

A New Construction of Strongly Regular Graphs with Parameters of the Complement Symplectic Graph

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Abstract

The symplectic graph $Sp(2d, q)$ is the collinearity graph of the symplectic space of dimension $2d$ over the finite field of order q . A k -regular graph on v vertices is a divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n , such that any two different vertices from the same class have λ_1 common neighbours, and any two vertices from different classes have λ_2 common neighbours whenever it is not complete or edgeless. In this paper we propose a new construction of strongly regular graphs with the parameters of the complement of the symplectic graph using divisible design graphs.

Mathematics Subject Classifications: 05C50, 05E30

1 Introduction

Definition 1. The *symplectic graph* $Sp(2d, q)$ is the collinearity graph of the symplectic space of dimension $2d$ over the finite field of order q .

For $d = 1$ the symplectic graph is a $(q + 1)$ -coclique. For $d \geq 2$ this graph is strongly regular, with parameters

$$\left(\frac{q^{2d} - 1}{q - 1}, \frac{q(q^{2d-2} - 1)}{q - 1}, \frac{q^2(q^{2d-4} - 1)}{q - 1} + q - 1, \frac{q^{2d-2} - 1}{q - 1} \right).$$

See, for example, [2, Chapter 2] for definitions and properties of finite polar spaces and their collinearity graphs.

In 2016, A. Abiad and W.H. Haemers [1] used Godsil-McKay switching to obtain strongly regular graphs with the same parameters as $Sp(2d, 2)$ for all $d \geq 3$. In 2017,

F. Ihringer [11] provided a new general construction of strongly regular graphs from the collinearity graph of a polar spaces of rank at least 3 over a finite field of order q . Recently A.E. Brouwer, F. Ihringer and W.M. Kantor [3] described a switching operation on the collinearity graphs of polar spaces to obtain new strongly regular graphs which satisfy the so-called 4-vertex condition if the original graph comes from a symplectic polar space.

The article is organized as follows. One type of divisible design graphs from [12] is presented in Section 2. In Section 3 we discuss a regular decomposition of strongly regular graph. We propose a new construction of strongly regular graphs with parameters

$$\left(\frac{q^{2d} - 1}{q - 1}, q^{2d-1}, q^{2d-2}(q - 1), q^{2d-2}(q - 1) \right)$$

which are the parameters of the complement of $Sp(2d, q)$ in Section 4. Another construction from Hoffman coloring of strongly regular graphs is given in Section 5. At the end, we discussed small examples, some open questions and the number of isomorphism classes of the graphs from our constructions.

2 Construction of divisible design graphs

Definition 2. A k -regular graph on v vertices is a *divisible design graph* with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n , such that any two different vertices from the same class have λ_1 common neighbours, and any two vertices from different classes have λ_2 common neighbours whenever it is not complete or edgeless.

The partition of a divisible design graph into classes is called a *canonical partition*.

Divisible design graphs were first introduced by W.H. Haemers, H. Kharaghani and M. Meulenberg in [9].

W.D. Wallis [17], D.G. Fon-Der-Flaass [6], and M. Muzychuk [14] proposed new constructions of strongly regular graphs based on affine designs. Similar idea is used in Construction DDG to obtain divisible design graphs.

Any d -dimensional affine space over a finite field of order q is a point-hyperplane affine design with q^d points. Any block contains q^{d-1} points, q^{d-2} points are in the intersection of any two different blocks, and the number of blocks containing any two different points is $(q^{d-1} - 1)/(q - 1)$.

Definition 3. A set \mathcal{Q} equipped with a binary operation \circ is called a *left quasigroup* if for all elements i and j in \mathcal{Q} there is a unique element h such that $i \circ h = j$. In other words, the mapping $h \mapsto i \circ h$ is a bijection of \mathcal{Q} for any $i \in \mathcal{Q}$.

This section presents a construction of divisible design graphs. This construction first appeared in [12, Construction 1]. For the construction we use a left quasigroup and affine designs.

Let $\mathcal{D}_1, \dots, \mathcal{D}_m$ be arbitrary affine designs all with parameters (q, q^{d-2}) , where $m = (q^d - 1)/(q - 1)$ is the number of parallel classes of blocks in each \mathcal{D}_i . For all $i \in [m] :=$

$\{1, 2, \dots, m\}$, let $\mathcal{D}_i = (\mathcal{P}_i, \mathcal{B}_i)$. Parallel classes in each \mathcal{D}_i are enumerated by integers from $[m]$ and j -th parallel class of \mathcal{D}_i is denoted by \mathcal{B}_i^j . For any $x \in \mathcal{P}_i$, the block in the parallel class \mathcal{B}_i^j which contains x is denoted by $B_i^j(x)$.

