A Tableau Formula for Vexillary Schubert Polynomials in Type C

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Abstract

Ikeda–Mihalcea–Naruse's double Schubert polynomials [Adv. Math. 226 (2011)] represent the equivariant cohomology classes of Schubert varieties in the type C flag varieties. The goal of this paper is to obtain a new tableau formula of these polynomials associated to *vexillary signed permutations* introduced by Anderson–Fulton. To achieve that goal, we introduce *flagged factorial (Schur) Q-functions*, combinatorially defined functions in terms of *marked shifted tableaux for flagged strict partitions*, and prove their Schur–Pfaffian formula. As an application, we also obtain a new combinatorial formula of factorial Q-functions of Ivanov in which *monomials* bijectively corresponds to flagged marked shifted tableaux.

Mathematics Subject Classifications: 05E05, 14M15

1 Introduction

Ikeda–Mihalcea–Naruse [14] introduced the *double Schubert polynomials* of type C (also B and D) by extending Billey–Haiman [5]'s construction for the single variable case. These polynomials represent the equivariant cohomology classes of Schubert varieties in type C flag varieties. By the work of the work of Kazarian [16] and Ikeda [12], Ikeda–Mihalcea–Naruse's double Schubert polynomials associated to Lagrangian signed permutations can be expressed by the Schur-Pfaffian and also coincide with the *factorial Schur Q-functions* of Ivanov [15] defined in terms of *marked shifted tableaux* of strict partitions (cf. [22]). The corresponding fact for the single variable case was established in the earlier work

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of Pragacz [23]. In [1, 2], Anderson–Fulton extended the Schur-Pfaffian formula for the Lagrangian elements to so-called *vexillary signed permutations*. In [3], Anderson–Fulton characterized the vexillary signed permutations in terms of pattern-avoidance and show that their vexillary signed permutations coincide with the type B vexillary elements originally introduced by Billey and Lam [6].

The goal of this paper is to give a new tableau formula for the double Schubert polynomials associated to vexillary signed permutations. For this purpose, we introduce *marked shifted tableaux for flagged strict partitions*, extending the notation of marked shifted tableaux of strict partitions. This leads us to define a new type of functions, called *flagged Schur Q-functions*, which generalize Ivanov's factorial Schur Q-functions. We prove that those flagged Schur Q-functions can be expressed by the Schur-Pfaffian formula written in terms of the flagged strict partitions. It turns out that this formula is identical to the Schur-Pfaffian formula obtained by Anderson–Fulton in [1, 2]. Thus we conclude that the double Schubert polynomials for vexillary signed permutations are flagged Schur-Q functions.

Our study is motivated by the analogy in type A. Lascoux–Schützenberger's double Schubert polynomials [18, 19, 20] represent the equivariant cohomology classes of Schubert varieties of type A flag varieties, due to Fulton [9]. The double Schubert polynomials associated to Grassmannians coincide with the factorial Schur polynomials, that are essentially introduced and studied by Biedenharn–Louck [4]. Lascoux then identified the family of permutations, now called *vexillary permutations*, such that that their associated double Schubert polynomials are given in a Jacobi–Trudi type determinant formula. The vexillary permutations include the Grassmannian elements and the associated double Schubert polynomials are further generalized to the flagged double (or factorial) Schur polynomials. These are defined either by Jacobi–Trudi type determinant or by flagged semistandard tableaux for a partition, by the work of Chen–Li–Louck [8] (for the single variable case, see Gessel–Viennot [10], and Wachs [27]).

Below we explain our main results in more detail. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a strict partition of length r, *i.e.*, a strictly decreasing sequence of r positive integers. We identify it with its *shifted Young diagram*, obtained from the usual Young diagram by shifting the i-th row $(i-1)$ boxes to the right, for each $i \geqslant 1$. Let $f = (f_1, \dots, f_r)$ be a sequence of nonnegative integers. We call f a *flagging* of λ and the pair (λ, f) a *flagged strict partition*. Consider the ordered set consisting of *unmarked* numbers $1, 2, \dots$ and *primed* numbers $1', 2', \dots$ with $1' < 1 < 2' < 2 < \cdots$. The classical *marked shifted tableau* T of λ is an assignment of a primed or unmarked number to each box of the diagram subject to the rules: (1) assigned elements are weakly increasing in each column and row; (2) unmarked numbers are strictly increasing in each column; (3) primed numbers are strictly increasing in each row. In order to extend this notion, we add, to the above ordered set, *circled* numbers $1^\circ < 2^\circ < \cdots$ which are greater than any unmarked and primed number. We denote this extended ordered set by $\mathbf{P}$. We define a *(flagged) marked shifted tableau* of (λ, f) to be an assignment of an element of $\mathbf{P}$ to each box of λ with rules: in addition to (1), (2), and (3), we require (4) circled numbers are strictly increasing in each row, and (5) each element in the i-th row is at most f_i°. See Example 3. We denote the set of all marked shifted tableaux of (λ, f) by $\mathrm{MST}(\lambda, f)$.

A *signed permutation* w is a permutation on the set $\{1, 2, \dots\} \cup \{-1, -2, \dots\}$ such that $w(i) \neq i$ for only finitely many i, and $\overline{w(i)} = w(\bar{i})$ where we denote $\bar{i} = -i$. Let $x = (x_i)_{i\in\mathbb{N}}, z = (z_i)_{i\in\mathbb{N}}, b = (b_i)_{i\in\mathbb{N}}$. The double Schubert polynomial associated to a signed permutation w is denoted by $\mathfrak{C}_w(x; z|b)$. Note that the variables b coincide with $-t$ in the notation of [14].

Anderson–Fulton [3] defined a signed permutation to be *vexillary* if it arises from a *triple* $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$. On the other hand, each triple reduces to a unique essential triple from which we can construct a flagged strict partition (λ, f). The correspondence between (essential) triples and flagged strict partitions will be explained in Section 5.2.

For each $T \in \mathrm{MST}(\lambda, f)$, we assign

$$(xz|b)^T = \prod_{k\in T}\left(x_k + b_{c(k)-r(k)}\right)\cdot\prod_{k'\in T}\left(x_k - b_{c(k')-r(k')}\right)\cdot\prod_{k^\circ\in T}\left(z_k + b_{k+r(k^\circ)-c(k^\circ)}\right),$$

where $r(\ )$ and $c(\ )$ denote the column and row indices of the entry respectively, and we set $b_{-i} := -b_{i+1}$ for all $i \geqslant 0$.

Our main result is as follows.

Theorem A (Theorem 24). *Let w be a vexillary signed permutation in the sense of Anderson–Fulton [3] and (λ, f) the corresponding flagged strict partition. Then we have*

$$\mathfrak{C}_w(x; z|b) = \sum_{T\in\mathrm{MST}(\lambda,f)} (xz|b)^T. \tag{1}$$

For a general flagged strict partition (λ, f), we denote by $Q_{\lambda,f}(x; z|b)$ the function defined by the right hand side of (1). We call it a *flagged factorial Q-function*, since it is nothing but the original definition of Ivanov's factorial Q-functions $Q_\lambda(x|b)$ when $f = (0, \dots, 0)$. The proof of Theorem A is based on the following Schur–Pfaffian formula of $Q_{\lambda,f}(x; z|b)$ which generalizes the corresponding formula of $Q_\lambda(x|b)$ when $f = (0, \dots, 0)$ in [15, Theorem 9.1].

Theorem B (Theorem 22). *Let (λ, f) be a flagged strict partition of length r. Assume that, if $r \geqslant 2$,* (a) *$\lambda_i - f_i \geqslant \lambda_j - f_j$ for all $1 \leqslant i < j \leqslant r$ and* (b) *$\lambda_{r-1} - f_{r-1} > 0$. Then we have*

$$Q_{\lambda,f}(x; z|b) = \mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]} q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]} \cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right],$$

where Pf *is the Schur–Pfaffian defined in §4, and the function $q_m^{[k|\ell]} = q_m^{[k|\ell]}(x; z|b)$ is defined by*

$$\sum_{m\geqslant 0} q_m^{[k|\ell]} u^m := \left(\prod_{i\geqslant 1}\frac{1 + x_i u}{1 - x_i u}\right) e_u^{[k]}(z) e_u^{[\ell]}(b), \qquad e_u^{[k]}(z) = \begin{cases}\displaystyle\prod_{i=1}^{k}(1 + z_i u) & (k \geqslant 0),\\ \displaystyle\prod_{i=1}^{|k|}\frac{1}{1 - z_i u} & (k \leqslant 0).\end{cases}$$

For a flagged strict partition (λ, f), if the inequality $\lambda_r - f_r > 0$ holds, as well as the assumption for (λ, f) in Theorem B, we can associate to it a vexillary signed permutation. Conversely, if $\lambda_r - f_r \leqslant 0$, we cannot associate to (λ, f) a vexillary signed permutation w such that $Q_{\lambda,f} = \mathfrak{S}_w$. For example, $Q_{(r),(f)}(x; z|b)$ is not the double Schubert polynomial of any vexillary signed permutation if $f \geqslant r$.

As an application of Theorem A, we obtain a new tableau formula of Ivanov's factorial Q-function $Q_\lambda(x|b)$. Ikeda–Mihalcea–Naruse [14] showed that $\mathfrak{C}_w(x; z|b) = \mathfrak{C}_{w^{-1}}(x; b|z)$ and Anderson–Fulton [3] showed that if w is vexillary, then so is w^{-1}. If w is Lagrangian with strict partition λ of length r, we can see that the strict partition of w^{-1} is also λ and its flag is $f = (\lambda_1 - 1, \ldots, \lambda_r - 1)$. All together we obtain

Theorem C (Theorem 25). *Let λ be a strict partition of length r and $f = (\lambda_1 - 1, \ldots, \lambda_r - 1)$, then we have*

$$Q_\lambda(x|b) = \sum_{T \in \mathrm{MST}(\lambda, f)} (xb)^T$$

where $(xb)^T$ is the monomial given by

$$(xb)^T = \prod_{k \in T} x_k \prod_{k' \in T} x_k \prod_{k^\circ \in T} b_k.$$

Anderson–Fulton [2] also introduced a larger family of *theta-vexillary signed permutations* (Lambert [17]), containing the k-Grassmannian signed permutations. They obtained the theta-polynomial (or raising operator, Pfaffian-sum) formula of double Schubert polynomials associated to such elements, extending the ones for k-Grassmannians ([7, 28, 13]). The combinatorial aspect of these signed permutation is far more complicated than the vexillary ones. In particular, there is a tableau formula of the corresponding *single variable* Schubert polynomials associated to k-Grassmannian signed permutations, due to Tamvakis [26]. Since some of those polynomials can also be given in terms of the tableaux introduced in this paper, it is an interesting problem to find the relation to these expressions and to extend the formula to all theta-vexillary double Schubert polynomials.

