Spectral extremal graphs for disjoint cliques

Zhenyu Ni*

Department of Mathematics
Hainan University
Haikou 570228, P.R. China

1051466287@qq.com

Jing Wang

College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, P.R. China

wj517062214@163.com

Liying Kang†

Department of Mathematics
Shanghai University
Shanghai 200444, PR China

lykang@shu.edu.cn

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Abstract

Let kK_{r+1} be the graph consisting of k vertex-disjoint copies of the complete graph K_{r+1}. Moon [Canad. J. Math. 20 (1968) 95–102] and Simonovits [Theory of Graphs (Proc. colloq., Tihany, 1996)] independently showed that if n is sufficiently large, then the join of a complete graph K_{k-1} and an r-partite Turán graph $T_{n-k+1,r}$ is the unique extremal graph for kK_{r+1}. In this paper we consider the graph which has the maximum spectral radius among all graphs without k disjoint cliques. We show that if G attains the maximum spectral radius over all n-vertex kK_{r+1}-free graphs for sufficiently large n, then G is isomorphic to the join of a complete graph K_{k-1} and an r-partite Turán graph $T_{n-k+1,r}$.

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1 Introduction

In this paper, we consider only simple and undirected graphs. For two vertex disjoint graphs G, H, the *union* of graph G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. In particular, we write kG the vertex-disjoint union of k copies of G. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding edges joining every vertex of G to every vertex of H. For two graphs G and F, G is called *F-free* if it does not contain a copy of F as a subgraph. For a fixed graph F, the Turán type extremal problem is to determine the maximum number of edges among all n-vertex F-free graphs, where the maximum number of edges is called the *Turán number*, denoted by $\mathrm{ex}(n, F)$. An F-free graph on n vertices is called an *extremal graph* for F if it has $\mathrm{ex}(n, F)$ edges, and the set of all extremal graphs is denoted by $\mathrm{Ex}(n, F)$.

Let $K_r(n_1, \ldots, n_r)$ be the complete r-partite graph with classes of sizes $n_1, \ldots, n_r$. If $\sum_{i=1}^{r} n_i = n$ and $|n_i - n_j| \leqslant 1$ for any $1 \leqslant i < j \leqslant r$, then $K_r(n_1, \ldots, n_r)$ is called an r-partite *Turán graph*, denoted by $T_{n,r}$. The well-known Turán Theorem states that the extremal graph corresponding to Turán number $\mathrm{ex}(n, K_{r+1})$ is $T_{n,r}$, i.e. $\mathrm{ex}(n, K_{r+1}) = |E(T_{n,r})|$. There are lots of researches on Turán type extremal problems (such as [3, 5, 10, 21]). Simonovits [20] and Moon [14] showed that if n is sufficiently large, then $K_{k-1} \vee T_{n-k+1,r}$ is the unique extremal graph for kK_{r+1}.

Theorem 1 ([20, 14])**.** *Let G be a graph of sufficiently large order n that does not contain kK_{r+1} as a subgraph. Then $e(G) \leqslant e(K_{k-1} \vee T_{n-k+1,r})$, and $K_{k-1} \vee T_{n-k+1,r}$ is the unique extremal graph for kK_{r+1}.*

The following spectral version of the Turán type problem was proposed in Nikiforov [19]: What is the maximum spectral radius of a graph G on n vertices without a subgraph isomorphic to a given graph F? Researches of the spectral Turán type extremal problem have drawn increasingly extensive interest (for example, see [16, 2, 15, 23, 24, 25]). Nikiforov [17] showed that if G is a K_{r+1}-free graph on n vertices, then $\rho(G) \leqslant \rho(T_{n,r})$, with equality if and only if $G = T_{n,r}$. Cioabă et al. [8] proved that the spectral extremal graphs for F_k belong to $\mathrm{Ex}(n, F_k)$, where F_k is the graph consisting of k triangles which intersect in exactly one common vertex. The family $\mathrm{Ex}(n, F_k)$ was uniquely determined for sufficiently large n by Zhai, Liu and Xue [26]. Desai et al. [9] generalized the result of [8] to $F_{k,r}$, where $F_{k,r}$ is the graph consisting of k copies of K_r which intersect in a single vertex. Cioabă et al. [7] investigated the largest spectral radius of an n-vertex graph that does not contain the odd-wheel graph W_{2k+1}. Moreover, they raised the following conjecture.

Conjecture 2 ([7])**.** Let F be any graph such that the graphs in $\mathrm{Ex}(n, F)$ are Turán graphs plus $O(1)$ edges. Then for sufficiently large n, a graph attaining the maximum spectral radius among all F-free graphs on n vertices is a member of $\mathrm{Ex}(n, F)$.

The results of Nikiforov [17], Cioabă et al. [8], Desai et al. [9] and Li et al. [13] tell us that Conjecture 2 holds for K_{r+1}, F_k, $F_{k,r}$ and $H_{s,k}$, where $H_{s,k}$ is the graph defined by intersecting s triangles and k odd cycles of length at least 5 in exactly one common vertex. Recently, Wang et al. [22] proved Conjecture 2 completely.

In this paper, we shall prove the following theorem.

Theorem 3. *For $k \geqslant 2$, $r \geqslant 2$, and sufficiently large n. Suppose that G has the maximum spectral radius among all kK_{r+1}-free graphs on n vertices, then G is isomorphic to $K_{k-1} \vee T_{n-k+1,r}$.*

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, $N(v)$ is the set of neighbors of v in G. The *degree* $d(v)$ of v is $|N(v)|$, and the minimum and maximum degrees are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We denote by $e(G)$ the number of edges in G. For $V_1, V_2 \subseteq V(G)$, $E(V_1, V_2)$ denotes the set of edges of G between V_1 and V_2, and $e(V_1, V_2) = |E(V_1, V_2)|$. For any $S \subseteq V(G)$, we write $N(S) = \cup_{u \in S} N(u)$, $d_S(v) = |N_S(v)| = |N(v) \cap S|$. Denote by $G \setminus S$ the graph obtained from G by deleting all vertices in S and their incident edges. $G[S]$ denotes the graph induced by S whose vertex set is S and whose edge set consists of all edges of G which have both ends in S. A set M of disjoint edges of G is called a *matching* in G. The *matching number*, denoted by $\nu(G)$, is the maximum cardinality of a matching in G. We call a matching with k edges a *k-matching*, denoted by M_k. For a matching M of G, each vertex incident with an edge of M is said to be *covered* by M.

