

Enumeration of Generalized Dyck Paths Based on the Height of Down-Steps Modulo k

Clemens Heuberger*

Sarah J. Selkirk[†]

Stephan Wagner[‡]

Department of Mathematics
University of Klagenfurt
Austria

Department of Mathematics
Uppsala Universitet
Sweden

{clemens.heuberger, sarah.selkirk}@aau.at

stephan.wagner@math.uu.se

Submitted: Apr 29, 2022; Accepted: Dec 3, 2022; Published: Feb 10, 2023

© The authors. Released under the CC BY license (International 4.0).

Abstract

For fixed non-negative integers k , t , and n , with $t < k$, a k_t -Dyck path of length $(k+1)n$ is a lattice path that starts at $(0,0)$, ends at $((k+1)n,0)$, stays weakly above the line $y = -t$, and consists of steps from the step-set $\{(1,1), (1,-k)\}$. We enumerate the family of k_t -Dyck paths by considering the number of down-steps at a height of i modulo k . Given a tuple (a_1, a_2, \dots, a_k) we find an exact enumeration formula for the number of k_t -Dyck paths of length $(k+1)n$ with a_i down-steps at a height of i modulo k , $1 \leq i \leq k$. The proofs given are done via bijective means or with generating functions.

Mathematics Subject Classifications: 05A15

1 Introduction

For a fixed positive integer k , a k -Dyck path with length $(k+1)n$ is a lattice path consisting of steps $\{(1,1), (1,-k)\}$ which starts at $(0,0)$, ends at $((k+1)n,0)$, and stays (weakly) above the x -axis. This is a natural extension of Dyck paths. The family of k -Dyck paths (or their reverse, with step-set $\{(1,k), (1,-1)\}$) has been the subject of a number of papers (see for example [6, 12, 13, 14, 20]). The family of k -Dyck paths is enumerated by the Fuss–Catalan numbers

$$\frac{1}{kn+1} \binom{(k+1)n}{n},$$

*Supported by the Austrian Science Fund (FWF): P 28466-N35 and DOC 78.

[†]Supported by the Austrian Science Fund (FWF): P 28466-N35 and DOC 78.

[‡]Supported by the Knut and Alice Wallenberg Foundation.

where references to OEIS sequences [15] for specific values of k can be found in the second row of Table 1. Bijections have been established between them and several other combinatorial objects, see for example [11] for a list of such objects (referred to as k -Catalan structures).

In [10] Gu, Prodinger and Wagner studied the family of k -plane trees: labelled plane trees where each vertex is labelled i with $i \in \{1, 2, \dots, k\}$ such that the sum of labels along any edge is at most $k + 1$, with the root labelled k . Here they established a relationship between these trees and a modified version of a k -Dyck paths, which have since been named k_t -Dyck paths, whose definition is given below.

Definition 1 (k_t -Dyck paths). For fixed non-negative integers k, t , and n , with $0 \leq t < k$, a k_t -Dyck path of length $(k + 1)n$ is a lattice path consisting of n steps of type $(1, -k)$ and kn steps of type $(1, 1)$, that starts at $(0, 0)$, ends at $((k + 1)n, 0)$, and stays weakly above the line $y = -t$.

The number of k_t -Dyck paths of length $(k + 1)n$ was shown to be equal to the Raney numbers (see [19]),

$$\frac{t + 1}{(k + 1)n + t + 1} \binom{(k + 1)n + t + 1}{n}, \tag{1}$$

which when specialized to $t = 0$ is equal to the sequence of Fuss–Catalan numbers. The Raney numbers enumerate tuples of k -ary trees [19], k -plane trees [10], types of planar embeddings [4], binary matrices used in coding theory [2], and most recently threshold sequences and Motzkin-like paths [18].

$t \setminus k$	1	2	3	4	5	6	7
0	A000108	A001764	A002293	A002294	A002295	A002296	A007556
1	X	A006013	A069271	A118969	A212071	A233832	A234461
2	X	X	A006632	A118970	A212072	A233833	A234462

Table 1: References to OEIS sequences [15] of equation (1) for fixed values of k and t .

The family of k_t -Dyck paths can also be seen as lattice paths on a rectangular grid from $(0, 0)$ to (kn, n) , consisting of steps $(1, 0)$ and $(0, 1)$, where the paths never go above the line $ky = x + t$. Here the $(1, 0)$ steps are up-steps and the $(0, 1)$ steps are down-steps in our model, respectively. The case where $t = 0$ has been studied in [3, 7, 11, 12, 17], for example.

In this paper, we will study the enumeration of k_t -Dyck paths based on the number of down-steps $(1, -k)$ at given heights modulo k . In the rectangular grid model, this is the number of $(0, 1)$ -steps with x -coordinates congruent to a given number modulo k . We will make use of the following notation to avoid excessive repetition of the same lengthy phrase.

Definition 2. Let k, t , and n be non-negative integers with $0 \leq t < k$. For each $1 \leq i \leq k$ let a_i be a non-negative integer, where $a_1 + \dots + a_k = n$. Then we define $\mathcal{K}_{k,t}^n(a_1, \dots, a_k)$ to be the set of all k_t -Dyck paths of length $(k + 1)n$ with a_i down-steps $(1, -k)$ such that the endpoints of the down-steps are at a height of i modulo k for $1 \leq i \leq k$.

As usual, we denote the number of elements in $\mathcal{K}_{k,t}^n(a_1, \dots, a_k)$ by $|\mathcal{K}_{k,t}^n(a_1, \dots, a_k)|$.

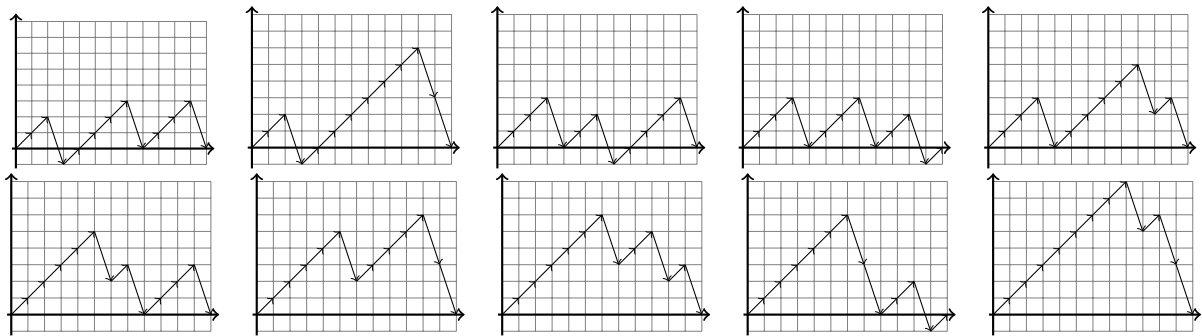


Figure 1: The ten 3_1 -Dyck paths in $\mathcal{K}_{3,1}^3(0, 1, 2)$ for $n = 3$, $k = 3$, and $t = 1$.

