K_S}(z_1) = n+k-2 = k(2r+2)$ noting that $n = 2kr + k + 2$, so it must be that $\gamma(z_1) = 2r+2$ and that every edge incident with z_1 is in a star in $\mathcal{D}$ centred at z_1. But this contradicts the fact that, for any vertex $y_1 \in V_2$ such that $\gamma(y_1) = 1$, the edge y_1z_1 must be in a star in $\mathcal{D}$ centred at y_1. $\square$

5 Proof of Theorem 1

From Lemma 14, it is not too difficult to prove Theorem 1 in the case where k is even. Note that in fact the argument in the proof also applies when k is odd.

Lemma 17. *Let $k \geqslant 2$ and $n \geqslant 1$ be integers. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s < (6-2\sqrt{2})k$.*

Proof. Let $\mathcal{D}$ be a partial k-star decomposition of K_n and L be its leave. Note that we will have $|E(L \vee K_s)| \equiv 0 \pmod{k}$ for any integer s such that $n+s \equiv 0 \pmod{2k}$. If $k = 2$ then we can choose $s \in \{1,2,3,4\}$ such that $n+s \equiv 0 \pmod{4}$ and $L \vee K_s$ will have a 2-star decomposition by Lemma 5, so we may assume $k \geqslant 3$. We consider three cases according to the value of n.

Case 1: Suppose that $n \geqslant 2\sqrt{2}k$. Let s be an integer such that $(4-2\sqrt{2})k \leqslant s < (6-2\sqrt{2})k$ and $n+s \equiv 0 \pmod{2k}$. By Lemma 14 there is a k-star decomposition of $L \vee K_s$ and hence the result is proved provided that (9) holds. The lower bound on s given by (9) can be seen to be decreasing in n, so it suffices to show that this bound is less than $(4-2\sqrt{2})k$ when $n = 2\sqrt{2}k$. Substituting $n = 2\sqrt{2}k$ into the bound gives

$$\left(1-2\sqrt{2}\right)k + \tfrac{1}{2} + \sqrt{9k^2 - 8k^{3/2} - k + \tfrac{1}{4}}$$

which is easily seen to be less than $(4-2\sqrt{2})k$ since the final term is less than $3k - \frac{1}{2}$.

Case 2: Suppose that $k+1 \leqslant n < 2\sqrt{2}k$. We show that we can embed $\mathcal{D}$ in a k-star decomposition of K_{4k}. Let $s = 4k-n$ and note that $k \leqslant s < (6-2\sqrt{2})k$ since $k+1 \leqslant n < 2\sqrt{2}k$ and that $\binom{n+s}{2} \equiv 0 \pmod{k}$. By Lemma 14 there is a k-star decomposition of $L \vee K_s$ and hence the result is proved provided that (9) holds. Now (9) holds if and only if

$$\left(3k - \tfrac{1}{2}\right)^2 > \left(n - \sqrt{2k}\right)^2 + k(k-3) + \tfrac{1}{4}$$

and this can in turn be shown to hold using $n < 2\sqrt{2}k$.

Case 3: Suppose that $1 \leqslant n \leqslant k$. Then $\mathcal{D}$ is empty and hence a k-star decomposition of K_{2k}, which exists by Theorem 4(a), is an embedding of $\mathcal{D}$. □

Lemma 18 below shows that if the constant $6-2\sqrt{2}$ in Theorem 1 were decreased then the result would fail to hold for each sufficiently large k that is 2 to some odd power. To see this, observe that the value of n in the statement of Lemma 18 is at most $2\sqrt{k(2k+1)} + 2\sqrt{2k}$ and hence is $2\sqrt{2}k + O(\sqrt{k})$ as k becomes large.