The *adjacent matrix* of G is $A(G) = (a_{ij})_{n \times n}$ with $a_{ij} = 1$ if $ij \in E(G)$, and $a_{ij} = 0$ otherwise. The *spectral radius* of G is the largest eigenvalue of $A(G)$, denoted by $\rho(G)$. For a connected graph G on n vertices, let $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ be an eigenvector of $A(G)$ corresponding to $\rho(G)$. Then $\mathbf{x}$ is a positive real vector, and

$$\rho(G) x_i = \sum_{ij \in E(G)} x_j, \text{ for any } i \in [n]. \tag{1}$$

Another useful result concerns the Rayleigh quotient:

$$\rho(G) = \max_{\mathbf{x} \in \mathbb{R}^n_+} \frac{\mathbf{x}^{\mathrm{T}} A(G) \mathbf{x}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}} = \max_{\mathbf{x} \in \mathbb{R}^n_+} \frac{2 \sum_{ij \in E(G)} x_i x_j}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}. \tag{2}$$

The following spectral version of Stability Theorem was given by Nikiforov [18].

Theorem 4 ([18])**.** *Let $r \geqslant 2, 1/\ln n < c < r^{-8(r+21)(r+1)}, 0 < \varepsilon < 2^{-36} r^{-24}$ and G be a graph on n vertices. If $\rho(G) > (1 - \frac{1}{r} - \varepsilon)n$, then one of the following statements holds:*
(a) G contains a $K_{r+1}(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-\sqrt{c}} \rceil)$;
(b) G differs from $T_{n,r}$ in fewer than $(\varepsilon^{1/4} + c^{1/(8r+8)})n^2$ edges.

From the above theorem, we can get the following result.

Lemma 5 ([9]). *Let F be a graph with chromatic number $\chi(F) = r+1$. For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that if G is an F-free graph on $n \geqslant n_0$ vertices with $\rho(G) \geqslant (1-\frac{1}{r}-\delta)n$, then G can be obtained from $T_{n,r}$ by adding and deleting at most εn^2 edges.*

Let G be a simple graph with matching number $\nu(G)$ and maximum degree $\Delta(G)$. For two given integers ν and Δ, define $f(\nu,\Delta) = \max\{e(G) : \nu(G) \leqslant \nu, \Delta(G) \leqslant \Delta\}$. In 1976, Chvátal and Hanson [6] obtained the following result.

Lemma 6 ([6]). *For every two integers $\nu \geqslant 1$ and $\Delta \geqslant 1$, we have*

$$f(\nu,\Delta) = \Delta\nu + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor \leqslant \Delta\nu + \nu.$$

The following lemma was given in [8].

Lemma 7 ([8]). *Let $V_1,\ldots,V_n$ be n finite sets. Then*

$$|V_1 \cap \cdots \cap V_n| \geqslant \sum_{i=1}^{n} |V_i| - (n-1)|\bigcup_{i=1}^{n} V_i|.$$

3 Proof of Theorem 3

In this section we shall give a proof of Theorem 3. Suppose that G has the maximum spectral radius among all kK_{r+1}-free graphs on n vertices, then we will prove G is isomorphic to $K_{k-1} \vee T_{n-k+1,r}$ for sufficiently large n. Clearly, G is connected. Let $\rho(G)$ be the spectral radius of G, $\mathbf{x}$ be a positive eigenvector of $\rho(G)$ with $\max\{x_i : i \in V(G)\} = 1$. Without loss of generality, we assume $x_z = 1$.

Lemma 8. *Let G be a kK_{r+1}-free graph on n vertices with maximum spectral radius. Then*

$$\rho(G) \geqslant \frac{r-1}{r}n + \frac{2(k-1)}{r} - \frac{1}{n}\left(\frac{(k-1)(r+k-1)}{r} + \frac{r}{2}\right).$$

Proof. Let $H = K_{k-1} \vee T_{n-k+1,r}$. Since $K_{k-1} \vee T_{n-k+1,r}$ is the unique extremal graph for kK_{r+1}, then

$$\begin{aligned} \mathrm{ex}(n, kK_{r+1}) &= e(T_{n-k+1,r}) + (k-1)(n-k+1) + \binom{k-1}{2} \\ &\geqslant e(T_{n,r}) + \frac{k-1}{r}n - \frac{(k-1)(r+k-1)}{2r} - \frac{r}{8}. \end{aligned} \tag{3}$$

According to (2) and (3),we have

$$\rho(G) \geqslant \rho(H) \geqslant \frac{\mathbf{1}^{\mathrm{T}} A(H)\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{1}} = \frac{2\mathrm{ex}(n, kK_{r+1})}{n}$$

$$\geqslant \frac{2}{n}\left(e(T_{n,r}) + \frac{k-1}{r}n - \frac{(k-1)(r+k-1)}{2r} - \frac{r}{8}\right)$$
$$\geqslant \frac{r-1}{r}n + \frac{2(k-1)}{r} - \frac{1}{n}\left(\frac{(k-1)(r+k-1)}{r} + \frac{r}{2}\right).$$

□

Lemma 9. *Let G be a kK_{r+1}-free graph on n vertices with maximum spectral radius. For every $\varepsilon > 0$, there is an integer n_0 such that if $n \geqslant n_0$, then*

$$e(G) \geqslant e(T_{n,r}) - \varepsilon n^2.$$

Furthermore, G has a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that the number of crossing edges of G (i.e. $\sum_{1\leqslant i<j\leqslant r} e(V_i, V_j)$) attains the maximum, and

$$\sum_{i=1}^{r} e(V_i) \leqslant \varepsilon n^2,$$

and for any $i \in [r]$

$$\frac{n}{r} - 3\sqrt{\varepsilon}n < |V_i| < \frac{n}{r} + 3\sqrt{\varepsilon}n.$$

