# On a characterization of lattice cubes via discrete isoperimetric inequalities

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**Abstract**

We obtain a characterization of lattice cubes as the only sets that reach equality in several discrete isoperimetric-type inequalities associated with the $L_\infty$ norm, including well-known results by Radcliffe and Veomett.

We furthermore provide a new isoperimetric inequality for the lattice point enumerator that generalizes previous results, and for which the aforementioned characterization also holds.

**Mathematics Subject Classifications:** 52C07, 11H06

## 1 Introduction and notation

We denote the $n$-dimensional Euclidean space by $\mathbb{R}^n$. The $n$-dimensional integer lattice will be denoted by $\mathbb{Z}^n$, and we will use $\mathbb{Z}^n_{\geqslant 0}$ to refer to the points with non-negative coordinates in $\mathbb{Z}^n$. For all $i = 1, \dots, n$, the $i$-th canonical unit vector will be denoted by $\mathrm{e}_i$. The family of *lattice cubes*, i.e., the intersection of an $n$-dimensional cube $[a, b]^n \subseteq \mathbb{R}^n$ with the integer lattice $\mathbb{Z}^n$, will play a special role in our main results.

For a measurable set $M \subseteq \mathbb{R}^n$, we write $\mathrm{vol}(M)$ to refer to its volume, this is, its $n$-dimensional Lebesgue measure. As discrete counterparts we use $|X|$ to denote the cardinality of any finite set $X \subseteq \mathbb{R}^n$, together with the lattice point enumerator $\mathrm{G}_n(K) = |K \cap \mathbb{Z}^n|$ for any bounded set $K \subseteq \mathbb{R}^n$.

The Brunn-Minkowski inequality arises as a natural connection between the notions of Minkowski addition and volume. It states, in one of its forms, that for $K, L \subseteq \mathbb{R}^n$

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non-empty compact convex sets, we have

$$\mathrm{vol}(K+L)^{1/n} \geqslant \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}, \tag{1}$$

with equality, if $\mathrm{vol}(K)\,\mathrm{vol}(L) > 0$, if and only if $K$ and $L$ are homothetic, i.e., $L = x + \lambda K$ for some $\lambda > 0$ and $x \in \mathbb{R}^n$. Here, we use $+$ to denote the Minkowski addition, i.e.,

$$A + B = \{a + b : a \in A, b \in B\},$$

for any non-empty sets $A, B \subseteq \mathbb{R}^n$.

The hypothesis of the Brunn-Minkowski inequality can be relaxed substantially, not only by getting rid of the convexity, but even the compactness, as the inequality can be obtained for bounded measurable sets. Comprehensive surveys on the topic may be found in [2, 7].

It is well known that if one replaces the volume functional in (1) by the cardinality (in the case of finite sets) or the lattice point enumerator, then the resulting inequality does not hold in general (see [12, 15]). Therefore, in order to obtain discrete Brunn-Minkowski type inequalities several approaches have been considered. In [8], Gardner and Gronchi defined an order in $\mathbb{Z}^n$ and, by combining this order with a technique of compressions, they proved an engaging discrete analogue of an equivalent version of (1), which improved previous results obtained by Ruzsa in [21, 22].

Another approach has been to consider a modification of the sets involved to get a valid inequality preserving the structure of (1). In this regard, one can find inequalities for the cardinality [9, 12, 15], functional extensions of them [11, 14, 15, 16, 24] and also discrete inequalities for the lattice point enumerator [11, 13, 15]. As a relevant example for the discussion, we highlight the following result:

**Theorem A.** *[15, Theorem 3.2] Let $X, Y \subseteq \mathbb{Z}^n$ be non-empty finite sets. Then*

$$\left|X + Y + \{0,1\}^n\right|^{1/n} \geqslant |X|^{1/n} + |Y|^{1/n}. \tag{2}$$

*Equality holds if both $X$ and $Y$ are lattice cubes.*

Perhaps one of the most elegant consequences of the Brunn-Minkowski inequality is the remarkable *isoperimetric inequality*, which characterizes the Euclidean balls as the only convex bodies that minimize the surface area (i.e. the Minkowski content) for prescribed positive volume. It can be expressed as

$$\left(\frac{\mathrm{S}(K)}{\mathrm{S}(B_n)}\right)^n \geqslant \left(\frac{\mathrm{vol}(K)}{\mathrm{vol}(B_n)}\right)^{n-1},$$

where $\mathrm{S}(\cdot)$ represents the surface area and $B_n$ is the Euclidean unit ball.

The isoperimetric inequality was already known in antiquity in two dimensions and, since then, a rich collection of analogs has been obtained in varied and diverse fields and branches of mathematics. For instance, a version for *mixed volumes* known as *Minkowski's first inequality* (cf. [23, Theorem 7.2.1]), as well as an equivalent analytic version due to

Sobolev (see e.g. [7, Section 5]). Other related inequalities, which can be consulted in [23, Section 7.2], include *Diskant's inequality* or the *Bonnesen-type inequalities* in the plane. Apart from the measure, the geometric space can also be modified. In this regard, isoperimetric inequalities in the spherical and hyperbolic spaces have been proved (cf. [4]). Also, the reverse isoperimetric inequality by Ball deserves a special mention (see e.g. [1]). We refer the reader to [19] for a thorough survey on the topic.

A more general version of the isoperimetric inequality is known as its "neighborhood" form (see, e.g., [17, Proposition 14.2.1]): it states that for any non-empty compact convex set $K \subseteq \mathbb{R}^n$, and all $t \geqslant 0$, we have

$$\mathrm{vol}(K + tB_n) \geqslant \mathrm{vol}(rB_n + tB_n), \tag{3}$$

where $r \geqslant 0$ satisfies $\mathrm{vol}(rB_n) = \mathrm{vol}(K)$.

Furthermore, the Brunn-Minkowski inequality also implies that given any $t \geqslant 0$ and non-empty compact convex sets $K, E \subseteq \mathbb{R}^n$ with positive volume, if $r > 0$ is such that $\mathrm{vol}(K) = \mathrm{vol}(rE)$, then we have

$$\mathrm{vol}(K + tE) \geqslant \mathrm{vol}(rE + tE), \tag{4}$$

with equality if and only if $K$ and $E$ are homothetic. Since the set $K + tB_n$ consists of all points having Euclidean distance at most $t$ to the set $K$, one could see (4) as an isoperimetric inequality where the "distance" involved is modified according to the set $E$. This allows to extend the isoperimetric inequality to metric spaces where there is a notion of measure, without needing to define the concept of surface area.

A brief survey on the neighborhood form of the isoperimetric inequality can be found in [17, Section 14.2], where different spaces are considered (e.g. the *Gauss space* and the $n$-dimensional discrete unit cube $\{0,1\}^n$). In [25] and [5], this type of inequalities are studied in $\mathbb{Z}^n$ endowed with the $L_1$ norm, characterizing the equality in some particular cases.

In [20], Radcliffe and Veomett proved an exceptional discrete isoperimetric inequality in the spirit of (4) for the integer lattice $\mathbb{Z}^n$ endowed with the $L_\infty$ norm considering the cardinality as the measure. To this end, the authors defined a complete order in $\mathbb{Z}^n$ (see Definition 4) to show that the initial segments $\mathcal{I}_r \subseteq \mathbb{Z}^n$ (i.e., the first $r$ points in the order, $r \in \mathbb{Z}_{>0}$) minimize the functional $|X + \{-1,0,1\}^n|$, among all sets $X \subseteq \mathbb{Z}^n$ with $|X| = r$:

**Theorem B.** *[20, Theorem 1] Let $X \subseteq \mathbb{Z}^n$ with $r = |X| \in \mathbb{Z}_{>0}$. Then*

$$\left|X + \{-1,0,1\}^n\right| \geqslant \left|\mathcal{I}_r + \{-1,0,1\}^n\right|. \tag{5}$$

The authors also considered the restriction to $\mathbb{Z}^n_{\geqslant 0}$ of the aforementioned order to show an analogous result for the corresponding initial segments $\mathcal{J}_r \subseteq \mathbb{Z}^n_{\geqslant 0}$:

**Theorem C.** *[20, Corollary 1] Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ with $r = |X| \in \mathbb{Z}_{>0}$. Then*

$$\left|\left(X + \{-1,0,1\}^n\right) \cap \mathbb{Z}^n_{\geqslant 0}\right| \geqslant \left|\left(\mathcal{J}_r + \{-1,0,1\}^n\right) \cap \mathbb{Z}^n_{\geqslant 0}\right|. \tag{6}$$

We note that for $r = (\rho+1)^n$, $\rho \in \mathbb{Z}_{\geqslant 0}$, both initial segments $\mathcal{I}_r \subseteq \mathbb{Z}^n$ and $\mathcal{J}_r \subseteq \mathbb{Z}^n_{\geqslant 0}$ are lattice cubes (cf. Remark 5). The authors of [20] already observed that initial segments do not characterize the equality case in (5) and (6).

Nevertheless, here we show, on the one hand, that lattice cubes can be characterized as the only sets attaining equality in Theorem B and Theorem C, for those cardinalities that allow lattice cubes:

**Theorem 1.** *Let $X \subseteq \mathbb{Z}^n$ with $|X| = (\rho+1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$. Then equality holds in* (5) *if and only if $X$ is a lattice cube.*

**Theorem 2.** *Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ with $|X| = (\rho+1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$. Then equality holds in* (6) *if and only if $X = \{0, \dots, \rho\}^n$.*

Observe that, in the latter characterization, the lattice cube must be anchored at the origin (see Remark 5 for examples and more explanations).

Furthermore, in Section 2 we connect both problems, by developing further understanding about the structure of initial segments in both settings, and showing that it suffices to consider $\mathbb{Z}^n_{\geqslant 0}$, having several advantages over $\mathbb{Z}^n$ which we shall discuss. In Section 3, we prove a stronger characterization of lattice cubes as the only minimizers of the functional $|X + \{0, \dots, s\}^n|$ for all $s \in \mathbb{Z}_{>0}$ (see Theorem 17). This further allows to characterize lattice cubes in a wider family of discrete inequalities, for instance, in the Brunn-Minkowski type inequality given in Theorem A (see Corollary 35), where we have equality if and only if $X$ is a lattice cube, provided that $Y$ is a lattice cube (and $|X| = (\rho+1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$).

On the other hand, using the initial segments $\mathcal{I}_r \subseteq \mathbb{Z}^n$, in [13], the authors defined the *extended cubes* $\mathcal{C}_{\mathcal{I}_r} \subseteq \mathbb{R}^n$ (see Definition 37): a uniparametric family of star-shaped sets characterized as the largest sets (with respect to inclusion) such that $\mathcal{C}_{\mathcal{I}_r} + (-1,1)^n \subseteq \mathcal{I}_r + (-1,1)^n$. Analogously, extended cubes $\mathcal{C}_{\mathcal{J}_r} \subseteq \mathbb{R}^n_{\geqslant 0}$ can be defined. The extended cubes $\mathcal{C}_{\mathcal{I}_r}$ allowed the authors to obtain an equivalent version of (5) for the lattice point enumerator $\mathrm{G}_n(\cdot)$ that can be used (see [13, Theorem 1.4]) to infer the neighborhood form (4) of the isoperimetric inequality when $E = [-1,1]^n$.

**Theorem D.** *[13, Theorem 1.2] Let $K \subseteq \mathbb{R}^n$ be a non-empty bounded set. If $r = \mathrm{G}_n(K) > 0$, then*

$$\mathrm{G}_n\big(K + t[-1,1]^n\big) \geqslant \mathrm{G}_n\big(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n\big) \tag{7}$$

*for all $t \geqslant 0$.*

Here, we prove an isoperimetric-type inequality (with the corresponding equality cases) for the lattice point enumerator that generalizes (7) (see Proposition 41) which, consequently, can also be used to obtain the neighborhood form (4) for $E = [0,1]^n$.

**Theorem 3.** *Let $K \subseteq \mathbb{R}^n$ be a non-empty bounded set. If $r = \mathrm{G}_n(K) > 0$, then*

$$\mathrm{G}_n\big(K + t[0,1]^n\big) \geqslant \mathrm{G}_n\big(\mathcal{C}_{\mathcal{J}_r} + t[0,1]^n\big) \tag{8}$$

*for all $t \geqslant 0$. When $\mathrm{G}_n(K) = (\rho+1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$, for each $t \geqslant 0$ equality holds if and only if $K \cap \mathbb{Z}^n$ is a lattice cube and we have*

$$(K + t[0,1]^n) \cap \mathbb{Z}^n = (K \cap \mathbb{Z}^n) + (t[0,1]^n \cap \mathbb{Z}^n).$$

The second equality condition is related to relevant problems in several lines of mathematics. For instance, in lattice polytope theory, this equality was discussed in [18]. Sufficient conditions for it were obtained in the plane in [6], using an algebraic geometry approach; and in [10], this result was then strengthened via a discrete geometry approach.

This paper is organized as follows. In Section 2 we compare the initial segments in $\mathbb{Z}^n$ and $\mathbb{Z}^n_{\geqslant 0}$, to obtain some initial results which will enable the comparison between Theorem B and Theorem C. Sections 3 and 4 are devoted to the characterization results for the lattice cubes, and the results for the lattice point enumerator, respectively.

## 2 Comparing the initial segments in $\mathbb{Z}^n$ and $\mathbb{Z}^n_{\geqslant 0}$

This section is devoted to further studying the structure of initial segments and their corresponding order, both in $\mathbb{Z}^n$ and $\mathbb{Z}^n_{\geqslant 0}$ (see, e.g., Remark 13), which will enable us to establish precise relations between them and, in turn, connect Theorem B and Theorem C (and thus, later, their respective characterizations, Theorems 1 and 2).