This paper is organized as follows. In Section 2, we introduce a few basic functions and set up an algebraic framework to study double Schubert polynomials and the combinatorially defined functions defined in this paper. In Section 3, we introduce flagged marked shifted tableaux and the functions defined by them. We prove a few basic formulas that will be used in the proof of Theorem B. In Section 4, we review the definition of the Schur-Pfaffian and prove Theorem B. In Section 5, we first recall the basic facts about double Schubert polynomials and vexillary signed permutations, following Ikeda–Mihalcea–Naruse [14] and Anderson–Fulton [3]. We explain how Theorem A and Theorem C follow from Theorem B. In the appendix, we give a proof of a Jacobi–Trudi type formula of row-strict skew Schur polynomials, extending the work of Wachs [27] and Chen–Li–Louck [8]. This formula is used in the proof of Theorem B.

2 Preliminary

Before we proceed with our main object of interest, we prepare the notations for a few basic functions. The goal is to set an algebraic framework in which we can study combinatorially defined functions. In particular, our Pfaffian formula of the vexillary double Schubert polynomials (and also the factorial flagged Q-functions) will be in terms of the basic functions that we review here. We use infinite sequences of variables, $x = (x_i)_{i\in\mathbb{N}}, z = (z_i)_{i\in\mathbb{N}}$, and $b = (b_i)_{i\in\mathbb{N}}$.

We define functions $q_m = q_m(x)$ in the x-variables for integers $m \geqslant 0$ by the generating function

$$q_u(x) = \sum_{m\geqslant 0} q_m(x)u^m := \prod_{i\geqslant 1} \frac{1 + x_i u}{1 - x_i u},$$

where u is a formal variable. For each integer k, we also define polynomials $e^{[k]}_m(b)$ in the b-variables for $m \geqslant 0$ by

$$e^{[k]}_u(b) = \sum_{m\geqslant 0} e^{[k]}_m(b)u^m := \begin{cases} \displaystyle\prod_{i=1}^{k}(1 + b_i u) & (k \geqslant 0) \\ \displaystyle\prod_{i=1}^{|k|} \frac{1}{1 - b_i u} & (k \leqslant 0). \end{cases}$$

The polynomials $e^{[k]}_m(b)$ and $e^{[-k]}_m(b)$ are nothing but the elementary and complete symmetric polynomials of degree m in $b_1, \dots, b_k$ respectively.

For integers $k, \ell \in \mathbb{Z}$, we set

$$\begin{aligned} e^{[k|\ell]}_u(z|b) &= \sum_{m\geqslant 0} e^{[k|\ell]}_m(z|b)u^m := e^{[k]}_u(z)e^{[\ell]}_u(b), \\ q^{[\ell]}_u(x|b) &= \sum_{m\geqslant 0} q^{[\ell]}_m(x|b)u^m := q_u(x)e^{[\ell]}_u(b), \\ q^{[k|\ell]}_u(x; z|b) &= \sum_{m\geqslant 0} q^{[k|\ell]}_m(x; z|b)u^m := q_u(x)e^{[k]}_u(z)e^{[\ell]}_u(b). \end{aligned}$$

We will also denote $h^{[k|\ell]}_m(z|b) := e^{[-k|-\ell]}_m(z|b)$. Moreover, we often suppress the variables when it is clear from the context, *e.g.*, $e^{[-k|-\ell]}_m = e^{[-k|-\ell]}_m(z|b)$, $q^{[k|\ell]}_m = q^{[k|\ell]}_m(x; z|b)$, and so on.

Occasionally we use the infinite sequence of variables $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$. With this extended sequence of b-variables in mind, we will use the following index shifting operator τ. For each integer $k \in \mathbb{Z}$, let $\tau^k(b)$ be the sequence of variables defined by

$$\tau^k(b) = (b_{1+k}, b_{2+k}, b_{3+k}, \dots).$$

Similarly $\tau^k(\mathbf{b})$ denotes the sequence of variables such that its i-th variable is b_{i+k} for $i \in \mathbb{Z}$.

We consider the ring $\Gamma = \mathbb{Z}[q_1, q_2, \ldots]$. We should note that this is not a polynomial ring since q_i's are not algebraically independent. It is well-known that Γ has a $\mathbb{Z}$-basis consisting of Schur Q-functions $Q_\lambda(x)$ (cf. [22]). Ivanov's factorial Q-functions $Q_\lambda(x|b)$ [15] form a $\mathbb{Z}[b]$-basis of the $\mathbb{Z}[b]$-algebra $\Gamma[b] := \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[b]$ where $\mathbb{Z}[b]$ denotes the polynomial ring in b-variables. All functions defined above are regarded as elements of

$$\Gamma[z, b] := \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[z] \otimes_{\mathbb{Z}} \mathbb{Z}[b].$$

3 Flagged factorial Q-functions

In this section, we introduce *flagged factorial Q-functions* $Q_{\lambda,f}(x; z|b)$ based on the notion of *marked shifted tableaux* of flagged strict partitions (λ, f). We will also discuss basic formulas that will be used in the proof of Schur–Pfaffian formula for $Q_{\lambda,f}(x; z|b)$ in the next section.

3.1 Definition of tableaux and functions

A *strict partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative integers such that $\lambda_i > \lambda_{i+1}$ if $\lambda_i \neq 0$ and the number of positive integers in λ, called the *length* of λ, is finite. We also denote a strict partition of length r as a finite sequence of r positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ and identify it with its *shifted Young digram*, obtained from the usual Young diagram by shifting the i-th row $(i-1)$ boxes to the right, for $1 \leqslant i \leqslant r$. Let $\mathcal{SP}$ be the set of all strict partitions and $\mathcal{SP}_r$ the set of all strict partition of length at most r.

Consider the ordered set $\mathbf{P}$ consisting of *unmarked* numbers $1, 2, \ldots$, *primed* numbers $1', 2', \ldots$, and *circled* numbers $1^\circ, 2^\circ, \ldots$, where the total order is given by

$$1' < 1 < 2' < 2 < 3' < 3 < \cdots < 1^\circ < 2^\circ < \cdots .$$

For a given strict partition λ of length r, a *flagging* of λ is a sequence $f = (f_1, \ldots, f_r)$ of non-negative integers. We call the pair (λ, f) a *flagged strict partition.*

Definition 1. A *(flagged) marked shifted tableau* of a flagged strict partition (λ, f) is a filling of the shifted Young diagram of λ which assigns an element in $\mathbf{P}$ to each box, subject to the rules

1. assigned elements are weakly increasing in each row and column,
2. unmarked numbers are strictly increasing in each column,
3. primed numbers are strictly increasing in each row,
4. circled numbers are strictly increasing in each row, and,
5. for $1 \leqslant i \leqslant r$, each element assigned in the i-th row is at most f_i°.

Remark 2. By the total order of $\mathbf{P}$ and the rule (1), the part consisting of unmarked and primed numbers forms the usual marked shifted tableaux of the shifted Young diagram of a strict partition μ contained in λ (cf. [22, p.256]). It is also clear from the order of $\mathbf{P}$ that the part consisting of circled numbers forms a row-strict semistandard Young tableau of a skew shape λ/μ.

Example 3. Let $\lambda = (5, 3, 1)$ and $f = (2, 1, 0)$. The following are examples of marked shifted tableaux of the flagged strict partition (λ, f):

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 2 & 3' \\ \hline & 2' & 3 & 4 & \\ \hline & & 4' & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 2 & 1^\circ \\ \hline & 2' & 3 & 1^\circ & \\ \hline & & 4' & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 1^\circ & 2^\circ \\ \hline & 2' & 3 & 1^\circ & \\ \hline & & 4' & & \\ \hline \end{array}$$

The following are non-examples due to rules (2), (5), (4), respectively:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 2 & 3' \\ \hline & 2' & 2 & 4 & \\ \hline & & 4' & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 2 & 1^\circ \\ \hline & 2' & 3 & 1^\circ & \\ \hline & & 1^\circ & & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 1^\circ & 1^\circ \\ \hline & 2' & 3 & 1^\circ & \\ \hline & & 4' & & \\ \hline \end{array}$$

We call the element of $\mathbf{P}$ assigned to a box of λ by T an *entry* of T, and denote it by $e \in T$. Abusing the notation slightly, we often write those entries by their assigned elements and denote the numeric value of an entry $e \in T$ by $|e|$, *i.e.*, $k, k', k^\circ \in T$ and $|k| = |k'| = |k^\circ| = k$. Let $c(e)$ and $r(e)$ be the column and row indices of an entry e respectively. Let $\mathrm{MST}(\lambda, f)$ be the set of all marked shifted tableaux of (λ, f). If $f = (0, \ldots, 0)$, we denote $\mathrm{MST}(\lambda)$ instead of $\mathrm{MST}(\lambda, f)$.