Lemma 18. *Let $k = 2^t$ for some odd integer $t \geqslant 3$, let $m = \sqrt{2k}$, let n be the smallest integer such that $n \equiv 0 \pmod{m}$ and $n > 2\sqrt{k(2k+1)} + \sqrt{2k}$, and let L be a graph of order n that is a vertex disjoint union of $\frac{n}{m}$ copies of K_m. A partial k-star decomposition of K_n whose leave is L exists and furthermore it cannot be embedded in a k-star decomposition of K_{n+s} for any $s < 6k-n$.*

Proof. Observe that $|E(L)| = \frac{n}{m}\binom{m}{2} = \frac{n}{2}(m-1)$. We first show that L is the leave of a partial k-star decomposition of K_n. Note that $\deg_{\overline{L}}(y) = n-m \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ since $n > 2\sqrt{k(2k+1)} + \sqrt{2k}$. Furthermore, $|E(\overline{L})| = \frac{n}{2}(n-m) \equiv 0 \pmod{k}$ because $n \equiv 0 \pmod{m}$. Therefore, by Theorem 8, a k-star decomposition of $\overline{L}$ exists.

Now suppose for a contradiction that a k-star decomposition of $L \vee K_S$ exists where $|S| = s$ for some nonnegative integer $s < 6k-n$. We must have $n+s \equiv 0 \pmod{2k}$ or $n+s \equiv 1 \pmod{2k}$ by Theorem 4(b). Therefore, because $0 \leqslant s < 6k-n$ and $n > 2k+1$, we have $s \in \{4k-n, 4k-n+1\}$.

Now $\alpha(L) = \frac{n}{m}$ because an independent set in L can contain at most one vertex from each copy of K_m. So we complete the proof by showing that $n+s-\frac{1}{k}(|E(L)|+ns+\binom{s}{2}) > \frac{n}{m}$ and hence concluding by Lemma 10 that there is no k-star decomposition of $L \vee K_S$. Using $|E(L)| = \frac{n}{2}(m-1)$ and $m = \sqrt{2k}$ and multiplying through by $2k$, this is equivalent to showing that

$$n\left(2k - 2\sqrt{2k} + 1\right) - s(s + 2n - 2k - 1) \tag{11}$$

is positive. Using $s \leqslant 4k - n + 1$, (11) is at least $n(n - 2\sqrt{2k}) - 2k(4k+1)$. In turn this can be shown to be positive using $n > 2\sqrt{k(2k+1)} + \sqrt{2k}$. □

In order to prove Theorem 1 when k is odd, we need to make a closer examination of leaves of partial k-star decompositions of K_n where $k < n \leqslant 2k$. It turns out that these leaves must contain a large clique and hence we can improve on the bound of Theorem 13(b) for their independence number using Theorem 13(a). Our first step is to improve on Theorem 13(b) in the case where the graph considered contains a large clique.

Lemma 19. *If L is a graph of order n such that L has a copy of K_r as a subgraph and $|E(L)| \leqslant \frac{1}{2}n(r-1)$, then*

$$\alpha(L) \geqslant 1 + \frac{(n-r)^2}{2|E(L)| + n - r^2}.$$

Proof. Let $V = V(L)$ and R be a subset of V such that $L[R]$ is a copy of K_r. Let $d = \frac{2|E(L)| - r(r-1)}{n-r}$ and note that $d \leqslant r-1$ since $|E(L)| \leqslant \frac{1}{2}n(r-1)$. By Theorem 13(a) we have that

$$\alpha(L) \geqslant \sum_{x \in V} \frac{1}{\deg_L(x) + 1}. \tag{12}$$

Observe that $\deg_L(x) \geqslant r-1$ for $x \in R$, that $|R| = r$, that $\sum_{x\in V} \deg_L(x) = 2|E(L)|$, and that $d \leqslant r-1$. By convexity, the minimum value of $\sum_{i=1}^{n} \frac{1}{x_i+1}$, where the x_i are nonnegative reals subject to the constraints $x_i \geqslant r-1$ for $i \in \{1,\dots,r\}$ and $\sum_{i=1}^{n} x_i = 2|E(L)|$, occurs when $x_i = r-1$ for each $i \in \{1,\dots,r\}$ and $x_i = d$ for each $i \in \{r+1,\dots,n\}$. Thus from (12) we have

$$\alpha(L) \geqslant \frac{r}{(r-1)+1} + \frac{n-r}{d+1} = 1 + \frac{(n-r)^2}{2|E(L)| + n - r^2}. \qquad \square$$

By combining Lemma 19 with Lemmas 11 and 12, we can improve on Lemma 14 in the special case where L is the leave of a partial k-star decomposition of K_n and $k < n \leqslant 2k$. Again the $k = 2$ case is covered by Lemma 5.