To this end, we show a new isoperimetric-type inequality in $\mathbb{Z}^n_{\geqslant 0}$ that generalizes both theorems B and C (see Corollary 11 and the observations thereafter). Next, we prove that both the initial segments in $\mathbb{Z}^n$ and in $\mathbb{Z}^n_{\geqslant 0}$ reach equality in this new inequality (see Corollary 16). This will allow us to indistinctly work with initial segments in either $\mathbb{Z}^n$ or $\mathbb{Z}^n_{\geqslant 0}$ (in the context of these discrete isoperimetric type inequalities). The new formulation in $\mathbb{Z}^n_{\geqslant 0}$ provides several additional advantages, such as being able to work with lattice cubes of odd length, or simplifying the definitions and results. This will make this setting our preferred choice for subsequent sections.

The following complete order was defined in [20], and it will be the keystone of the subsequent development.

**Definition 4.** If $n = 1$, the order $\prec$ in $\mathbb{Z}$ is defined by

$$0 \prec 1 \prec -1 \prec 2 \prec -2 \prec \cdots \prec m \prec -m \prec \dots$$

For $n \geqslant 2$ and for any vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$, let

$$m_w = \max_{\prec}\{w_i : i = 1, \dots, n\},\ i_w = \min\bigl\{i :\ w_i = m_w\bigr\} \text{ and}$$
$$w' = (w_1 \dots, w_{i_w-1}, w_{i_w+1}, \dots, w_n) \in \mathbb{Z}^{n-1}.$$

Then, $\prec$ is defined recursively as follows: for any $u, v \in \mathbb{Z}^n$ with $u \neq v$, one has $u \prec v$ if

i) $m_u \prec m_v$ or

ii) $m_u = m_v$ and then either $i_v < i_u$ or ($i_v = i_u$ and) $u' \prec v'$.

Moreover, we write $u \preceq v$ if either $u \prec v$ or $u = v$.

We note that, in order to define $\prec$ in $\mathbb{Z}^n_{\geqslant 0}$, one could see that order as the restriction of the order in $\mathbb{Z}^n$ to the subset $\mathbb{Z}^n_{\geqslant 0}$, or as the generalization of the usual order in $\mathbb{Z}_{\geqslant 0}$, to which one applies the same process described in Definition 4.

For any $r \in \mathbb{Z}_{>0}$, we denote by $\mathcal{I}_r$ (resp., $\mathcal{J}_r$) the *initial segment* in $\mathbb{Z}^n$ (resp., $\mathbb{Z}^n_{\geqslant 0}$) of cardinality $r$, that is, the set of the first $r$ points with respect to the order $\prec$ of $\mathbb{Z}^n$ (resp., $\mathbb{Z}^n_{\geqslant 0}$).

*Remark* 5. It can be easily verified from the definition of $\prec$ that for $r = (\rho + 1)^n$, with $\rho \in \mathbb{Z}_{\geqslant 0}$, the initial segments $\mathcal{I}_r \subseteq \mathbb{Z}^n$ and $\mathcal{J}_r \subseteq \mathbb{Z}^n_{\geqslant 0}$ are both lattice cubes. More precisely, $\mathcal{I}_r = \{-\rho/2, \ldots, \rho/2\}^n$ for $\rho$ even and $\mathcal{I}_r = \{-(\rho+1)/2 + 1, \ldots, (\rho+1)/2\}^n$ for $\rho$ odd, whereas $\mathcal{J}_r = \{0, \ldots, \rho\}^n$ for all $\rho \in \mathbb{Z}_{\geqslant 0}$. See Remark 13 for a more precise description of the structure of the initial segments.

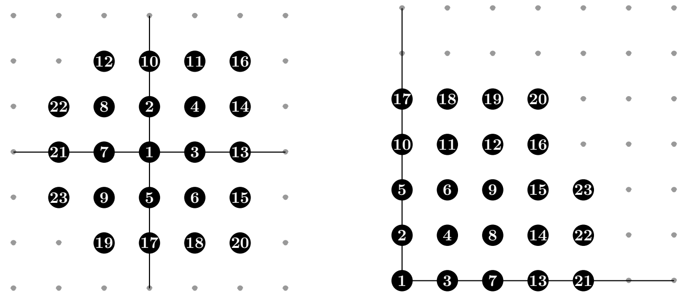


Figure 1: The initial segments $\mathcal{I}_{23}$ (left) and $\mathcal{J}_{23}$ (right) for $n = 2$.

For any given $x \in \mathbb{Z}^n$ we denote its *rank*, i.e., its position with respect to the order $\prec$ in $\mathbb{Z}^n$, by $r(x) \in \mathbb{Z}_{\geqslant 0}$. Furthermore, for any non-empty finite set $X \subseteq \mathbb{Z}^n$, the rank of $X$ is defined as

$$r(X) = \sum_{x \in X} r(x).$$

We will use the same notation when working with the order $\prec$ in $\mathbb{Z}^n_{\geqslant 0}$, without specifying whenever there is no ambiguity.

**Definition 6.** [20, Definition 2] A non-empty set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$, $n > 1$, is said to be downward compressed in the $i$-th coordinate, $i = 1, \ldots, n$, with respect to $x = (x_1, \ldots, x_{n-1}) \in \mathbb{Z}^{n-1}_{\geqslant 0}$ if the set

$$\{y \in \mathbb{Z}_{\geqslant 0} : (x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1}) \in X\}$$

is either empty or of the form $\{y \in \mathbb{Z}_{\geqslant 0} : 0 \leqslant y \leqslant a\}$ for some $a \in \mathbb{Z}_{\geqslant 0}$.

Moreover, we say that $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is downward compressed in the $i$-th coordinate if it is downward compressed in the $i$-th coordinate with respect to every $x \in \mathbb{Z}^{n-1}_{\geqslant 0}$. Finally, we say that $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is downward compressed if it is downward compressed in the $i$-th coordinate for all $i = 1, \ldots, n$.

*Remark* 7. Let $x, y, z \in \mathbb{Z}^n$ (resp. $\mathbb{Z}^n_{\geqslant 0}$). Then:

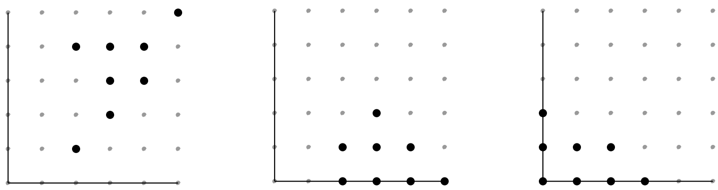

Figure 2: From left to right: a finite set, a downward compressed set in the 2nd coordinate and a downward compressed set.

i) If for some $i \in \{1, \ldots, n\}$ we have $x_i \prec y_i$ and $x_j = y_j$ for all $j \neq i$, then $x \prec y$.

ii) In particular, if $x_i \preceq y_i$ for all $i = 1, \ldots, n$, then $x \preceq y$.

iii) If $x \prec y$, then $x + z \prec y + z$.

On the one hand, a straightforward consequence of the above observation is that every initial segment $\mathcal{J}_r \subseteq \mathbb{Z}^n_{\geqslant 0}$ is downward compressed. And since it is clear that every downward compressed set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ satisfies

$$(X + \{-1, 0, 1\}^n) \cap \mathbb{Z}^n_{\geqslant 0} = X + \{0, 1\}^n, \tag{9}$$

then so does $\mathcal{J}_r$ for all $r \in \mathbb{Z}_{\geqslant 0}$.

On the other hand, in [20, page 11], the authors show that

$$|(\mathcal{J}_r + \{-1, 0, 1\}^n) \cap \mathbb{Z}^n_{\geqslant 0}| + 2^{\mathrm{l}(x)} = |(\mathcal{J}_{r+1} + \{-1, 0, 1\}^n) \cap \mathbb{Z}^n_{\geqslant 0}|,$$

where $x \in \mathbb{Z}^n_{\geqslant 0}$ satisfies $\mathcal{J}_r \cup \{x\} = \mathcal{J}_{r+1}$ and $\mathrm{l}(x) \in \{0, \ldots, n\}$ is the number of coordinates equal to zero in $x$. Putting all this together yields the following result.

**Lemma 8.** *Let $r \in \mathbb{Z}_{\geqslant 0}$ and let $x \in \mathbb{Z}^n_{\geqslant 0}$ be such that $\mathcal{J}_r \cup \{x\} = \mathcal{J}_{r+1}$. Then*

$$|\mathcal{J}_r + \{0, 1\}^n| + 2^{\mathrm{l}(x)} = |\mathcal{J}_{r+1} + \{0, 1\}^n|.$$

Following the ideas from [25, 20], we now prove a new discrete isoperimetric type inequality. We will later show that it is, in fact, equivalent to Theorem C (see Proposition 12).

**Lemma 9.** *Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set with $|X| = r$. Then*

$$\big|X + \{0, 1\}^n\big| \geqslant \big|\mathcal{J}_r + \{0, 1\}^n\big|. \tag{10}$$

*Proof.* If $n = 1$, since $\mathcal{J}_r = \{0, \ldots, r - 1\} \subseteq \mathbb{Z}_{\geqslant 0}$, by applying Theorem A to the sets $X, \{0\} \subseteq \mathbb{Z}_{\geqslant 0}$ we immediately get

$$|X + \{0, 1\}| \geqslant r + 1 = |\mathcal{J}_r + \{0, 1\}|. \tag{11}$$

Let $n > 1$. If $X$ is downward compressed then the result is a direct consequence of Theorem C, together with the fact that $\mathcal{J}_r$ is also downward compressed and (9).

If $X$ is not downward compressed, it is enough to show that we can find a downward compressed set $Z \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $|X| = |Z|$ and $|X + \{0,1\}^n| \geqslant |Z + \{0,1\}^n|$, and apply the previous case.

So, we assume that $X$ is not downward compressed in the $i$-th coordinate, for some $i \in \{1,\dots,n\}$, and we define the set $Y \subseteq \mathbb{Z}^n_{\geqslant 0}$ as

$$Y = \bigcup_{x\in X} \{(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n) \in \mathbb{Z}^n_{\geqslant 0} : 0 \leqslant t < |(x+\ell_i)\cap X|\},$$

where $\ell_i = \operatorname{lin}\{e_i\}$. The set $Y$ is downward compressed in the $i$-th coordinate and satisfies $|(x+\ell_i)\cap X| = |(x+\ell_i)\cap Y|$ for all $x \in \mathbb{Z}^n_{\geqslant 0}$. Therefore, $|Y| = |X|$. Furthermore, since $Y$ is downward compressed in the $i$-th coordinate, then for all $x \in \mathbb{Z}^n_{\geqslant 0}$ one has that $(x+\ell_i)\cap Y$ has no "holes", i.e., it is formed by consecutive points of $\mathbb{Z}^n_{\geqslant 0}$ in $x+\ell_i$. Hence, $|(x+\ell_i)\cap(Y+\{0,1\}^n)| \leqslant |(x+\ell_i)\cap(X+\{0,1\}^n)|$, and therefore, $|Y+\{0,1\}^n| \leqslant |X+\{0,1\}^n|$.

We also note that, by repeatedly "compressing" the set $X$ with respect to different coordinates as many times as necessary, we eventually get a downward compressed set $Z \subseteq \mathbb{Z}^n_{\geqslant 0}$ after a finite number of steps. Indeed, by looking at the ranks of $X$ and $Y$, we note that $r(X) \geqslant r(Y)$ with a strict inequality if $X \neq Y$ (cf. Remark 7), and so it is a consequence of the fact that $r(X)$ is bounded from below. □

*Remark* 10. We note that $\mathcal{J}_r + \{0,1\}^n$ is an initial segment, which follows from (9) and the fact that $(\mathcal{J}_r + \{-1,0,1\}^n)\cap\mathbb{Z}^n_{\geqslant 0}$ is an initial segment (cf. [20, page 11]).

As a consequence of the previous remark, and by iterating Lemma 9, one gets the following corollary.

**Corollary 11.** *Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set with $|X| = r$. Then*

$$|X + \{0,\dots,s\}^n| \geqslant |\mathcal{J}_r + \{0,\dots,s\}^n| \tag{12}$$

*for all $s \in \mathbb{Z}_{\geqslant 0}$.*

We emphasize that a similar process was already developed in [13] to get

$$|X + \{-s,\dots,s\}^n| \geqslant |\mathcal{I}_r + \{-s,\dots,s\}^n| \tag{13}$$

for any $X \subseteq \mathbb{Z}^n$ and all $s \in \mathbb{Z}_{\geqslant 0}$, which is equivalent to Theorem B.

We note that, just like $\mathcal{I}_r$, the initial segments $\mathcal{J}_r$ also give equality in (13) for any $s \in \mathbb{Z}_{\geqslant 0}$, and thus, also in (5). Indeed, it suffices to apply Corollary 11 with the cube $\{0,\dots,2s\}^n$, together with the translation invariance of the cardinality.

It is also easy to show that Corollary 11 and Theorem C are equivalent.