Definition 4. Consider the infinite sequence of variables $x = (x_i)_{i\in\mathbb{N}}, z = (z_i)_{i\in\mathbb{N}}, b = (b_i)_{i\in\mathbb{N}}$ as before. Let (λ, f) be a flagged strict partition. To each $T \in \mathrm{MST}(\lambda, f)$, we assign the *weight*

$$(xz|b)^T = \prod_{k\in T} \left(x_k + b_{c(k)-r(k)}\right) \cdot \prod_{k'\in T} \left(x_k - b_{c(k')-r(k')}\right) \cdot \prod_{k^\circ\in T} \left(z_k + b_{k+r(k^\circ)-c(k^\circ)}\right)$$

where we set $b_{-i} := -b_{i+1}$ for all $i \geqslant 0$. We define the *flagged factorial Q-function* $Q_{\lambda,f}(x; z|b)$ by

$$Q_{\lambda,f}(x; z|b) = \sum_{T\in\mathrm{MST}(\lambda,f)} (xz|b)^T.$$

Remark 5. When $f = (0, \ldots, 0)$, the z-variables are not involved and $Q_{\lambda,f}(x; z|b)$ coincides with Ivanov's factorial Q-function $Q_\lambda(x|b)$ [15]:

$$Q_\lambda(x|b) = \sum_{T\in\mathrm{MST}(\lambda)} (x|b)^T, \quad (x|b)^T = \prod_{k\in T} \left(x_k + b_{c(k)-r(k)}\right) \cdot \prod_{k'\in T} \left(x_k - b_{c(k')-r(k')}\right).$$

Furthermore, in view of Remark 2, $Q_{\lambda,f}(x; z|b)$ can be expanded in terms of $Q_\mu(x|b)$ for strict partitions $\mu \subset \lambda$. This expansion, as we see in the example below, will be discussed in the next subsection.

Example 6. Let $\lambda = (3,1)$ and $f = (1,0)$. In this case, $\mathrm{MST}(\lambda, f)$ can be divided into two families of tableaux

$$\begin{array}{|c|c|c|}\hline \ast & \ast & \ast \\ \hline \multicolumn{1}{c|}{} & \ast & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline \ast & \ast & 1^\circ \\ \hline \multicolumn{1}{c|}{} & \ast & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

where the part with $\ast$ consists of unmarked and primed numbers. Thus we have

$$Q_{\lambda,f}(x; z|b) \;=\; Q_{31}(x|b) + Q_{21}(x|b)(z_1 - b_2)$$

Similarly, if $\lambda = (5,3,1)$ and $f = (2,1,0)$, we have

$$\begin{array}{ccccc}
\ast\,\ast\,\ast\,\ast\,\ast & \ast\,\ast\,\ast\,\ast\,k^\circ & \ast\,\ast\,\ast\,\ast\,\ast & \ast\,\ast\,\ast\,\ast\,k^\circ & \ast\,\ast\,\ast\,1^\circ\,2^\circ \\
\;\;\ast\,\ast\,\ast\;\; & \;\;\ast\,\ast\,\ast\;\; & \;\;\ast\,\ast\,1^\circ\;\; & \;\;\ast\,\ast\,1^\circ\;\; & \;\;\ast\,\ast\,1^\circ\;\; \\
\ast & \ast & \ast & \ast & \ast
\end{array}$$

where $k = 1$ or 2 so that

$$\begin{aligned} Q_{\lambda,f}(x|b) \;=\;& Q_{531}(x|b) + Q_{431}(x|b)(z_1 - b_4 + z_2 - b_3) + Q_{521}(x|b)(z_1 - b_2) \\ &+ Q_{421}(x|b)(z_1 - b_4 + z_2 - b_3)(z_1 - b_2) + Q_{321}(x|b)(z_1 - b_3)(z_2 - b_3)(z_1 - b_2). \end{aligned}$$

3.2 Decomposition into Q-functions and skew Schur polynomials

We can expand $Q_\lambda(x; z|b)$ in terms of Ivanov's factorial Q functions in x and b where the coefficients are a variant of row-strict flagged skew Schur polynomials considered by Wachs [27, p.288] in z and b.

A *partition* λ is a weakly decreasing finite sequence of positive integers and we identify it with its Young diagram. The length of λ is r if it consists of r positive integers. We denote the set of all partitions by $\mathcal{P}$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be partitions of length at most r such that $\mu \subset \lambda$, and λ/μ the corresponding skew diagram. A flagging $f = (f_1, \ldots, f_r)$ of λ/μ is a sequence of non-negative integers. We call the pair $(\lambda/\mu, f)$ a *flagged skew diagram.* A *row-strict (flagged) tableau* T of $(\lambda/\mu, f)$ is a filling of the skew diagram λ/μ which assigns a positive integer to each box of λ/μ subject to the rules:

- numbers increase strictly from left to right along rows,
- numbers increase weakly from top to bottom along columns, and
- for each $i = 1, \ldots, r$, the numbers used in the i-th row are at most f_i.

Let $\mathrm{SST}^*(\lambda/\mu, f)$ be the set of all row-strict tableaux of the flagged skew diagram $(\lambda/\mu, f)$.

Definition 7. Let $\mathbf{b} = (b_i)_{i\in\mathbb{Z}}$. We define the row-strict flagged skew factorial Schur polynomial of a flagged shape $(\lambda/\mu, f)$ by

$$\widetilde{s}_{\lambda/\mu,f}(z|\mathbf{b}) = \sum_{T\in \mathrm{SST}^*(\lambda/\mu,f)} (z|\mathbf{b})^T$$

where we assign the weight for each T by

$$(z|\mathbf{b})^T = \prod_{e\in T} \left(z_{|e|} + b_{|e|+r(e)-c(e)}\right).$$

Note that here we *do not* assume $b_{-i} = -b_{i+1}$ for $i \geqslant 0$. In the case when $\mu = \varnothing$, then we denote the corresponding polynomial by $\widetilde{s}_{\lambda,f}(z|\mathbf{b})$.

Remark 8. In §6, we will obtain a Jacobi–Trudi type determinant formula for the row-strict flagged skew factorial Schur polynomials (Theorem 29). Our proof is analogous to the lattice path method given in [8], although their set-up is technically different. Namely, our factorial (or double) Schur polynomials are row-strict flagged, or equivalently, column-strict and column-flagged, while theirs are column-strict and row-flagged (cf. [27, Theorem 3.5 and Theorem 3.5*]).

Proposition 9. *Let λ be a strict partition of length r. For a strict partition $\mu \subset \lambda$, let $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_r)$ be the sequence defined by $\bar{\mu}_i = \mu_i + i - 1$ for $i = 1, \ldots, r$. Assume that, if $r \geqslant 2$, then $\lambda_{r-1} > f_{r-1}$ or $\lambda_r \geqslant f_r$. Then we have*

$$Q_{\lambda,f}(x; z|b) = \sum_{\substack{\mu\in\mathcal{SP}\\ \mu\subset\lambda\\ \bar{\mu}\in\mathcal{P}}} Q_\mu(x|b) \cdot \widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b})^\star, \tag{2}$$

where $\star$ is the substitution $b_{-i} \mapsto -b_{i+1}$ for all $i \geqslant 0$.

Proof. The circled numbers form a row-strict flagged skew tableau of a skew *shifted* diagram λ/μ since the entries must be weakly increasing in each row and column. Furthermore, the assumption assures that this skew *shifted* diagram is indeed a skew *unshifted* diagram $\bar{\lambda}/\bar{\mu}$, *i.e.*, $\bar{\mu}$ is a partition contained in the partition $\bar{\lambda}$. Indeed the assumption $\lambda_{r-1} > f_{r-1}$ or $\lambda_r \geqslant f_r$ implies that the circled numbers can appear in the j-th column for $j \geqslant r$. Therefore we have $\bar{\mu} \in \mathcal{P}$. Thus we see that there is an obvious bijection

$$\mathrm{MST}(\lambda, f) \cong \bigsqcup_{\substack{\mu\in\mathcal{SP}\\ \mu\subset\lambda\\ \bar{\mu}\in\mathcal{P}}} \mathrm{MST}(\mu) \times \mathrm{SST}^*(\bar{\lambda}/\bar{\mu}, f), \qquad T \mapsto (T', T^\circ)$$

where T' is the part of T with unmarked and primed numbers and T° is the part of T with circled numbers. This bijection apparently preserves the weights after the substitution $\star$, and hence we obtain the desired formula. □

Remark 10. Proposition 9 implies that $Q_{\lambda,f}(x; z|b)$ is an element of $\Gamma[z, b]$ defined in §2. Indeed, it follows from the facts that the summation in (2) is finite, and that both $Q_\mu(x|b)$ and $\widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b})^\star$ are elements of $\Gamma[z, b]$.

3.3 One row case

In this section, we describe $Q_{\lambda,f}(x;z|b)$ in the case λ has only one row.

Lemma 11. *Let r, t and f be nonnegative integers such that $r-t \geqslant 0$. We have*

$$\widetilde{s}_{(r)/(t),(f)}(z|\mathbf{b}) = e_{r-t}^{[f|r-t-f-1]}(z|\tau^{-t}b).$$

In particular, if $r-t > f$, both sides of the equation are zero.