Lemma 20. *Let k, n and s be integers such that $s \geqslant k \geqslant 3$, $2k \geqslant n > k$ and $\binom{n+s}{2} \equiv 0 \pmod{k}$. Any partial k-star decomposition of K_n can be embedded in a k-star* *decomposition of K_{n+s} if*

$$s > k - n + \frac{1}{2} + \sqrt{4k\left(\sqrt{n-k} - \tfrac{1}{\sqrt{2}}\right)^2 + k(k-3) + \tfrac{1}{4}} \tag{13}$$

Proof. Observe that the right hand side of (13) is real because $k \geqslant 3$. Suppose that (13) holds. Let $\mathcal{D}$ be a partial k-star decomposition of K_n. We may assume that $\mathcal{D}$ is maximal for otherwise we can greedily add k-stars to $\mathcal{D}$ until it is maximal and then apply the proof. Let L be the leave of $\mathcal{D}$ and note that L has maximum degree at most $k-1$. Let $b = \frac{1}{k}|E(L \vee K_s)|$, note that b is an integer because $|E(L \vee K_s)| \equiv 0 \pmod{k}$ since $\binom{n+s}{2} \equiv 0 \pmod{k}$ and L is the leave of a partial k-star decomposition of K_n. If $b \geqslant n+s$, then a k-star decomposition of $L \vee K_s$ exists by Lemma 12, so we may assume that $b < n+s$. By Lemma 11 it suffices to show that $\alpha(L) \geqslant n+s-b$.

Let V_0 be the set of vertices in $V(L)$ that have no star in $\mathcal{D}$ centred at them. No star in $\mathcal{D}$ can contain an edge between a pair of vertices in V_0 and hence $L[V_0]$ must be a complete graph. Because $\mathcal{D}$ contains $\frac{1}{k}(\binom{n}{2} - |E(L)|)$ stars, $|V_0| \geqslant r$ where $r = n - \frac{1}{k}(\binom{n}{2} - e)$ and $e = |E(L)|$. Note that $r \geqslant 1$ since $k \geqslant \frac{n}{2}$ from our hypotheses. So L contains a copy of K_r as a subgraph. Also, it follows from the definition of r that $e = \binom{n}{2} - k(n-r)$ and hence, because $k \geqslant \frac{n}{2}$, that $e \leqslant \frac{1}{2}n(r-1)$. Thus, by Lemma 19 we have $\alpha(L) \geqslant 1 + \frac{(n-r)^2}{2e+n-r^2}$.

So, because $\alpha(L)$ and $n+s-b$ are both integers, it is enough to show that $1+\frac{(n-r)^2}{2e+n-r^2} > n+s-b-1$. Using $b = \frac{1}{k}(e + ns + \binom{s}{2})$ and multiplying through by $2k$, this is equivalent to showing that

$$s^2 + (2n-2k-1)s - 2kn + 4k + 2e + \frac{2k(n-r)^2}{2e+n-r^2} \tag{14}$$

is positive. Using $e = \binom{n}{2} - k(n-r)$, (14) is equal to

$$s^2 + (2n-2k-1)s - (4k-n)(n-1) + 2k\left(r + \frac{n-r}{n+r-2k}\right). \tag{15}$$

Because L contains a copy of K_r as a subgraph, we have that $e \geqslant \binom{r}{2}$ or equivalently, using $e = \binom{n}{2} - k(n-r)$, that $\frac{1}{2}(n-r)(n+r-2k-1) \geqslant 0$. This implies that $2k+1-n \leqslant r \leqslant n$. Considered as a function of a real variable r where $2k+1-n \leqslant r \leqslant n$, (15) is minimised when $r = 2k-n+\sqrt{2n-2k}$ and, substituting this value for r and rearranging, we have that (15) is at least

$$s^2 + (2n-2k-1)s - (6k-n)(n-1) + 4k\left(k + \sqrt{2n-2k} - 1\right).$$

Considering this last expression as a quadratic in s, we can see that it is positive when (13) holds. Thus (14) is positive and $\alpha(L) \geqslant n+s-b$, as required. $\square$

We now finish the proof of Theorem 1 by considering the case where k is odd.

