Degree 2 Boolean Functions on Grassmann Graphs

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Abstract

We investigate the existence of Boolean degree d functions on the Grassmann graph of k-spaces in the vector space $\mathbb{F}_q^n$. For $d = 1$ several non-existence and classification results are known, and no non-trivial examples are known for $n \geqslant 5$. This paper focusses on providing a list of examples on the case $d = 2$ in general dimension and in particular for $(n, k) = (6, 3)$ and $(n, k) = (8, 4)$.

We also discuss connections to the analysis of Boolean functions, regular sets/equitable bipartitions/perfect 2-colorings in graphs, q-analogs of designs, and permutation groups. In particular, this represents a natural generalization of Cameron-Liebler line classes.

Mathematics Subject Classifications: 05B25

1 Introduction

The research presented here is motivated by a variety of open problems in only loosely related areas such as finite geometry, Boolean function analysis, association schemes and design theory. Since it seems reasonable to assume that most readers are not familiar with concepts and conventions in all of these areas, we provide a relatively long introduction. We refer to [26] for a more detailed discussion of degree 1 functions.

More technical details and definitions are omitted from the the introduction and can be found in Section 2.

1.1 Low Degree Boolean Functions

It is a well-known fact that one can write any $0,1$-valued (Boolean) function on the hypercube $\{0,1\}^n$ as a real, multilinear polynomial of degree at most n. The study of such functions which we can write as a polynomial of some bounded degree d has been very fruitful. For instance, it has been observed countless times that a Boolean degree one function on the hypercube is of the form 0, 1, x_i, or $1 - x_i$ for some $i \in \{1, \ldots, n\}$:

Theorem 1 (Folklore)**.** *A Boolean degree* 1 *function on the hypercube depends on at most one coordinate.*

One of the fundamental results in *Boolean function analysis* (see [40] for a detailed introduction) is a characterization by Nisan and Szegedy of Boolean degree d functions in [39].

Theorem 2 (Nisan, Szegedy (1994))**.** *A Boolean degree d function on the hypercube depends on at most $d2^{d-1}$ coordinates.*

Let $\gamma(d)$ denote the optimal upper bound for given d, that is there exists a Boolean degree d function depending on $\gamma(d)$ coordinates, but not depending on $\gamma(d)+1$ coordinates. Nisan and Szegedy showed that $\gamma(d) \leqslant d2^{d-1}$. They also described a Boolean degree d function with $2^d - 1$ relevant variables. Recently, better upper and lower bounds of magnitude $O(2^d)$ were found, see [11] and subsequent work. The better lower bound was, in a different context, first observed in [12].

In the last few years, there has been interest in comparable results on domains different from the hypercube. For instance see [13] or [26] and the references therein. One example is the *Johnson graph* $J(n,k)$, also known as *slice* of the hypercube, which consists of all k-subsets of $\{1, \ldots, n\}$, two subsets adjacent when their intersection has size $k-1$. For instance, a classification of Boolean degree 1 functions on the Johnson graph has been obtained several times independently, see [24] and [36]. Note that k and $n-k$ have to be at least 2 as otherwise all functions have degree 1.

Theorem 3. *Let $n-k, k \geqslant 2$. A Boolean degree* 1 *function on the Johnson graph $J(n,k)$ depends on at most one coordinate.*

Filmus and the third author generalized the result by Nisan and Szegedy to the Johnson graph [27].

Theorem 4 (Filmus et al. (2019))**.** *There exists a constant C such that the following holds. If $C^d \leqslant k \leqslant n - C^d$ and $f : \binom{\{1,\ldots,n\}}{k} \to \{0,1\}$ has degree d, then f depends on at most $\gamma(d)$ coordinates.*

As before, bounds on k are necessary here, but $C^d \leqslant k \leqslant n - C^d$ seems overly generous. Our (very) limited investigation here suggests that for $d = 2$, we only need to exclude $(n, k) = (6, 3)$ and $k < 2d$, see Section 3. In fact, recently Filmus showed in [25] that if we do not insist on the upper bound $\gamma(d)$ derived from the hypercube, but just some upper bound in $O(1)$, then $2d \leqslant k \leqslant n - 2d$ suffices.