Let \mathcal{Q} be a left quasigroup on $[m]$ equipped with a binary operation \circ .

For every pair i, j choose an arbitrary bijection

$$\sigma_{i,j} : \mathcal{B}_i^{i \circ j} \rightarrow \mathcal{B}_j^{j \circ i}.$$

We require that $\sigma_{i,j} = \sigma_{j,i}^{-1}$ for $i \neq j$ and $\sigma_{i,i}$ is identity for all $i, j \in [m]$.

Construction DDG.

Let Γ be a graph defined as follows:

- The vertex set of Γ is $V = \bigcup_{i=1}^m \mathcal{P}_i$.
- Two different vertices $x \in \mathcal{P}_i$ and $y \in \mathcal{P}_j$ are adjacent in Γ if and only if

$$y \notin \sigma_{ij}(B_i^{i \circ j}(x)) \quad \text{for all } i, j \in [m].$$

Theorem 4. *If Γ is a graph from Construction DDG, then Γ is a divisible design graph with parameters*

$$\begin{aligned} v &= q^d(q^d - 1)/(q - 1), & k &= q^{d-1}(q^d - 1), \\ \lambda_1 &= q^{d-1}(q^d - q^{d-1} - 1), & \lambda_2 &= q^{d-2}(q - 1)(q^d - 1), \\ m &= (q^d - 1)/(q - 1), & n &= q^d. \end{aligned}$$

Other known examples of affine designs are Hadamard 3-designs with $q = 2$. Due to the peculiarities of Construction DDG, such designs cannot be used in it.

3 Regular decomposition of strongly regular graphs

Let the set of vertices of a strongly regular graph Γ admit a partition $V(\Gamma) = X_1 \cup X_2$ such that Γ_1 on X_1 and Γ_2 on X_2 are the induced subgraphs of Γ . This decomposition is called regular if Γ_1 and Γ_2 are regular, and strongly regular if Γ_1 and Γ_2 are respectively strongly regular, a clique, or a coclique. There is the incident structure \mathcal{D} having block set X_1 and point set X_2 , and incidence given by adjacency in Γ . Strongly regular graphs with strongly regular decomposition were studied by W.H. Haemers and D.G. Higman in [8]. One special case from [8, Theorem 2.4] was especially noted by E.R. van Dam when studying graphs with four distinct eigenvalues [4, Section 4.5.1]. Let Γ be a primitive strongly regular graph on v vertices, with spectrum $\{k^1, r^f, s^g\}$. If Γ_1 on X_1 is a regular graph and Γ_2 on X_2 is a coclique of size c meeting the Delsarte-Hoffman bound, i.e.,

$c = vs/(s - k)$, then the induced subgraph $\Gamma \setminus C$ is a regular, connected graph with spectrum

$$\{(k + s)^1, r^{f-c+1}, (r + s)^{c-1}, s^{g-c}\},$$

so it has four distinct eigenvalues if $c < g$.

The spectrum of any divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ can be calculated using its parameters as follows:

$$\{k^1, \sqrt{k - \lambda_1}^{f_1}, -\sqrt{k - \lambda_1}^{f_2}, \sqrt{k^2 - \lambda_2 v}^{g_1}, -\sqrt{k^2 - \lambda_2 v}^{g_2}\}.$$

Moreover, $f_1 + f_2 = m(n - 1)$ and $g_1 + g_2 = m - 1$ [9, Lemma 2.1].

Therefore, the spectrum of any divisible design graph from Theorem 4 have four distinct eigenvalues

$$\{q^d(q^{d-1} - 1), q^{d-1}, 0, -q^{d-1}\},$$

with multiplicities

$$\left\{ 1, \frac{(q^d - 1)(q^d - q)}{2(q - 1)}, \frac{q^d - q}{q - 1}, \frac{(q^d - 1)(q^d + q - 2)}{2(q - 1)} \right\}.$$

Note that the eigenvalue 0 multiplicity is $(q^d - q)/(q - 1)$. This observation allows us to use any divisible design graph from Construction DDG as Γ_1 and a coclique of size $c = (q^d - q)/(q - 1) + 1 = (q^d - 1)/(q - 1)$ as Γ_2 to construct a new strongly regular graph.