Proof. If $r-t > f$, then the left hand side is zero, since the tableaux are row-strict. The right hand side is also zero, since it is the $(r-t)$-th elementary symmetric polynomial in $r-t-1$ variables. Suppose $r-t \leqslant f$. If $t=0$, then we have

$$\begin{aligned}\widetilde{s}_{(r),(f)}(z|\mathbf{b}) &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant f} (z_{i_1}+b_{i_1})(z_{i_2}+b_{i_2-1})\cdots(z_{i_r}+b_{i_r+1-r}) \\ &= \sum_{1\leqslant j_1\leqslant\cdots\leqslant j_r\leqslant f+1-r} (b_{j_1}+z_{j_1})(b_{j_2}+z_{j_2+1})\cdots(b_{j_r}+z_{j_r+r-1}).\end{aligned}$$

Since this is the usual one-row factorial Schur polynomial, we have

$$\widetilde{s}_{(r),(f)}(z|\mathbf{b}) = h_r^{[f+1-r|-f]}(b|z) = e_r^{[f|r-f-1]}(z|b).$$

In the general case $t \geqslant 0$, let $m := r-t$, then we have

$$\begin{aligned}\widetilde{s}_{(r)/(t),(f)}(z|\mathbf{b}) &= \sum_{1\leqslant i_1<\cdots<i_m\leqslant f} (z_{i_1}+b_{i_1-t})(z_{i_2}+b_{i_2-1-t})\cdots(z_{i_m}+b_{i_m+1-m-t}) \\ &= \widetilde{s}_{(m),(f)}(z|\tau^{-t}\mathbf{b}) = e_m^{[f|m-f-1]}(z|\tau^{-t}b).\end{aligned}$$

This completes the proof of the formula. □

Remark 12. Suppose that $0 \leqslant r-t \leqslant f$. The b-variables appearing in $\widetilde{s}_{(r)/(t),(f)}(z|\mathbf{b})$ are

$$b_{1-t}, b_{2-t}, \ldots, b_{f-r}, b_{f-r+1}.$$

If $r > f$, then $t > 0$ and the indices of those b_i's are all nonpositive.

Proposition 9 and Lemma 11 imply the following.

Proposition 13. *For nonnegative integers r and f, we have*

$$Q_{(r),(f)}(x;z|b) = \sum_{k=0}^{f} q_{r-k}^{[r-k-1]}(x|b)\cdot e_k^{[f|k-f-1]}(z|\tau^{k-r}b)^{\star},$$

where $\star$ is the substitution $b_{-i} = -b_{i+1}$ for all $i \geqslant 0$.

Proof. Proposition 9 implies that

$$Q_{(r),(f)}(x;z|b) = \sum_{k=0}^{r} Q_{(r-k)}(x|b) \cdot \widetilde{s}_{(r)/(r-k),(f)}(z|\mathbf{b})^{\star}.$$

It is known that $Q_{(m)}(x|b) = q_m^{[m-1]}(x|b)$ (see [14, §11]) and thus together with Lemma 11 we have

$$Q_{(r),(f)}(x;z|b) = \sum_{k=0}^{r} q_{r-k}^{[r-k-1]}(x|b) \cdot e_k^{[f|k-f-1]}(z|\tau^{k-r}b)^{\star}.$$

The upper bound for k in the summation can be f instead of r: if $r < f$, the claim holds since $q_{r-k}^{[r-k-1]}(x|b) = 0$ for $r < k \leqslant f$; if $f < r$, the claim holds since $e_k^{[f|k-f-1]}(z|\tau^{k-r}b) = 0$ for $f < k \leqslant r$. Thus we have proved the desired formula. □

3.4 Other formulas

In the rest of the section, we prove Proposition 15 below. It will be used in the proof of Theorem 22 in the next section. Let $\star$ denote the substitution $b_{-i} \mapsto -b_{i+1}$ for all $i \geqslant 0$ as before. We start with the following lemma.

Lemma 14. *Let $s, t, m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geqslant 0}$. For each $s \in \mathbb{Z}$, we have*

$$\sum_{\ell \leqslant s} q_\ell^{[m]} \cdot e_{t-\ell}^{[-n-1]}(\tau^{-m}b)^{\star} = q_s^{[m-1]} \cdot e_{t-s}^{[-n-1]}(\tau^{-m}b)^{\star} + \sum_{\ell \leqslant s-1} q_\ell^{[m-1]} \cdot e_{t-\ell}^{[-n]}(\tau^{1-m}b)^{\star}.$$

Proof. By definition, we have $q_u^{[m]} = q_u^{[m-1]} \cdot (1 + b_m^{\star} u)$ so that

$$q_\ell^{[m]} = q_\ell^{[m-1]} + q_{\ell-1}^{[m]} \cdot b_m^{\star} \quad (\ell \in \mathbb{Z}). \tag{3}$$

Similarly, we have $e_u^{[-n]}(\tau^{1-m}b)^{\star} = e_u^{[-n-1]}(\tau^{-m}b)^{\star} \cdot (1 + b_m^{\star} u)$ so that

$$e_\ell^{[-n]}(\tau^{1-m}b)^{\star} = e_\ell^{[-n-1]}(\tau^{-m}b)^{\star} + e_{\ell-1}^{[-n-1]}(\tau^{-m}b)^{\star} \cdot b_m^{\star} \qquad (\ell \in \mathbb{Z}). \tag{4}$$

Using these identities, we can compute:

$$\begin{aligned}
&\sum_{\ell \leqslant s} q_\ell^{[m]} \cdot e_{t-\ell}^{[-n-1]}(\tau^{-m}b)^{\star} \\
\overset{(3)}{=}\; & \sum_{\ell \leqslant s} q_\ell^{[m-1]} \cdot e_{t-\ell}^{[-n-1]}(\tau^{-m}b)^{\star} + \sum_{\ell \leqslant s} q_{\ell-1}^{[m-1]} \cdot b_m^{\star} \cdot e_{t-\ell}^{[-n-1]}(\tau^{-m}b)^{\star} \\
=\; & q_s^{[m-1]} \cdot e_{t-s}^{[-n-1]}(\tau^{-m}b)^{\star} + \sum_{\ell \leqslant s-1} q_\ell^{[m-1]} \cdot e_{t-\ell}^{[-n-1]}(\tau^{-m}b)^{\star} \\
& \qquad + \sum_{\ell \leqslant s-1} q_\ell^{[m-1]} \cdot b_m^{\star} \cdot e_{t-\ell-1}^{[-n-1]}(\tau^{-m}b)^{\star} \\
\overset{(4)}{=}\; & q_s^{[m-1]} \cdot e_{t-s}^{[-n-1]}(\tau^{-m}b)^{\star} + \sum_{\ell \leqslant s-1} q_\ell^{[m-1]} \cdot e_{t-\ell}^{[-n]}(\tau^{1-m}b)^{\star}.
\end{aligned}$$

Thus we obtain the desired formula. □

Proposition 15. *For integers $r, f \geqslant 0$ and an integer a, we have*

$$q_{r+a}^{[f|r-f-1]}(x;z|b) = \sum_{k=0}^{f} q_{r-k+a}^{[r-k-1]}(x|b) \cdot e_k^{[f|k-1-f]}(z|\tau^{k-r}b)^{\star}.$$

In particular, we have

$$Q_{(r),(f)}(x;z|b) = q_r^{[f|r-f-1]}(x;z|b) \tag{5}$$

in view of Proposition 13.

Proof. First we observe that $e_u^{[r-1-f]}(b) = e_u^{[r]}(b) \cdot e_u^{[-1-f]}(\tau^{-r}b)^{\star}$. Indeed, if $r > f$, then

$$\begin{aligned} e_u^{[r]}(b) \cdot e_u^{[-1-f]}(\tau^{-r}b)^{\star} &= e_u^{[r]}(b_1,\ldots,b_r) \cdot e_u^{[-1-f]}(b_{1-r}, b_{2-r}, \ldots, \ldots, b_{f+1-r})^{\star} \\ &= e_u^{[r]}(b_1,\ldots,b_r) \cdot e_u^{[-1-f]}(-b_r, -b_{r-1}, \ldots, -b_{r-f}) \\ &= e_u^{[r]}(b_1,\ldots,b_r) \cdot e_u^{[f+1]}(b_{r-f}, \cdots, b_{r-1}, b_r)^{-1} \\ &= e_u^{[r-1-f]}(b_1,\ldots,b_{r-f-1}) = e_u^{[r-1-f]}(b). \end{aligned}$$

If $r \leqslant f$, then

$$\begin{aligned} e_u^{[r]}(b) \cdot e_u^{[-1-f]}(\tau^{-r}b)^{\star} &= e_u^{[r]}(b_1,\ldots,b_r) \cdot e_u^{[-1-f]}(\underbrace{b_{1-r}, b_{2-r}, \ldots, b_{-1}, b_0}_{r}, b_1 \ldots, b_{f+1-r})^{\star} \\ &= e_u^{[r]}(b_1,\ldots,b_r) \cdot e_u^{[-1-f]}(\underbrace{-b_r, -b_{r-1}, \ldots, -b_2, -b_1}_{r}, b_1 \ldots, b_{f+1-r}) \\ &= e_u^{[r-1-f]}(b_1,\ldots,b_{f+1-r}) = e_u^{[r-1-f]}(b). \end{aligned}$$

Thus $q_u^{[f|r-f-1]} = q_u^{[r]} \cdot e_u^{[f|-1-f]}(z|\tau^{-r}b)^{\star}$. In particular, we have

$$q_{r+a}^{[f|r-f-1]} = \sum_{\ell \leqslant r+a} q_\ell^{[r]} \cdot e_{r+a-\ell}^{[f|-1-f]}(z|\tau^{-r}b)^{\star}. \tag{6}$$

On the other hand, by setting $s = r + a - k$, $m = r - k$, $t = r + a$, $n = f - k$ for $k = 0, \ldots, f$ in the identity of Lemma 14, we obtain

$$\begin{aligned} &\sum_{\ell \leqslant r+a-k} q_\ell^{[r-k]} \cdot e_{r+a-\ell}^{[f|k-1-f]}(z|\tau^{k-r}b)^{\star} \\ &= q_{r+a-k}^{[r-k-1]} \cdot e_k^{[f|k-1-f]}(z|\tau^{k-r}b)^{\star} + \sum_{\ell \leqslant r+a-k-1} q_\ell^{[r-k-1]} \cdot e_{r+a-\ell}^{[f|k-f]}(z|\tau^{k+1-r}b)^{\star}. \end{aligned}$$