Okoth and Wagner have studied a similar statistic in [16], where they determined the enumeration of k -plane trees based on the number of vertices labelled i , as well as a similar enumeration of non-crossing trees. However, the bijection between k -plane trees and k_t -Dyck paths does not preserve this statistic, and the counting formula is different for the k -plane trees statistic, and thus one cannot derive a formula for $|\mathcal{K}_{k,t}^n(a_1, \dots, a_k)|$ from this relationship.

In [5], Burstein considers the distribution of peak heights modulo k and double descents in k -Dyck paths, which is closely related to this work since k_0 -Dyck paths are k -Dyck paths, and a peak is an up-step followed by a down-step, so the height of peaks has a relationship with the height of down-steps.

In this paper, we will make use of bijective and generating function methods to show the following main result along with related results.

Theorem 6. Let k , n and t be non-negative integers with $0 \leq t \leq k - 1$. For each $1 \leq i \leq k$ let a_i be a non-negative integer, where $a_1 + \dots + a_k = n$. Then the number of k_t -Dyck paths of length $(k + 1)n$ with a_i down-steps at a height of i modulo k , $1 \leq i \leq k$, is

$$|\mathcal{K}_{k,t}^n(a_1, \dots, a_k)| = \frac{a_{k-t} + \dots + a_k}{n(n+1)} \prod_{i=k-t}^k \binom{n+a_i}{a_i} \prod_{i=1}^{k-t-1} \binom{n+a_i-1}{a_i}.$$

Thereafter we establish bijective relationships between some special cases of the set $\mathcal{K}_{k,t}^n(a_1, \dots, a_k)$ and other combinatorial objects such as peaks and valleys in Dyck paths, Dyck paths themselves, and the sum of the indices of the down-steps in all valleys in Dyck paths.

2 Enumeration of k_t -Dyck paths according to the heights of down-steps modulo k

We begin by enumerating a special case of k_t -Dyck path based on height of down-steps modulo k , when $t = k - 1$, and will later use this to enumerate the general case. The proof

of the special case is done bijectively, and the remaining cases are proved via generating function methods.

Lemma 3. *When $t = k - 1$, $|\mathcal{K}_{k,t}^n(a_1, \dots, a_k)|$ is symmetric in a_1, \dots, a_k . That is, if (a_1, \dots, a_k) is a permutation of (a_1, \dots, a_k) , then*

$$|\mathcal{K}_{k,k-1}^n(a_1, \dots, a_k)| = |\mathcal{K}_{k,k-1}^n(\overline{a_1, \dots, a_k})|.$$

Proof. When $k = 1$ it is clear that the lemma holds, since any permutation of a_1 is trivial. Therefore we assume that $k \geq 2$ for the remainder of this proof.

Firstly, note that because down-steps $(1, -k)$ do not contribute to height modulo k , the height of a down-step modulo k is entirely dependent on the number of up-steps $(1, 1)$ that occur before it in the path. The central concept of this proof is that we can swap two (maximal) sets of down-steps at different heights modulo k . A simple case of this is that one can shift down-steps at a height of k modulo k to a height of $k - 1$ modulo k (and vice versa) by shifting the down-steps at a height of k modulo k one up-step to the left (vice versa: right) in the path. This is allowed because by definition ($t < k$), a down-step (or a consecutive sequence thereof) at a height of k modulo k must end at a height of $y = 0$ or above and cannot occur at the beginning of a path, and thus the shifted down-step(s) will end at a height of -1 or above, and thus be (weakly) above $y = -t = -k + 1$. Similarly, a down-step at a height of $k - 1$ modulo k will not occur directly at the end of a path, and thus it will be followed by at least one up-step which makes the shift one up-step to the right permissible. Note that consecutive sequences of down-steps that are shifted are all shifted one up-step to the left (vice versa: right) and thus remain consecutive. Exchanging a_{k-1} and a_k (simultaneously) using these shifts as demonstrated in Figure 2 is a bijection, and thus $|\mathcal{K}_{k,k-1}^n(a_1, \dots, a_{k-2}, a_{k-1}, a_k)| = |\mathcal{K}_{k,k-1}^n(a_1, \dots, a_{k-2}, a_k, a_{k-1})|$.



Figure 2: A 3_2 -Dyck path of length 12. The down-steps $(1, -3)$ at a height of 3 modulo 3 (marked in red) are ‘exchanged’ with those at a height of 2 modulo 3 (marked in blue).

This shifting can be generalised:

- Let $1 \leq j < i \leq k$. To shift down-steps from a height of i modulo k to j modulo k , down-steps should be shifted $i - j$ up-steps to the left.
- Let $1 \leq i < j \leq k$. To shift down-steps from a height of i modulo k to j modulo k , down-steps should be shifted $j - i$ up-steps to the right.

The lowest possible endpoint for a down-step (or a consecutive sequence thereof) at a height of i modulo k is $y = -k + i$, therefore exchanges of a_i to any other a_j are possible

as long as a shift to the left of i (or more) positions does not take place (otherwise the path would not stay weakly above $y = -(k - 1)$). Indeed, this is prevented since any left shift is $i - j$ up-steps and from the condition on left shifts, $0 < i - j \leq i - 1$. Lastly, there will always be (at least) $i - j$ up-steps to the left of a down-step (or a consecutive sequence thereof) at a height of i modulo k since $i - j \leq i$. Similarly, a shift to the right is restricted by the number of up-steps to the right of a down-step. Since the total number of up-steps is kn and thus congruent to k modulo k , there are a minimum of $k - i$ up-steps after a down-step (or a consecutive sequence thereof) at a height of i modulo k , and thus exchanges of a_i to any other a_j are possible as long as a shift to the right of $k - i + 1$ (or more) places does not take place, and the condition for steps to the right ensures that $j - i \leq k - i$.

Since left and right shifts have been shown to be legal, any pair a_i and a_j can be simultaneously exchanged uniquely (which steps shift left and right is clearly defined), meaning

$$|\mathcal{K}_{k,k-1}^n(a_1, \dots, a_i, \dots, a_j, \dots, a_k)| = |\mathcal{K}_{k,k-1}^n(a_1, \dots, a_j, \dots, a_i, \dots, a_k)|.$$

Simultaneous exchange can be thought of as composition of the two shifts, which are independent of the positioning of other down-steps, keeping track of the original heights of the steps to avoid shifting one height of steps twice. Since any permutation of (a_1, \dots, a_k) can be obtained via a finite number of exchanges of two elements, we have proved the result. \square

Remark 4. The symmetry described in Lemma 3 and the proof thereof can be applied to the more general case within (a_{k-t}, \dots, a_k) and (a_1, \dots, a_{k-t-1}) , of which $t = k - 1$ is a special case.