**Proposition 12.** *The discrete isoperimetric inequalities* (6) *and* (12) *are equivalent.*

*Proof.* Since the proof of Corollary 11 uses Lemma 9 (and thus, Theorem C), we only need to show that (12) implies (6). But this is a direct consequence of the fact that

$$|(X + \{-1,0,1\}^n)\cap\mathbb{Z}^n_{\geqslant 0}| \geqslant |X+\{0,1\}^n| \geqslant |\mathcal{J}_r + \{0,1\}^n| = |(\mathcal{J}_r + \{-1,0,1\}^n)\cap\mathbb{Z}^n_{\geqslant 0}|,$$

for all finite sets $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ with $|X| \geqslant r$ (cf. (9)). □

In order to further compare the initial segments in $\mathbb{Z}^n$ and $\mathbb{Z}^n_{\geqslant 0}$, we will analyze their $(n-1)$-dimensional sections, providing a description which will become crucial in subsequent sections. From now on, the following notation will be used: Let $n > 1$ and let $X \subseteq \mathbb{Z}^n$ be a non-empty finite set. Given $m \in \mathbb{Z}$ and $i \in \{1, \ldots, n\}$, we denote by $X^i(m)$ the section of $X$ at height $m$ orthogonal to $e_i$, i.e.,

$$X^i(m) = \{(x_1, \ldots, x_{n-1}) \in \mathbb{Z}^{n-1} : (x_1, \ldots, x_{i-1}, m, x_i, \ldots, x_{n-1}) \in X\}.$$

We note that, for any $x \in \mathbb{Z}^n$ (resp. $\mathbb{Z}^n_{\geqslant 0}$), all but the "last" section of $\mathcal{I}_{r(x)}$ (resp. $\mathcal{J}_{r(x)}$) are uniquely determined by $m_x$ and $i_x$. Indeed:

*Remark* 13. For $x \in \mathbb{Z}^n$, let $r = r(x)$. Then, from the fact that $\mathcal{I}_r = \{z \in \mathbb{Z}^n : z \preceq x\}$ we get that the only non-empty $(n-1)$-dimensional sections of $\mathcal{I}_r$ (with respect to $e_{i_x}$) are

$$(\mathcal{I}_r)^{i_x}(m) = \{t \in \mathbb{Z} : t \prec m_x\}^{i_x - 1} \times \{t \in \mathbb{Z} : t \preceq m_x\}^{n-i_x} \tag{14}$$

for all $m \prec m_x$, and

$$(\mathcal{I}_r)^{i_x}(m_x) = \{z \in \mathbb{Z}^{n-1} : z \preceq x'\} \subseteq \{t \in \mathbb{Z} : t \prec m_x\}^{i_x-1} \times \{t \in \mathbb{Z} : t \preceq m_x\}^{n-i_x} \tag{15}$$

(see Figure 3).

Now, let $y \in \mathbb{Z}^n_{\geqslant 0}$ and $r = r(y)$. Then, since $\mathcal{J}_r = \{z \in \mathbb{Z}^n_{\geqslant 0} : z \preceq y\} \subseteq \mathbb{Z}^n_{\geqslant 0}$ and $\{t \in \mathbb{Z}_{\geqslant 0} : t \prec m_y\} = \{0, \ldots, m_y - 1\}$, the prior relations translate into

$$(\mathcal{J}_r)^{i_y}(m) = \{0, \ldots, m_y - 1\}^{i_y - 1} \times \{0, \ldots, m_y\}^{n-i_y} \tag{16}$$

for all $0 \leqslant m < m_y$, and

$$(\mathcal{J}_r)^{i_y}(m_y) = \{z \in \mathbb{Z}^{n-1}_{\geqslant 0} : z \preceq y'\} \subseteq \{0, \ldots, m_y - 1\}^{i_y-1} \times \{0, \ldots, m_y\}^{n-i_y}. \tag{17}$$

(a) The initial segment $\mathcal{I}_{44} \subseteq \mathbb{Z}^3$, whose last point is $x = (-1, 2, 1)$ with $m_x = 2$ and $i_x = 2$.

(b) The initial segment $\mathcal{J}_{44} \subseteq \mathbb{Z}^3_{\geqslant 0}$, whose last point is $y = (2, 3, 1)$ with $m_y = 3$ and $i_y = 2$.

Figure 3: The sections of the initial segments $\mathcal{I}_{44}$ and $\mathcal{J}_{44}$.

We already know that the initial segments $\mathcal{J}_r \subseteq \mathbb{Z}^n_{\geqslant 0}$ give equality in (13) (and thus in (5)). In order to conclude this section, we show that, accordingly, the initial segments $\mathcal{I}_r \subseteq \mathbb{Z}^n$ also attain the equality in (12) (and thus in (6)).

Before doing that, however, it is convenient to make an observation that will be useful throughout the manuscript.

*Remark* 14. We note that the sequence

$$\begin{array}{rrcrr} 1, & 2, & \dots, & 2^{n-2}, & 2^{n-1}, \\ 2^n, & 2^{n-1}\cdot 3, & \dots, & 2^2\cdot 3^{n-2}, & 2\cdot 3^{n-1}, \\ 3^n, & 3^{n-1}\cdot 4, & \dots, & 3^2\cdot 4^{n-2}, & 3\cdot 4^{n-1}, \\ & & \vdots & & \end{array}$$

is strictly increasing, and therefore,

$$\left\{ \left[ s^i(s+1)^{n-i}, s^{i-1}(s+1)^{n-i+1} \right) \cap \mathbb{Z}_{\geqslant 0} : s \in \mathbb{Z}_{>0}, i = 1, \dots, n \right\}$$

is a partition of $\mathbb{Z}_{>0}$.

**Lemma 15.** *Let $r \in \mathbb{Z}_{>0}$. Then $|\mathcal{I}_r + \{0,1\}^n| = |\mathcal{J}_r + \{0,1\}^n|$.*

*Proof.* We proceed by induction on the dimension $n$. The case $n = 1$ is immediate since we have $|\mathcal{I}_r + \{0,1\}| = r + 1 = |\mathcal{J}_r + \{0,1\}|$.

Now, let $n > 1$, and assume that the $(n-1)$-dimensional case is already proved. Let $x \in \mathbb{Z}^n$, $y \in \mathbb{Z}^n_{\geqslant 0}$ be the last points in the order $\prec$ of $\mathcal{I}_r$ and $\mathcal{J}_r$, respectively (so, $r(x) = r = r(y)$), and let

$$s = |\{m \in \mathbb{Z} : m \prec m_x\}|,$$

i.e., the number of sections of $\mathcal{I}_r$ of the form (14). Then, using (14) and (15), we have

$$s^{i_x}(s+1)^{n-i_x} < r \leqslant s^{i_x-1}(s+1)^{n-i_x+1}. \tag{18}$$

Analogously, from (16) and (17), we get

$$m_y^{i_y}(m_y+1)^{n-i_y} < r \leqslant m_y^{i_y-1}(m_y+1)^{n-i_y+1}. \tag{19}$$

Therefore, Remark 14, together with (18) and (19), implies that $s = m_y$ and $i_x = i_y$ and, consequently,

$$\left|\{z \in \mathbb{Z}^{n-1} : z \preceq x'\}\right| = r - s^{i_x}(s+1)^{n-i_x} = \left|\left\{z \in \mathbb{Z}^{n-1}_{\geqslant 0} : z \preceq y'\right\}\right|.$$

Remark 13 also yields that both initial segments are the union of a lattice box of cardinality $s^{i_x}(s+1)^{n-i_x}$ with an $(n-1)$-dimensional initial segment of cardinality $r - s^{i_x}(s+1)^{n-i_x}$, in their respective orders. Moreover, we have

$$|\mathcal{I}_r + \{0,1\}^n| = (s+1)^{i_x}(s+2)^{n-i_x} + \left|\{z \in \mathbb{Z}^{n-1} : z \preceq x'\} + \{0,1\}^{n-1}\right|$$

and

$$|\mathcal{J}_r + \{0,1\}^n| = (s+1)^{i_x}(s+2)^{n-i_x} + \left|\left\{z \in \mathbb{Z}^{n-1}_{\geqslant 0} : z \preceq y'\right\} + \{0,1\}^{n-1}\right|.$$

This concludes the proof since the induction hypothesis implies that

$$\left|\{z \in \mathbb{Z}^{n-1} : z \preceq x'\} + \{0,1\}^{n-1}\right| = \left|\left\{z \in \mathbb{Z}^{n-1}_{\geqslant 0} : z \preceq y'\right\} + \{0,1\}^{n-1}\right|. \qquad \square$$

In [20], the authors already noted that for every $r \in \mathbb{Z}_{>0}$, the set $\mathcal{I}_r + \{-1, 0, 1\}^n$ is an initial segment in $\mathbb{Z}^n$ (see [20, Lemma 1]). Since $\mathcal{J}_r + \{0, 1\}^n$ is also an initial segment in $\mathbb{Z}^n_{\geqslant 0}$ (see Remark 10), by iterating these properties, using (12), (13) and Lemma 15, and due to the translation invariance of the cardinality, we have the following result:

**Corollary 16.** *Let $r \in \mathbb{Z}_{>0}$. Then $|\mathcal{I}_r + \{0, \ldots, s\}^n| = |\mathcal{J}_r + \{0, \ldots, s\}^n|$ for all $s \in \mathbb{Z}_{\geqslant 0}$.*

## 3 Characterization of lattice cubes

This section is devoted to characterize the lattice cubes as the only minimizers of the functional $|X + \{0, \ldots, s\}^n|$ for any $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ and $s \in \mathbb{Z}_{\geqslant 0}$:

**Theorem 17.** *Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ with $|X| = (\rho + 1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$ and let $s \in \mathbb{Z}_{>0}$. If*

$$\left|X + \{0, \ldots, s\}^n\right| = \left|\mathcal{J}_{(\rho+1)^n} + \{0, \ldots, s\}^n\right|,$$

*then $X$ is a lattice cube.*

We first set further new definitions and get some initial results. The next subsections address separately the 2-dimensional and the general case of Theorem 17 for $s = 1$. An additional inductive argument then shows Theorem 17 in its full generality. Finally, as a consequence, we obtain Theorems 1 and 2. We refer the reader to [25] and [5] for similar studies with other norms.

**Definition 18.** We say that a non-empty finite set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is minimal if for all $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ with $|A| = |X|$ we have $|A + \{0, 1\}^n| \geqslant |X + \{0, 1\}^n|$.

**Definition 19.** Given a finite set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$, we define the ($n$-dimensional) neighborhood of $X$ as $\mathrm{N}^n_X = (X + \{0, 1\}^n) \setminus X$ if $X \neq \emptyset$, and $\mathrm{N}^n_X = \emptyset$ if $X = \emptyset$. Moreover, its cardinality will be denoted by $\mathrm{n}(X) = |\mathrm{N}^n_X|$.

We note that the minimality of a finite set can be defined in terms of the functional $\mathrm{n}(\cdot)$, since any set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is minimal if and only if $\mathrm{n}(A) \geqslant \mathrm{n}(X)$ for each $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ with $|A| = |X|$.

**Lemma 20.** *Let $n > 1$ and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set. If $|X| > (\rho + 1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$, then $|X + \{0, 1\}^n| > (\rho + 2)^n$ and $\mathrm{n}(X) > (\rho + 2)^n - (\rho + 1)^n$.*

*Proof.* Let $\mathcal{J}_a \subseteq \mathcal{J}_b \subseteq \mathcal{J}_c \subseteq \mathbb{Z}^n_{\geqslant 0}$ be the initial segments in $\mathbb{Z}^n_{\geqslant 0}$ of cardinalities $a = (\rho+1)^n$, $b = (\rho + 1)^n + 1$ and $c = |X|$. Then, $\mathcal{J}_a = \{0, \ldots, \rho\}^n$ and $\mathcal{J}_b = \mathcal{J}_a \cup (0, \ldots, 0, \rho + 1)$ (see Remark 5), and Lemma 9 yields

$$|X + \{0, 1\}^n| \geqslant |\mathcal{J}_c + \{0, 1\}^n| > |\mathcal{J}_a + \{0, 1\}^n| = (\rho + 2)^n.$$

In the following, we show that for any $r \in \mathbb{Z}_{>0}$, we have

$$\mathrm{n}(\mathcal{J}_{r+1}) \geqslant \mathrm{n}(\mathcal{J}_r). \tag{20}$$

Let $x_0 \in \mathbb{Z}^n_{\geqslant 0}$ be the last point of $\mathcal{J}_{r+1}$ in the order $\prec$. Then, using Remark 7 and since $x \preceq (1,\ldots,1)$ for all $x \in \{0,1\}^n$, we get

$$z + x \preceq z + (1,\ldots,1) \prec x_0 + (1,\ldots,1),$$

for all $z \prec x_0$. This implies that $x_0 + (1,\ldots,1) \in (\mathcal{J}_{r+1} + \{0,1\}^n) \setminus (\mathcal{J}_r + \{0,1\}^n)$. Consequently, $|\mathcal{J}_{r+1} + \{0,1\}^n| \geqslant |\mathcal{J}_r + \{0,1\}^n| + 1$, and, since $|\mathcal{J}_{r+1}| = |\mathcal{J}_r| + 1$, we deduce that

$$\mathrm{n}(\mathcal{J}_{r+1}) = |\mathcal{J}_{r+1} + \{0,1\}^n| - |\mathcal{J}_{r+1}| \geqslant |\mathcal{J}_r + \{0,1\}^n| - |\mathcal{J}_r| = \mathrm{n}(\mathcal{J}_r),$$

proving (20). This concludes the proof, since we also have

$$\mathrm{n}(\mathcal{J}_a) = (\rho+2)^n - (\rho+1)^n \quad \text{and} \quad \mathrm{n}(\mathcal{J}_b) = (\rho+2)^n + 2^{n-1} - ((\rho+1)^n + 1)$$

(see, e.g., Lemma 8), and therefore, by (20),

$$\mathrm{n}(X) = |X + \{0,1\}^n| - |X| \geqslant |\mathcal{J}_c + \{0,1\}^n| - c = \mathrm{n}(\mathcal{J}_c) \geqslant \mathrm{n}(\mathcal{J}_b) > (\rho+2)^n - (\rho+1)^n,$$

where the strict inequality follows from the fact that $n > 1$. □

**Definition 21.** We say that a non-empty finite set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is connected if for each $x, y \in X$, each $i \in \{1,\ldots,n\}$, and any $m \in \mathbb{Z}_{\geqslant 0}$ such that $x_i < m < y_i$, there exists $z \in X$ satisfying $z_i = m$.