We apply this to the right hand side of (6) consecutively from $k=0$ to $k=f$, and obtain

$$\begin{aligned}
q_{r+a}^{[f|r-f-1]} &= \sum_{\ell\leqslant r+a} q_\ell^{[r]}\cdot e_{r+a-\ell}^{[f|-1-f]}(z|\tau^{-r}b)^\star\\
&= q_{r+a}^{[r-1]}\cdot e_0^{[f|-1-f]}(z|\tau^{-r}b)^\star + \sum_{\ell\leqslant r+a-1} q_\ell^{[r-1]}\cdot e_{r+a-\ell}^{[f|-f]}(z|\tau^{1-r}b)^\star\\
&= \sum_{k=0}^{1} q_{r+a-k}^{[r-1-k]}\cdot e_k^{[f|k-1-f]}(z|\tau^{k-r}b)^\star + \sum_{\ell\leqslant r+a-2} q_\ell^{[r-2]}\cdot e_{r+a-\ell}^{[f|1-f]}(z|\tau^{2-r}b)^\star\\
&= \cdots\\
&= \sum_{k=0}^{f} q_{r+a-k}^{[r-1-k]}\cdot e_k^{[f|k-1-f]}(z|\tau^{k-r}b)^\star + \sum_{\ell\leqslant r+a-f-1} q_\ell^{[r-f-1]}\cdot e_{r+a-\ell}^{[f|0]}(z|\tau^{f+1-r}b)^\star.
\end{aligned}$$

The last summation is zero since $r+a-\ell>f$ and $e_u^{[f|0]}$ is a degree f polynomial in u. Thus we obtain the desired equation. □

Remark 16. The identity of Proposition 15 can be also written as

$$q_{r+a}^{[f|r-f-1]}(x;z|b) = \sum_{k\in\mathbb{Z}} q_{r-k+a}^{[r-k-1]}(x|b)\cdot e_k^{[f|k-1-f]}(z|\tau^{k-r}b)^\star.$$

since, if $k>f$, then $e_u^{[f|k-1-f]}(z|\tau^{k-r}b)$ is a degree $k-1$ polynomial in u so that $e_k^{[f|k-1-f]}(z|\tau^{k-r}b)=0$.

4 Schur-Pfaffian formula

In this section, we review the basic properties of Schur-Pfaffian and then prove a Pfaffian formula of the flagged factorial Q-functions $Q_\lambda(x;z|b)$.

4.1 Schur-Pfaffian and factorial Q-functions

Let $\alpha=(\alpha_1,\ldots,\alpha_r)\in\mathbb{Z}^r$ be a sequence of integers. Consider the Laurent series in variables $t_1,\ldots,t_r$

$$f^\alpha(t) = t_1^{\alpha_1}\cdots t_r^{\alpha_r}\prod_{1\leqslant i<j\leqslant r}\frac{1-t_i/t_j}{1+t_i/t_j}$$

where we expand $\frac{1}{1+t_i/t_j}$ as the series $\sum_{m\geqslant 0}(-t_it_j^{-1})^m$. Consider sequences of indeterminants

$$c^{(i)} = \left(\, c_m^{(i)}\,\right)_{m\in\mathbb{Z}} \qquad (i=1,\ldots,r).$$

The *Schur-Pfaffain* Pf $\left[c_{\alpha_1}^{(1)}\cdots c_{\alpha_r}^{(r)}\right]$ associated to α is defined by replacing each monomial $t_1^{m_1}\cdots t_r^{m_r}$ in $f^\alpha(t)$ by $c_{m_1}^{(1)}\cdots c_{m_r}^{(r)}$.

Below in Lemma 17 and 19, we list well-known properties without proofs (cf. [13, §4]).

Lemma 17.

(1) *If* $\mathrm{Pf}[c^{(i)}_{\alpha_i}c^{(j)}_{\alpha_j}] + \mathrm{Pf}[c^{(j)}_{\alpha_j}c^{(i)}_{\alpha_i}] = 0$ *for all* $1 \leqslant i, j \leqslant r$*, then for any* $w \in S_r$*, we have*

$$\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_r}\right] = \mathrm{sign}(w)\,\mathrm{Pf}\left[c^{(1)}_{\alpha_{w(1)}}\cdots c^{(r)}_{\alpha_{w(r)}}\right].$$

(2) *If* $c^{(i)}_m = ka_m + \ell b_m$ *with variables* $a = (a_m)_{m\in\mathbb{Z}}$ *and* $b = (b_m)_{m\in\mathbb{Z}}$*, we have*

$$\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(i)}_{\alpha_i}\cdots c^{(r)}_{\alpha_r}\right] = k\,\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots a_{\alpha_i}\cdots c^{(r)}_{\alpha_r}\right] + \ell\,\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots b_{\alpha_i}\cdots c^{(r)}_{\alpha_r}\right].$$

(3) *If* r *is even, then*

$$\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_r}\right] = \mathrm{Pf}\left(\mathrm{Pf}[c^{(i)}_{\alpha_i}c^{(j)}_{\alpha_j}]\right)_{1\leqslant i<j\leqslant r}$$

where the right hand side is the Pfaffian of the $r \times r$ *skew symmetric matrix given by*

$$\left(\mathrm{Pf}[c^{(i)}_{\alpha_i}c^{(j)}_{\alpha_j}]\right)_{1\leqslant i<j\leqslant r}$$

with (i,j)*-entry*

$$\mathrm{Pf}[c^{(i)}_{\alpha_i}c^{(j)}_{\alpha_j}] = c^{(i)}_{\alpha_i}c^{(j)}_{\alpha_j} + 2\sum_{k\geqslant 1}(-1)^k c^{(i)}_{\alpha_i+k}c^{(j)}_{\alpha_j-k}$$

for $i < j$*.*

Remark 18. Lemma 17 (3) follows from the identity

$$\prod_{1\leqslant i<j\leqslant r}\frac{1-t_i/t_j}{1+t_i/t_j} = \mathrm{Pf}\left(\frac{1-t_i/t_j}{1+t_i/t_j}\right)_{1\leqslant i<j\leqslant r}.$$

for r even, which is due to Schur [24].

Lemma 19. *We denote the substitution* $c^{[i]}_m = 0$ *for all* $m < 0$ *and* $i = 1, \ldots, r$ *by* $\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_r}\right]_{\geqslant 0}$*. We have*

(1) $\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_r}\right]_{\geqslant 0}$ *is a polynomial in* $c^{(i)}_m$*'s* $(m \geqslant 0)$*.*

(2) *If* $\alpha_r = 0$*, then* $\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_r}\right]_{\geqslant 0} = \mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_{r-1}}\right]_{\geqslant 0}$*.*

If $\alpha_r < 0$*, then* $\mathrm{Pf}\left[c^{(1)}_{\alpha_1}\cdots c^{(r)}_{\alpha_r}\right]_{\geqslant 0} = 0$*.*

By the work of Kazarian [16] and Ikeda [12], it is known that the factorial Q-functions $Q_\lambda(x|b)$ of Ivanov [15] can be expressed as a Schur-Pfaffian: for a strict partition $\lambda = (\lambda_1, \ldots, \lambda_r)$,

$$Q_\lambda(x|b) = \mathrm{Pf}\left[q^{[\lambda_1-1]}_{\lambda_1}q^{[\lambda_2-1]}_{\lambda_2}\cdots q^{[\lambda_r-1]}_{\lambda_r}\right] := \mathrm{Pf}\left[c^{(1)}_{\lambda_1}c^{(2)}_{\lambda_2}\cdots c^{(r)}_{\lambda_r}\right]\Big|_{c^{(i)}_m = q^{[\lambda_i-1]}_m,\forall i,m}.$$

Lemma 20. *For $k, \ell \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\mathrm{Pf}\left[q_k^{[k-1]} q_\ell^{[\ell-1]}\right] + \mathrm{Pf}\left[q_\ell^{[\ell-1]} q_k^{[k-1]}\right] = 0.$$

Proof. The left hand side equals to $2\sum_{r\in\mathbb{Z}}(-1)^r q_{k+r}^{[k-1]} q_{\ell-r}^{[\ell-1]}$, which is the coefficient of $u^{k+\ell}$ in $2q_{-u}^{[k-1]} q_u^{[\ell-1]} = 2e_{-u}^{[k-1]}(b) e_u^{[\ell-1]}(b)$, a polynomial in u of degree $k+\ell-2$. Therefore it is zero. $\square$

Lemma 17 (1) and Lemma 20 imply the following.

Lemma 21. *For a sequence of positive integers $(\alpha_1, \ldots, \alpha_r)$ and $w \in S_r$, we have*

$$\mathrm{Pf}\left[q_{\alpha_1}^{[\alpha_1-1]} \cdots q_{\alpha_r}^{[\alpha_r-1]}\right] = \mathrm{sgn}(w)\,\mathrm{Pf}\left[q_{\alpha_{w(1)}}^{[\alpha_1-1]} \cdots q_{\alpha_{w(r)}}^{[\alpha_r-1]}\right].$$

If particular, if $\alpha_i = \alpha_j$ for some $i \neq j$, we have $\mathrm{Pf}\left[q_{\alpha_1}^{[\alpha_1-1]} \cdots q_{\alpha_r}^{[\alpha_r-1]}\right] = 0$.

4.2 Schur-Pfaffian formula of flagged factorial Q-functions

In this section, we prove the Schur–Pfaffian formula of flagged Q-functions. The idea of the proof is to separate the part of the function given in the z-variables and to identify it as a (factorial) skew Schur polynomials. This allows us to apply the Schur-determinant formula of skew Schur polynomials from the appendix below and the Schur-Pfaffian formula of Schur Q-functions.