Proposition 5. *For k a positive integer,*

$$|\mathcal{K}_{k,k-1}^n(a_1, \dots, a_k)| = \frac{1}{n+1} \prod_{i=1}^k \binom{n+a_i}{a_i}. \quad (2)$$

Proof. Consider any k_{k-1} -Dyck path, and consider the weak right-to-left minima of the path (in this paper, a weak right-to-left minimum is an end-point of a down-step $(1, -k)$ that is no higher than any points that are further right). See Figure 3 for a depiction of this. Let M be the set of these points. For each point in M , we consider the first return to the x -axis that comes after it (i.e., the first point to the right of it that lies on the x -axis; for a weak right-to-left minimum on the x -axis, that is just the point itself). Let R be the set of these first returns.

We cut the k_{k-1} -Dyck path at the points in R to divide it into a number of segments (remark: for $k = 1$, this is the standard decomposition of Dyck paths into a sequence of arches). Each of them contains exactly one point of M either inside or at its right end (we do not count the left end even if it also belongs to M). This is because given two consecutive right-to-left minima at heights i and h where $1 - k \leq i \leq 0$ and $i \leq h \leq 0$, there must be at least k up-steps between the two minima for the down-step ending at

height h to occur. Thus either one of the k up-steps has an endpoint on the x -axis or both right-to-left minima are on the x -axis. If a segment contains a point in M whose y -coordinate is $j - k$, then its shape is

- j steps up, followed by a (vertically shifted) k_{k-1} -Dyck path, followed by a down-step $(1, -k)$ and $k - j$ steps up.

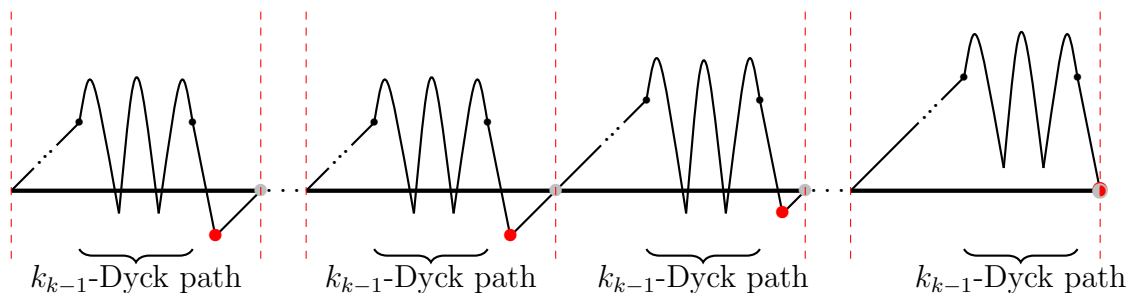


Figure 3: The decomposition of a k_{k-1} -Dyck path into segments based on weak right-to-left minima M (marked in red) and the corresponding returns to the x -axis R (marked in gray).

So we have a decomposition into shorter k_{k-1} -Dyck paths (and some additional steps), which can be translated to a generating function identity.

Let $F_{k-1}(z, x_1, \dots, x_k)$ be the generating function for k_{k-1} -Dyck paths where z represents $(1, -k)$ steps, and x_j represents down-steps at height j modulo k . We note that although the total number of down-steps can be recovered from the number of down-steps in each congruence class, we include z for readability. For ease of notation, let $\bar{x} = (z, x_1, \dots, x_k)$. The weak right-to-left minima's y -coordinates read from left to right are a (possibly empty) sequence of $1 - k$ values, followed by a (possibly empty) sequences of $2 - k$ values, and continues in this way until it ends with a (possibly empty) sequence of 0 values. Let $\mathcal{K}_{k,t}$ be the class of k_t -Dyck paths, $/$ be a $(1, 1)$ step, and \backslash be a $(1, -k)$ step. Then using the notation of [8, Chapter I], the symbolic decomposition of $\mathcal{K}_{k,k-1}$ according to right-to-left minima is given by

$$\text{SEQ}(\{/\} \times \mathcal{K}_{k,k-1} \times \{\backslash\} \times (\{/\})^{k-1}) \times \dots \times \text{SEQ}(\{/\}^k \times \mathcal{K}_{k,k-1} \times \{\backslash\} \times \{/\}^0)$$

From this decomposition of k_{k-1} -Dyck paths, we obtain the functional equation

$$F_{k-1}(\bar{x}) = \prod_{i=1}^k \frac{1}{1 - z x_i F_{k-1}(\bar{x})}. \quad (3)$$

Although the k_{k-1} -Dyck paths in the decomposition are shifted vertically, we can consider all height shifts of them as equal due to Lemma 3: the shifted paths are symmetric and thus have the same generating function.

We now apply Lagrange Inversion to obtain the coefficients of the generating function (3) which will prove the formula in (2). Note that the Lagrange Inversion Formula [8, Appendix A.6] states that if a power series y satisfies the equation $y(z) = z\Phi(y(z))$, where Φ is a formal power series with $\Phi(0) \neq 0$, then

$$[z^n]y(z) = \frac{1}{n}[t^{n-1}]\Phi(t)^n$$

for $n \geq 1$. To apply this, set $F_{k-1}^*(\bar{x}) = zF_{k-1}(\bar{x})$. Then for $\Phi(t) = \prod_{i=1}^k \frac{1}{1-x_it}$,

$$F_{k-1}^*(\bar{x}) = z \prod_{i=1}^k \frac{1}{1-x_i F_{k-1}^*(\bar{x})} = z\Phi(F_{k-1}^*(\bar{x})),$$

which is amenable to Lagrange Inversion, resulting in

$$[z^n]F_{k-1}(\bar{x}) = [z^{n+1}]F_{k-1}^*(\bar{x}) = \frac{1}{n+1}[t^n] \prod_{i=1}^k \frac{1}{(1-x_it)^{n+1}},$$

and thus for $n = a_1 + a_2 + \dots + a_k$,

$$[z^n x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}]F_{k-1}(\bar{x}) = \frac{1}{n+1} [(tx_1)^{a_1} (tx_2)^{a_2} \dots (tx_k)^{a_k}] \prod_{i=1}^k \frac{1}{(1-x_it)^{n+1}}.$$

Thus we can consider just $[x_1^{a_1} \dots x_k^{a_k}]$ to obtain more specific information involving a_i -values while implicitly keeping track of the t^n term. It is clear that

$$[x_j^{a_j}] \prod_{i=1}^k \frac{1}{(1-x_it)^{n+1}} = \binom{n+a_j}{a_j} t^{a_j} \prod_{\substack{i=1 \\ i \neq j}}^k \frac{1}{(1-x_it)^{n+1}},$$

and thus

$$[z^n x_1^{a_1} \dots x_k^{a_k}]F_{k-1}(\bar{x}) = \frac{1}{n+1} \prod_{i=1}^k \binom{n+a_i}{a_i}. \quad \square$$