Recently, Theorem 4 has been extended to several other structures, for instance the multislice [28] by Filmus, O'Donnell and Wu, and to the perfect matching scheme by Dafni, Filmus, Lifshitz, Lindzey, and Vinayls [13].

1.2 Cameron-Liebler Line Classes and Boolean Functions on the Grassmann Graph

Our main focus are low degree Boolean functions on the *Grassmann graph* $J_q(n, k)$ which consists of all k-subspaces of an n-dimensional vector space over the finite field of order q, two subspaces adjacent when their meet has dimension $k - 1$. Let $\mathcal{F}$ be a family of k-spaces of $V := \mathbb{F}_q^n$. We read $\mathcal{F}$ as a Boolean function over the reals, that is we identify it with the function f from all k-spaces of V to the reals, where $f(S) = 1$ if $S \in \mathcal{F}$ and $f(S) = 0$ otherwise. Let T be a subspace of V. Let x_T denote the family of all k-spaces which are incident with T. We say that a (not necessarily: Boolean) function f has *degree* d if we can write f as a linear combination (over the reals) of all x_T with $\dim(T) = d$.

The study of Boolean degree 1 functions, limited to $k = 2$, under the name of *Cameron-Liebler line classes* is actually older than most of the aforementioned results. In the Grassmann graph, 1-spaces (in projective notation: *points*) or, equivalently, $(n-1)$-spaces (*hyperplanes*) are a natural choice for variables, see §2.2 for details. It was conjectured by Cameron and Liebler in [9] that, as for the hypercube and the Johnson graph, all degree 1 examples are the trivial ones:

Conjecture 5 (Cameron, Liebler (1982))**.** Let $n \geqslant 4$ and $k = 2$. If f is a Boolean degree 1 function on the Grassmann graph $J_q(n, k)$, then f depends on at most one point and one hyperplane.

More explicitly, the conjecture suggests that f is one of $0, x_P, x_H, x_P + x_H, 1, 1 - x_P, 1 - x_H, 1 - x_P - x_H$ for some 1-space P and some $(n-1)$-space H with $P \not\subseteq H$. In the terminology of finite geometry, they suggested that an example either consists of none of the lines (2-spaces), a point-pencil, a dual point-pencil, the union of a point-pencil and a dual point-pencil or the complement of any of these examples. In a breakthrough result, Drudge showed in his PhD thesis [19] that this conjecture fails for $(n, k, q) = (4, 2, 3)$. Nowadays many counterexamples to the Conjecture of Cameron and Liebler are known if $(n, k) = (4, 2)$ and $q \geqslant 3$; see [8, 14, 22, 23, 31, 33]. The general case of $k > 2$ has been investigated more recently, for instance see [4, 15, 16, 26, 35, 41]. Indeed, it has been shown in [26] that Conjecture 5 holds for $k \geqslant 2$ and $q \in \{2, 3, 4, 5\}$ if we exclude the case $(n, k) = (4, 2)$.

In analogy to Theorem 4, it seems natural to assume that when $n - k$ and k are large enough, that is $n - k, k \geqslant C(d)$ for some $C(d)$ independent of n and k, then all degree d

functions on the Grassmann graph $J_q(n,k)$ only depend on very few coordinates x_P and x_H, that is on the intersection with very few points and hyperplanes. Our results here give an indication of what $C(d)$ could be.

In Section 5.1 and in Section 5.2 we construct Boolean degree 2 functions which depend on many coordinates. For this, we use finite symplectic and orthogonal geometries.