Proof. Since G is kK_{r+1}-free, by Lemmas 5 and 8, for sufficiently large n, there exists a partition of $V(G) = U_1 \cup \cdots \cup U_r$ such that $e(G) \geqslant e(T_{n,r}) - \varepsilon n^2$, $\sum_{i=1}^{r} e(U_i) \leqslant \varepsilon n^2$, and $\lfloor \frac{n}{r} \rfloor \leqslant |U_i| \leqslant \lceil \frac{n}{r} \rceil$ for each $i \in [r]$. Therefore, G has a partition $V(G) = V_1 \cup \ldots \cup V_r$ such that the number of crossing edges of G attains the maximum, and

$$\sum_{i=1}^{r} e(V_i) \leqslant \sum_{i=1}^{r} e(U_i) \leqslant \varepsilon n^2.$$

Let $a = \max\left\{\left||V_j| - \frac{n}{r}\right|, j \in [r]\right\}$. Without loss of generality, we may assume that $\left||V_1| - \frac{n}{r}\right| = a$. Then

$$\begin{aligned}
e(G) &\leqslant \sum_{1\leqslant i<j\leqslant r} |V_i||V_j| + \sum_{i=1}^{r} e(V_i) \\
&\leqslant |V_1|(n - |V_1|) + \sum_{2\leqslant i<j\leqslant r} |V_i||V_j| + \varepsilon n^2 \\
&= |V_1|(n - |V_1|) + \frac{1}{2}\Big((\sum_{j=2}^{r} |V_j|)^2 - \sum_{j=2}^{r} |V_j|^2\Big) + \varepsilon n^2 \\
&\leqslant |V_1|(n - |V_1|) + \frac{1}{2}(n - |V_1|)^2 - \frac{1}{2(r-1)}(n - |V_1|)^2 + \varepsilon n^2 \\
&< -\frac{r}{2(r-1)}a^2 + \frac{r-1}{2r}n^2 + \varepsilon n^2,
\end{aligned}$$

where the last second inequality holds by Hölder's inequality, and the last inequality holds since $\left||V_1| - \frac{n}{r}\right| = a$. On the other hand, since $e(G) \geqslant e(T_{n,r}) - \varepsilon n^2$, we have

$$e(G) \geqslant e(T_{n,r}) - \varepsilon n^2 \geqslant \frac{r-1}{2r}n^2 - \frac{r}{8} - \varepsilon n^2 > \frac{r-1}{2r}n^2 - 2\varepsilon n^2.$$

Therefore, $\frac{r}{2(r-1)}a^2 < 3\varepsilon n^2$, which implies that $a < \sqrt{\frac{6(r-1)\varepsilon}{r}n^2} < 3\sqrt{\varepsilon}n$. The proof is completed. □

Lemma 10. *Suppose ε and θ are two sufficiently small constants with $\theta < \frac{1}{20kr^4(r+1)}$ and $\varepsilon \leqslant \theta^2$. Let*

$$W := \cup_{i=1}^{r}\{v \in V_i : d_{V_i}(v) \geqslant 2\theta n\}.$$

Then $|W| \leqslant \theta n$.

Proof. For all $i \in [r]$, let $W_i = W \cap V_i$. Then

$$2e(V_i) = \sum_{u\in V_i} d_{V_i}(u) \geqslant \sum_{u\in W_i} d_{V_i}(u) \geqslant 2|W_i|\theta n.$$

Combining with Lemma 9, we have

$$\varepsilon n^2 \geqslant \sum_{i=1}^{r} e(V_i) \geqslant |W|\theta n,$$

which implies that $|W| \leqslant \frac{\varepsilon n}{\theta} \leqslant \theta n$. □

Lemma 11. *Suppose ε_1 is a sufficiently small constant with $\sqrt{\varepsilon} < \varepsilon_1 \ll \theta$. Let*

$$L := \{v \in V(G) : d(v) \leqslant (1 - \frac{1}{r} - \varepsilon_1)n\}.$$

Then $|L| \leqslant \varepsilon_2 n$, where $\varepsilon_2 \ll \varepsilon_1$ is a sufficiently small constant satisfying $\varepsilon - \varepsilon_1\varepsilon_2 + \frac{r-1}{2r}\varepsilon_2^2 < 0$.

Proof. Suppose to the contrary that $|L| > \varepsilon_2 n$, then there exists $L' \subseteq L$ with $|L'| = \lfloor \varepsilon_2 n \rfloor$. Therefore,

$$\begin{aligned} e(G \setminus L') &\geqslant e(G) - \sum_{v\in L'} d(v) \\ &\geqslant e(T_{n,r}) - \varepsilon n^2 - \varepsilon_2 n(1 - \frac{1}{r} - \varepsilon_1)n \\ &= e(T_{n,r}) - \varepsilon n^2 - \frac{r-1}{r}\varepsilon_2 n^2 + \varepsilon_1\varepsilon_2 n^2 \\ &> \frac{r-1}{2r}(n - \lfloor \varepsilon_2 n \rfloor)^2 + \frac{k-1}{r}(n - \lfloor \varepsilon_2 n \rfloor) - \frac{(k-1)(k+r-1)}{2r} \end{aligned}$$

$$\begin{aligned} &\geqslant e(T_{n',r}) + \frac{(k-1)n'}{r} - \frac{(k-1)(k+r-1)}{2r} \\ &= \mathrm{ex}(n', kK_{r+1}), \end{aligned}$$

where $n' = n - \lfloor \varepsilon_2 n \rfloor$. Since $e(G \setminus L') > \mathrm{ex}(n - |L'|, kK_{r+1})$, $G \setminus L'$ contains a kK_{r+1} as subgraph. This contradicts the fact that G is kK_{r+1}-free. $\square$

Lemma 12. *For any $i \in [r]$, if uv is an edge of $G[V_i \setminus (W \cup L)]$, then G has $k(r+1)$ copies of K_{r+1} which have only one common edge uv.*