Theorem 6. *Let k, n and t be non-negative integers with $0 \leq t \leq k-1$. For each $1 \leq i \leq k$ let a_i be a non-negative integer, where $a_1 + \dots + a_k = n$. Then the number of k_t -Dyck paths of length $(k+1)n$ with a_i down-steps at a height of i modulo k , $1 \leq i \leq k$, is*

$$|\mathcal{K}_{k,t}^n(a_1, \dots, a_k)| = \frac{a_{k-t} + \dots + a_k}{n(n+1)} \prod_{i=k-t}^k \binom{n+a_i}{a_i} \prod_{i=1}^{k-t-1} \binom{n+a_i-1}{a_i}. \quad (4)$$

Proof. We decompose k_t -Dyck paths in an analogous way to that of the proof of Proposition 5: by determining the weak right-to-left minima, and splitting the path into segments according to the first return of the path to the x -axis after the minima. The decomposition for the case $t = k-1$ can be seen in Figure 3. Analogously, using $F_t(\bar{x})$ to denote the

generating function for k_t -Dyck paths where z counts the number of $(1, -k)$ steps and x_j counts the number of x_j steps at height j modulo k , we obtain the functional equation

$$F_t(\bar{x}) = \prod_{i=k-t}^k \frac{1}{1 - zx_i F_{k-1}(\bar{x})}. \quad (5)$$

We now apply Lagrange Inversion in the form as given in [9, Equation 2.1.1], which states that if $y(z) = z\Phi(y(z))$ where Φ is a formal power series with $\Phi(0) \neq 0$, then for a power series g , we have that

$$[z^n]g(y(z)) = \frac{1}{n}[s^{n-1}]g'(s)\Phi(s)^n, \quad \text{for } n \geq 1.$$

This is satisfied in the following way: Set $y(z) = F_{k-1}^*(\bar{x}) := zF_{k-1}(\bar{x})$ and setting

$$g(s) = \prod_{i=k-t}^k \frac{1}{1 - x_i s},$$

we see that $F_t(\bar{x}) = g(F_{k-1}^*(\bar{x}))$. Additionally, Φ is the same as in Proposition 5. We differentiate g to obtain

$$g'(y) = \sum_{j=k-t}^k \frac{x_j}{1 - x_j y} \prod_{i=k-t}^k \frac{1}{1 - x_i y},$$

and thus we find that the coefficients are

$$[z^n x_1^{a_1} \cdots x_k^{a_k}]g(y) = \frac{1}{n}[s^{n-1} x_1^{a_1} \cdots x_k^{a_k}] \sum_{j=k-t}^k \frac{x_j}{1 - x_j s} \prod_{i=k-t}^k \frac{1}{(1 - x_i s)^{n+1}} \prod_{i=1}^{k-t-1} \frac{1}{(1 - x_i s)^n}. \quad (6)$$

In the equations below we can ignore the s^{n-1} term after the second equality and thenceforth set $s = 1$. This is because we once again note that

$$\begin{aligned} [s^{n-1} x_1^{a_1} \cdots x_{j-1}^{a_{j-1}} x_j^{a_j-1} x_{j+1}^{a_{j+1}} \cdots x_k^{a_k}] \\ = [(x_1 s)^{a_1} \cdots (x_{j-1} s)^{a_{j-1}} (x_j s)^{a_j-1} (x_{j+1} s)^{a_{j+1}} \cdots (x_k s)^{a_k}], \end{aligned}$$

since the sum of all a_i terms for $1 \leq i \leq k$ is $n - 1$. By (6), we have

$$\begin{aligned} n[z^n x_1^{a_1} \cdots x_k^{a_k}]g(y) \\ = [s^{n-1} x_1^{a_1} \cdots x_k^{a_k}] \sum_{j=k-t}^k \frac{x_j}{1 - x_j s} \prod_{i=k-t}^k \frac{1}{(1 - x_i s)^{n+1}} \prod_{i=1}^{k-t-1} \frac{1}{(1 - x_i s)^n} \\ = \sum_{j=k-t}^k [x_1^{a_1} \cdots x_{j-1}^{a_{j-1}} x_j^{a_j-1} x_{j+1}^{a_{j+1}} \cdots x_k^{a_k}] \frac{1}{(1 - x_j)^{n+2}} \prod_{\substack{i=k-t \\ i \neq j}}^k \frac{1}{(1 - x_i)^{n+1}} \prod_{i=1}^{k-t-1} \frac{1}{(1 - x_i)^n} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=k-t}^k \binom{n+a_j}{a_j-1} \prod_{\substack{i=k-t \\ i \neq j}}^k \binom{n+a_i}{a_i} \prod_{i=1}^{k-t-1} \binom{n+a_i-1}{a_i} \\
&= \sum_{j=k-t}^k \frac{a_j}{n+1} \binom{n+a_j}{a_j} \prod_{\substack{i=k-t \\ i \neq j}}^k \binom{n+a_i}{a_i} \prod_{i=1}^{k-t-1} \binom{n+a_i-1}{a_i} \\
&= \frac{1}{n+1} \prod_{i=k-t}^k \binom{n+a_i}{a_i} \prod_{i=1}^{k-t-1} \binom{n+a_i-1}{a_i} \sum_{j=k-t}^k a_j.
\end{aligned}$$

Altogether for $n \geq 1$ this gives

$$[z^n x_1^{a_1} \cdots x_k^{a_k}] F_t(\bar{x}) = \frac{\sum_{j=k-t}^k a_j}{n(n+1)} \prod_{i=k-t}^k \binom{n+a_i}{a_i} \prod_{i=1}^{k-t-1} \binom{n+a_i-1}{a_i},$$

which is what we wanted to prove. \square

The result obtained is demonstrated in Table 2 for $k = 3$ and $n = 4$. Note that the last row is independent of permutations of $a_1 a_2 a_3$, as proved in Lemma 3. Additionally, summing across rows, it can be checked that the Raney number (1) for $k = 3$, $n = 4$, and t as given in the corresponding row is obtained.

$a_1 a_2 a_3$	004	040	400	013	031	103	130	301	310	022	202	220	112	121	211
$t = 0$	14	0	0	21	5	21	0	5	0	15	15	0	24	10	10
$t = 1$	14	14	0	35	35	21	21	5	5	45	15	15	45	45	25
$t = 2$	14	14	14	35	35	35	35	35	35	45	45	45	75	75	75

Table 2: The number of k_t -Dyck paths of length 16 with $k = 3$ which have a_i down-steps at a height of i modulo k .