1.3 Equitable Bipartitions

A *regular set* or *equitable bipartition* of a k-regular graph Γ is a subset S of vertices of Γ such that there exists constants a and b such that a vertex in S has precisely a neighbors in S and such that a vertex not in S has precisely b neighbors in S. The eigenvalues of the *quotient matrix*

$$\begin{pmatrix} a & k-a \\ b & k-b \end{pmatrix}$$

are also eigenvalues of the adjacency matrix of Γ [32, Lemma 9.3.1]. Equitable bipartitions are known under various other names, for instance *perfect 2-colorings*, *completely regular codes*, or *intriguing sets*, see also [26] and the references therein.

Boolean degree 1 functions in some classical lattices, in particular Johnson and Grassmann graphs, are equitable bipartitions (for instance, this follows from our discussion in §2.4). More generally, equitable bipartitions correspond to Boolean degree d functions which (in the terminology of Boolean function analysis) have no weights on degrees in $\{1,\ldots,d-1\}$. Regular sets on the hypercube are well-investigated, primarly due to Fon-der-Flaass [29]. There has been recent work on the Johnson graph, most notably the equitable bipartitions of the Johnson graph $J(n,3)$ have been classified for n odd, see [30]. Recently, this attracted much research: regular sets of degree 2 have been classified in the Johnson graph by Vorob'ev in [44]; Metsch and De Winter investigated small equitable bipartitions in the the Grassmann graph of planes $J_q(n,3)$ [17]; Mogilnykh surveyed equitable bipartitions in $J_2(6,3)$ and $J_q(n,2)$ [37].

2 Preliminaries

2.1 Projective Geometry

Using projective notation, in $\mathbb{F}_q^n$ we denote 1-space as *points*, 2-spaces as *lines*, 3-spaces as *planes*, 4-spaces as *solids*, $(n-2)$-spaces as *colines*, and $(n-1)$-spaces as *hyperplanes.* For a vector space V, let $\begin{bmatrix} V \\ k \end{bmatrix}$ denote its k-dimensional subspaces. We denote the q-binomial (or: Gaussian) coefficient by $\begin{bmatrix} n \\ k \end{bmatrix}_q := |\begin{bmatrix} V \\ k \end{bmatrix}|$ for $V = \mathbb{F}_q^n$. We write $[n]_q := \begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$. Usually, we do not put the q and write $\begin{bmatrix} n \\ k \end{bmatrix}$ and $[n]$. Note that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \prod_{i=0}^{k-1} \frac{[n-i]}{[k-i]}.$$

Several of our constructions use well-known finite simple groups, namely the symplectic group $\mathrm{Sp}(n,q)$ and orthogonal group $O^\varepsilon(n,q)$. We try to keep this reasonably self-contained by providing explicit equations, but we refer to standard literature on finite geometry and the classical finite simple groups for more details, see [42].

2.2 Analysis of Boolean Functions

Recall the concept of a Booleean degree d function in $\mathbb{F}_q^n$ on k-spaces from §1.2. We say that a Boolean function f is a *j-junta* if there is a set J of points and hyperplanes with $|J| = j$ such that we can write f as a polynomial in x_R, $R \in J$. A rather trivial example of a 2-junta is the set f of k-spaces inside a fixed hyperplane π or trough a fixed point $p \notin \pi$. As we can write $f = x_p + \alpha \sum_{r \in \left[{\pi \atop 1}\right]} x_r + \beta \sum_{r \in \left[{V \atop 1}\right] \setminus \left[{\pi \atop 1}\right]} x_r$ for suitable real constants α and β, this is a Boolean degree 1 function. It only depends on x_p and x_π, thus it is a 2-junta.

We summarize easy, well-known facts in the following lemma. It shows that for fixed degree d, induction on n and k is feasible. Therefore it motivates our study of small n and k to establish an inductive basis.