Proof. For any $i \in [r]$, and any vertex $w \in V_i \setminus (W \cup L)$, we have $d(w) > (1 - \frac{1}{r} - \varepsilon_1)n$, $d_{V_i}(w) < 2\theta n$. Then for any $j \in [r]$ and $j \neq i$,

$$\begin{aligned} d_{V_j}(w) &\geqslant d(w) - d_{V_i}(w) - (r-2)(\frac{n}{r} + 3\sqrt{\varepsilon} n) \\ &> (1 - \frac{1}{r} - \varepsilon_1)n - 2\theta n - (r-2)(\frac{n}{r} + 3\sqrt{\varepsilon} n) \\ &> \frac{n}{r} - 3(r-1)\theta n. \end{aligned}$$

Without loss of generality, let uv be an edge of $G[V_1 \setminus (W \cup L)]$. We consider the common neighbors of u, v in $V_2 \setminus (W \cup L)$. Combining with Lemma 7, we have

$$\begin{aligned} &|N_{V_2}(u) \cap N_{V_2}(v) \setminus (W \cup L)| \\ \geqslant\ & d_{V_2}(u) + d_{V_2}(v) - |V_2| - |W| - |L| \\ >\ & 2(\frac{n}{r} - 3(r-1)\theta n) - (\frac{n}{r} + 3\sqrt{\varepsilon} n) - \theta n - \varepsilon_2 n \\ >\ & \frac{n}{r} - 6r\theta n \\ >\ & k(r+1). \end{aligned}$$

So there exist $k(r+1)$ vertices $u_{2,1}, \ldots, u_{2,k(r+1)}$ in $V_2 \setminus (W \cup L)$ such that the subgraph induced by two partitions $\{u, v\}$ and $\{u_{2,1}, \ldots, u_{2,k(r+1)}\}$ is a complete bipartite graph. For an integer s with $2 \leqslant s \leqslant r-1$, suppose that there are vertices $u_{s,1}, \ldots, u_{s,k(r+1)} \in V_s \setminus (W \cup L)$ such that $\{u, v\}$, $\{u_{2,1}, \ldots, u_{2,k(r+1)}\}$, $\ldots$, $\{u_{s,1}, \ldots, u_{s,k(r+1)}\}$ induce a complete s-partite subgraph. We next consider the common neighbors of the above $(s-1)k(r+1)+2$ vertices in $V_{s+1} \setminus (W \cup L)$. By Lemma 7, we have

$$\begin{aligned} &|N_{V_{s+1}}(u) \cap N_{V_{s+1}}(v) \cap (\cap_{i \in [s]\setminus\{1\}, j \in [k(r+1)]} N_{V_{s+1}}(u_{i,j})) \setminus (W \cup L)| \\ \geqslant\ & d_{V_{s+1}}(u) + d_{V_{s+1}}(v) + \sum_{i=2}^{s} \sum_{j=1}^{k(r+1)} d_{V_{s+1}}(u_{i,j}) - ((s-1)k(r+1)+1)|V_{s+1}| - |W| - |L| \\ >\ & ((s-1)k(r+1)+2)\left(\frac{n}{r} - 3(r-1)\theta n\right) - ((s-1)k(r+1)+1)\left(\frac{n}{r} + 3\sqrt{\varepsilon} n\right) \\ & -\theta n - \varepsilon_2 n \end{aligned}$$

$$
\begin{aligned}
&> \frac{n}{r} - 12skr(r+1)\theta n \\
&> k(r+1).
\end{aligned}
$$

Then we can find $k(r+1)$ vertices $u_{s+1,1}, \ldots, u_{s+1,k(r+1)} \in V_{s+1} \setminus (W \cup L)$, which together with $\{u, v\}$, $\{u_{2,1}, \ldots, u_{2,k(r+1)}\}$, $\ldots$, $\{u_{s,1}, \ldots, u_{s,k(r+1)}\}$ forms a complete $(s+1)$-partite subgraph in G. Therefore, for every $2 \leqslant i \leqslant r$, there exist $k(r+1)$ vertices in $V_i \setminus (W \cup L)$ such that $\{u_{2,1}, \ldots, u_{2,k(r+1)}\}$, $\ldots$, $\{u_{r,1}, \ldots, u_{r,k(r+1)}\}$ induce a complete $(r-1)$-partite subgraph in G, and u, v are adjacent to all the above $k(r-1)(r+1)$ vertices. Hence G has $k(r+1)$ copies of K_{r+1} which have only one common edge uv. $\square$

Lemma 13. *For each $i \in [r]$, there exists an independent set $I_i \subseteq V_i \setminus (W \cup L)$ such that $|I_i| \geqslant |V_i \setminus (W \cup L)| - 2(k-1)$.*

Proof. We first claim that $G[V_i \setminus (W \cup L)]$ is M_k-free for any $i \in [r]$. Suppose to the contrary that there exists $i_0 \in [r]$ such that $G[V_{i_0} \setminus (W \cup L)]$ contains a copy of M_k. Then we can find a kK_{r+1} by Lemma 12, and this contradicts the fact that G is kK_{r+1}-free. For every $i \in [r]$, let M^i be a maximum matching of $G[V_i \setminus (W \cup L)]$, and B^i be the set of vertices covered by M^i. Since $G[V_i \setminus (W \cup L)]$ is M_k-free, $|B^i| \leqslant 2(k-1)$. Therefore, there exists an independent set $I_i \subseteq V_i \setminus (W \cup L)$ by deleting all vertices of B^i, and $|I_i| \geqslant |V_i \setminus (W \cup L)| - 2(k-1)$. $\square$

Lemma 14. *For any $i \in [r]$ and any $v \in V_i \setminus (W \cup L)$, $d_{V_i \setminus (W \cup L)}(v) < k(r+1)$.*