An important observation is that any minimal set (see Definition 18) is connected. Indeed, an analogous argument to the one in [5, Proposition 1.4], translating one connected component next to the boundary of another one (without overlapping) to strictly decrease the functional $\mathrm{n}(\cdot)$, shows it. We include the proof here for completeness.

**Proposition 22.** *If $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is minimal, then $X$ is connected.*

*Proof.* If $X$ is not connected, then there exist $i \in \{1,\ldots,n\}$ and $m \in \mathbb{Z}_{>0}$ such that we can partition $X$ into

$$X_1 = \big\{x \in X : \langle x, \mathrm{e}_i\rangle < m\big\} \neq \emptyset \quad \text{and} \quad X_2 = \big\{x \in X : \langle x, \mathrm{e}_i\rangle > m\big\} \neq \emptyset$$

satisfying that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. Then, we may apply an integer translation to $X_1$ in the direction of $\mathrm{e}_i$ (and, possibly, in another canonical direction as well) such that the new set, $X_1' \subseteq \mathbb{Z}^n_{\geqslant 0}$, satisfies $X_1' \cap X_2 = \emptyset$ and $\big(X_1' + \{0,1\}^n\big) \cap X_2 \neq \emptyset$. Clearly, denoting by $X' = X_1' \cup X_2 \subseteq \mathbb{Z}^n_{\geqslant 0}$, one has $|X'| = |X|$ and $\mathrm{n}(X') < \mathrm{n}(X)$, contradicting the minimality of $X$. □

### 3.1 Characterization in dimension 2

The 2-dimensional case of Theorem 17 is based on the fact that any connected set $X$ can be enlarged up to a suitable lattice box without increasing the functional $\mathrm{n}(X)$.

Thus, for a finite non-empty set $X \subseteq \mathbb{Z}^2_{\geqslant 0}$, we denote by $\mathcal{B}(X) \subseteq \mathbb{Z}^2_{\geqslant 0}$ the smallest lattice box (with respect to set inclusion) such that $X \subseteq \mathcal{B}(X)$, i.e., $\mathcal{B}(X) = \big([a_1, b_1] \times [a_2, b_2]\big) \cap \mathbb{Z}^2_{\geqslant 0}$, where

$$a_i = \min\{x_i : (x_1, x_2) \in X\} \text{ and}$$
$$b_i = \max\{x_i : (x_1, x_2) \in X\}$$

for $i = 1, 2$.

**Lemma 23.** *Let $X \subseteq \mathbb{Z}^2_{\geqslant 0}$ be a non-empty connected finite set. Then*

$$\mathrm{n}(\mathcal{B}(X)) \leqslant \mathrm{n}(X).$$

*Proof.* We may assume, by applying a translation to $X$ if necessary, that

$$\mathcal{B}(X) = \big([1, b_1] \times [1, b_2]\big) \cap \mathbb{Z}^2_{\geqslant 0}$$

for some $b = (b_1, b_2) \in \mathbb{Z}^2_{\geqslant 0}$. Then,

$$|\mathcal{B}(X)| = b_1 b_2, \qquad |\mathcal{B}(X) + \{0,1\}^n| = (b_1 + 1)(b_2 + 1)$$

and therefore $\mathrm{n}(\mathcal{B}(X)) = (b_1 + 1)(b_2 + 1) - b_1 b_2 = b_1 + b_2 + 1$.

Let $X_1, X_2 \subseteq (X + \{0,1\}^2) \setminus X$ be defined as

$$X_1 = \{(x_1, x_2) \in X : \nexists (m, x_2) \in X \text{ with } m > x_1\} + \mathrm{e}_1 \text{ and}$$
$$X_2 = \{(x_1, x_2) \in X \cup X_1 : \nexists (x_1, m) \in X \cup X_1 \text{ with } m > x_2\} + \mathrm{e}_2.$$

We note that $X_1 \subseteq (X + \mathrm{e}_1) \setminus X$ and $X_2 \subseteq (X + \{0,1\}^2) \setminus (X + \mathrm{e}_1)$. Therefore $X_1 \cap X_2 = \emptyset$ and, since $X$ is connected, $|X_1| = b_2$ and $|X_2| = b_1 + 1$. Altogether we conclude the proof since we have

$$\mathrm{n}(\mathcal{B}(X)) = b_1 + b_2 + 1 = |X_1| + |X_2| \leqslant \mathrm{n}(X). \qquad \square$$

We are now under the conditions to prove the following lemma, which corresponds to the 2-dimensional case of Theorem 17 for $s = 1$.

**Lemma 24.** *Let $X \subseteq \mathbb{Z}^2_{\geqslant 0}$ be a non-empty finite set with $|X| = (\rho+1)^2$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$. If $|X + \{0,1\}^2| = (\rho + 2)^2$, then $X$ is a lattice cube.*

*Proof.* By the hypothesis on $X$, we deduce that $X$ is minimal and, consequently, by Proposition 22, it is connected. Furthermore, it must satisfy $X = \mathcal{B}(X)$. Indeed, otherwise we would get $|\mathcal{B}(X)| > |X| = (\rho + 1)^2$ and, by Lemma 23, $\mathrm{n}(\mathcal{B}(X)) \leqslant \mathrm{n}(X) = (\rho + 2)^2 - (\rho + 1)^2$, which would contradict Lemma 20.

We assume, by applying a translation to $X$ if necessary, that

$$X = \mathcal{B}(X) = \big([0, b_1] \times [0, b_2]\big) \cap \mathbb{Z}^2_{\geqslant 0}$$

for some $b = (b_1, b_2) \in \mathbb{Z}^2_{\geqslant 0}$. Then, we have

$$(\rho + 1)^2 = |X| = (b_1 + 1)(b_2 + 1) \quad \text{and} \quad 2(\rho + 1) + 1 = \mathrm{n}(X) = b_1 + b_2 + 3,$$

and thus,

$$\rho + 1 = \sqrt{(b_1 + 1)(b_2 + 1)} = \frac{(b_1 + 1) + (b_2 + 1)}{2}.$$

Hence, the equality condition in the well-known arithmetic-geometric mean inequality (see, e.g., [3, page 71]) implies $\rho = b_1 = b_2$, as desired. $\square$

### 3.2 Characterization in general dimension

The proof when $n > 2$ in Theorem 17 is based on a process that we call "normalization" (see Definition 28). It extends the process of normalization first introduced in [25, Section 4] and adapts it to the $L_\infty$ setting.

The following lemma shows that the functional $\mathrm{n}(\cdot)$ can be estimated in terms of the sections of the set.

**Lemma 25.** *Let $n > 1$ and $i \in \{1, \dots, n\}$, and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set. Then*

$$\mathrm{n}(X) \geqslant \left| \Big( \bigcup_{m \in \mathbb{Z}_{\geqslant 0}} X^i(m) \Big) + \{0,1\}^{n-1} \right| + \sum_{m \in \mathbb{Z}_{\geqslant 0}} \left| \mathrm{N}^{n-1}_{X^i(m)} \right| . \tag{21}$$

*Furthermore, if the sections $X^i(m)$ form a decreasing sequence, namely, $X^i(0) \supset X^i(1) \supset \dots$, then equality holds in* (21).

*Proof.* In order to prove (21), we consider the sets

$$Y_m = \{y \in X : y_i = m\} + \big(\{0,1\}^{i-1} \times \{0\} \times \{0,1\}^{n-i}\big)$$

for all $m \in \mathbb{Z}_{\geqslant 0}$ such that $\{y \in X : y_i = m\} \neq \emptyset$, and

$$Y = \{y \in X + \{0,1\}^n : y + k\mathrm{e}_i \notin X + \{0,1\}^n \text{ for all } k > 0\}.$$

Clearly, the sets $Y_m$ are pairwise disjoint and do not intersect with $Y$. Furthermore, $Y_m \subseteq X + \{0,1\}^n$ and $|Y_m| = |X^i(m) + \{0,1\}^{n-1}|$ for all $m \in \mathbb{Z}_{\geqslant 0}$ with $X^i(m) \neq \emptyset$. Moreover, observe that $|Y| = \big|Y|\mathrm{e}_i^\perp\big|$ and

$$Y|\mathrm{e}_i^\perp = (X + \{0,1\}^n)|\mathrm{e}_i^\perp = \Big( \bigcup_{m \in \mathbb{Z}_{\geqslant 0}} X^i(m) \Big) + \{0,1\}^{n-1}.$$

Therefore,

$$|X + \{0,1\}^n| \geqslant |Y| + \sum_{m \in \mathbb{Z}_{\geqslant 0}} |Y_m| = \left| \Big( \bigcup_{m \in \mathbb{Z}_{\geqslant 0}} X^i(m) \Big) + \{0,1\}^{n-1} \right| + \sum_{\substack{m \in \mathbb{Z}_{\geqslant 0} \\ X^i(m) \neq \emptyset}} \left| X^i(m) + \{0,1\}^{n-1} \right| .$$

By subtracting $|X| = \sum_{m \in \mathbb{Z}_{\geqslant 0}} |X^i(m)|$ we conclude the proof of (21).

If we have $m_0 \in \mathbb{Z}_{\geqslant 0}$ such that $X^i(m) = \emptyset$ for all $m > m_0$ and

$$X^i(0) \supset X^i(1) \supset X^i(2) \supset \dots \supset X^i(m_0) \neq \emptyset,$$

then

$$\left| \{y \in X + \{0,1\}^n : y_i = 0\} \right| = \left| X^i(0) + \{0,1\}^{n-1} \right|$$

and

$$\left| \{y \in X + \{0,1\}^n : y_i = m+1\} \right| = \left| X^i(m) + \{0,1\}^{n-1} \right|$$

for all $0 \leqslant m \leqslant m_0$. So, we conclude that

$$\begin{aligned} |X+\{0,1\}^n| &= \sum_{m=0}^{m_0+1} \Big|\{y \in X+\{0,1\}^n : y_i = m\}\Big| \\ &= \Big|X^i(0)+\{0,1\}^{n-1}\Big| + \sum_{m=0}^{m_0} \Big|X^i(m)+\{0,1\}^{n-1}\Big| \\ &= \left|\Big(\bigcup_{m\in\mathbb{Z}_{\geqslant 0}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{\substack{m\in\mathbb{Z}_{\geqslant 0}\\ X^i(m)\neq\emptyset}} \Big|X^i(m)+\{0,1\}^{n-1}\Big|, \end{aligned}$$

as desired. □

We note that any minimal set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ must reach equality in Lemma 25: indeed, simply by changing each section $X^i(m)$ by an initial segment in $\mathbb{Z}^{n-1}_{\geqslant 0}$ of the same cardinality, and then rearranging the sections in decreasing order, we get a new set $Z \subseteq \mathbb{Z}^n_{\geqslant 0}$ that gives equality in Lemma 25. Therefore, for some $m_0 \in \mathbb{Z}_{\geqslant 0}$,

$$\begin{aligned} \mathrm{n}(Z) &= \left|\Big(\bigcup_{m\in\mathbb{Z}_{\geqslant 0}} Z^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \Big|\mathrm{N}^{n-1}_{Z^i(m)}\Big| \\ &= \big|Z^i(0)+\{0,1\}^{n-1}\big| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \Big|\mathrm{N}^{n-1}_{Z^i(m)}\Big| \\ &\leqslant \big|X^i(m_0)+\{0,1\}^{n-1}\big| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \Big|\mathrm{N}^{n-1}_{X^i(m)}\Big| \qquad (22) \\ &\leqslant \left|\Big(\bigcup_{m\in\mathbb{Z}_{\geqslant 0}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \Big|\mathrm{N}^{n-1}_{X^i(m)}\Big| \\ &\leqslant \mathrm{n}(X), \end{aligned}$$

and thus $\mathrm{n}(Z) = \mathrm{n}(X)$ due to the minimality of $X$. This allows us to deduce the following result.

**Corollary 26.** *Let $n > 1$ and $i \in \{1, \dots, n\}$, and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a minimal set. Then the sections $X^i(m)$ are minimal (as $(n-1)$-dimensional sets) and $X$ satisfies*

$$\mathrm{n}(X) = \left|\Big(\bigcup_{m\in\mathbb{Z}_{\geqslant 0}} X^i(m)\Big)+\{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \Big|\mathrm{N}^{n-1}_{X^i(m)}\Big|. \qquad (23)$$

We note that the converse is not true: there are examples of non-minimal sets satisfying (23) for all $i = 1, \dots, n$, and having all $(n-1)$-dimensional sections minimal (see Figure 4).