Lemma 6. *Let $n - d \geqslant k \geqslant d \geqslant 1$. Let f, g be Boolean degree d functions on $J_q(n,k)$ with $d \geqslant 1$. Let P be a 1-space and H a hyperplane of $V \coloneqq \mathbb{F}_q^n$. Then all of the following have degree d:*

(a) The (not necessarily Boolean) functions 0, 1, $f+g$, $f-g$, and $1-f$.

(b) The set $\{S \in \left[{H \atop k}\right] : f(S) = 1\}$.

(c) The set $\{S/P \in \left[{V/P \atop k-1}\right] : f(S) = 1\}$.

Proof. Clearly, any linear combination of two degree d functions has degree d. This shows (a).

For (b): Write f as

$$f = \sum_{T \in \left[{V \atop d}\right]} c_T x_T.$$

For a k-space $S \subseteq H$, we have that $f(S) = 1$ if and only if $h(S) = 1$, where

$$h \coloneqq \sum_{T \in \left[{H \atop d}\right]} c_T x_T.$$

So h has degree d and is the characteristic function of the set.

The statements (b) and (c) are dual. □

We will show in Lemma 8 that there exists a relatively easy upper bound which shows that in the Grassmann graph any function depends on at most $C_{n,k} q^{n-k}$ coordinates.

Lemma 7. *Let $n \geqslant 2k \geqslant 2$. Then there exists a q_0 such that for all $q \geqslant q_0$ there exists a family $\mathcal{H}$ of $(n-k+1)$-spaces in $\mathbb{F}_q^n$ with $|\mathcal{H}| = k^2(n-k+1)$ such that for each k-space S we have that $\langle S \cap \bigcup_{H \in \mathcal{H}} H \rangle = S$.*

Proof. For a k-space S and $(n-k+1)$-spaces $H_1, \dots, H_m$ with $m \leqslant k^2(n-k)$ put $T_m(S) = \langle S \cap \bigcup_{i=1}^m H_i \rangle$. Consider the property (P) that $\dim(T_m(S)) < m$. First we calculate the probability p_m that random $H_1, \dots, H_m$ have property (P). We claim that $p_m < (1+o(1))q^{m-k-1}$ (as $q \to \infty$) and our proof proceeds by induction on m.

Clearly, $p_1 = 0$ as H_1 intersects S nontrivially. For $m > 1$, the probability that $H_1, \dots, H_{m-1}$ satisfy (P) is p_{m-1}. Suppose that $\dim(T_{m-1}(S)) \geqslant m-1$. If $\dim(T_{m-1}(S)) > m-1$, then $\dim(T_m(S)) \geqslant m$, so property (P) is not satisfied. If $\dim(T_{m-1}(S)) = m-1$, then there are $[k]-[m-1] = q^{m-1}[k-m+1]$ points in $S \setminus T_{m-1}(S)$ and $[m-1]$ points in $T_{m-1}(S)$. Hence, the probability that H_m meets S only in $T_{m-1}(S)$ is at most $\frac{[m-1]}{[k]}$ (as H_m meets S nontrivially and this is the probability of a point of S being in $T_{m-1}(S)$). Hence, by the union bound for the two cases $\dim(T_{m-1}(S)) < m-1$ and $\dim(T_{m-1}(S)) \geqslant m-1$,

$$p_m < p_{m-1} + \frac{[m-1]}{[k]} < (1+o(1))q^{m-k-1}.$$

This completes the proof of the claim.