Proof. We will prove this lemma by contradiction. Without loss of generality, suppose that there exists a vertex $u \in V_1 \setminus (W \cup L)$ such that $d_{V_1 \setminus (W \cup L)}(u) \geqslant k(r+1)$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{uw : uw \notin E(G)\}$. It follows from $u \in V_1 \setminus (W \cup L)$ that $E(G) \subset E(G')$. By the maximum of $\rho(G)$, G' contains kK_{r+1}, say F_1, as a subgraph. From the construction of G', we see that $u \in V(F_1)$, and there is a $(k-1)K_{r+1}$, say F_2, in $F_1 \setminus \{u\}$. Obviously, $F_2 \subseteq G$. Thus F_2 is a $(k-1)K_{r+1}$ copy of G, and $u \notin V(F_2)$. Since $d_{V_1 \setminus (W \cup L)}(u) \geqslant k(r+1)$, there exists a vertex $v \in N_{V_1 \setminus (W \cup L)}(u)$ such that $v \notin V(F_2)$. Then we can find $k(r+1)$ copies of K_{r+1} which have only one common edge uv by Lemma 12. Thus, we can find a K_{r+1}, say F_3, such that $V(F_3) \cap V(F_2) = \emptyset$. Thus $F_2 \cup F_3$ is a kK_{r+1} copy of G, which contradicts the fact that G is kK_{r+1}-free. $\square$

Lemma 15. *For any $u \in W \setminus L$, G contains $k(r+1)$ copies of K_{r+1} which intersect only in u.*

Proof. For any $u \in W \setminus L$, without loss of generality, we may assume that $u \in V_1$. Combining with Lemmas 10 and 11, we have $d(u) > (1 - \frac{1}{r} - \varepsilon_1)n$, and

$$
\begin{aligned}
d_{V_1 \setminus (W \cup L)}(u) &\geqslant d_{V_1}(u) - |W \cup L| \\
&\geqslant 2\theta n - \theta n - \varepsilon_2 n \\
&> k(r+1).
\end{aligned}
$$

Let $u_{1,1}, \ldots, u_{1,k(r+1)}$ be the neighbors of u in $V_1 \setminus (W \cup L)$. Then for every $i \in [k(r+1)]$, we have $d(u_{1,i}) > (1 - \frac{1}{r} - \varepsilon_1)n$, $d_{V_1}(u_{1,i}) < 2\theta n$, and

$$\begin{aligned} d_{V_2}(u_{1,i}) &\geqslant d(u_{1,i}) - d_{V_1}(u_{1,i}) - (r-2)(\frac{n}{r} + 3\sqrt{\varepsilon} n) \\ &> \frac{n}{r} - \varepsilon_1 n - 2\theta n - 3(r-2)\sqrt{\varepsilon} n \\ &> \frac{n}{r} - 3(r-1)\theta n. \end{aligned} \tag{4}$$

Since $V(G) = V_1 \cup \cdots \cup V_r$ is the vertex partition that maximizes the number of crossing edges of G, we have $d_{V_1}(u) \leqslant \frac{1}{r} d(u)$. Therefore

$$\begin{aligned} d_{V_2}(u) &\geqslant d(u) - d_{V_1}(u) - (r-2)(\frac{n}{r} + 3\sqrt{\varepsilon} n) \\ &> \frac{r-1}{r}(1 - \frac{1}{r} - \varepsilon_1)n - (r-2)(\frac{n}{r} + 3\sqrt{\varepsilon} n) \\ &> \frac{n}{r^2} - \varepsilon_1 n - 3(r-2)\sqrt{\varepsilon} n \\ &> \frac{n}{r^2} - (3r+5)\varepsilon_1 n. \end{aligned} \tag{5}$$

We consider the common neighbors of $u, u_{1,1}, \ldots, u_{1,k(r+1)}$ in $V_2 \setminus (W \cup L)$. Combining with Lemma 7, we have

$$\begin{aligned} & |N_{V_2}(u) \cap (\cap_{i \in [k(r+1)]} N_{V_2}(u_{1,i})) \setminus (W \cup L)| \\ \geqslant\ & d_{V_2}(u) + \sum_{i=1}^{k(r+1)} d_{V_2}(u_{1,i}) - k(r+1)|V_2| - |W| - |L| \\ >\ & \frac{n}{r^2} - (3r+5)\varepsilon_1 n + k(r+1)(\frac{n}{r} - 3(r-1)\theta n) - k(r+1)(\frac{n}{r} + 3\sqrt{\varepsilon} n) - \theta n - \varepsilon_2 n \\ >\ & \frac{n}{r^2} - 16kr(r+1)\theta n \\ >\ & k(r+1). \end{aligned}$$

Let $u_{2,1}, \ldots, u_{2,k(r+1)}$ be the common neighbors of $u, u_{1,1}, \ldots, u_{1,k(r+1)}$ in $V_2 \setminus (W \cup L)$. For an integer $2 \leqslant s \leqslant r-1$, suppose that $u_{s,1}, \ldots, u_{s,k(r+1)}$ are the common neighbors of $\{u, u_{i,1}, \ldots, u_{i,k(r+1)} : 1 \leqslant i \leqslant s-1\}$ in $V_s \setminus (W \cup L)$. We next consider the common neighbors of $\{u, u_{i,1}, \ldots, u_{i,k(r+1)} : 1 \leqslant i \leqslant s\}$ in $V_{s+1} \setminus (W \cup L)$. Using the similar method as in the proof of (4) and (5), for every $i \in [s]$ and $j \in [k(r+1)]$, we have

$$d_{V_{s+1}}(u_{i,j}) > \frac{n}{r} - 3(r-1)\theta n,$$

and

$$d_{V_{s+1}}(u) > \frac{n}{r^2} - (3r+5)\varepsilon_1 n.$$

By Lemma 7, we have

$$\begin{aligned}
& |N_{V_{s+1}}(u) \cap (\cap_{i\in[s],j\in[k(r+1)]} N_{V_{s+1}}(u_{i,j})) \setminus (W \cup L)| \\
\geqslant\ & d_{V_{s+1}}(u) + \sum_{i=1}^{s}\sum_{j=1}^{k(r+1)} d_{V_{s+1}}(u_{i,j}) - sk(r+1)|V_{s+1}| - |W| - |L| \\
>\ & \frac{n}{r^2} - (3r+5)\varepsilon_1 n + sk(r+1)\left(\frac{n}{r} - 3(r-1)\theta n\right) - sk(r+1)\left(\frac{n}{r} + 3\sqrt{\varepsilon} n\right) \\
& -\theta n - \varepsilon_2 n \\
>\ & \frac{n}{r^2} - 16skr(r+1)\theta n \\
>\ & k(r+1).
\end{aligned}$$