Proof of Theorem 1. When k is even the result follows from Lemma 17, so we may assume that k is odd. Let $\mathcal{D}$ be a partial k-star decomposition of K_n and L be its leave. Note that we will have $|E(L \vee K_s)| \equiv 0 \pmod{k}$ for any integer s such that $n+s \equiv 0 \pmod{k}$. We consider four cases according to the value of n.

Case 1: Suppose that $n \geqslant 2\sqrt{2}k$. Let s be an integer such that $\frac{5}{4}k \leqslant s < \frac{9}{4}k$ and $n+s \equiv 0 \pmod{k}$. We saw in Case 1 of the proof of Lemma 17 that the right hand side of (9) is less than $(4-2\sqrt{2})k$ when $n \geqslant 2\sqrt{2}k$. So by Lemma 14 there is a k-star decomposition of $L \vee K_s$ and hence the result is proved, because $s \geqslant \frac{5}{4}k > (4-2\sqrt{2})k$.

Case 2: Suppose that $\frac{7}{4}k < n < 2\sqrt{2}k$. We show that we can embed $\mathcal{D}$ in a k-star decomposition of K_{4k}. Let $s = 4k-n$ and note that $k \leqslant s < \frac{9}{4}k$ since $\frac{7}{4}k < n < 2\sqrt{2}k$ and that $\binom{n+s}{2} \equiv 0 \pmod{k}$. We showed in Case 2 of the proof of Lemma 17 that (9) holds when $s = 4k-n$ and $n < 2\sqrt{2}k$. So by Lemma 14 there is a k-star decomposition of $L \vee K_s$.

Case 3: Suppose that $k+1 \leqslant n \leqslant \frac{7}{4}k$. We show that we can embed $\mathcal{D}$ in a k-star decomposition of K_{3k}. Let $s = 3k-n$ and note that $k \leqslant s < \frac{9}{4}k$ since $k+1 \leqslant n \leqslant \frac{7}{4}k$ and that $\binom{n+s}{2} \equiv 0 \pmod{k}$. Then (13) holds if and only if

$$\left(2k - \tfrac{1}{2}\right)^2 > 4k\left(n - \sqrt{2n-2k}\right) - k(3k+1) + \tfrac{1}{4}. \tag{16}$$

For $n \geqslant k+1$, the right hand side of (16) is increasing in n and hence (16) can be shown to hold for $k+1 \leqslant n \leqslant \frac{7}{4}k$ by substituting $n = \frac{7}{4}k$. So by Lemma 20 there is a k-star decomposition of $L \vee K_s$.

Case 4: Suppose that $1 \leqslant n \leqslant k$. Then $\mathcal{D}$ is empty and hence a k-star decomposition of K_{2k}, which exists by Theorem 4(a), is an embedding of $\mathcal{D}$. □

Finally, we prove Lemma 21, which shows that if the constant $\frac{9}{4}$ in Theorem 1 were decreased then the result would fail to hold for each sufficiently large k that is a power of an odd prime. To see this, observe that the definition of n in the statement of Lemma 21 can be rephrased as $n = \frac{1}{2}a + k$ where a is the smallest even perfect square that is greater than $\frac{3}{2}k + \sqrt{6k+6} + \frac{5}{2}$. Clearly then, $a = \frac{3}{2}k + O(\sqrt{k})$ and hence $n = \frac{7}{4}k + O(\sqrt{k})$ as k becomes large.