Now let us pick $k^2(n-k+1)$ random $(n-k+1)$-spaces. Let X denote the random variable which counts the number of k-spaces S with $\dim(T_d(S)) < k$. Recall that $\begin{bmatrix} n \\ k \end{bmatrix} < (1+o(1))q^{k(n-k)}$. Then

$$\mathbb{E}(X) = \begin{bmatrix} n \\ k \end{bmatrix} \cdot p_k^{k(n-k+1)} < (1+o(1))q^{k(n-k)} \cdot q^{-k(n-k+1)} < 1.$$

Hence, by linearity of expectation, there exists a choice of $k^2(n-k+1)$ $(n-k+1)$-spaces with $T_m(S) = S$ for all k-spaces S. □

Lemma 8. *Let $1 < k < n-1$. Then there exists a q_0 such that for all $q \geqslant q_0$ we have that any Boolean function on $J_q(n,k)$ is a $k^2(n-k+1)\frac{q^{n-k+1}-1}{q-1}$-junta.*

Proof. By duality, we assume that $k \leqslant n/2$. Put $V = \mathbb{F}_q^n$. By Lemma 7, we can find a set $\mathcal{H}$ of $k^2(n-k+1)$ subspaces of dimension $n-k+1$ such that any k-space T contains at least k points in $\bigcup_{H \in \mathcal{H}} H$ which span T. For each T, let us denote k such points by $\mathcal{P}(T)$. Then

$$f = \sum_{f(T)=1} \prod_{P \in \mathcal{P}(T)} x_P.$$

Hence, we see that f depends on at most $k^2(n-k+1) \cdot \frac{q^{n-k+1}-1}{q-1}$ points. □

2.3 The Spectra of Johnson and Grassmann Graphs

Let $n \geqslant 2k$. Consider the eigenvalues of the adjacency matrices of the Johnson graph $J(n,k)$ and the Grassmann graph $J_q(n,k)$. These are well-understood objects, for instance see Chapter 9 in [7]. Both graphs have $k+1$ eigenspaces $V_0, V_1, \dots, V_k$ with a natural ordering: the eigenspace V_d has dimension $\begin{bmatrix} n \\ d \end{bmatrix} - \begin{bmatrix} n \\ d-1 \end{bmatrix}$ (where we read $\begin{bmatrix} n \\ d \end{bmatrix} = \binom{n}{d}$ for the Johnson graph). Corollary 3.2.3 in [43] implies

Lemma 9. *Let $n \geqslant 2k$ and consider $J_q(n,k)$, then we have for every $0 \leqslant d \leqslant k$ that $V_0 + \cdots + V_d = \langle x_D : \dim D = d \rangle$.*

In particular, the first eigenspace is spanned by the all-ones vector (or, as we identify vectors and functions here, $f = 1$). Note that the eigenvalue of $J_q(n,k)$ corresponding to the eigenspace V_j is

$$q^{j+1}[k-j][n-k-j] - [j].$$

Asking for an equitable bipartitions is the same as asking for a set with characteristic function in $V_0 + V_d$ for some d, see [32, §9.3].

2.4 Some Equivalent Definitions

The next result emphasizes the (well-known) fact that there are three ways of looking at degree d functions: We can see them as degree d polynomials, we can see them as functions in certain eigenspaces, or we can see them as a certain type of functions in posets. We include some minor variants which we consider useful. We assume that $d \leqslant k$.

The *d-space-to-k-space incidence matrix* $A = (a_{ij})$ of $J_q(n,k)$ is the $(\begin{bmatrix} n \\ d \end{bmatrix} \times \begin{bmatrix} n \\ k \end{bmatrix})$-matrix indexed by d-spaces and k-spaces of $\mathbb{F}_q^n$ where $a_{ij} = 1$ if the i-th d-space lies in the j-th k-space and $a_{ij} = 0$ otherwise.

Proposition 10. *Let $n \geqslant 2k$. For f a real function on $J_q(n,k)$ the following are equivalent:*

(a) The function f has degree d.

(b) The function f lies in $V_0 + \cdots + V_d$.

(c) The function f is orthogonal to $V_{d+1} + \cdots + V_n$.

(d) There exists a weighting $wt \colon \begin{bmatrix} V \\ d \end{bmatrix} \to \mathbb{R}$ such that for all $S \in \begin{bmatrix} V \\ k \end{bmatrix}$ we have

$$f(S) = \sum_{D \in \begin{bmatrix} S \\ d \end{bmatrix}} wt(D).$$

(e) The function f lies in the image of the d-space-to-k-space incidence matrix.