Let $u_{s+1,1}, \ldots, u_{s+1,k(r+1)}$ be the common neighbors of $\{u, u_{i,1}, \ldots, u_{i,k(r+1)} : 1 \leqslant i \leqslant s\}$ in $V_{s+1} \setminus (W \cup L)$. Therefore, for every $i \in [r]$, there exist $k(r+1)$ vertices, denoted by $\{u_{i,1}, \ldots, u_{i,k(r+1)}\}$, in $V_i \setminus (W \cup L)$ such that $\{u_{1,1}, \ldots, u_{1,k(r+1)}\}$, $\{u_{2,1}, \ldots, u_{2,k(r+1)}\}$, $\ldots$, $\{u_{r,1}, \ldots, u_{r,k(r+1)}\}$ form a complete r-partite subgraph in G, and u is adjacent to the above $kr(r+1)$ vertices. Hence we can find $k(r+1)$ copies of K_{r+1} in G which intersect only in u. □

Lemma 16. $|W \setminus L| \leqslant k - 1$.

Proof. Suppose to the contrary that $|W \setminus L| \geqslant k$. By Lemma 15, for any $u \in W \setminus L$, we can find $k(r+1)$ copies of K_{r+1} in G which intersect only in u. Therefore, we can find at least k disjoint K_{r+1} in G. This is a contradiction to the fact that G is kK_{r+1}-free. □

Lemma 17. $L = \emptyset$.

Proof. Let $x_{v_0} = \max\{x_v : v \in V(G) \setminus W\}$. Recall that $x_z = \max\{x_v : v \in V(G)\} = 1$, then

$$\rho(G) = \rho(G)x_z \leqslant |W| + (n - |W|)x_{v_0}.$$

By Lemmas 11 and 16, we have

$$|W| = |W \cap L| + |W \setminus L| \leqslant |L| + k - 1 \leqslant \varepsilon_2 n + k - 1. \tag{6}$$

Combining with Lemma 8, we have

$$x_{v_0} \geqslant \frac{\rho(G) - |W|}{n - |W|} \geqslant \frac{\rho(G) - |W|}{n} \geqslant 1 - \frac{1}{r} - \varepsilon_2 - \frac{O(1)}{n} > 1 - \frac{2}{r}. \tag{7}$$

Therefore, we have

$$\begin{aligned}
\rho(G)x_{v_0} &= \sum_{vv_0\in E(G)} x_v = \sum_{v\in W, vv_0\in E(G)} x_v + \sum_{v\notin W, vv_0\in E(G)} x_v \\
&\leqslant |W| + (d(v_0) - |W|)x_{v_0},
\end{aligned}$$

which implies that

$$\begin{aligned}
d(v_0) &\geqslant \rho(G)+|W|-\frac{|W|}{x_{v_0}}\\
&\geqslant \rho(G)-\frac{2|W|}{r-2}\\
&\geqslant \frac{r-1}{r}n+\frac{2(k-1)}{r}-\frac{1}{n}\left(\frac{(k-1)(r+k-1)}{r}+\frac{r}{2}\right)-\frac{2\varepsilon_2 n}{r-2}-\frac{2(k-1)}{r-2}\\
&> (1-\frac{1}{r}-\varepsilon_1)n,
\end{aligned}$$

where the last inequality holds as $\varepsilon_2 \ll \varepsilon_1$. Thus we have $v_0 \notin L$, that is $v_0 \in V(G)\setminus(W\cup L)$. Without loss of generality, we assume that $v_0 \in V_1 \setminus (W \cup L)$. Combining with Lemmas 13 and 14, we have

$$\begin{aligned}
\rho(G)x_{v_0} &= \sum_{\substack{v\in W\cup L,\\ vv_0\in E(G)}} x_v+\sum_{\substack{v\in V_1\setminus(W\cup L),\\ vv_0\in E(G)}} x_v+\sum_{\substack{v\in(\cup_{i=2}^r V_i)\setminus(W\cup L),\\ vv_0\in E(G)}} x_v\\
&< |W|+|L|x_{v_0}+k(r+1)x_{v_0}+\sum_{\substack{v\in\cup_{i=2}^r I_i,\\ vv_0\in E(G)}} x_v+\sum_{\substack{v\in(\cup_{i=2}^r V_i\setminus I_i)\setminus(W\cup L),\\ vv_0\in E(G)}} x_v\\
&\leqslant |W|+|L|x_{v_0}+k(r+1)x_{v_0}+2(k-1)(r-1)x_{v_0}+\sum_{v\in\cup_{i=2}^r I_i} x_v,
\end{aligned}$$

which implies that

$$\sum_{v\in\cup_{i=2}^r I_i} x_v \geqslant (\rho(G)-|L|-k(3r-1)+2(r-1))x_{v_0}-|W|. \tag{8}$$

Next we will prove $L=\emptyset$. Suppose to the contrary that there is a vertex $u_0 \in L$, then $d(u_0) \leqslant (1-\frac{1}{r}-\varepsilon_1)n$. Let G' be the graph with $V(G')=V(G)$ and $E(G')=E(G\setminus\{u_0\})\cup\{wu_0 : w\in\cup_{i=2}^r I_i\}$. It is obvious that G' is kK_{r+1}-free. Combining with Lemmas 8, 11, (6), (7) and (8), we have

$$\begin{aligned}
\rho(G')-\rho(G) &\geqslant \frac{\mathbf{x}^T\left(A(G')-A(G)\right)\mathbf{x}}{\mathbf{x}^T\mathbf{x}}=\frac{2x_{u_0}}{\mathbf{x}^T\mathbf{x}}\left(\sum_{w\in\cup_{i=2}^r I_i} x_w-\sum_{uu_0\in E(G)} x_u\right)\\
&\geqslant \frac{2x_{u_0}}{\mathbf{x}^T\mathbf{x}}\Big((\rho(G)-|L|-k(3r-1)+2(r-1))x_{v_0}-2|W|-(d(u_0)-|W|)x_{v_0}\Big)\\
&= \frac{2x_{u_0}}{\mathbf{x}^T\mathbf{x}}\Big((\rho(G)-|L|-k(3r-1)+2(r-1)-d(u_0)+|W|)x_{v_0}-2|W|\Big)\\
&\geqslant \frac{2x_{u_0}}{\mathbf{x}^T\mathbf{x}}\Big(\frac{r-2}{r}(\varepsilon_1 n-\varepsilon_2 n-O(1))-2|W|\Big)
\end{aligned}$$