Theorem 22. *Let (λ, f) be a flagged strict partition of length r. Assume that, if $r \geqslant 2$,* (a) *$\lambda_i - f_i \geqslant \lambda_j - f_j$ for all $1 \leqslant i < j \leqslant r$ and* (b) *$\lambda_{r-1} - f_{r-1} > 0$. We have*

$$Q_{\lambda,f}(x, z|b) = \mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]} q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]} \cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right].$$

Proof. Let $(\nu_1, \ldots, \nu_r) \in \mathbb{Z}^r$. By Proposition 15 (and Remark 16), we have

$$q_{\lambda_i+\nu_i}^{[f_i|\lambda_i-f_i-1]} = \sum_{\alpha_i\in\mathbb{Z}} q_{\alpha_i+\nu_i}^{[\alpha_i-1]} \cdot e_{\lambda_i-\alpha_i}^{[f_i|\lambda_i-\alpha_i-f_i-1]}(z|\tau^{-\alpha_i}b)^\star \qquad (i = 1, \ldots, r).$$

By linearity (Lemma 17 (2)),

$$\begin{aligned}&\mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]} q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]} \cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right]\\ =& \sum_{(\alpha_1,\ldots,\alpha_r)\in\mathbb{Z}^r} \left(\prod_{i=1}^r e_{\lambda_i-\alpha_i}^{[f_i|\lambda_i-\alpha_i-f_i-1]}(z|\tau^{-\alpha_i}b)^\star\right) \mathrm{Pf}\left[q_{\alpha_1}^{[\alpha_1-1]} q_{\alpha_2}^{[\alpha_2-1]} \cdots q_{\alpha_r}^{[\alpha_r-1]}\right].\end{aligned}$$

Suppose that $\lambda_r - f_r > 0$. In this case, by the assumption (a), we have $\lambda_i - f_i > 0$ for all $i = 1, \ldots, r$ so that $e_{\lambda_i-\alpha_i}^{[f_i|\lambda_i-\alpha_i-f_i-1]}(z|\tau^{-\alpha_i}b)^\star = 0$ for $\alpha_i \leqslant 0$. Thus we have

$$\begin{aligned}&\mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]} q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]} \cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right]\\ =& \sum_{(\alpha_1,\ldots,\alpha_r)\in(\mathbb{Z}_{>0})^r} \left(\prod_{i=1}^r e_{\lambda_i-\alpha_i}^{[f_i|\lambda_i-\alpha_i-f_i-1]}(z|\tau^{-\alpha_i}b)^\star\right) \mathrm{Pf}\left[q_{\alpha_1}^{[\alpha_1-1]} q_{\alpha_2}^{[\alpha_2-1]} \cdots q_{\alpha_r}^{[\alpha_r-1]}\right].\end{aligned}$$

By Lemma 21, we have

$$
\begin{aligned}
&\mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]}q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]}\cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right]\\
=&\sum_{\substack{\mu\in\mathcal{SP}_r\\ \mu_r>0}}\left(\sum_{w\in S_r}\mathrm{sgn}(w)\prod_{i=1}^{r}e_{\lambda_i-\mu_{w(i)}}^{[f_i|\lambda_i-\mu_{w(i)}-f_i-1]}(z|\tau^{-\mu_{w(i)}}b)^{\star}\right)Q_\mu(x|b)\\
=&\sum_{\substack{\mu\in\mathcal{SP}_r\\ \mu_r>0}}\det\left(e_{\lambda_i-\mu_j}^{[f_i|\lambda_i-\mu_j-f_i-1]}(z|\tau^{-\mu_j}b)^{\star}\right)_{1\leqslant i,j\leqslant r}Q_\mu(x|b).
\end{aligned}
$$

It is easy to see that the determinant vanishes if there is k such that $\lambda_k-\mu_k<0$. Thus the sum runs over all $\mu\in\mathcal{SP}_r$ such that $\mu_r>0$ and $\mu\subset\lambda$. In particular, $\bar{\mu}$ is a partition since $\mu_r>0$. Finally, by Theorem 29, we have

$$
\begin{aligned}
\det\left(e_{\lambda_i-\mu_j}^{[f_i|\lambda_i-\mu_j-f_i-1]}(z|\tau^{-\mu_j}b)\right)_{1\leqslant i,j\leqslant r} &= \det\left(e_{\bar{\lambda}_i-\bar{\mu}_j+j-i}^{[f_i|\bar{\lambda}_i-\bar{\mu}_j+j-i-f_i-1]}(z|\tau^{j-\bar{\mu}_j-1}b)\right)_{1\leqslant i,j\leqslant r}\\
&= \widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b}),
\end{aligned}
$$

where the assumption (a) implies the inequalities that must be satisfied by $(\overline{\lambda},f)$. Thus we obtain

$$
\mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]}q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]}\cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right]=\sum_{\substack{\mu\in\mathcal{SP}\\ \mu\subset\lambda\\ \bar{\mu}\in\mathcal{P}}}\widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b})^{\star}\cdot Q_\mu(x|b),
$$

and finally the claim follows from Proposition 9. Here note that $\widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b})^{\star}=0$ if $\mu_r=0$ since $\lambda_r-f_r>0$.

Suppose that $\mu_r-f_r\leqslant 0$. In this case, we have

$$
\begin{aligned}
&\mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]}q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]}\cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right]\\
=&\sum_{(\alpha_1,\dots,\alpha_r)\in(\mathbb{Z}_{>0})^r}\left(\prod_{i=1}^{r}e_{\lambda_i-\alpha_i}^{[f_i|\lambda_i-\alpha_i-f_i-1]}(z|\tau^{-\alpha_i}b)^{\star}\right)\mathrm{Pf}\left[q_{\alpha_1}^{[\alpha_1-1]}q_{\alpha_2}^{[\alpha_2-1]}\cdots q_{\alpha_r}^{[\alpha_r-1]}\right]\\
&+\sum_{(\alpha_1,\dots,\alpha_{r-1})\in(\mathbb{Z}_{>0})^{r-1}}e_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}(z|b)^{\star}\\
&\qquad\qquad\times\left(\prod_{i=1}^{r-1}e_{\lambda_i-\alpha_i}^{[f_i|\lambda_i-\alpha_i-f_i-1]}(z|\tau^{-\alpha_i}b)^{\star}\right)\mathrm{Pf}\left[q_{\alpha_1}^{[\alpha_1-1]}q_{\alpha_2}^{[\alpha_2-1]}\cdots q_{\alpha_{r-1}}^{[\alpha_{r-1}-1]}\right]\\
=&\sum_{\substack{\mu\in\mathcal{SP}\\ \mu\subset\lambda\\ \mu_r>0}}\widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b})^{\star}\cdot Q_\mu(x|b)\\
&\qquad+\sum_{\substack{\mu\in\mathcal{SP}\\ \mu\subset\lambda\\ \mu_r=0}}e_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}(z|b)^{\star}\det\left(e_{\lambda_i-\mu_j}^{[f_i|\lambda_i-\mu_j-f_i-1]}(z|\tau^{-\mu_j}b)^{\star}\right)_{1\leqslant i,j\leqslant r-1}Q_\mu(x|b).
\end{aligned}
$$

Since $e_{\lambda_i-\mu_r}^{[f_i|\lambda_i-\mu_r-f_i-1]}(z|\tau^{-\mu_r}b)^\star = 0$ for all $i = 1, \dots, r-1$, we have

$$\begin{aligned} & e_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}(z|b)^\star \det\left(e_{\lambda_i-\mu_j}^{[f_i|\lambda_i-\mu_j-f_i-1]}(z|\tau^{-\mu_j}b)^\star\right)_{1\leqslant i,j\leqslant r-1} \\ = \ & \det\left(e_{\lambda_i-\mu_j}^{[f_i|\lambda_i-\mu_j-f_i-1]}(z|\tau^{-\mu_j}b)^\star\right)_{1\leqslant i,j\leqslant r}. \end{aligned}$$

Thus we obtain

$$\mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]}q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]}\cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right] = \sum_{\substack{\mu\in\mathcal{SP} \\ \mu\subset\lambda \\ \bar{\mu}\in\mathcal{P}}} \widetilde{s}_{\bar{\lambda}/\bar{\mu},f}(z|\mathbf{b})^\star \cdot Q_\mu(x|b),$$

and finally the claim follows from Proposition 9. $\square$

5 Vexillary double Schubert polynomials of type C

5.1 Double Schubert polynomials of type C

In this section, we briefly recall the double Schubert polynomials of Ikeda–Mihalcea–Naruse. Please see [14] for more detail.

Let W_∞ be the infinite hyperoctahedral group, *i.e.*, the Weyl group of type C_∞ (or B_∞). It is given as the group defined by generators (simple reflections) $\{s_i \mid i = 0, 1, 2, \dots\}$ and relations

$$s_i^2 = e \ (i \geqslant 0), \ \ s_1s_0s_1s_0 = s_0s_1s_0s_1, \ \ s_is_{i+1}s_i = s_{i+1}s_is_{i+1}(i \geqslant 1), \ \ s_is_j = s_js_i(|i-j| \geqslant 2),$$

where e is the identity element. We identify W_∞ with the group of *signed permutations*, *i.e.*, permutations w of the set $\{1, 2, \dots\} \cup \{-1, -2, \dots\}$ such that $w(i) \neq i$ for only finitely many i, and $\overline{w(i)} = w(\bar{i})$ where we denote $\bar{i} = -i$. Each element of W_∞, therefore, can be specified by the sequence $(w(1)w(2)w(3)\cdots)$ which we call the one-line notation of w. The simple reflections are identified with the transpositions $s_0 = (1, \bar{1})$ and $s_i = (i, i+1)(\bar{i}, \overline{i+1})$ for $i \geqslant 1$. Similarly we consider the group W_n of signed permutations of $\{1, 2, \dots, n\} \cup \{\bar{1}, \bar{2}, \dots, \bar{n}\}$ and naturally regard it as a subgroup of W_∞. If a signed permutation w in W_n, its one line notation is also denoted by the finite sequence $(w(1)w(2)\cdots w(n))$.

To each $w \in W_\infty$, Ikeda–Mihalcea–Naruse [14] associated a unique function $\mathfrak{C}_w = \mathfrak{C}_w(x; z|b)$ in the ring $\Gamma[z, b]$[1]. They are characterized by left and right divided difference operators δ_i and ∂_i with $i = 0, 1, 2, \dots$. Namely there is a unique family of elements $\mathfrak{C}_w(x; z|b) \in \Gamma[z, b]$ $(w \in W_\infty)$, satisfying

$$\partial_i\mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise}, \end{cases} \qquad \delta_i\mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_iw} & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise}, \end{cases}$$

for all $i = 0, 1, 2, \dots$, and such that $\mathfrak{C}_w$ has no constant term except for $\mathfrak{C}_e = 1$.