Proof. Lemma 9 shows the equivalence of (a) and (b). The equivalence of (b) and (c) follows from the fact that the common eigenspaces of the association scheme $J_q(n,k)$ are pairwise orthogonal (as its adjacency matrices are symmetric). Further, (a) and (d) are equivalent: If (a) holds, then we can write f as

$$f = \sum_{D \in \begin{bmatrix} V \\ d \end{bmatrix}} c_D x_D.$$

Take $\mathrm{wt}(D) = c_D$ to obtain (d). Conversely, if (d) holds, then take $c_D = \mathrm{wt}(D)$ to see that f has degree d. Note that $D \in \begin{bmatrix} V \\ d \end{bmatrix}$ lies on some $S \in \begin{bmatrix} V \\ k \end{bmatrix}$. Let A denote the d-space-to-k-space incidence matrix. Then (d) states that $f = A^T \cdot \mathrm{wt}$. Here we see wt as a vector of weights. Hence, (d) and (e) are equivalent. □

2.5 Boolean Functions and Designs

Classical designs live in the Johnson graph. Let $n \geqslant 2k \geqslant 2d \geqslant 0$. A (classical) d-(n,k,λ) design in the Johnson graph $J(n,k)$ is a family $\mathcal{D}$ of k-sets such that each d-set lies in exactly λ elements of $\mathcal{D}$. A d-(n,k,λ) design in the Grassmann graph $J_q(n,k)$ is a family $\mathcal{D}$ of k-spaces such that each d-space lies in exactly λ elements of $\mathcal{D}$. The existence of these q-analogs of classical designs was settled (at least in some weak sense) asymptotically by Fazeli, Lovett, and Vardy in 2014 [21], but for small parameters deciding existence is notoriosly hard. Maybe most prominently, a classical 2-$(7,3,1)$ design is well-known as the Fano plane. The existence of a 2-$(7,3,1)$ design in $J_q(7,3)$, the so-called q-analog, is a long-standing open problem.

Let $\mathcal{D}$ be a d-(n,k,λ) design of the Grassmann graph $J_q(n,k)$. By a standard double counting argument, $|\mathcal{D}| = \lambda \begin{bmatrix} n \\ d \end{bmatrix} / \begin{bmatrix} k \\ d \end{bmatrix}$. For any family $\mathcal{F}$ such that the characteristic function f of $\mathcal{F}$ has degree d, then $|\mathcal{D} \cap \mathcal{F}|$ only depends on $|\mathcal{D}|$ and $|\mathcal{F}|$. Indeed, Boolean degree d functions are precisely the objects with this property. It is a case of what Delsarte called *design-orthogonality*, see also [18].

Corollary 11. *Let $n \geqslant 2k$. Consider a d-(n,k,λ) design $\mathcal{D}$ of $J_q(n,k)$ with characteristic function g. If $\mathcal{F}$ is a degree d subset of $J_q(n,k)$, then $|\mathcal{F} \cap \mathcal{D}| = |\mathcal{F}| \cdot |\mathcal{D}| / \begin{bmatrix} n \\ k \end{bmatrix}$. If also $\langle g^\gamma : \gamma \in \mathrm{P\Gamma L}(n,q) \rangle = V_0 + V_{d+1} + \cdots + V_k$, then the converse holds too.*

Hence, if $\mathcal{F}$ has degree d, then $|\mathcal{D}| \cdot |\mathcal{F}| / \begin{bmatrix} n \\ k \end{bmatrix}$ is an integer. This is a well-known generalization of the fact that if k divides n, then the size of a Boolean degree 1 function is divisible by $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ (using 1-(n,k,λ) designs, that is spreads). We list the divisibility conditions which derive from the known designs in §B.