$$\geqslant \frac{2x_{u_0}}{\mathbf{x}^T\mathbf{x}}\Big(\frac{r-2}{r}(\varepsilon_1 n - \varepsilon_2 n - O(1)) - 2(\varepsilon_2 n + k - 1)\Big) > 0$$

where the last inequality holds since $\varepsilon_2 \ll \varepsilon_1$. This contradicts the fact that G has the largest spectral radius over all kK_{r+1}-free graphs, so L must be empty. $\square$

Lemma 18. *For any $v \in V(G)$, $x_v \geqslant 1 - \frac{1}{r-1}$.*

Proof. Since $L = \emptyset$, then $|W| = |W \setminus L| \leqslant k-1$ by Lemma 16. Let $x_{v_0} = \max\{x_v : v \in V(G) \setminus W\}$. Recall that $x_z = \max\{x_v : v \in V(G)\} = 1$, then

$$\rho(G) = \rho(G)x_z \leqslant |W| + (n - |W|)x_{v_0}.$$

Combining with Lemma 8, we have

$$x_{v_0} \geqslant \frac{\rho(G) - |W|}{n - |W|} \geqslant \frac{\rho(G) - |W|}{n} \geqslant 1 - \frac{1}{r} - \frac{O(1)}{n}. \tag{9}$$

Using the similar method as in the proof of (8), we have

$$\sum_{v \in \cup_{i=2}^r I_i} x_v \geqslant (\rho(G) - k(r+3) + 2)x_{v_0} - (k-1).$$

Suppose to the contrary that there exists $u \in V(G)$ such that $x_u < 1 - \frac{1}{r-1}$. Let G' be the graph with $V(G') = V(G)$ and $E(G') = E(G \setminus \{u\}) \cup \{uw : w \in \cup_{i=2}^r I_i\}$. It is obvious that G' is kK_{r+1}-free. Therefore, we have

$$\begin{aligned}
\rho(G') - \rho(G) &\geqslant \frac{\mathbf{x}^T\left(A(G') - A(G)\right)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{2x_u}{\mathbf{x}^T\mathbf{x}}\left(\sum_{w \in \cup_{i=2}^r I_i} x_w - \sum_{uv \in E(G)} x_v\right) \\
&\geqslant \frac{2x_u}{\mathbf{x}^T\mathbf{x}}\Big((\rho(G) - k(r+3) + 2)x_{v_0} - (k-1) - \rho(G)x_u\Big) \\
&> \frac{2x_u}{\mathbf{x}^T\mathbf{x}}\Big((\rho(G) - k(r+3) + 2)(1 - \frac{1}{r} - \frac{O(1)}{n}) - (k-1) - \rho(G)(1 - \frac{1}{r-1})\Big) \\
&> \frac{2x_u}{\mathbf{x}^T\mathbf{x}}\Big(\frac{n}{r^2} - O(1)\Big) > 0.
\end{aligned}$$

This contradicts the fact that G has the largest spectral radius over all kK_{r+1}-free graphs. $\square$

Lemma 19. *$|W| = k-1$, and $V_i \setminus W$ is an independent set for any $i \in [r]$.*

Proof. Let $|W| = s$. Then $s \leqslant k-1$ by Lemmas 16 and 17.

Claim 20. $\nu(\cup_{i=1}^r G[V_i \setminus W]) \leqslant k - 1 - s$.

Proof of Claim 20. Otherwise, $\nu(\cup_{i=1}^r G[V_i \setminus W]) \geqslant k-s$. By Lemma 12, we can find a $(k-s)K_{r+1}$, denoted by F_1. Since $|W|=s$, by Lemma 15, we can find a sK_{r+1}, denoted by F_2, such that $V(F_1)\cap V(F_2)=\emptyset$. Therefore, $F_1 \cup F_2$ is a copy of kK_{r+1} in G, a contradiction. □

Suppose to the contrary that $s<k-1$. By Lemmas 14 and 17, we have $\Delta(\cup_{i=1}^r G[V_i \setminus W]) < k(r+1)$. Combining with Lemma 6, we have

$$\begin{aligned} e(\cup_{i=1}^r G[V_i \setminus W]) &\leqslant f(\nu(\cup_{i=1}^r G[V_i \setminus W]), \Delta(\cup_{i=1}^r G[V_i \setminus W])) \\ &\leqslant f(k-s-1, k(r+1)) \\ &\leqslant k(k-s)(r+1). \end{aligned}$$

Take $S \subseteq V_1 \setminus W$ with $|S| = k-s-1$. Let G' be the graph with $V(G')=V(G)$ and $E(G') = E(G)\setminus \{uv : uv \in \cup_{i=1}^r E(G[V_i \setminus W])\} \cup \{uv : u \in S, v \in (V_1\setminus W)\setminus S\}$. It is obvious that G' is kK_{r+1}-free. Therefore,

$$\begin{aligned} &\rho(G') - \rho(G) \\ &\geqslant \frac{\mathbf{x}^T\left(A(G')-A(G)\right)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ &= \frac{2}{\mathbf{x}^T\mathbf{x}}\left(\sum_{ij\in E(G')} x_ix_j - \sum_{ij \in E(G)} x_ix_j\right) \\ &\geqslant \frac{2}{\mathbf{x}^T\mathbf{x}}\left((k-s-1)(|V_1|-|W|-k+s+1)(1-\frac{1}{r-1})^2 - k(k-s)(r+1)\right) \\ &\geqslant \frac{2}{\mathbf{x}^T\mathbf{x}}\Big((k-s-1)(\frac{n}{r}-3\sqrt{\varepsilon}n-k+1)(1-\frac{1}{r-1})^2-k(k-s)(r+1)\Big) \\ &> 0. \end{aligned}$$