The following result shows, roughly speaking, that in order to minimize the expression $\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b)$ for $a, b \in \mathbb{Z}_{\geqslant 0}$ with $a + b$ fixed, one may begin by choosing $a, b$ such

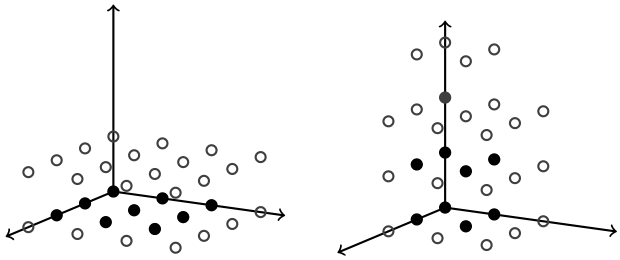

Figure 4: Left: A set $X \subseteq \mathbb{Z}^3_{\geqslant 0}$ (in black) and $X + \{0,1\}^3$ (in white). Right: $\mathcal{J}_9 \subseteq \mathbb{Z}^3_{\geqslant 0}$ (in black) and $\mathcal{J}_9 + \{0,1\}^3$ (in white). $X$ satisfies (23) and its 2-dimensional sections are minimal, but $\mathrm{n}(X) = 23 > 22 = \mathrm{n}(\mathcal{J}_9)$.

that one of the resulting initial segments is the largest possible lattice box of the form $\{0,\dots,\rho-1\}^j \times \{0,\dots,\rho\}^{n-j}$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$ and $j \in \{1,\dots,n\}$. Furthermore, it shows that a single initial segment $\mathcal{J}_{a+b}$ does never exceed this minimum.

**Lemma 27.** *Let $a, b, c \in \mathbb{Z}_{>0}$ with $\max\{a,b\} < c < a+b$ and such that $c = \rho^j(\rho+1)^{n-j}$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$ and $j \in \{1,\dots,n\}$. Then*

$$\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b) \geqslant \mathrm{n}(\mathcal{J}_{a+b-c}) + \mathrm{n}(\mathcal{J}_c). \tag{24}$$

*Moreover,*

$$\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b) > \mathrm{n}(\mathcal{J}_{a+b}). \tag{25}$$

*Proof.* We proceed by induction on the dimension $n$. Both inequalities are clear for $n = 1$ since for every initial segment $\mathcal{J}_r \subseteq \mathbb{Z}_{\geqslant 0}$ we have $\mathrm{n}(\mathcal{J}_r) = 1$.

Assume now that $n > 1$ and that the lemma holds for every dimension up to $n-1$. It suffices to show that if $1 < a \leqslant b < c = \rho^j(\rho+1)^{n-j}$, then it is possible to find $d \in \mathbb{Z}_{\geqslant 0}$ with $0 < d < a$ and $d \leqslant c - b$ such that

$$\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b) \geqslant \mathrm{n}(\mathcal{J}_{a-d}) + \mathrm{n}(\mathcal{J}_{b+d}), \tag{26}$$

and iterating this process will prove the lemma. Indeed, notice on the one hand that the conditions above imply that $1 \leqslant a - d < b + d \leqslant c$, and if $c < a + b$, it is easy to check that the process will necessarily conclude when the upper bound is reached, i.e., (24). On the other hand, if $c \geqslant a + b$, then it will necessarily conclude when the lower bound is reached, i.e.,

$$\mathrm{n}(\mathcal{J}_a) + \mathrm{n}(\mathcal{J}_b) \geqslant \mathrm{n}(\mathcal{J}_1) + \mathrm{n}(\mathcal{J}_{a+b-1}),$$

which implies (25) since Lemma 8 yields

$$2^n - 1 + \mathrm{n}(\mathcal{J}_{a+b-1}) > \mathrm{n}(\mathcal{J}_{a+b}).$$

Now, in order to prove (26), we let $x, y \in \mathbb{Z}^n_{\geqslant 0}$ be the last points with respect to $\prec$ in $\mathcal{J}_a$, $\mathcal{J}_b$, respectively. Also, for the sake of brevity, we denote by $G, H \subseteq \mathbb{Z}^{n-1}_{\geqslant 0}$ the last non-empty sections of $\mathcal{J}_a, \mathcal{J}_b \subseteq \mathbb{Z}^n_{\geqslant 0}$, i.e., $G = (\mathcal{J}_a)^{i_x}(m_x)$ and $H = (\mathcal{J}_b)^{i_y}(m_y)$.

We note that since $1 < a$, the set $\mathcal{J}_a$ has at least two non-empty sections (with respect to the direction $e_{i_x}$), and therefore $|G| < a$. Using Remark 13 (in particular, (16) and (17)), we know that

$$a = m_x^{i_x}(m_x+1)^{n-i_x} + |G| \quad \text{and} \quad b = m_y^{i_y}(m_y+1)^{n-i_y} + |H| \tag{27}$$

with

$$|G| \leqslant m_x^{i_x-1}(m_x+1)^{n-i_x} \quad \text{and} \quad |H| \leqslant m_y^{i_y-1}(m_y+1)^{n-i_y}. \tag{28}$$

Also, since $a \leqslant b$ then $x \preceq y$, and thus either $m_x < m_y$, or $m_x = m_y$ with $i_x \geqslant i_y$. This implies (see Remark 14) that

$$|G| \leqslant m_x^{i_x-1}(m_x+1)^{n-i_x} \leqslant m_y^{i_y-1}(m_y+1)^{n-i_y}. \tag{29}$$

Likewise, since

$$m_y^{i_y}(m_y+1)^{n-i_y} < b < c = \rho^j(\rho+1)^{n-j},$$

then Remark 14 implies that $c \geqslant m_y^{i_y-1}(m_y+1)^{n-i_y+1}$. This, together with (27), shows that

$$m_y^{i_y-1}(m_y+1)^{n-i_y} - |H| \leqslant c - b. \tag{30}$$

Now, we consider the following cases, which are exhaustive as a consequence of (28):

(i) $|G| > |H|$.

(ii) $|G| \leqslant |H| < m_y^{i_y-1}(m_y+1)^{n-i_y}$.

(iii) $|G| \leqslant |H| = m_y^{i_y-1}(m_y+1)^{n-i_y}$ and $i_y > 1$.

(iv) $|G| \leqslant |H| = m_y^{i_y-1}(m_y+1)^{n-i_y}$ and $i_y = 1$.

In case (i) we choose $d = |G| - |H|$, and so we may, roughly speaking, interchange the last sections $G$ and $H$, i.e., we have $H = (\mathcal{J}_{a-d})^{i_x}(m_x)$ and $G = (\mathcal{J}_{b+d})^{i_y}(m_y)$. The rest of the sections (and their union) remain the same, i.e., $(\mathcal{J}_{a-d})^{i_x}(m) = (\mathcal{J}_a)^{i_x}(m)$ for all $0 \leqslant m < m_x$ and $(\mathcal{J}_{b+d})^{i_y}(m) = (\mathcal{J}_b)^{i_y}(m)$ for all $0 \leqslant m < m_y$. Therefore, by using Corollary 26, we get (26) with equality. Clearly $0 < d \leqslant |G| < a$, and $d \leqslant c - b$ follows from (29) and (30).

In case (ii) we use the induction hypothesis (in dimension $n-1$) with $\bar{a} = |G|$, $\bar{b} = |H|$ and $\bar{c} = m_y^{i_y-1}(m_y+1)^{n-i_y}$. We choose $d = \min(\bar{a}, \bar{c} - \bar{b})$ and so, we get

$$\left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}}}\right| + \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{b}}}\right| \geqslant \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}+\bar{b}-\bar{c}}}\right| + \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{c}}}\right|$$

if $\bar{c} < \bar{a} + \bar{b}$, and

$$\left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}}}\right| + \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{b}}}\right| > \left|\mathrm{N}^{n-1}_{\mathcal{J}_{\bar{a}+\bar{b}}}\right|$$

if $\bar{c} \geqslant \bar{a} + \bar{b}$. Again, from Remark 13 we get $(\mathcal{J}_{a-d})^{i_x}(m) = (\mathcal{J}_a)^{i_x}(m)$ for all $0 \leqslant m < m_x$ and $(\mathcal{J}_{b+d})^{i_y}(m) = (\mathcal{J}_b)^{i_y}(m)$ for all $0 \leqslant m < m_y$, and the union of the sections remains likewise unchanged. Thus, an application of Corollary 26 yields (26). We again trivially have $0 < d \leqslant |G| < a$, and $d \leqslant c - b$ follows directly from (30).

In case (iii) we have $\mathcal{J}_b = \{0, \ldots, m_y - 1\}^{i_y - 1} \times \{0, \ldots, m_y\}^{n - i_y + 1}$ (see Remark 13). Therefore, we may choose $d = |G|$, and, by applying Remark 13 again, we get that the only non-empty sections of $\mathcal{J}_{a-d}$ are $(\mathcal{J}_{a-d})^{i_x}(m) = (\mathcal{J}_a)^{i_x}(m)$ for all $0 \leqslant m < m_x$. Moreover, the only non-empty sections of $\mathcal{J}_{b+d}$ are $(\mathcal{J}_{b+d})^{i_y - 1}(m) = (\mathcal{J}_b)^{i_y-1}(m)$ for all $0 \leqslant m < m_y$ and $(\mathcal{J}_{b+d})^{i_y-1}(m_y) = G$. So, since the union of all these sections has not changed, by using Corollary 26 we obtain (26), once more with equality. It is straightforward that $0 < d = |G| < a$, and since $b = m_y^{i_y - 1}(m_y + 1)^{n - i_y + 1}$ and $b < c$, Remark 14 implies $c \geqslant m_y^{i_y - 2}(m_y + 1)^{n - i_y + 2}$, and so from (29) it follows that $d \leqslant c - b$.

Finally, in case (iv) we have $\mathcal{J}_b = \{0, \ldots, m_y\}^n$. Again, we may choose $d = |G|$, which yields the same sections for $\mathcal{J}_{a-d}$ as in the previous case, whereas for the non-empty sections of $\mathcal{J}_{b+d}$ we have $(\mathcal{J}_{b+d})^n(m) = (\mathcal{J}_b)^n(m)$ for all $0 \leqslant m \leqslant m_y$ and $(\mathcal{J}_{b+d})^n(m_y + 1) = G$. Once more, Corollary 26 yields (26) with equality. It is again trivial that $0 < d = |G| < a$, and this time, since $b = (m_y + 1)^n$ and $b < c$, Remark 14 implies $c \geqslant (m_y + 1)^{n-1}(m_y + 2)$, and so $d \leqslant c - b$ follows from (29).

This completes the proof of (26), and thus, of the result. □

Now, for any $a, n \in \mathbb{Z}_{>0}$, let $C \subseteq \mathcal{J}_a \subseteq \mathbb{Z}^n_{\geqslant 0}$ be the largest lattice box (with respect to the cardinality) of the form $C = \{0, \ldots, \rho - 1\}^j \times \{0, \ldots, \rho\}^{n-j}$ for some $\rho := \rho(a, n)$, $j := j(a, n) \in \mathbb{Z}_{\geqslant 0}$, $1 \leqslant j \leqslant n$. Then, we denote by $c(a, n) = |C| = \rho^j(\rho + 1)^{n-j}$. Furthermore, for any $i \in \{1, \ldots, n\}$ and any non-empty set $X \subseteq \mathbb{Z}^n_{\geqslant 0}$, we denote by $c^i(X) = \max_{m \in \mathbb{Z}_{\geqslant 0}} c(|X^i(m)|, n - 1)$.

We proceed to define the notion of "normalization", which extends the normalization process defined in [25] and also utilized, among others, in [20].

**Definition 28.** Let $n > 1$ and $k \in \{1, \ldots, n\}$, and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set. Let $\rho \in \mathbb{Z}_{\geqslant 0}$ and $j \in \{1, \ldots, n - 1\}$ be such that $c^k(X) = \rho^j(\rho + 1)^{n-1-j}$. The $k$-normalization of $X$, denoted by $\overline{X}_k \subseteq \mathbb{Z}^n_{\geqslant 0}$, is the result of the following process:

(i) Replacing each non-empty section $X^k(m)$, $m \in \mathbb{Z}_{\geqslant 0}$, by the $(n - 1)$-dimensional initial segment of the same cardinality.

(ii) Reordering the sections in decreasing order (with respect to set inclusion) such that the largest section corresponds to $m = 0$.

(iii) Starting with $m_1 = 1$ and $m_2 = \max\{m \in \mathbb{Z}_{\geqslant 0} : X^k(m) \neq \emptyset\}$, and while $m_1 \leqslant \rho < m_2$, we repeat both of these steps:

1. If $|X^k(m_1)| < c^k(X)$, we replace the sections $X^k(m_1)$ and $X^k(m_2)$ by the initial segments of cardinality $|X^k(m_1)| + h$ and $|X^k(m_2)| - h$, respectively, where $h = \min\{|X^k(m_2)|, c^k(X) - |X^k(m_1)|\}$.
2. If $|X^k(m_2)| = 0$, we decrease $m_2$ by 1, whereas if $|X^k(m_1)| = c^k(X)$, we increase $m_1$ by 1.