3 Degree 2 in Hypercube and Johnson Scheme

Consider the hypercube $\{0,1\}^n$. Nisan and Szegedy (Theorem 2) showed that a Boolean degree d function on the hypercube depends on at most $d2^{d-1}$ variables, so a Boolean degree 2 functions depends on at most 4 variables. Hence, one can obtain a complete list by considering the first four input variable x, y, z, w: Up to permutation and negation of the input, Boolean degree 2 functions are

$$\begin{aligned}
&0, \quad x, \quad x \text{ AND } y = xy, \quad x \text{ XOR } y = x + y - xy, \quad xy + (1-x)z, \\
&\mathrm{Ind}(x{=}y{=}z) = xy + xz + yx - x - y - z + 1, \\
&\mathrm{Ind}(x{\leqslant}y{\leqslant}z{\leqslant}w \text{ OR } x{\geqslant}y{\geqslant}z{\geqslant}w).
\end{aligned}$$

Here $\mathrm{Ind}(B)$ is the indicator of B. This list was first obtained by Camion, Carlet, Charpin, and Sendrier in [10]. For $d = 3$ a Boolean degree 2 function depends on at most 10 variables, see [45]. Note that $3 \cdot 2^2 = 12 > 10$.

Now consider the Johnson graph $J(n,k)$ with $n \geqslant 2k$. For $k = 3$ there are countless examples for degree 2 functions which depends on an arbitrary amount of coordinates, see [25, 27] for more details and conjectures. All equitable bipartitions of degree 2 are classified for $k = 3$ [20, 30]. For n divisible by d and $k \leqslant 2d - 1$, we can find a partition $\mathcal{L}$ of $\{1, \ldots, n\}$ into d-sets. Then

$$f(x) = \sum_{S \in \mathcal{L}} \prod_{i \in S} x_i$$

has degree d. This function corresponds to the family of k-sets containing one of the d-sets of $\mathcal{L}$. This is a very special case of what Martin calls *groupwise complete design*, see [34].

For $J(8, 4)$ we found an example that depends on 5 variables. Identify $J(8, 4)$ as a subset of $\{0, 1\}^8$ and take all vertices which start with one of

$$11000, 01100, 00110, 00011, 10001,$$
$$11100, 01110, 00111, 10011, 11001.$$

For an alternative description, let Z be the cyclic group of order 5 with its natural action on $\{1, \ldots, 5\}$. Then we can take any 4-set which intersects $\{1, \ldots, 5\}$ in one of the orbits $\{1, 2\}^Z$ or $\{1, 2, 3\}^Z$. It is an equitable bipartition with quotient matrix

$$\begin{pmatrix} 8 & 8 \\ 6 & 10 \end{pmatrix}.$$

Recall that Boolean degree 2 function on the hypercube depends on at most 4 coordinates. Thus, the behavior of the Johnson graph is notably different from the hypercube.

4 Examples for General Degree

4.1 Trivial Examples

Let $2d \leqslant 2k \leqslant n$ and let $\perp$ be some polarity of $\mathbb{F}_q^n$. For a d-space T, let $x_{T,i}$ denote all k-spaces S with $\dim(S \cap T) = d - i$, and $x_{T^\perp,i}$ denote all k-spaces S with $\dim(S \cap T^\perp) = d - i$. We call these examples *trivial*. We also call all examples trivial which one can obtain from these examples by taking unions, differences, and complements.

For our main interest, $d = 2$, there are three examples to emphasize.

Example 12. (a) The set of all k-spaces through a fixed 2-space L: $x_L = x_P x_Q$. Here P and Q are points which span L.

(b) The set of all k-spaces in a fixed $(n - 2)$-space C: $x_C = x_H x_K$. Here H and K are hyperplanes which intersect in C.

(c) The set of all k-spaces through a fixed 1-space P in a fixed $(n - 1)$-space H: $x_P x_H$. Here $P \subseteq H$.