This contradicts the fact that G has the largest spectral radius over all kK_{r+1}-free graphs. Therefore, $|W|=s=k-1$. Then it follows from the claim that $\nu(\cup_{i=1}^r G[V_i \setminus W]) \leqslant k-1-s=0$ for any $i\in[r]$. So $V_i \setminus W$ is an independent set. □

Lemma 21. *For any $u \in W$, $d(u)=n-1$.*

Proof. Suppose to the contrary that there exists $u\in W$ such that $d(u)<n-1$. Let $v\in V(G)$ be a vertex such that $uv \notin E(G)$. Let G' be the graph with $V(G')=V(G)$ and $E(G')=E(G)\cup\{uv\}$. We claim that G' is kK_{r+1}-free. Otherwise, G' contains a copy of kK_{r+1}, say F_1, as a subgraph, and $uv \in E(F_1)$. Let F_2 be the K_{r+1} of F_1 which contains uv. Then G contains a $(k-1)K_{r+1}$, denoted by F_3, as a subgraph, with $V(F_2)\cap V(F_3)=\emptyset$ and $u\notin V(F_3)$. Since $u\in W$, by Lemmas 15 and 17, we can find a K_{r+1}, denoted by F_4, which contains u and $V(F_4)\cap V(F_3)=\emptyset$. Thus $F_3\cup F_4$ is a copy of kK_{r+1} in G, a contradiction. Therefore, G' is kK_{r+1}-free. By the construction of G', we have $\rho(G')>\rho(G)$, which contradicts the assumption that G has the maximum spectral radius among all kK_{r+1}-free graphs on n vertices. □

Proof of Theorem 3. Now we prove that G is isomorphic to $K_{k-1} \vee T_{n-k+1,r}$. For any $i \in [r]$, let $|V_i \setminus W| = n_i$. By Lemmas 19 and 21, there exists an r-partite graph H with classes of size $n_1, n_2, \ldots, n_r$ such that $G \cong K_{k-1} \vee H$. By the maximum of $\rho(G)$, $H \cong K_r(n_1, n_2, \ldots, n_r)$. It suffices to show that $|n_i - n_j| \leqslant 1$ for any $1 \leqslant i < j \leqslant r$. Suppose $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$. We prove the assertion by contradiction. Assume that there exist i_0, j_0 with $1 \leqslant i_0 < j_0 \leqslant r$ such that $n_{i_0} - n_{j_0} \geqslant 2$. Let $H' = K_r(n_1, \ldots, n_{i_0} - 1, \ldots, n_{j_0} + 1, \ldots, n_r)$, and $G' = K_{k-1} \vee H'$.

Recall that $\mathbf{x}$ is the eigenvector of G corresponding to $\rho(G)$, by the symmetry we may assume $\mathbf{x} = (\underbrace{x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, \ldots, x_2}_{n_2}, \ldots, \underbrace{x_r, \ldots, x_r}_{n_r}, \underbrace{x_{r+1}, \ldots, x_{r+1}}_{k-1})^{\mathrm{T}}$. Thus by (1), we have

$$\rho(G)x_i = \sum_{j=1}^{r} n_j x_j - n_i x_i + (k-1)x_{r+1}, \text{ for any } i \in [r], \tag{10}$$

and

$$\rho(G)x_{r+1} = \sum_{j=1}^{r} n_j x_j + (k-2)x_{r+1}. \tag{11}$$

Combining (10) and (11), we have $x_i = \frac{\rho(G)+1}{\rho(G)+n_i} x_{r+1}$ for any $i \in [r]$, which implies that $x_{r+1} = \max\{x_v : v \in V(G)\}$. Recall that $\max\{x_v : v \in V(G)\} = 1$, then $x_{r+1} = 1$, and $x_i = \frac{\rho(G)+1}{\rho(G)+n_i}$ for any $i \in [r]$. Let $u_{i_0} \in V_{i_0} \setminus W$ be a fixed vertex. Then G' can be obtained from G by deleting all edges between u_{i_0} and $V_{j_0} \setminus W$, and adding all edges between u_{i_0} and $V_{i_0} \setminus (W \cup \{u_{i_0}\})$. According to (2), we deduce that

$$\begin{aligned}
\rho(G') - \rho(G) &\geqslant \frac{\mathbf{x}^{\mathrm{T}}(A(G') - A(G))\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \\
&= \frac{2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \left((n_{i_0} - 1)x_{i_0}^2 - n_{j_0} x_{i_0} x_{j_0} \right) \\
&= \frac{2x_{i_0}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \left((n_{i_0} - 1)\frac{\rho(G)+1}{\rho(G)+n_{i_0}} - n_{j_0}\frac{\rho(G)+1}{\rho(G)+n_{j_0}} \right) \\
&= \frac{2x_{i_0}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \frac{(\rho(G)+1)(n_{i_0}\rho(G) - n_{j_0}\rho(G) - \rho(G) - n_{j_0})}{(\rho(G)+n_{i_0})(\rho(G)+n_{j_0})} \\
&\geqslant \frac{2x_{i_0}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \frac{(\rho(G)+1)(\rho(G) - n_{j_0})}{(\rho(G)+n_{i_0})(\rho(G)+n_{j_0})} > 0,
\end{aligned}$$

where the last second inequality holds as $n_{i_0} - n_{j_0} \geqslant 2$, and the last inequality holds since $\rho(G) \geqslant \frac{r-1}{r}n + \frac{2(k-1)}{r} - \frac{1}{n}\left(\frac{(k-1)(r+k-1)}{r} + \frac{r}{2}\right)$, and $n_{j_0} = |V_{j_0} \setminus W| \leqslant \frac{n}{r} + 3\sqrt{\varepsilon}n - (k-1)$. This contradicts the assumption that G has the maximum spectral radius among all n-vertex kK_{r+1}-free graphs. Therefore, G is isomorphic to $K_{k-1} \vee T_{n-k+1,r}$. $\square$

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