Remark 7 (On Schur-positivity). In Lemma 3, it was proven that $|\mathcal{K}_{k,k-1}^n(a_1, \dots, a_k)|$ is symmetric in a_1, \dots, a_k . Equivalently, the polynomial

$$[z^n] F_{k-1}(\bar{x})$$

is a symmetric polynomial in x_1, \dots, x_k . As the proof of Proposition 5 shows, this polynomial can be expressed as

$$[z^n] F_{k-1}(\bar{x}) = \frac{1}{n+1} [t^n] \prod_{i=1}^k \frac{1}{(1-x_i t)^{n+1}}.$$

Generating functions of this form are familiar from the theory of Schur polynomials. Indeed, with s_λ denoting the Schur polynomial associated with a partition λ , we have (see e.g. [1, Corollary 8.16])

$$\sum_{\lambda} s_{\lambda}(x_1, x_2, \dots) s_{\lambda}(y_1, y_2, \dots) = \prod_{i,j \geq 1} \frac{1}{1-x_i y_j}.$$

Setting $x_{k+1} = x_{k+2} = \dots = y_{n+2} = y_{n+3} = \dots = 0$ and $y_1 = y_2 = \dots = y_{n+1} = t$, we find that

$$[z^n]F_{k-1}(\bar{x}) = \frac{1}{n+1} [t^n] \sum_{\lambda} s_{\lambda}(x_1, x_2, \dots, x_k) s_{\lambda}(t, t, \dots, t).$$

Note here that $[t^n]s_{\lambda}(t, t, \dots, t)$ is non-zero only when λ is a partition of n . In this case, it is exactly the number of semistandard Young tableaux of shape λ with entries in $\{1, 2, \dots, n+1\}$. This representation shows in particular that the polynomial $[z^n]F_{k-1}(\bar{x})$ is Schur-positive.

Theorem 8. For fixed non-negative integers k, t , and n with $0 \leq t < k$, in k_t -Dyck paths of length $(k+1)n$ the total number of down-steps $(1, -k)$ at a height of i modulo k is:

1. if $k-t \leq i \leq k$,

$$\frac{(t+1)n+k+1}{n-1} \binom{(k+1)n+t}{n-2}.$$

2. if $1 \leq i \leq k-t-1$,

$$(t+1) \binom{(k+1)n+t}{n-2}.$$

Proof. Let $F_t(\bar{x})$ be defined as in the proof of Theorem 6. From (6), we have that

$$[z^n]F_t(\bar{x}) = \frac{1}{n} [s^{n-1}] \sum_{j=k-t}^k \frac{x_j}{1-x_j s} \prod_{i=k-t}^k \frac{1}{(1-x_i s)^{n+1}} \prod_{i=1}^{k-t-1} \frac{1}{(1-x_i s)^n}.$$

Note that by determining the partial derivative of $F_t(\bar{x})$ with respect to x_i and then setting $x_1 = x_2 = \dots = x_k = 1$, we obtain a generating function for the total number of down-steps $(1, -k)$ at a height of i modulo k with respect to number of down-steps. Thus we begin by determining an expression for the partial derivative of $F_{k-1}^*(\bar{x})$ with respect to x_i , where i is fixed and $1 \leq i \leq k$. So we differentiate with respect to x_i and plug in 1 for x_1, \dots, x_k . For $i \leq k-t-1$, this yields

$$\begin{aligned} [z^n] \frac{\partial}{\partial x_i} F_t(\bar{x}) \Big|_{x_1=\dots=x_k=1} &= \frac{1}{n} [s^{n-1}] \frac{t+1}{1-s} \cdot \frac{1}{(1-s)^{(t+1)(n+1)}} \cdot \frac{1}{(1-s)^{(k-t-2)n}} \cdot \frac{sn}{(1-s)^{n+1}} \\ &= [s^{n-2}] \frac{(t+1)}{(1-s)^{kn+t+3}} \\ &= (t+1) \binom{(k+1)n+t}{n-2}. \end{aligned}$$

For $i \geq k-t$, we can either use the same approach or make use of the fact that the number must be the same for all such i by symmetry, coupled with the fact that the total number of down-steps in all k_t -Dyck paths of length $(k+1)n$ is

$$n \cdot \frac{(t+1)}{(k+1)n+t+1} \binom{(k+1)n+t+1}{n} = (t+1) \binom{(k+1)n+t}{n-1}$$

by (1). Thus for $i \geq k - t$, the total number of down-steps at a height of i modulo k is

$$\frac{(t+1)\binom{(k+1)n+t}{n-1} - (k-t-1) \cdot (t+1)\binom{(k+1)n+t}{n-2}}{t+1},$$

which simplifies to

$$\frac{(t+1)n+k+1}{n-1} \binom{(k+1)n+t}{n-2}. \quad \square$$

Remark 9. Note that this proof could also be done by multiplying the formula in Theorem 6 by a_i and then summing over all possible k -tuples (a_1, a_2, \dots, a_k) and using the Vandermonde identity to simplify the summation.

Corollary 10. *For fixed non-negative integers k, t , and n , with $0 \leq t < k$, the average number of down-steps $(1, -k)$ at a height of i modulo k in k_t -Dyck paths of length $(k+1)n$ is:*

1. if $k - t \leq i \leq k$,

$$\frac{n((t+1)n+k+1)}{(kn+t+2)(t+1)}.$$

2. if $1 \leq i \leq k - t - 1$,

$$\frac{n(n-1)}{kn+t+2}.$$

Proof. This follows from Theorem 8, dividing each quantity by the total number of k_t -Dyck paths of length $(k+1)n$ given in (1). \square

3 Related sequences

In this section we use the phrase “ k_t -Dyck paths with (a_1, \dots, a_k) ” to mean the set of k_t -Dyck paths with down-step statistics at heights modulo k of (a_1, \dots, a_k) , where the i -th entry of the (1-indexed) tuple is the number of down-steps at a height of i modulo k .