4 Construction of strongly regular graphs

In this section, a construction of strongly regular graphs with parameters

$$((q^{2d} - 1)/(q - 1), q^{2d-1}, q^{2d-2}(q - 1), q^{2d-2}(q - 1))$$

is given. These parameters are known as the parameters of the complement of $Sp(2d, q)$.

Let Γ^* be a divisible design graph with parameters

$$v^* = q^d(q^d - 1)/(q - 1), \quad k^* = q^{d-1}(q^d - 1), \\ \lambda_1^* = q^{d-1}(q^d - q^{d-1} - 1), \quad \lambda_2^* = q^{d-2}(q - 1)(q^d - 1)$$

on the vertex set V^* . The canonical partition of Γ^* consists of $m^* = (q^d - 1)/(q - 1)$ classes which have the size q^d . By [9, Theorem 3.1] the canonical partition of the adjacency matrix of Γ^* is equitable. By [9, Theorem 3.3] its quotient matrix R is exactly the same as the quotient matrix of a divisible design graph from Construction DDG, i.e. $R = q^{d-1}(q - 1)J$, where J is the all-one matrix.

Let $\mathcal{D}^* = (\mathcal{P}^*, \mathcal{B}^*)$ be a symmetric $2-((q^d - 1)/(q - 1), q^{d-1}, q^{d-2}(q - 1))$ design.

Let ϕ be an arbitrary bijection from the set of all canonical classes of Γ^* to the set of blocks \mathcal{B}^* .

Construction SRG1.

Let Γ be a graph defined as follows:

- The vertex set of Γ is $V = V^* \cup \mathcal{P}^*$.
- Two different vertices from V^* are adjacent in Γ if and only if they are adjacent in Γ^* . The set \mathcal{P}^* is a coclique in Γ . A vertex x in V^* from any canonical class \mathcal{P}_i is adjacent to a vertex y in \mathcal{P}^* if and only if y belongs to the block $\phi(\mathcal{P}_i)$, where $i \in [m^*]$.

Theorem 5. *If Γ is a graph from Construction SRG1, then Γ is a strongly regular graph with parameters*

$$((q^{2d} - 1)/(q - 1), q^{2d-1}, q^{2d-2}(q - 1), q^{2d-2}(q - 1)).$$

Proof. Let Γ be a graph from Construction SRG1. The number of vertices in Γ is equal to

$$q^d(q^d - 1)/(q - 1) + (q^d - 1)/(q - 1) = (q^{2d} - 1)/(q - 1).$$

If x is a vertex of Γ from \mathcal{P}_i , where $i \in [m^*]$, then there are

$$k^* + |\phi(\mathcal{P}_i)| = q^{d-1}(q^d - 1) + q^{d-1} = q^{2d-1}$$

vertices in $\Gamma(x)$. If y is a vertex of Γ from \mathcal{P}^* , then $\Gamma(y)$ consists of q^{d-1} canonical classes of size q^d from $V \setminus \mathcal{P}^*$. Hence, Γ is a regular graph of degree q^{2d-1} .

Let x and y be two different vertices from Γ and $\lambda(x, y)$ be the number of their common neighbours in Γ . This number depends only on the relative position of x and y in Γ^* and V^* . Consider all the possibilities of placing x and y in Γ .

- (1) If x and y belong to the same canonical class of Γ^* , then x and y have $\lambda_1^* = q^{d-1}(q^d - q^{d-1} - 1)$ common neighbours in V^* and q^{d-1} common neighbours in \mathcal{P}^* . Thus,

$$\lambda(x, y) = q^{d-1}(q^d - q^{d-1} - 1) + q^{d-1} = q^{2d-2}(q - 1).$$

- (2) If x and y belong to different canonical classes of Γ^* , then x and y have $\lambda_2^* = q^{d-2}(q - 1)(q^d - 1)$ common neighbours in V^* and $q^{d-2}(q - 1)$ common neighbours in \mathcal{P}^* . Thus,

$$\lambda(x, y) = q^{d-2}(q - 1)(q^d - 1) + q^{d-2}(q - 1) = q^{2d-2}(q - 1).$$

- (3) If x and y belong to \mathcal{P}^* , then x and y have $q^{d-2}(q - 1)$ times q^d common neighbours in V^* , and they have no common neighbours in \mathcal{P}^* . Thus,