[1] Note that the parameters $t = (t_1, t_2, \dots)$ in [14] are replaced by $-b = (-b_1, -b_2, \dots)$ in this paper.

5.2 Vexillary signed permutations

We follow Anderson–Fulton [3]. A *triple* is a three r-tuples of positive integers, $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, with $\mathbf{k} = (0 < k_1 < \cdots < k_s)$, $\mathbf{p} = (p_1 \geqslant \cdots \geqslant p_s > 0)$, and $\mathbf{k} = (q_1 \geqslant \cdots \geqslant q_s > 0)$, satisfying the inequality

$$(*) \qquad k_{i+1} - k_i \leqslant p_i - p_{i+1} + q_i - q_{i+1} \qquad (1 \leqslant i \leqslant s-1).$$

A triple is *essential* if the inequality $(*)$ is strict for all i. Each triple reduces to a unique essential triple by successively removing (k_i, p_i, q_i) such that the equality holds in $(*)$ and two triples are equivalent if they reduce to the same essential triple.

Anderson–Fulton explained how to construct a signed permutation $w = w(\tau)$ in [3, §2]. They define a signed permutation to be *vexillary* if it arises from a triple in such a way. Equivalent triples give the same vexillary signed permutation ([3, Lemma 2.2]). An essential triple τ determines a strict partition $\lambda(\tau)$ of length $r := k_s$, by setting $\lambda_{k_i} = p_i + q_i - 1$ for $i = 1, \ldots, s$, and filling in the remaining λ_k minimally so that $\lambda_1 > \cdots > \lambda_r$. Similarly, we introduce a flag $f(\tau) = (f_1, \ldots, f_r)$ associated to an essential triple τ by setting $f_{k_i} := p_i - 1$ for $i = 1, \ldots, s$, and filling in the remaining f_k minimally so that $f_1 \geqslant \cdots \geqslant f_r$. In this way, we can assign a unique flagged strict partition to each vexillary signed permutation. For example, consider an (essential) triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q}) = (234, 421, 663)$. The corresponding vexillary signed permutation is $w = (\bar{3}\bar{8}1\bar{7}\bar{6}245)$. The associated flagged strict partition is $\lambda = (10, 9, 7, 3)$ with $f = (3, 3, 1, 0)$.

A flagged strict partition under a certain condition gives rise to an (essential) triple and hence a vexillary signed permutation. For a flagged strict partition (λ, f), let $\mathbf{k} = (k_1 < \cdots < k_s)$ be the row indices of the south-east corners of the shifted Young diagram of λ. Suppose that the following condition holds:

$$f_{k_i} < \lambda_{k_i} \text{ for } i = 1, \ldots, s \text{ and } f_{k_i} - f_{k_{i+1}} \leqslant \lambda_{k_i} - \lambda_{k_{i+1}} \text{ for } i = 1, \ldots, s-1. \tag{7}$$

The corresponding triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$ is given by setting $p_i = f_{k_i} + 1$ and $q_i = \lambda_{k_i} - p_i + 1 = \lambda_{k_i} - f_{k_i}$. Note that f_{k_i}'s form nothing but the labeling (m_i) of $\lambda(\tau)$ given in [3, §4]. On the other hand, we cannot associate a triple to a flagged strict partition that does not satisfy the condition (7). In this point of view, our flagged strict partitions generalize the labelled Young diagrams in [3] which are in bijection with vexillary signed permutations.

One of the characterizations obtained by Anderson–Fulton relates vexillary signed permutations to vexillary permutations. A permutation $u \in S_m$ of $\{1, \ldots, m\}$ is called *vexillary* if it is 2143-avoiding, i.e. there is no $i < j < k < l$ such that $u(j) < u(i) < u(l) < u(k)$. Anderson–Fulton show that $w \in W_n$ is vexillary if and only if $\iota(w)$ is 2143-avoiding where ι is the obvious embedding of W_n into the symmetric group S_{2n+1} obtained by setting $\iota(w)(i) = w(i-n-1)$ and $w(0) := 0$. It is easy to check, by this characterization, that the above example $w = (\bar{3}\bar{8}1\bar{7}\bar{6}245)$ is a vexillary signed permutation.

From the work of Anderson-Fulton [1, 2], it follows that the double Schubert polynomials associated to vexillary signed permutations can be given in the following Pfaffian formula.

Theorem 23 (Anderson-Fulton [1, 2]). *Let w be a vexillary signed permutation and (λ, f) the associated flagged strict partition. Then we have*

$$\mathfrak{C}_w(x;z|b) = \mathrm{Pf}\left[q_{\lambda_1}^{[f_1|\lambda_1-f_1-1]}q_{\lambda_2}^{[f_2|\lambda_2-f_2-1]}\cdots q_{\lambda_r}^{[f_r|\lambda_r-f_r-1]}\right].$$

By construction, the flagged strict partition (λ, f) associated to a vexillary signed permutation w satisfies the requirement in Theorem 22. Thus we obtain the following theorem.

Theorem 24. *Let w be a vexillary signed permutation and (λ, f) the associated flagged strict partition. Then we have $\mathfrak{C}_w(x;z|b) = Q_{\lambda,f}(x;z|b)$.*

In Example 6, we considered the flagged strict partition $(\lambda, f) = ((3,1),(1,0))$ and $(\lambda, f) = ((5,3,1),(2,1,0))$. These examples give rise to triples $\tau = (12, 21, 21)$ and $\tau = (123, 321, 321)$ respectively. The corresponding vexillary signed permutations are $w = (\bar{1}\bar{2})$ and $(\bar{1}\bar{2}\bar{3})$ respectively. These are the longest elements in W_2 and W_3. By Theorem 24, we have

$$\mathfrak{C}_{\bar{1}\bar{2}}(x;z|b) = Q_{(3,1),(1,0)}(x;z|b), \qquad \mathfrak{C}_{\bar{1}\bar{2}\bar{3}}(x;z|b) = Q_{(5,3,1),(2,1,0)}(x;z|b).$$

and we can further decompose the right hand side into Schur Q-functions and skew Schur polynomials as in Example 6. As mentioned in above, the condition (7) is necessary to associate a vexillary signed permutation to a flagged strict partition. For example, $Q_{(1),(f)}(x;z|b)$ is not the double Schubert polynomial of a vexillary signed permutation if $f \geqslant 1$ because the vexillary signed permutation w of length 1 is only s_0 by [3, Proposition 3.1] and $\mathfrak{S}_{s_0} = Q_{(1),(0)}$.

5.3 A new tableau formula of Ivanov's factorial Q functions

A signed permutation w is *Lagrangian* if $w(1) < w(2) < \cdots < w(r) < 0 < w(r+1) < \cdots$ for some integer $r \geqslant 1$. A Lagrangian signed permutation is vexillary. Indeed, we can define a triple τ from which w is constructed by setting $k_i = i, p_i = 1$, and $q_i = \overline{w(i)}$ for $i = 1, \ldots, r$. The associated flagged strict partition (λ, f) is given by $\lambda_i = \overline{w(i)}$ and $f_i = 0$ for $i = 1, \ldots, r$.

Anderson–Fulton showed that, if w is vexillary, then w^{-1} is also vexillary. In fact, for a triple $\tau = (\mathbf{k}, \mathbf{p}, \mathbf{q})$, we have $w(\tau)^{-1} = w(\tau^*)$ where $\tau^* = (\mathbf{k}, \mathbf{q}, \mathbf{p})$ ([3, Lemma 2.4]). From this, we can deduce that if w is Lagrangian with the strict partition λ of length r (and the flag $f = (0, \ldots, 0)$), then w^{-1} is a vexillary signed permutation with the strict partition λ and the flag $f = (\lambda_1 - 1, \ldots, \lambda_r - 1)$.

Due to the work of Kazarian [16] and Ikeda [12], we have $\mathfrak{C}_w(x;z|b) = Q_\lambda(x|b)$ for a Lagrangian signed permutation w with the associated strict partition λ. On the other hand, we have $\mathfrak{C}_w(x;z|b) = \mathfrak{C}_{w^{-1}}(x;b|z)$ for any signed permutation w by [14, Theorem 8.1 (3)]. Combining these facts with Theorem 24, we obtain that $\mathfrak{C}_w(x;z|b) = Q_{(\lambda,f)}(x;b|z)$ for a Lagrangian signed permutation w with the associated strict partition λ where we set $f = (\lambda_1 - 1, \ldots, \lambda_r - 1)$. Thus we can conclude that $Q_\lambda(x|b) = Q_{(\lambda,f)}(x;b|z)$, which shows that the function on the right hand side does not depend on the z-variables. Now by applying Theorem 22, we obtain the following theorem.

Theorem 25. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a strict partition of length r, and $f = (\lambda_1 - 1, \ldots, \lambda_r - 1)$. Then Ivanov's factorial Q function associated to λ can be expressed as*

$$Q_\lambda(x|b) = \sum_{T \in \mathrm{MST}(\lambda, f)} (xb)^T, \qquad (xb)^T = \prod_{k \in T} x_k \prod_{k' \in T} x_k \prod_{k^\circ \in T} b_k.$$

Remark 26. From Theorem 25 and Proposition 9, we can also write

$$Q_\lambda(x|b) = \sum_{\substack{\mu \in \mathcal{SP} \\ \mu \subset \lambda \\ \bar{\mu} \in \mathcal{P}}} Q_\mu(x) \cdot \widetilde{s}_{\bar{\lambda}/\bar{\mu}, f}(b),$$

for a strict partition λ of length r where $f = (\lambda_1 - 1, \ldots, \lambda_r - 1)$. In view of Theorem 29, this recovers [15, Theorem 10.2].