Lemma 21. *Let k be a sufficiently large integer that is a power of an odd prime and let n be the smallest integer such that $n > \frac{7}{4}k + \frac{1}{2}\sqrt{6k+6} + \frac{5}{4}$ and $\sqrt{2n-2k}$ is an integer. Let $m = \sqrt{2n-2k}$ and $r = 2k-n+m$, and let L be a graph of order n that is a vertex disjoint union of $m-1$ copies of K_m and a copy of K_r. A partial k-star decomposition of K_n whose leave is L exists and furthermore it has no embedding in a k-star decomposition of K_{n+s} for any $s < 4k-n$.*

Proof. Observe that, for sufficiently large k, $r = \frac{k}{4} + O(\sqrt{k})$ because $n = \frac{7}{4}k + O(\sqrt{k})$ as noted in the paragraph before the lemma. We first show that L is the leave of a partial k-star decomposition. Let V_0 be the vertex set of the copy of K_r in L and let $V_1, \ldots, V_{m-1}$ be the vertex sets of the copies of K_m in L. Let $\gamma : V(L) \to \mathbb{Z}^{\geqslant 0}$ be defined by $\gamma(x) = 0$ for each $x \in V_0$ and $\gamma(y) = 1$ for each $y \in V(L) \setminus V_0$. Then γ is a precentral function for $\overline{L}$, because we have $\frac{1}{k}(\binom{n}{2} - |E(L)|) = m(m-1)$ using $|E(L)| = \binom{r}{2} + (m-1)\binom{m}{2}$, the definition of r and $n = \frac{1}{2}m^2 + k$. Let $\mathcal{G}$ be $\overline{L}$ equipped with γ and let $T \in \mathcal{T}$. We will show that $\Delta_T = 0$ and hence that a k-star decomposition of $\overline{L}$ exists. For each $i \in \{1, \ldots, m-1\}$, we have $V_i \subseteq T$ or $T \cap V_i = \emptyset$ by Lemma 7 with $U = V_i$. So without loss of generality we can assume that $T = V_1 \cup \cdots \cup V_t$ for some $t \in \{0, \ldots, m-1\}$. Then $\Delta_T^+ = \binom{t}{2}m^2 + mt(n - mt)$ and $\Delta_T^- = kmt$. Thus, using $n = \frac{1}{2}m^2 + k$ and simplifying,

$$\Delta_T = \tfrac{1}{2}tm^2(m - 1 - t)$$

which is nonnegative since $t \in \{0, \ldots, m-1\}$. Thus $\Delta_T = 0$ and a k-star decomposition of $\overline{L}$ exists.

Now suppose for a contradiction that a k-star decomposition of $L \vee K_S$ exists where $|S| = s$ for some nonnegative integer $s < 4k - n$. We must have $n + s \equiv 0 \pmod{k}$ or $n + s \equiv 1 \pmod{k}$ by Theorem 4(c). Therefore, because $0 \leqslant s < 4k - n$ and $n > k + 1$, we have $s \in \{2k - n, 2k - n + 1, 3k - n, 3k - n + 1\}$.

Now $\alpha(L) = m$ because an independent set in L can contain at most one vertex from the copy of K_r and at most one vertex from each copy of K_m. So we complete the proof by showing that $n + s - \frac{1}{k}(|E(L)| + ns + \binom{s}{2}) > m$ and hence concluding by Lemma 10 that there is no k-star decomposition of $L \vee K_s$. Using $|E(L)| = \binom{r}{2} + (m-1)\binom{m}{2}$, the definitions of r and m, and multiplying through by $2k$, this is equivalent to showing that

$$n(6k - n + 1) - 4k\left(k + \sqrt{2n - 2k}\right) - s(s + 2n - 2k - 1) \tag{17}$$

is positive. Using $s \leqslant 3k - n + 1$, (17) is at least $k(4n - 7k - 4\sqrt{2n - 2k} - 1)$. In turn this can be shown to be positive using $n > \frac{7}{4}k + \frac{1}{2}\sqrt{6k + 6} + \frac{5}{4}$. □

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