Note that there is a bijection between k_t -Dyck paths with (a_1, \dots, a_k) and $(k+1)_t$ -Dyck paths with $(0, a_1, \dots, a_k)$. This is because of a simple correspondence: inserting an up-step $(1, 1)$ before every up-step at a height of k modulo k and extending all down-steps to be $(1, -(k+1))$ converts a k_t -Dyck path with (a_1, \dots, a_k) to a $(k+1)_t$ -Dyck path with $(0, a_1, \dots, a_k)$, and the reverse follows plainly. This bijection can be extended inductively. Therefore we will state cases of interesting sequences in their simplest form, but analogous results can be extracted for higher values than k . Additionally, Lemma 3 provides a basis for reordering a_i 's. The following sequences of (a_1, \dots, a_k) give rise to interesting relationships between the height of down-steps modulo k in k_t -Dyck paths and other combinatorial objects. Some of these relationships will be explored in the subsections that follow, the first being a simple case and the second being more involved to show the range of application:

- 3.1 In this section three straightforward bijections between k_t -Dyck paths with a prescribed (a_1, a_2) and other combinatorial objects are given. Sequence OEIS [A001700](#) with offset¹ 1 corresponds to 2_1 -Dyck paths of length $3n$ with $(1, n - 1)$. A bijection between these paths and the number of peaks in Dyck paths of length $2n$ is given in detail. Additionally, a bijection between 2_0 -Dyck paths of length $3n$ with $(1, n - 1)$ and the number of valleys in all Dyck paths of length $2n$ is described. These objects have counting sequence OEIS [A002054](#) with offset 1. Finally, we show that 2_0 -Dyck paths of length $3n$ with $(n - 1, 1)$ are in bijection with Dyck paths of length $2n - 2$, which are counted by the sequence OEIS [A000108](#) (Catalan numbers) with offset 1.
- 3.2 OEIS [A002740](#), $(2, n - 2)$ for $t = 0$ with offset 0. Simplifying this case in (4), it becomes $\frac{n-2}{2} \binom{2n-2}{n}$ for $n \geq 2$ and 0 otherwise. According to the OEIS, this sequence counts the sum of the indices of the down-steps in all valleys (a down-step followed immediately by an up-step)² in Dyck paths of length $2n - 2$.

Several other special cases of (4) occur in the OEIS without combinatorial interpretations at present, including OEIS [A110609](#) with offset 1 for 3_1 -Dyck paths of length $3n$ with $(1, 1, n - 2)$, and OEIS [A188681](#) with offset 0 for 2_1 -Dyck paths of length $6n$ with (n, n) .

3.1 Three straightforward bijective relationships between 2_t -Dyck paths with a prescribed (a_1, a_2) and other combinatorial objects

There is a simple bijection between 2_1 -Dyck paths of length $3n$ with $(a_1, a_2) = (1, n - 1)$ and peaks in Dyck paths of length $2n$. Note that other correspondences follow quite readily, for example, peaks in Dyck paths of length $2n$ under the so-called ‘‘glove bijection’’ translate into leaves in rooted ordered trees with n edges. See Figure 4 for an example of the mapping.

The correspondence goes as follows: since there is exactly one down-step $(1, -2)$ at height 1 modulo 2, this step is preceded and followed by an up-step $(1, 1)$. This sequence of up-down-up can then be exchanged for a sequence of up-up-down, marking the down-step. In this way we have created a 2_0 -Dyck path which consists only of down-steps at a height of 0 modulo 2, and thus can reduce every two up-steps to one up-step $(1, 1)$, and every down-step $(1, -2)$ to a shorter down-step $(1, -1)$. The marked down-step is by construction preceded by an up-step, and uniquely marks a peak of the resulting Dyck path. This mapping is easily reversible.

A similar bijection holds between OEIS [A002054](#) with offset 1, which is the sequence for the number of valleys in all Dyck paths of length $2n$, and 2_0 -Dyck paths of length $3n$ with $(1, n - 1)$. The down-step $(1, -2)$ is shifted one up-step $(1, 1)$ to the left, creating a sequence down-up-up, which can be marked as a valley. Then ‘reduce’ pairs of up-steps and down-steps as described previously to get a Dyck path with a marked valley.

¹By offset [value] we mean that the sequence in the OEIS is shifted [value] positions in order to obtain the sequence we refer to. For example, the sequence $1, 1, 1, \dots$ is the sequence $0, 0, 1, 1, 1, \dots$ with offset 2.

²This is something like a restricted analogue of the major index for Dyck paths.

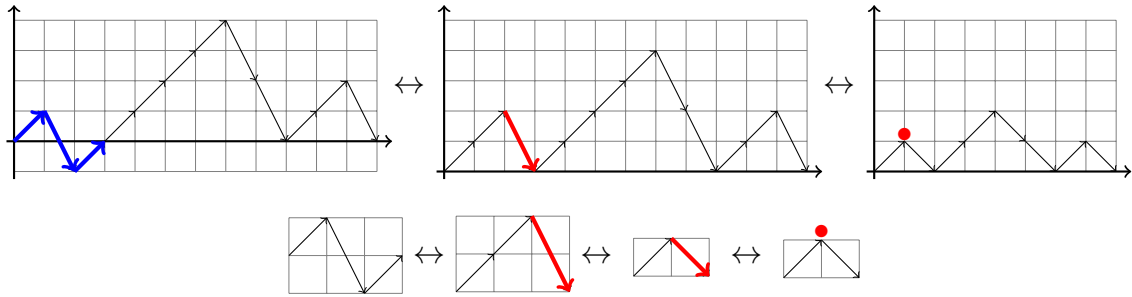


Figure 4: The mapping between a 2_1 -Dyck path of length 12 with $(a_1, a_2) = (1, 3)$ and a peak in a Dyck path of length 8.

The final closely related bijection relates 2_0 -Dyck paths of length $3n$ with $(n - 1, 1)$ and Dyck paths of length $2n - 2$ (OEIS [A000108](#) with offset 1). The single down-step at height 0 modulo 2 must be at the end of the path for it to end on the x -axis. We remove the first up-step in the path, and also remove the final up-step and down-step (the final up-step is immediately before the final down-step), to obtain a path of length $3n - 3$ with all down-steps at a height of 0 modulo 2. Again ‘reduce’ pairs of up-steps and down-steps to obtain a Dyck path of length $2n - 2$.

3.2 The family of 2_0 -Dyck paths of length $3n$ with $(a_1, a_2) = (2, n - 2)$ is enumerated by the sum of indices of the down-steps in all valleys in Dyck paths of length $2(n - 1)$

Since the sum of indices of the down-steps in all valleys (down-steps followed by up-steps) in Dyck paths of length $2n - 2$ is being considered, we represent the total sum using double-marked Dyck paths of length $2n - 2$. That is, if a Dyck path has a valley whose down-step occurs at index i , then we mark the valley in question, and create i copies of the path where exactly one (unique) step in the path is marked in each copy. See Figure 5 below for an example of this marking.

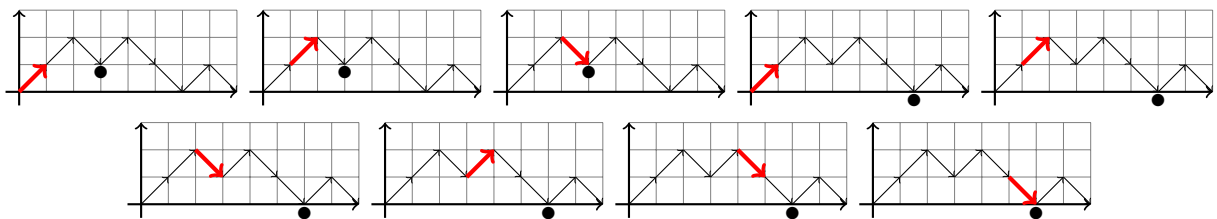


Figure 5: The sum of indices of the down-steps in the valleys of the Dyck path given by step-sequence $1, 1, -1, 1, -1, -1, 1, -1$ is 9, and the 9 double-marked Dyck paths which represent this are drawn, with valley-markings (circle) and unique step-markings (red, thick) depicted.