$$\lambda(x, y) = q^d q^{d-2}(q - 1) = q^{2d-2}(q - 1).$$

- (4) If $x \in V^*$ and $y \in \mathcal{P}^*$, then x and y have $q^d - q^{d-1}$ multiplied by q^{d-1} common neighbours in V^* and have no common neighbours in \mathcal{P}^* . Thus,

$$\lambda(x, y) = q^{d-1}q^d - q^{d-1} = q^{2d-2}(q - 1).$$

Therefore, in all cases the number of common neighbours of x and y in Γ equals $q^{2d-2}(q - 1)$. Hence, Γ is a strongly regular graph with parameters

$$((q^{2d} - 1)/(q - 1), q^{2d-1}, q^{2d-2}(q - 1), q^{2d-2}(q - 1)).$$

This completes the proof of Theorem 5. □

5 Construction from Hoffman coloring

Let Γ be a strongly regular graph of valency k and smallest eigenvalue s . By P. Delsarte [5] clique in Γ of size $1 - k/s$ is called a *Delsarte clique*.

A natural question arises: is it possible to construct a strongly regular graph using a Delsarte clique instead of a coclique on \mathcal{P}^* in Construction SRG1. The answer is yes, but we must use a different divisible design graph.

A.J. Hoffman [10] proved that the chromatic number of any graph with largest eigenvalue k and smallest eigenvalue s is at least $vs/(s - k)$. The coloring of a graph meeting this bound is called a Hoffman coloring. D. Panasenko and L. Shalaginov [15] found divisible design graphs using Hoffman coloring of strongly regular graphs with parameters $(v, k, \mu + 2, \mu)$.

Suppose Δ is a strongly regular graph with parameters $(v, k, \mu + 2, \mu)$ and has a Hoffman coloring with Hoffman cocliques of size n . Let A be the adjacency matrix of Δ . Let $K = K_{(m,n)}$ and $I = I_v$. By [15, Construction 16] $A + K - I$ is the adjacency matrix of a divisible design graph with parameters

$$(mn, k + n - 1, n + \mu - 2, 2k/(m - 1) + \mu, m, n).$$

Let Γ^* be a divisible design graph with parameters $(v^*, k^*, \lambda_1^*, \lambda_2^*, m^*, n^*)$ on the vertex set V^* from [15, Construction 16]. Let $\mathcal{D}^* = (\mathcal{P}^*, \mathcal{B}^*)$ be a symmetric 2- $(v^\infty, k^\infty, \lambda^\infty)$ design.

Let $v^\infty = m^*$ and ϕ be an arbitrary bijection from the set of canonical classes of Γ^* to the set of blocks \mathcal{B}^* .

Construction SRG2. Let Γ be a graph defined as follows:

- The vertex set of Γ is $V = V^* \cup \mathcal{P}^*$.
- Two different vertices from V^* are adjacent in Γ if and only if they are adjacent in Γ^* . The set \mathcal{P}^* is a clique in Γ . A vertex x in V^* from the canonical class \mathcal{P}_i , where $i \in [m^*]$, is adjacent to a vertex y in \mathcal{P}^* if and only if y belongs to the block $\phi(\mathcal{P}_i)$.

If the following equalities hold

$$\sqrt{k - \mu + 1} + n + \mu - 2 = m - 2 + n\lambda^\infty = 2k/(m - 1) + \mu + \lambda^\infty,$$

then from Construction SRG2 we have a strongly regular graph with parameters

$$(m(n + 1), k + \sqrt{k - \mu + 1} + n - 1, \sqrt{k - \mu + 1} + n + \mu - 2, \sqrt{k - \mu + 1} + n + \mu - 2).$$

6 Small examples

6.1 Examples from Construction SRG1

If $q = 2$ and $d = 2$, then, by Theorem 4, a divisible design graph has parameters $(12, 6, 2, 3, 3, 4)$. The line graph of the octahedron is a unique graph with such parameters.

By Theorem 5, we have the triangular graph $T(6)$ which is a strongly regular graph with parameters $(15, 8, 4, 4)$.

D. Panasenko and L. Shalaginov [15] found all divisible design graphs up to 39 vertices by direct computer calculations, except for the three tuples of parameters: $(32, 15, 6, 7, 4, 8)$, $(32, 17, 8, 9, 4, 8)$, $(36, 24, 15, 16, 4, 9)$. These cases proved to be very difficult to handle.