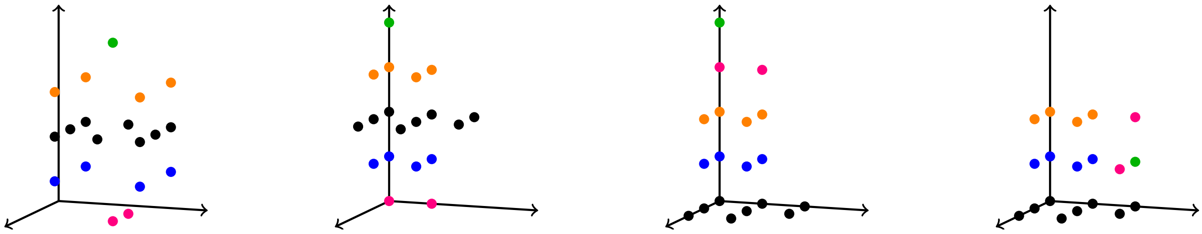

Figure 5: From left to right: a finite set, together with the same set after each step of the 3-normalization is applied.

Furthermore, we say that $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ is stable if $X = \overline{X}_k$ for all $k = 1, \dots, n$.

*Remark* 29. We note that the end result $\overline{X}_k$ of a $k$-normalization, $k \in \{1, \dots, n\}$, is a set such that its non-empty sections, $(\overline{X}_k)^k(m)$, $m \in \mathbb{Z}_{\geqslant 0}$, are $(n-1)$-dimensional initial segments ordered in decreasing order, i.e., $(\overline{X}_k)^k(0) \supset (\overline{X}_k)^k(1) \supset (\overline{X}_k)^k(2) \supset \dots$, and we have either $(\overline{X}_k)^k(\rho+1) = \emptyset$ or $|(\overline{X}_k)^k(m)| = c^k(X)$ for all $m \in \{1, \dots, \rho\}$ (where $\rho$ is as in Definition 28).

Next we show that both the rank and the functional $\mathrm{n}(\cdot)$ do not increase under the normalization process.

**Lemma 30.** *Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set. If $X \neq \overline{X}_k$, for some $k \in \{1, \dots, n\}$, then $r(X) > r(\overline{X}_k)$.*

*Proof.* We proceed by proving that if any of the 3 steps of the normalization changes the set, then the rank of $X$ strictly decreases.

First, it is a straightforward computation from the definition of the order $\prec$ that if some section is not an ($(n-1)$-dimensional) initial segment, then the ($n$-dimensional) rank of the set will decrease under step 1.

Next, since all sections are initial segments, $|X^k(m)| < |X^k(m')|$ for some $m < m'$ implies $X^k(m) \subseteq X^k(m')$. Therefore, interchanging them is equivalent to translating the points $\big(X^k(m') \setminus X^k(m)\big) \times \{m'\}$ by reducing their $k$-th coordinate by $m' - m$, which decreases the rank strictly due to Remark 7.

In the third step, if we move a point $z$ with $m_z \geqslant z_k > \rho$ (with $\rho$ as specified in Definition 28) to a point $y \in \mathbb{Z}^n_{\geqslant 0}$ with $m_y \leqslant \rho$, then again it is clear from the definition of the order $\prec$ that the rank strictly decreases. ☐

**Lemma 31.** *Let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set. Then $\mathrm{n}(X) \geqslant \mathrm{n}(\overline{X}_k)$ for all $k = 1, \dots, n$.*

*Proof.* Let $k \in \{1, \dots, n\}$. To begin with, we prove that the first two steps of the normalization process do not increase $\mathrm{n}(\cdot)$. Let us denote this resulting intermediate set by $Z$. By construction, we know there exists a permutation $\sigma : \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ such that, for every $m \in \mathbb{Z}^n_{\geqslant 0}$, $Z^k(m)$ is either empty or an initial segment with $|Z^k(m)| = |X^k(\sigma(m))|$. Then, the minimality of the initial segments (cf. Lemma 9) implies that $|\mathrm{N}^{n-1}_{Z^k(m)}| \leqslant |\mathrm{N}^{n-1}_{X^k(\sigma(m))}|$

for every $m \in \mathbb{Z}_{\geqslant 0}$, and, taking into account that the sections of $Z$ form a decreasing sequence, Lemma 25 yields $\mathrm{n}(Z) \leqslant \mathrm{n}(X)$ (cf. (22)).

To finish, we prove that the third step of the normalization process does not increase $\mathrm{n}(\cdot)$ either. We observe that the equality case in Lemma 25 gives

$$\begin{aligned} \mathrm{n}(Z) &= \left|Z^k(0) + \{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \left|\mathrm{N}^{n-1}_{Z^k(m)}\right| \text{ and} \\ \mathrm{n}(\overline{X}_k) &= \left|\left(\overline{X}_k\right)^k(0) + \{0,1\}^{n-1}\right| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \left|\mathrm{N}^{n-1}_{\left(\overline{X}_k\right)^k(m)}\right|. \end{aligned} \tag{31}$$

Note that $Z^k(0) = \left(\overline{X}_k\right)^k(0)$. Now, we let $m_1$, $m_2$ and $h$ be as in the third step of Definition 28, and we set $a = |Z^k(m_1)|$, $b = |Z^k(m_2)|$ and $c = c^k(Z)$. So, clearly $h = \min\{b, c-a\}$. Then, on the one hand, if $h = b \leqslant c - a$, we have $|Z^k(m_1)| + h = a + b$ and $|Z^k(m_2)| - h = 0$, and thus (25) (in dimension $n-1$) ensures that this step strictly decreases the sum of the cardinalities of the above $(n-1)$-dimensional neighborhoods. On the other hand, if $h = c - a < b$, then $c < a + b$, and as per Definition 28 we also clearly have $\max\{a, b\} < c$. Therefore (24) (in dimension $n-1$) again yields that this step does not increase the sum of the cardinalities of the $(n-1)$-dimensional neighborhoods above. Consequently, from (31) we conclude that $\mathrm{n}(Z) \geqslant \mathrm{n}(\overline{X}_k)$, as desired. $\square$

The stability property allows us to decompose the set in a precise way:

**Lemma 32.** *Let $n \geqslant 3$, $\rho \geqslant 1$ and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set with $|X| = (\rho+1)^n$. If $X$ is stable, then there exist $A, B \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that*

$$A \subseteq \{0, \dots, \rho-1\}^{n-1} \times \{\rho+1\}, \qquad \emptyset \neq B \subseteq \{\rho\} \times \{0, \dots, \rho\}^{n-1}$$

*and*

$$X = A \cup B \cup \left(\{0, \dots, \rho-1\} \times \{0, \dots, \rho\}^{n-1}\right).$$

*Proof.* For the sake of brevity we write $C = \{0, \dots, \rho\}^n$. If $X = C$, then the result holds by taking $A = \emptyset$ and $B = \{\rho\} \times \{0, \dots, \rho\}^{n-1}$, and so we assume that $X \neq C$.

For any $i \in \{1, \dots, n\}$, since $X$ is stable, we know that the non-empty sections $X^i(m)$ are initial segments verifying

$$X^i(0) \supset X^i(1) \supset X^i(2) \supset \dots, \tag{32}$$

and so $(\rho, \dots, \rho) \notin X$ because $X \neq C$.

First, we show that if $X^i(m) \neq \emptyset$ for some $i \in \{1, \dots, n\}$ and $m \geqslant 2$, then we must have $\{0, \dots, m-2\}^n \subseteq X$. To see this, let $x \in X$ with $x_i = m$. Then, for any $j \neq i$, since $X^j(x_j)$ is an initial segment, we get $\{0, \dots, m-1\}^{n-1} \subseteq X^j(x_j)$. Fixing such an index $j \neq i$, this implies in particular that $(m-1, \dots, m-1, x_j, m-1, \dots, m-1) \in X$, and so just like before, for any $k \neq j$, we obtain that $\{0, \dots, m-2\}^{n-1} \subseteq X^k(m-1)$, since $X^k(m-1)$ is an initial segment (observe how it is crucial in this step that $n \geqslant 3$).

This in particular implies that $(m-2,\dots,m-2)\in X$, which together with (32) yields $\{0,\dots,m-2\}^n\subseteq X$, as desired.

It is easy to check that the previous property applied to $m=\rho+1$ and $m=\rho+2$, respectively, together with the fact that $X\neq C$, yields

$$\{0,\dots,\rho-1\}^n\subseteq X\subseteq\{0,\dots,\rho+1\}^n.$$

In fact, we further have

$$X\subseteq\{0,\dots,\rho\}^{n-1}\times\{0,\dots,\rho+1\}. \tag{33}$$

Indeed, if $X^i(\rho+1)\neq\emptyset$ for some $i<n$, then there exists $x\in X$ such that $x_i=\rho+1$. Fixing any $j\neq i,n$ and using a very similar argument to the previous one, exploiting that $X^j(x_j)$ is an initial segment, we obtain that $\rho\,\mathrm{e}_i+(\rho+1)\mathrm{e}_n\in X$. But this yields $\{0,\dots,\rho\}^{n-1}\subseteq X^i(\rho)$ (since $X^i(\rho)$ is an initial segment), contradicting that $(\rho,\dots,\rho)\notin X$. This proves (33), as desired.

Now, on the one hand, (33) yields $X^1(\rho)\neq\emptyset$, since otherwise we would have

$$|X|\leqslant|\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-2}\times\{0,\dots,\rho+1\}|=\rho(\rho+1)^{n-2}(\rho+2)<(\rho+1)^n,$$

a contradiction. Since $X^1(\rho)$ is an initial segment then $(0,\dots,0)\in X^1(\rho)$, and thus $(\rho,0,\dots,0)\in X$. Hence, $(\rho,0,\dots,0)\in X^n(0)$, and therefore, since $X^n(0)$ is an initial segment, we have

$$\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-2}\subseteq X^n(0). \tag{34}$$

Both (33) and (34), together with the fact that $X\neq C$ and thus $X^n(0)\neq\{0,\dots,\rho\}^{n-1}$, yield the bounds

$$\rho(\rho+1)^{n-2}\leqslant|X^n(0)|<(\rho+1)^{n-1}.$$

This implies that $c^n(X)=\rho(\rho+1)^{n-2}$.

On the other hand, since $X\neq C$, it follows from (33) that $X^n(\rho+1)\neq\emptyset$. Therefore, as $X$ is stable (and thus $X=\overline{X}_n$), Remark 29 for $k=n$ implies that, for all $m=0,\dots,\rho$, we have $|X^n(m)|\geqslant\rho(\rho+1)^{n-2}$. Consequently, since $X^n(m)$ is an initial segment, we have

$$\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-2}\subseteq X^n(m) \tag{35}$$

for all $m=0,\dots,\rho$.

Finally, we note that, in fact,

$$X^n(\rho+1)\subseteq\{0,\dots,\rho-1\}^{n-1}. \tag{36}$$

Indeed, if $x\in X^n(\rho+1)$ with $x_i=\rho$ for $i<n$, then since $X^i(\rho)$ is an initial segment, we would have $\{0,\dots,\rho\}^{n-1}\subseteq X^i(\rho)$, contradicting that $(\rho,\dots,\rho)\notin X$.

This concludes the proof by setting $A=X^n(\rho+1)\times\{\rho+1\}$ and $B=\{\rho\}\times X^1(\rho)$, as a consequence of (33), (35) and (36). ☐

Finally, we prove that in order to characterize the lattice cubes we only need stability and, either cardinality $2^n$ or minimality.

**Corollary 33.** *Let $n \geqslant 3$ and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set with $|X| = 2^n$. If $X$ is stable, then $X = \{0,1\}^n$.*

*Proof.* Let $A, B \subseteq \mathbb{Z}^n_{\geqslant 0}$ be the sets arising from Lemma 32 for $\rho = 1$. We notice that $X = \{0,1\}^n$ if and only if $A = \emptyset$. Therefore, if $X \neq \{0,1\}^n$, then we must have $|A| = 1$ and, since $|A| + |B| = 2^{n-1}$, we also have $|B| = 2^{n-1} - 1$. Moreover, since $X$ is stable, $B$ is an $(n-1)$-dimensional initial segment, and so we have

$$X = (\{0,1\}^n \setminus (1,\dots,1)) \cup (0,\dots,0,2).$$

This contradicts the stability of $X$ since $\overline{X}_n = \{0,1\}^n \neq X$. ☐

**Lemma 34.** *Let $n \geqslant 3$, $\rho \geqslant 2$ and let $X \subseteq \mathbb{Z}^n_{\geqslant 0}$ be a non-empty finite set with $|X| = (\rho+1)^n$. If $X$ is minimal and stable, then $X = \{0,\dots,\rho\}^n$.*

*Proof.* Assume that $X \neq \{0,\dots,\rho\}^n$ and let $A, B \subseteq \mathbb{Z}^n_{\geqslant 0}$ be the sets arising from Lemma 32. Observe that, since $X$ is not a lattice cube, the set $A \neq \emptyset$. Then, $|A| > 0$ and

$$|A| + |B| = (\rho+1)^{n-1}. \tag{37}$$

If $(A+\{0,1\}^n)\cap(B+\{0,1\}^n) = \emptyset$ then, since $A = X^n(\rho+1)\times\{\rho+1\}$ and $B = \{\rho\}\times X^1(\rho)$, we clearly have, on the one hand, that

$$\mathrm{n}(X) = \left|\mathrm{N}^n_{\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-1}}\right| + \left|\mathrm{N}^{n-1}_{X^n(\rho+1)}\right| + \left|\mathrm{N}^{n-1}_{X^1(\rho)}\right|.$$

On the other hand, the minimality of $X$ yields

$$\mathrm{n}(X) = \mathrm{n}(\{0,\dots,\rho\}^n) = \left|\mathrm{N}^n_{\{0,\dots,\rho-1\}\times\{0,\dots,\rho\}^{n-1}}\right| + \left|\mathrm{N}^{n-1}_{\{0,\dots,\rho\}^{n-1}}\right|,$$

a contradiction because (25) for $a = \left|X^n(\rho+1)\right|$ and $b = \left|X^1(\rho)\right|$ implies (see also (37)) that

$$\left|\mathrm{N}^{n-1}_{X^n(\rho+1)}\right| + \left|\mathrm{N}^{n-1}_{X^1(\rho)}\right| > \left|\mathrm{N}^{n-1}_{\{0,\dots,\rho\}^{n-1}}\right|.$$

Now, if $\big(A+\{0,1\}^n\big)\cap\big(B+\{0,1\}^n\big) \neq \emptyset$, then $(\rho-1,0,\dots,0,\rho+1) \in A$. This, together with the fact that $A = X^n(\rho+1)\times\{\rho+1\}$ is an $(n-1)$-dimensional initial segment, implies, on the one hand, that there are $(\rho-1)\rho^{n-2}$ points in $X^n(\rho+1)$ strictly smaller, in the order $\prec$, than $(\rho-1,0,\dots,0) \in X^n(\rho+1)$, and therefore, $|A| > (\rho-1)\rho^{n-2}$. And, on the other hand, they ensure that $(\rho-2,\rho-1,\dots,\rho-1,\rho+1) \in A \subseteq X$.