6 Appendix: Lattice path method for row-strict Schur polynomials

In this section, we prove a Jacobi–Trudi type formula (Theorem 29 below) for the row-strict flagged skew factorial Schur polynomials from Definition 7. It is a factorial generalization of Theorem 3.5* in [27]. We prove it by interpreting the tableaux as lattice paths and applying [25, Theorem 1.2] (cf. [21, 10, 11]). The analogous proof in the column-strict case has been obtained in [8] (cf. Theorem 3.5 in [27]). One of the main differences from [8] is that their formula is given by a determinant with entries in (double) *complete* symmetric polynomials while Theorem 29 is given in terms of (double) *elementary* symmetric polynomials.

First we recall the basic notations from [25]. Let $D = (V, E)$ be an acyclic oriented graph without multiple edges: V is the set of vertices and E is the set of edges in D. For vertices u and v, a path from u to v is a sequence of edges $e_1, \ldots, e_m$ such that the source of e_1 is u, the target of e_m is v, and the target of e_i coincides with the source of e_{i+1} for all $i = 1, \ldots, m-1$. Let $\mathscr{P}(u, v)$ be the set of all paths from u to v. Let $w : E \to R$ be a weight function where R is some commutative ring. For a path P, we also denote $w(P)$ the product of the weights of all edges in P. Let

$$GF\left[\mathscr{P}(u, v)\right] = \sum_{P \in \mathscr{P}(u,v)} w(P).$$

Let $\mathbf{u} = (u_1, \ldots, u_r)$ and $\mathbf{v} = (v_1, \ldots, v_r)$ be ordered sets of vertices of D. Let $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$ is the set of all non-intersecting r-tuples of paths, $\mathbf{P} = (P_1, \ldots, P_r)$, with $P_i \in \mathscr{P}(u_i, v_i)$. We denote

$$GF\left[\mathscr{P}_0(\mathbf{u}, \mathbf{v})\right] = \sum_{\mathbf{P} \in \mathscr{P}_0(\mathbf{u}, \mathbf{v})} w(\mathbf{P})$$

where we set $w(\mathbf{P}) = w(P_1) w(P_2) \cdots w(P_r)$. Finally, we say that $\mathbf{u}$ is D-compatible with $\mathbf{v}$ if a path $P \in \mathscr{P}(u_i, v_j)$ intersects with a path $Q \in \mathscr{P}(u_k, v_l)$ whenever $i < k$ and $j > l$.

Theorem 27 (Theorem 1.2, [25])**.** *Let* $\mathbf{u} = (u_1, \ldots, u_r)$ *and* $\mathbf{v} = (v_1, \ldots, v_r)$ *be ordered sets of vertices such that* $\mathbf{u}$ *is* D*-compatible with* $\mathbf{v}$*. Then*

$$GF\left[\mathscr{P}_0(\mathbf{u},\mathbf{v})\right] = \det\left(GF\left[\mathscr{P}(u_i, v_j)\right]\right)_{1\leqslant i,j\leqslant r}.$$

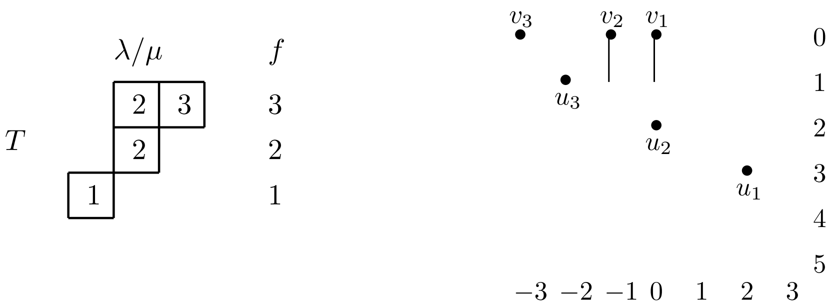
In order to apply Theorem 27 to the row-strict flagged Schur polynomials, we introduce an acyclic directed graph D as follows: its vertex set V is $\mathbb{Z} \times \mathbb{Z}_{\geqslant 0}$ and there is an edge $(u,v) \in E$ from the source u to the target v if $u - v$ is $(0,1)$ or $(1,1)$. We call an edge (u,v) *diagonal* if $u - v = (1,1)$, and *vertical* if $u - v = (0,1)$.

We define a weight function $w : E \to \mathbb{Z}[z, \mathbf{b}]$ by setting $w(e) = 1$ if e is horizontal and $w(e) = z_t + b_{t-s}$ if e is a diagonal edge with its source at (s,t).

Let λ/μ is a skew (unshifted) diagram of length at most r and f its flag. Consider the ordered sets of vertices $\mathbf{u} = (u_1, \ldots, u_r)$ and $\mathbf{v} = (v_1, \ldots, v_r)$ where

$$u_i = (\lambda_i - i, f_i), \quad v_i = (\mu_i - i, 0).$$

There is a bijection between $\mathrm{SST}^*(\lambda/\mu, f)$ and $\mathscr{P}_0(\mathbf{u},\mathbf{v})$ defined as follows. Let $T \in \mathrm{SST}^*(\lambda/\mu, f)$. Let $\mathbf{P} = (P_1, \ldots, P_r)$ be the corresponding r-tuple of paths defined as follows. If $j_m < \cdots < j_1$ are the entries of i-th row of T where $m = \lambda_i - \mu_i$, then we define P_i to be the unique path from u_i to v_i such that the k-th diagonal edge has its source at $(\lambda_i - i - k + 1, j_k)$ for $k = 1, \ldots, m$. For example, let $\lambda = (3,2,1)$, $\mu = (1,1,0)$ and $f = (3,2,1)$. The following is an example of a tableaux T in $\mathrm{SST}^*(\lambda/\mu, f)$ and the corresponding triple of non-intersecting paths.



It is not difficult to see that this defines a bijection from $\mathrm{SST}^*(\lambda/\mu, f)$ to $\mathscr{P}_0(\mathbf{u},\mathbf{v})$. Moreover, this bijection preserves the weights. Namely, suppose that T corresponds to $\mathbf{P}$. Let $j_m < \cdots < j_1$ be the entries of the i-th row of T. The column index of the entry j_k is $\lambda_i - k + 1$ and thus its corresponding weight is $z_{j_k} + b_{j_k + i - (\lambda_i - k + 1)}$. On the other hand, P_i's k-th diagonal edge has its sources at $(\lambda_i - i - k + 1, j_k)$ and thus its weight is also $z_{j_k} + b_{j_k + i - (\lambda_i - k + 1)}$. For example, the weights of the above examples of a tableau and the corresponding paths are both $(x_2 + b_1)(x_3 + b_1) \cdot (x_2 + b_2) \cdot (x_1 + b_3)$. Thus we have

$$\widetilde{s}_{\lambda/\mu, f}(z|\mathbf{b}) = \sum_{T \in \mathrm{SST}^*(\lambda/\mu, f)} (z|\mathbf{b})^T = GF\left[\mathscr{P}_0(\mathbf{u},\mathbf{v})\right]. \tag{8}$$

The following is an extension of Lemma 11 in view of the lattice path interpretation and will be used in the proof of Theorem 29 below.

Lemma 28. *Let $u = (s-1, f)$ and $v = (t-1, 0)$ where $s, t \in \mathbb{Z}$ and $f \in \mathbb{Z}_{\leqslant 0}$, then we have*

$$GF\left[\mathscr{P}(u,v)\right] = e_{s-t}^{[f|s-t-f-1]}(z|\tau^{-t}b).$$

In particular, this identity is trivially zero unless $0 \leqslant s-t \leqslant f$.

Proof. If $s-t < 0$, clearly the identity is zero. If $0 \leqslant f < s-t$, then $\mathscr{P}(u,v) = \varnothing$ so that $GF\left[\mathscr{P}(u,v)\right] = 0$. Furthermore, $e_u^{[f|s-t-f-1]}$ is a polynomial in u of degree $s-t-1$ so that $e_{s-t}^{[f|s-t-f-1]} = 0$. Below we suppose that $0 \leqslant s-t \leqslant f$.

If $t \geqslant 0$, the claim follows from Lemma 11. If $t < 0$, consider $u' = (s-1+n, f)$ and $v' = (t-1+n, 0)$ for some n such that $t+n \geqslant 0$, and then we have, also by Lemma 11,

$$GF\left[\mathscr{P}(u',v')\right] = e_{s-t}^{[f|s-t-f-1]}(z|\tau^{-t-n}b).$$

Since the paths in $\mathscr{P}(u,v)$ are obtained from the paths in $\mathscr{P}(u',v')$ by shifting horizontally to the left by n units, we obtain $GF\left[\mathscr{P}(u,v)\right]$ from $GF\left[\mathscr{P}(u',v')\right]$ by adding n to all indices of b variables. Thus the claim follows. □

Theorem 29. *Let $(\lambda/\mu, f)$ be a flagged skew partition where λ is a partition of length r. Assume that $\lambda_i - i - f_i \geqslant \lambda_j - j - f_j$ for all $i < j$. Then we have*

$$\widetilde{s}_{\lambda/\mu,f}(z|\mathbf{b}) = \det\left(e_{\lambda_i-\mu_j+j-i}^{[f_i|\lambda_i-\mu_j+j-i-f_i-1]}(z|\tau^{j-\mu_j-1}b)\right)_{1\leqslant i,j\leqslant r}.$$

Proof. By the assumption, it follows that $\mathbf{u}$ is D-compatible with $\mathbf{v}$. Thus we can apply Theorem 27 to the right hand side of (8), and obtain

$$\widetilde{s}_{\lambda/\mu,f}(z|\mathbf{b}) = \det\left(GF\left[\mathscr{P}(u_i,v_j)\right]\right)_{1\leqslant i,j\leqslant r}.$$

Now the claim follows by applying Lemma 28 with $u = u_i = (\lambda_i - i, f_i)$ and $v = v_j = (\mu_j - j, 0)$ so that $f = f_i$, $s - t = \lambda_i - \mu_j + j - i$, and $t = \mu_j - j + 1$. □

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