If $q = 3$ and $d = 2$, then, by Theorem 4, we have parameters $(36, 24, 15, 16, 4, 9)$. By Theorem 5, from any divisible design graph with parameters $(36, 24, 15, 16, 4, 9)$ and a $2-(4, 3, 2)$ design we can construct a strongly regular graph with parameters $(40, 27, 18, 18)$. The complement of this graph has parameters $(40, 12, 2, 4)$. All strongly regular graphs with parameters $(40, 12, 2, 4)$ were found by E. Spence [16]. There are exactly 28 non-isomorphic strongly regular graphs with parameters $(40, 12, 2, 4)$. Only the first one of them does not have 4-cliques. If strongly regular graph Γ with parameters $(40, 12, 2, 4)$ has a regular 4-clique C , then the induced subgraph on $V(\Gamma) \setminus C$ in the complement of Γ is a divisible design graph with parameters $(36, 24, 15, 16, 4, 9)$. It turns out that there are 87 non-isomorphic divisible design graphs with such parameters. The adjacency matrices of all these graphs were calculated by D. Panasenko by checking the graphs found by E. Spence [16]. All of them are available on the web page <http://alg.imm.uran.ru/dezagraphs/ddgtab.html>.

6.2 Examples from Construction SRG2

There are 56 divisible design graphs with parameters $(28, 15, 6, 8, 7, 4)$ and spectrum $\{15^1, 3^7, 1^6, -3^{14}\}$ from either the Triangular graph $T(8)$ or one of the Chang graphs [15, Construction 16]. Construction SRG2 gives us strongly regular graphs with parameters $(35, 18, 9, 9)$. Remark that strongly regular graphs with parameters $(35, 18, 9, 9)$ were complete enumerated by B.D. McKay and E. Spence in [13]. There are 3854 strongly regular graphs with parameters $(35, 18, 9, 9)$ and 499 of them can be obtained from Construction SRG2.

7 The number of isomorphism classes of graphs from Construction DDG and Construction SRG1

In this section we discuss how many non-isomorphic divisible design graphs and strongly regular graphs we can get by Construction DDG and Construction SRG1, respectively, for given q and d ?

Using the same arguments as M. Muzychuk in [14, Proposition 3.5] we obtain a lower bound for the number of non-isomorphic divisible design graphs from Construction DDG as follows

$$\frac{(q!)^m}{(q^d m^2)^{q^d m} (q^{d+1} m)^{m-1}},$$

where $m = (q^d - 1)/(q - 1)$. Let D_1 be the number of non-isomorphic symmetric $2-\left(\frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}, \frac{q^{d-2}-1}{q-1}\right)$ designs. If Γ^* is a divisible design graph with parameters from

Construction DDG, then there are at least D_1 pairwise non-isomorphic strongly regular graphs, by Construction SRG1, with Γ^* as an induced subgraph. To obtain the number of isomorphism classes of strongly regular graphs from Construction SRG1, we need to estimate the number of Delsarte cliques in a graph with the symplectic graph parameters. For $q \gg 1$, $d \gg 1$ this number is at most $D_2 = q^{2d-1}(q^{d-2})^{q^{d-2}}$. Hence, the number of isomorphism classes of strongly regular graphs from Construction SRG1 is at least

$$\frac{D_1(q!)^m}{D_2(q^d m^2)^{q^d m} (q^{d+1} m)^{m-1}}.$$

8 Open questions

Question 6. Is it possible to obtain all non-isomorphic divisible design graphs with parameters

$$\begin{aligned} v &= q^d(q^d - 1)/(q - 1), k = q^{d-1}(q^d - 1), \\ \lambda_1 &= q^{d-1}(q^d - q^{d-1} - 1), \lambda_2 = q^{d-2}(q - 1)(q^d - 1), \\ m &= (q^d - 1)/(q - 1), \quad n = q^d \end{aligned}$$

from Construction DDG for given q and d ?

A similar question for strongly regular graphs from Construction SRG1 has a “no” answer. There is one strongly regular graph with parameters $(40, 12, 2, 4)$ which we cannot obtain by Construction SRG1.

Question 7. Is it true that among all graphs with the same parameters as the symplectic graph, the symplectic graph contains the maximum possible number of Delsarte cliques?

If $d = 2$, then the answer is “yes” since the symplectic graph $Sp(4, 3)$ is one of two generalized quadrangles $GQ(3, 3)$.

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