From now on we will write $(x_1,\dots,\widehat{x_i},\dots,x_n)$ to indicate that the $i$-th coordinate $x_i$ does not appear in the point $(x_1,\dots,x_n)$, being henceforth a point in $\mathbb{R}^{n-1}$. Thus, considering the section $X^{n-1}(\rho-1)$, which is also an $(n-1)$-dimensional initial segment, one has that

$$(\rho,\dots,\rho,\widehat{\rho-1},\rho) \prec (\rho-2,\rho-1,\dots,\widehat{\rho-1},\rho+1) \quad \text{in} \quad X^{n-1}(\rho-1),$$

and hence $(\rho,\dots,\rho,\widehat{\rho-1},\rho) \in X^{n-1}(\rho-1)$, i.e., $(\rho,\dots,\rho,\rho-1,\rho) \in X$.

Next we observe that, since $X$ is stable and $X^n(\rho+1) \neq \emptyset$, then the set

$$D = \{(\rho, \dots, \rho, m) \in \mathbb{Z}^n_{\geqslant 0} : m = 0, \dots, \rho\} \subseteq \{\rho\} \times \{0, \dots, \rho\}^{n-1}$$

satisfies that $D \cap X = \emptyset$, since otherwise we would have $c^n(X) = (\rho+1)^{n-1}$ and thus, by Remark 29, that $X = \{0, \dots, \rho\}^n$, a contradiction. Furthermore,

$$B = \{\rho\} \times X^1(\rho) \subseteq \{\rho\} \times \{0, \dots, \rho\}^{n-1},$$

which yields $|B| \leqslant |\{\rho\} \times \{0, \dots, \rho\}^{n-1}| - |D| = (\rho+1)^{n-1} - (\rho+1)$, and we are going to see that, in fact, equality holds. Firstly, it is easy to see that if

$$(\widehat{\rho}, \rho, \dots, \rho, \rho - 1, \rho) \prec x_0 \preceq (\widehat{\rho}, \rho, \dots, \rho),$$

for some $x_0 \in \mathbb{Z}^n_{\geqslant 0}$, then $x_0 = (\widehat{\rho}, \rho, \dots, \rho, m)$ for some $m \in \{0, \dots, \rho\}$. Since $r((\widehat{\rho}, \rho, \dots, \rho)) = (\rho+1)^{n-1}$, this implies that $r((\widehat{\rho}, \rho, \dots, \rho, \rho-1, \rho)) = (\rho+1)^{n-1} - (\rho+1)$. Now, given that $(\widehat{\rho}, \rho, \dots, \rho, \rho-1, \rho) \in X^1(\rho)$, and that $X^1(\rho)$ is an $(n-1)$-dimensional initial segment, we know that $|X^1(\rho)| \geqslant r((\widehat{\rho}, \rho, \dots, \rho, \rho-1, \rho))$. Consequently, we have that

$$|B| = \left|X^1(\rho)\right| \geqslant r((\widehat{\rho}, \rho, \dots, \rho, \rho-1, \rho)) = (\rho+1)^{n-1} - (\rho+1).$$

To sum up, $|B| = (\rho+1)^{n-1} - (\rho+1)$, and by (37), $|A| = \rho+1$. This contradicts the fact that $|A| > (\rho-1)\rho^{n-2}$, except when $n = 3$ and $\rho = 2$. In this case, the set $X$ is shown in Figure 6, and a direct computation allows us to see that $X$ is not minimal. $\square$

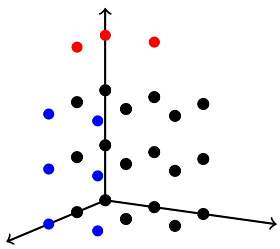

Figure 6: The stable set $X \subseteq \mathbb{Z}^3_{\geqslant 0}$ from the proof of Lemma 34. The sets $A$ and $B$ are shown in red and blue, respectively.

We are now in the position to prove Theorem 17.

*Proof of Theorem 17.* We proceed by induction on $s \in \mathbb{Z}_{>0}$. Let $s = 1$. Since $\mathcal{J}_{(\rho+1)^n} = \{0, \dots, \rho\}^n$, then $\left|\mathcal{J}_{(\rho+1)^n} + \{0,1\}^n\right| = \left|\{0, \dots, \rho+1\}^n\right| = (\rho+2)^n$, and so we have to prove that

$$\textit{if } \left|X + \{0,1\}^n\right| = (\rho+2)^n, \textit{ then } X \textit{ is a lattice cube.} \tag{38}$$

If $\rho = 0$ the result is trivial. Thus, we assume $\rho \geqslant 1$ and we proceed by induction on the dimension. If $n = 1$ then, in order to have $\left|(X + \{0,1\}) \backslash X\right| = 1$, necessarily it must be $X = \{0, \dots, \rho\}$ up to translations, i.e., a lattice cube. The case $n = 2$ is Lemma 24.

So we assume $n \geqslant 3$. Then, there exists a sequence of sets $\{X_j\}_{j=1}^r$ given recursively by $X_{j+1} = \overline{(X_j)}_{i_j}$ for some $i_j \in \{1,\dots,n\}$, $j = 1,\dots,r-1$, with $X_1 = X$, such that $X_r$ is stable (we recall here that $\overline{Y}_k$ is the $k$-normalization of $Y \subseteq \mathbb{Z}^n_{\geqslant 0}$, see Definition 28). Indeed, since the normalization process either leaves the set unchanged or strictly decreases its rank (see Lemma 30), which is bounded from below, such a sequence always exists.

By Lemma 9 the set $X$ is minimal, and so Lemma 31 ensures that all $X_j$ are also minimal for $j = 1,\dots,r$. Therefore, if $\rho = 1$, Corollary 33 ensures that $X_r$ is the lattice cube $\{0,1\}^n$, whereas for $\rho \geqslant 2$, Lemma 34 shows that $X_r = \{0,\dots,\rho\}^n$.

Let us now focus on $X_{r-1}$. Since $X_{r-1}$ is minimal, Corollary 26 yields

$$\mathrm{n}(X_{r-1}) = \left| \Big( \bigcup_{m\in\mathbb{Z}_{\geqslant 0}} (X_{r-1})^{i_{r-1}}(m) \Big) + \{0,1\}^{n-1} \right| + \sum_{m\in\mathbb{Z}_{\geqslant 0}} \left| \mathrm{N}^{n-1}_{(X_{r-1})^{i_{r-1}}(m)} \right|. \tag{39}$$

Moreover, we have

$$\overline{(X_{r-1})}_{i_{r-1}} = X_r = \{0,\dots,\rho\}^n.$$

We show next that this last normalization procedure does not involve the third step of the normalization process. Indeed, since all non-empty sections of the lattice cube $X_r = \{0,\dots,\rho\}^n$ are of the form $\{0,\dots,\rho\}^{n-1}$, applying step (iii) of the normalization process to $X_{r-1}$ would imply the existence of a section $(X_{r-1})^{i_{r-1}}(m_0)$, for some $m_0 \in \mathbb{Z}_{\geqslant 0}$, that becomes empty during such a step. But then, due to (39), an analogous argument to the one of the proof of Lemma 31 would show that $\mathrm{n}(X_{r-1}) > \mathrm{n}(X_r)$, contradicting the minimality of $X_{r-1}$.

Therefore, only the steps (i) and (ii) in Definition 28 are used in the last normalization $\overline{(X_{r-1})}_{i_{r-1}}$, which ensures that $X_{r-1}$ has exactly $\rho+1$ non-empty sections $(X_{r-1})^{i_{r-1}}(m)$, each of them with cardinality $(\rho+1)^{n-1}$. We also know that all these (non-empty) sections $(X_{r-1})^{i_{r-1}}(m)$ are minimal sets in $\mathbb{Z}^{n-1}_{\geqslant 0}$ (see Corollary 26), and so

$$\left| (X_{r-1})^{i_{r-1}}(m) + \{0,1\}^{n-1} \right| = (\rho+2)^{n-1}.$$

Thus, the induction hypothesis allows us to conclude that every (non-empty) section $(X_{r-1})^{i_{r-1}}(m)$ is an $(n-1)$-dimensional lattice cube. Furthermore, since $X_{r-1}$ is minimal, Proposition 22 ensures it is connected, and hence all these sections are consecutive. Finally, they must all be equal as well: indeed, otherwise, for any non-empty section $(X_{r-1})^{i_{r-1}}(m_0)$, $m_0 \in \mathbb{Z}_{\geqslant 0}$, we would have

$$(X_{r-1})^{i_{r-1}}(m_0) + \{0,1\}^{n-1} \subsetneq \Big( \bigcup_{m\in\mathbb{Z}_{\geqslant 0}} (X_{r-1})^{i_{r-1}}(m) \Big) + \{0,1\}^{n-1},$$

and we could translate the sections such that for all non-empty $(X_{r-1})^{i_{r-1}}(m)$, $m \in \mathbb{Z}_{\geqslant 0}$, we had $(X_{r-1})^{i_{r-1}}(m) = (X_{r-1})^{i_{r-1}}(m_0)$, strictly reducing the functional $\mathrm{n}(\cdot)$ (see (39)); this would contradict the minimality of $X_{r-1}$. Therefore, $X_{r-1}$ is itself a lattice cube. Since this argumentation does not depend on the index, but only on the fact that $X_r$ is a lattice cube, we can argue inductively for $j = r, \dots, 1$. In particular, $X = X_1$ is a lattice cube, which concludes the proof of the case $s = 1$. Thus we have shown (38).

Assume now that $s > 1$ and that the result holds for $s - 1$. On the one hand, Corollary 11 ensures that $|X + \{0, \dots, s-1\}^n| \geqslant |\mathcal{J}_{(\rho+1)^n} + \{0, \dots, s-1\}^n| = (\rho + s)^n$. On the other hand, if $|X + \{0, \dots, s-1\}^n| > (\rho + s)^n$, then Lemma 20 would imply that $|X + \{0, \dots, s\}^n| > (\rho + s + 1)^n = |\mathcal{J}_{(\rho+1)^n} + \{0, \dots, s\}^n|$, a contradiction. Therefore, $|X + \{0, \dots, s-1\}^n| = |\mathcal{J}_{(\rho+1)^n} + \{0, \dots, s-1\}^n|$, and thus, the induction hypothesis yields that $X$ is a lattice cube, as desired. ☐

### 3.3 Characterizations of lattice cubes via discrete isoperimetric and Brunn-Minkowski type inequalities

We are now in the position to prove Theorems 1 and 2:

*Proof of Theorem 1.* Let $r = (\rho+1)^n$. By the translation invariance of the cardinality we may assume, without loss of generality, that $X \subseteq \mathbb{Z}^n_{\geqslant 0}$. Then Corollaries 11 and 16 yield

$$|X+\{-1,0,1\}^n| = |X+\{0,1,2\}^n| \geqslant |\mathcal{J}_r+\{0,1,2\}^n| = |\mathcal{I}_r+\{0,1,2\}^n| = |\mathcal{I}_r+\{-1,0,1\}^n|.$$

Thus, if $|X+\{-1,0,1\}^n| = |\mathcal{I}_r+\{-1,0,1\}^n|$, we get that $|X+\{0,1,2\}^n| = |\mathcal{J}_r+\{0,1,2\}^n|$, and Theorem 17 shows that $X$ is a lattice cube. The converse is obvious. ☐

*Proof of Theorem 2.* Let $r = (\rho + 1)^n$. By Corollary 11 and (9) for $\mathcal{J}_r$, we have

$$|(X + \{-1,0,1\}^n) \cap \mathbb{Z}^n_{\geqslant 0}| \geqslant |X + \{0,1\}^n| \geqslant |\mathcal{J}_r + \{0,1\}^n| = |(\mathcal{J}_r + \{-1,0,1\}^n) \cap \mathbb{Z}^n_{\geqslant 0}|.$$

Thus, if equality holds in (6), we get, in particular, that $|X + \{0,1\}^n| = |\mathcal{J}_r + \{0,1\}^n|$, and Theorem 17 shows that $X$ is a lattice cube. Furthermore, it is clear that in order to have $|(X + \{-1,0,1\}^n) \cap \mathbb{Z}^n_{\geqslant 0}| = |X + \{0,1\}^n|$, it must in fact be $X = \{0, \dots, \rho\}^n$, as desired. The converse is obvious. ☐

Furthermore, as a consequence of Theorem 17, we can characterize the equality case in (2) in some particular cases:

**Corollary 35.** *Let $X \subseteq \mathbb{Z}^n$ be a finite set with $|X| = (\rho + 1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$ and let $Y$ be a lattice cube. Then*

$$\left|X + Y + \{0,1\}^n\right|^{1/n} = |X|^{1/n} + |Y|^{1/n}$$

*if and only if $X$ is a lattice cube.*

*Proof.* If $X$ is a lattice cube then Theorem A shows the result. So we assume that $\left|X + Y + \{0,1\}^n\right|^{1/n} = |X|^{1/n} + |Y|^{1/n}$, and let $Y = \{0, \dots, s\}^n$ for some $s \in \mathbb{Z}_{\geqslant 0}$. Then, by applying Corollary 11, we have

$$(\rho + s + 2)^n = |X + Y + \{0,1\}^n| \geqslant |\mathcal{J}_{(\rho+1)^n} + Y + \{0,1\}^n| = (\rho + s + 2)^n.$$

Thus, $|X + \{0, \dots, s+1\}^n| = |\mathcal{J}_{(\rho+1)^n} + \{0, \dots, s+1\}^n|$ and Theorem 17 concludes the proof. ☐

*Remark* 36. We note that there are examples of sets (even with the cardinality of a lattice cube) reaching equality in Theorem A which are not lattice cubes (see Figure 7). We point out that we have not been able to find an example in this regard different from a lattice box.