In this section we will provide a bijection between 2_0 -Dyck paths with $(2, n - 2)$ and these double-marked Dyck paths of length $2n - 2$.

Consider an arbitrary 2_0 -Dyck path of length $3n$ with $(2, n - 2)$. Since the number of down-steps at a height of 1 modulo 2 is fixed at 2, there are two possibilities for the relative positioning of these two down-steps: (A) they are separated by at least two up-steps, and (B) they are adjacent in the path. To avoid repetition, we refer to the two down-steps at a height of 1 modulo 2 as “odd down-steps”. The steps of the mapping to double-marked Dyck paths are given as follows, where examples of the different cases can be seen in Figures 6, 7, and 8. The reasons for each step being permissible is given in italics:

1. Mark the two odd down-steps. If case A holds, shift the rightmost odd down-step in the path left one up-step. If case B holds, shift both odd down-steps to the left one up-step.

At this point we have a 2_0 -Dyck path of length $3n$ with either one odd down-step remaining (case A), or no odd down-steps remaining (case B). The left shift is possible in both cases, as the odd down-steps must start at a height of 1 or above, and so a left shift of one up-step will stay weakly above the x -axis, and there will always be at least one up-step between odd down-steps and the other down-steps so a shift by one will not change the order of occurrence of down-steps.

2. Mark the endpoint of the rightmost (previously) odd down-step with a circle, and remove its previous marking. This new marking will be the *valley-marking*.

Since the rightmost (previously) odd down-step is always followed by an up-step, the shift to the left one up-step has created the step-sequence down-up-up, thus introducing a valley.

3. If case A held (e.g. Figure 6), then remove the leftmost (unshifted) odd down-step as well as the up-steps immediately to the left and right of it, marking the step to the left of the removed steps. This is the *step-marking*.

The step-sequence removed is up-down-up, and thus no height shift takes place and all conditions of a valid 2_0 -Dyck path are still met. The marking stores where this step was removed, and removing this step removes the last odd down-step left in the path.

4. If case B held, we consider a return to the x -axis to be any point in the path where the endpoint of a step is on the x -axis, and there are two possibilities:

- (a) The section of the path to the left of the leftmost odd down-step contains no returns (e.g. Figure 7). In this case, remove the first two up-steps in the path and the leftmost (previously) odd down-step, and mark the (previously) odd down-step in the valley with a step-marking.

Such a section of path which contains no returns and has all down-steps at a height of 0 modulo 2 must begin with a sequence of at least four consecutive up-steps – an odd number would not be possible because of the condition on the down-steps, and a maximal sequence of two consecutive up-steps would result

in a return. Therefore removing two up-steps at the start of the path and a down-step before another return of the path would not cause the path to go below the x -axis. Since the down-step in the valley cannot be preceded by an up-down-up as in the case above, this provides a way to mark this step.

- (b) The section of the path to the left of the leftmost (previously) odd down-step contains at least one return (e.g. Figure 8). In this case, give the down-step immediately before the rightmost return to the left of the leftmost (previously) odd down-step the step-marking, and remove the two up-steps to the right of this return. Additionally, remove the (previously) leftmost odd down-step.

In this case the first return to the left must be a down-step followed by at least four up-steps, where we can make the required removals as before.

5. What we now have is a 2_0 -Dyck path of length $3n - 3$ where all down-steps are at a height of 0 modulo 2. From left to right in the path we reduce every two up-steps to one up-step $(1, 1)$, and every down-step $(1, -2)$ to a down-step $(1, -1)$, step-marking the resulting step if one of the steps to be reduced has the step-marking. We also preserve the relative position of the valley-marking.

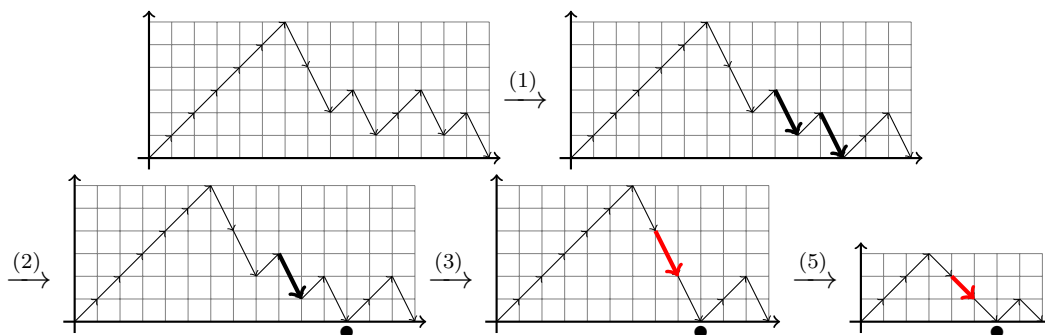


Figure 6: An example of the mapping in Case A: where the odd down-steps are separated by at least two up-steps in the path.

From the description of the mapping, it is clear that a 2_0 -Dyck path with $(2, n - 2)$ is mapped to a double-marked Dyck path. We now prove bijectivity.

It is clear that each of the three possible cases (described in bullets 3, 4a, 4b) for the mapping is independently injective, what remains to show is that no pair of cases yields the same double-marked Dyck path.

- **Case 3 and 4a:** In Case 3 the step to the left of a step-sequence of up-down-up becomes marked, where the down-step in that sequence is the leftmost odd down-step. However, in Case 4a the rightmost (previously) odd down-step, which is in the marked valley, becomes marked. It is therefore not possible to mark the same step under both mappings.

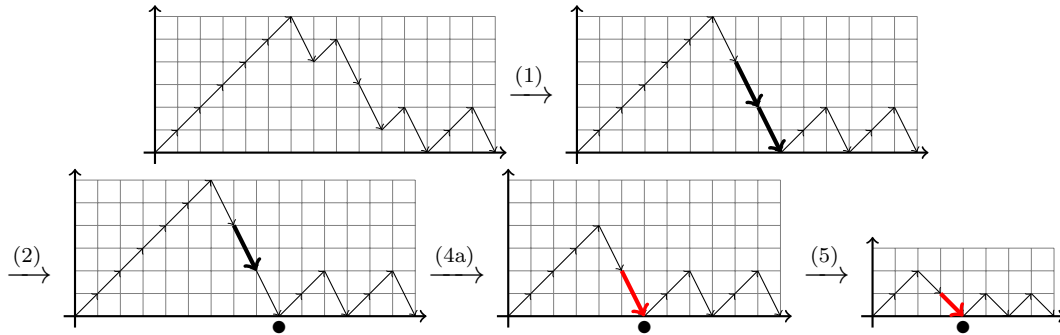


Figure 7: An example of the mapping in Case B: where the odd down-steps are adjacent and there are no returns to the left of the leftmost odd down-step in the path.