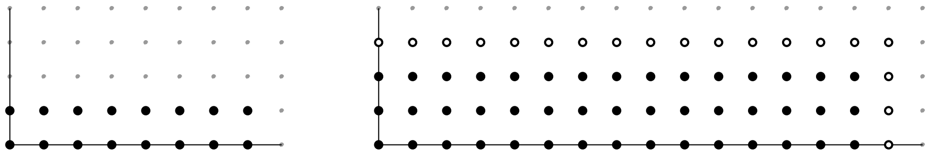

Figure 7: A set $X \subseteq \mathbb{Z}^2$ with $|X| = 16$ which is not a lattice cube (left), and $X+X+\{0,1\}^2$ (right), satisfying the equality in (2): $|X + X + \{0,1\}^2|^{1/2} = 8 = 2|X|^{1/2}$.

## 4 Isoperimetric inequality for the lattice point enumerator

This section is devoted to show the isoperimetric type inequality for the lattice point enumerator $\mathrm{G}_n(\cdot)$ given in Theorem 3. To this end, we recall the definition of extended cubes first introduced in [13]:

**Definition 37.** For a non-empty bounded set $M \subseteq \mathbb{R}^n$, we write

$$\mathcal{C}_M = \{(\lambda_1 x_1, \dots, \lambda_n x_n) \in \mathbb{R}^n : (x_1, \dots, x_n) \in M, \lambda_i \in [0,1] \text{ for } i = 1, \dots, n\}.$$

For the sake of brevity, we just write $\mathcal{C}_x := \mathcal{C}_{\{x\}}$ for any $x \in \mathbb{R}^n$.

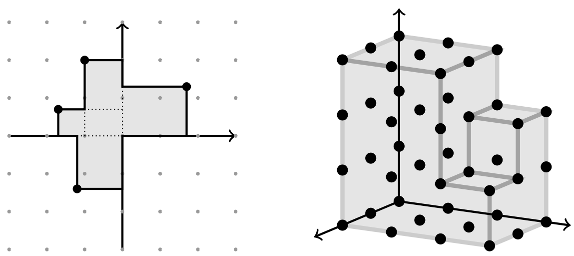

Figure 8: Left: A set $X \subseteq \mathbb{R}^2$ (four black dots) and $\mathcal{C}_X \subseteq \mathbb{R}^2$ (grey set). Right: $\mathcal{C}_{\mathcal{J}_{44}} \subseteq \mathbb{R}^3$.

As usual in the literature, we will denote the integer part of $t \geqslant 0$ by $\lfloor t \rfloor$.

*Proof of Theorem 3.* First, we show that for every $\lambda \in [0,1)$ we have

$$\mathcal{C}_{\mathcal{J}_r} + [0,\lambda]^n \subseteq \mathcal{J}_r + [0,1)^n. \tag{40}$$

Indeed, if $y \in \mathcal{C}_{\mathcal{J}_r} + [0,\lambda]^n$, then $y \in \mathcal{C}_x + [0,\lambda]^n$ for some $x \in \mathcal{J}_r$. Hence $y_i \leqslant x_i + \lambda$ and $x_i \in \mathbb{Z}_{\geqslant 0}$ for all $i = 1, \dots, n$, and so $\lfloor y_i \rfloor \leqslant x_i$, $i = 1, \dots, n$. Then Remark 7 implies that $(\lfloor y_1 \rfloor, \dots, \lfloor y_n \rfloor) \preceq x$, and therefore

$$y \in (\lfloor y_1 \rfloor, \dots, \lfloor y_n \rfloor) + [0,1)^n \subseteq \mathcal{J}_r + [0,1)^n.$$

Let $t > 0$ (the case $t = 0$ is trivial). By applying (40) with $\lambda = t - \lfloor t \rfloor$ and adding the cube $[0, \lfloor t \rfloor]^n$, we immediately get

$$\mathcal{C}_{\mathcal{J}_r} + t[0,1]^n \subseteq \mathcal{J}_r + (1 + \lfloor t \rfloor)[0,1)^n.$$

This completes the proof of the claimed inequality since, by applying Corollary 11 with $s = \lfloor t \rfloor$, we get

$$\begin{aligned} \mathrm{G}_n\left(K + t[0,1]^n\right) &\geqslant \mathrm{G}_n\left((K \cap \mathbb{Z}^n) + t[0,1]^n\right) = \left|(K \cap \mathbb{Z}^n) + \{0, \dots, \lfloor t \rfloor\}^n\right| \\ &\geqslant \left|\mathcal{J}_r + \{0, \dots, \lfloor t \rfloor\}^n\right| = \mathrm{G}_n(\mathcal{J}_r + (1 + \lfloor t \rfloor)[0,1)^n) \\ &\geqslant \mathrm{G}_n\left(\mathcal{C}_{\mathcal{J}_r} + t[0,1]^n\right). \end{aligned} \tag{41}$$

Now, assume that $\mathrm{G}_n(K) = (\rho + 1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$. In order to characterize the equality in (41), we first note that we have

$$\begin{aligned} (\mathcal{J}_r + (1 + \lfloor t \rfloor)[0,1)^n) \cap \mathbb{Z}^n = \mathcal{J}_r + \{0, \dots, \lfloor t \rfloor\}^n &= (\mathcal{C}_{\mathcal{J}_r} \cap \mathbb{Z}^n) + (t[0,1]^n \cap \mathbb{Z}^n) \\ &\subseteq (\mathcal{C}_{\mathcal{J}_r} + t[0,1]^n) \cap \mathbb{Z}^n, \end{aligned}$$

which gives equality in the last inequality of (41).

So we have equality in (41) if and only if the relations

$$\mathrm{G}_n\left(K + t[0,1]^n\right) = \mathrm{G}_n\left((K \cap \mathbb{Z}^n) + t[0,1]^n\right)$$

and

$$\left|(K \cap \mathbb{Z}^n) + \{0, \dots, \lfloor t \rfloor\}^n\right| = \left|\mathcal{J}_r + \{0, \dots, \lfloor t \rfloor\}^n\right|$$

hold. The first one is equivalent to

$$(K + t[0,1]^n) \cap \mathbb{Z}^n = (K \cap \mathbb{Z}^n) + (t[0,1]^n \cap \mathbb{Z}^n),$$

whereas the second one holds if and only if we have equality in Corollary 11, i.e., when $K \cap \mathbb{Z}^n$ is a lattice cube (see Theorem 17), as desired. □

*Remark* 38. In order to find a global minimal set, i.e., a set attaining equality in Theorem 3 for all values of $t \geqslant 0$, we observe that if we have a non-empty bounded set $K \subseteq \mathbb{R}^n$ such that $(K + [0,1)^n) \cap \mathbb{Z}^n = K \cap \mathbb{Z}^n$, then, by repeatedly adding the lattice cube $\{0,1\}^n$, one gets

$$(K + t[0,1]^n) \cap \mathbb{Z}^n = (K \cap \mathbb{Z}^n) + (t[0,1]^n \cap \mathbb{Z}^n)$$

for all $t \geqslant 0$. This shows that we have equality in Theorem 3 for all $t \geqslant 0$ if and only if $K \cap \mathbb{Z}^n$ is a lattice cube and $K$ satisfies $(K + [0,1)^n) \cap \mathbb{Z}^n = K \cap \mathbb{Z}^n$, provided $\mathrm{G}_n(K) = (\rho + 1)^n$ for some $\rho \in \mathbb{Z}_{\geqslant 0}$.

*Remark* 39. We note that the role of the set $\mathcal{C}_{\mathcal{J}_r}$ in Theorem 3 can also be played by any non-empty bounded set $M \subseteq \mathbb{R}^n$ satisfying $\mathrm{G}_n(M) = r$ and $M + [0,1)^n \subseteq \mathcal{J}_r + (-1,1)^n$. Nevertheless, $\mathcal{C}_{\mathcal{J}_r}$ are the largest sets (with respect to set inclusion) contained in $\mathbb{R}^n_{\geqslant 0}$ such that both of these properties hold. Indeed, in [13, Remark 2.1] the authors showed that the sets $\mathcal{C}_{\mathcal{I}_r}$ are the largest ones (with respect to set inclusion) satisfying $\mathrm{G}_n(\mathcal{C}_{\mathcal{I}_r}) = r$ and $\mathcal{C}_{\mathcal{I}_r} + (-1,1)^n \subseteq \mathcal{I}_r + (-1,1)^n$, and so by intersecting with the positive cone $\mathbb{R}^n_{\geqslant 0}$ we get the desired properties.

*Remark* 40. Theorem 3 also holds for an arbitrary lattice $\Lambda \subseteq \mathbb{R}^n$: if $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis of $\Lambda$, we denote by $\mathrm{G}_\Lambda(M) = |M \cap \Lambda|$ for any $M \subseteq \mathbb{R}^n$ and by $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the linear (bijective) map given by $\varphi(x) = \sum_{i=1}^n x_i v_i$ for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then Theorem 3 yields

$$\mathrm{G}_\Lambda\Big(K + t\varphi\big([0,1]^n\big)\Big) \geqslant \mathrm{G}_\Lambda\Big(\varphi(\mathcal{C}_{\mathcal{J}_r}) + t\varphi\big([0,1]^n\big)\Big)$$

for any bounded set $K \subseteq \mathbb{R}^n$ with $\mathrm{G}_\Lambda(K) = r > 0$ and all $t \geqslant 0$.

We conclude this section by showing that Theorem 3 implies Theorem D. Consequently, and due to the homogeneity and translation invariance of the volume, it will also imply the neighborhood form (4) of the isoperimetric inequality for the cube $E = [0,1]^n$.

**Proposition 41.** *The discrete inequality* (8) *implies* (7).

*Proof.* Fix $t \geqslant 0$ and $r > 0$, and let $r' = \mathrm{G}_n(C_{\mathcal{J}_r} + t[0,1]^n)$. It is clear from the definition that, if $A, B \subseteq \mathbb{R}^n_{\geqslant 0}$, then $\mathcal{C}_{A+B} = \mathcal{C}_A + \mathcal{C}_B$. In particular, since $\mathcal{J}_{r'} = \mathcal{J}_r + \{0, \dots, \lfloor t \rfloor\}^n$, we have $\mathcal{C}_{\mathcal{J}_{r'}} = \mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor [0,1]^n$. Furthermore, Corollary 16 implies that

$$\mathrm{G}_n(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n) = \mathrm{G}_n(\mathcal{C}_{\mathcal{I}_r} + \lfloor t \rfloor[-1,1]^n) = \mathrm{G}_n(\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor[-1,1]^n).$$

Therefore,

$$\begin{aligned}\mathrm{G}_n(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n) &= \mathrm{G}_n(\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor[-1,1]^n) = \mathrm{G}_n(-\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor[-1,1]^n)\\ &= \mathrm{G}_n(-\mathcal{C}_{\mathcal{J}_r} + \lfloor t \rfloor[-1,0]^n + \lfloor t \rfloor[0,1]^n) = \mathrm{G}_n(-\mathcal{C}_{\mathcal{J}_{r'}} + \lfloor t \rfloor[0,1]^n)\\ &\leqslant \mathrm{G}_n(\mathcal{C}_{\mathcal{J}_{r'}} + t[0,1]^n).\end{aligned}$$

Finally, using Theorem 3 with the set $-K$, we have

$$r' = \mathrm{G}_n(C_{\mathcal{J}_r} + t[0,1]^n) \leqslant \mathrm{G}_n(-K + t[0,1]^n) = \mathrm{G}_n(K + t[-1,0]^n).$$

Consequently, Theorem 3 applied now to the set $K + t[-1,0]^n$ yields

$$\begin{aligned}\mathrm{G}_n(\mathcal{C}_{\mathcal{I}_r} + t[-1,1]^n) \leqslant \mathrm{G}_n(\mathcal{C}_{\mathcal{J}_{r'}} + t[0,1]^n) &\leqslant \mathrm{G}_n(K + t[-1,0]^n + t[0,1]^n)\\ &= \mathrm{G}_n(K + t[-1,1]^n),\end{aligned}$$

as desired. □

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