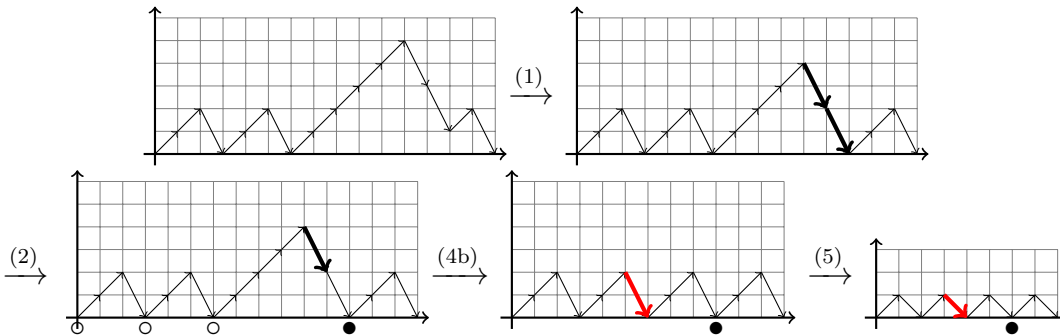


Figure 8: An example of the mapping in Case B: where the odd down-steps are adjacent and there is at least one return to the left of the leftmost odd down-step in the path. Returns to the left of the leftmost odd down-step are marked with an empty circle.

- **Case 3 and 4b:** The characteristic property of the resulting path from Case 4b is that the marked step is a down-step which ends on the x -axis. It is not possible for a path satisfying Case 3 to result in such a path, as in order for this to happen, the step-sequence up-down-up must occur immediately to the right of a return to the x -axis, which would violate the condition that a 2_0 -Dyck path stays weakly above the x -axis.
- **Case 4a and 4b:** In Case 4a the down-step in the marked valley is marked, and by definition of Case 4b, the marked down-step occurs strictly to the left of the down-step associated with the valley.

Therefore we have established injectivity of our function. Now, to establish surjectivity, we obtain the reverse mapping as (briefly) follows:

1. Exchange every up-step $(1, 1)$ for two up-steps $(1, 1)$, marking the second of the two if the original step was marked, and exchange down-steps $(1, -1)$ for down-steps $(1, -2)$, marking the down-step if it was originally marked.
2. Consider the marked step which results from (1):

- If it is in the marked valley, reverse the procedure described in (4a) – add two up-steps to the start of the path and a down-step immediately to the left of the marked step.
- Else if it is a down-step which has an endpoint on the x -axis, reverse the procedure described in (4b) – add two up-steps immediately to the right of the marked step, and one down-step immediately to the left of the down-step in the valley.
- Else insert a step-sequence of up-down-up immediately to the right of the marked step.

Mark the newly inserted down-step.

3. Remove the valley-marking and mark the down-step in the previously marked valley. Shift the rightmost marked down-step and any adjacent marked down-steps one up-step to the right. Remove the markings. What results is a 2_0 -Dyck path that satisfies our conditions.

Acknowledgements

The authors would like to thank the anonymous referee for their valuable comments.

References

- [1] Martin Aigner. *A course in enumeration*, volume 238 of *Graduate Texts in Mathematics*. Springer, Berlin, 2007.
- [2] Andrei Asinowski, Benjamin Hackl, and Sarah J. Selkirk. Down-step statistics in generalized Dyck paths. *Discrete Math. Theor. Comput. Sci.*, 24(1), 2022.
- [3] Cyril Banderier and Michael Wallner. *The Kernel Method for Lattice Paths Below a Line of Rational Slope*, pages 119–154. Springer International Publishing, 2019.
- [4] Jonathan E. Beagley and Paul Drube. The Raney generalization of Catalan numbers and the enumeration of planar embeddings. *Australasian Journal of Combinatorics*, 63.1:130–141, 2015.
- [5] Alexander Burstein. Distribution of peak heights modulo k and double descents on k -Dyck paths. [arXiv:2009.00760](https://arxiv.org/abs/2009.00760), 2020.
- [6] Naomi T. Cameron and Jillian E. McLeod. Returns and hills on generalized Dyck paths. *J. Integer Seq.*, 19(2):3, 2016.
- [7] Philippe Duchon. On the enumeration and generation of generalized Dyck words. *Discrete Mathematics*, 225(1):121–135, 2000. FPSAC'98.
- [8] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
- [9] Ira M. Gessel. Lagrange inversion. *J. Combin. Theory Ser. A*, 144:212–249, 2016.

- [10] Nancy S. S. Gu, Helmut Prodinger, and Stephan Wagner. Bijections for a class of labeled plane trees. *European J. Combin.*, 31(3):720–732, 2010.
- [11] Silvia Heubach, Nelson Y. Li, and Toufik Mansour. Staircase tilings and k -Catalan structures. *Discrete Mathematics*, 308:5954–5964, 2008.
- [12] Peter Hilton and Jean Pedersen. Catalan numbers, their generalization, and their uses. *Math. Intelligencer*, 13(2):64–75, 1991.
- [13] Matthieu Josuat-Vergès and Jang Soo Kim. Generalized Dyck tilings. *European J. Combin.*, 51:458–474, 2016.
- [14] Henri Mühle and Myrto Kallipoliti. The m -cover posets and the strip-decomposition of m -Dyck paths. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AT, 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014):393–404, 2014.
- [15] The On-Line Encyclopedia of Integer Sequences. <http://oeis.org>, 2022.
- [16] Isaac O. Okoth and Stephan Wagner. Refined enumeration of k -plane trees and k -noncrossing trees. [arXiv:2205.01002](https://arxiv.org/abs/2205.01002), 2022.
- [17] D. G. Rogers. A Schröder triangle: three combinatorial problems. In *Combinatorial mathematics, V (Proc. Fifth Austral. Conf., Roy. Melbourne Inst. Tech., Melbourne, 1976)*, pages 175–196. Lecture Notes in Math., Vol. 622. 1977.
- [18] Irena Rusu. Raney numbers, threshold sequences and Motzkin-like paths. *Discrete Math.*, 345(11):Paper No. 113065, 8, 2022.
- [19] Sarah J. Selkirk. On a generalisation of k -Dyck paths. Master’s thesis, Stellenbosch University, 2019.
- [20] Guoce Xin and Yingrui Zhang. On the sweep map for \vec{k} -Dyck paths. *Electron. J. Combin.*, 26(3):#P3.63, 2019.