Note that the last example is particularly interesting. Let C be an $(n-2)$-space. Let $\mathcal{H}$ be the set of $q+1$ hyperplanes through C. For each hyperplane $H \in \mathcal{H}$, pick a point $P_H \subseteq H$ outside of C. Then

$$f = \sum_{H \in \mathcal{H}} x_H x_{P_H}$$

is a Boolean degree 2 function of size $(q+1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. It is a $2(q+1)$-junta.

4.2 A (Partial) Spread

Here we describe a union of trivial examples which we consider noteworthy. Let $\mathcal{S}$ be a family of d-spaces of $\mathbb{F}_q^n$ which are pairwise disjoint. Such a family is called a *partial spread*. Clearly, $|\mathcal{S}| \leqslant [n]/[d]$. In case of equality $\mathcal{S}$ is called a *spread*. Indeed, spreads exist if and only if d divides n, see [2]. The maximal size of $\mathcal{S}$ when d does not divide n was determined recently in [38]. Clearly,

$$f = \sum_{S \in \mathcal{S}} x_S$$

is a Boolean degree d function for k-spaces if $k \leqslant 2d-1$. It shows that any type of Nisan-Szegedy theorem (for which we assume q fixed and $n \to \infty$) needs to exclude the case $k \leqslant 2d-1$.

4.3 Free Constructions from the Hypercube

Let $h : \{0,1\}^m \to \{0,1\}$ be a Boolean degree d function on the hypercube. Further, take a linear independent set $B = \{b_1, \ldots, b_m\}$ in $\mathbb{F}_q^n$ (here $n \geqslant m$). Define a Boolean degree d function f on the subspaces S of $\mathbb{F}_q^n$ by putting

$$f(S) = h((x_{\langle b_i \rangle}(S))_{i \in \{1,\ldots,m\}}).$$

In [11] a Boolean degree d function is described which depends on $m = \ell(d) := 3 \cdot 2^{d-1} - 2$ variables. Hence, a Boolean degree d function on $J_q(n,k)$ can depend on $\ell(d)$ variables. If B is not linear independent, then f depends on less than $\ell(d)$ variables.

For q fixed and d sufficiently large, this is the best construction for low degree Boolean functions on $J_q(n,k)$ which we are aware of.

For $(n,k) = (8,4)$ we describe a Boolean degree 2 function in §5.2 which seems to depend on more than $C(q^4+q^3+q^2+q+1)$ variables (we can only show that it depends on at least q^3+q^2+q+1 variables).

5 Global Degree 2 Examples from Polar Spaces

The most famous example for a nontrivial Boolean degree 1 function exists in $J_q(4,2)$ (for q odd) and is closely related to the elliptic quadric $O^-(4,q)$. For degree 2 we went through all polar spaces in small dimensions.

5.1 Examples for Planes

We consider examples on planes, that is $k = 3$.

5.1.1 Symplectic Spaces

Let $n \geqslant 6$. Consider a (possibly degenerate) symplectic form σ on $\mathbb{F}_q^n$. If n is even, then $\sigma(x, y) = x_1y_2 - x_2y_1 + \cdots + x_{n-1}y_n - x_ny_{n-1}$ is a nondegenerate choice for σ. We say that x, y are orthogonal if $\sigma(x, y) = 0$. Let S be a subspace. We write $S^\perp$ for the subspace of vectors orthogonal to S. The radical of S is $S \cap S^\perp$. We say that S is isotropic if its radical is S, and that S is nondegenerate if its radical is trivial.

There are two types of 2-spaces with respect to σ: Let $\mathcal{L}_1$ denote the set of isotropic 2-space, and let $\mathcal{L}_2$ denote the set of nonisotropic 2-spaces.

There are also two types of 3-spaces with respect to σ: Let Π_1 denote the set of isotropic planes, and let Π_2 denote the set of planes with a point as a radical.

We claim that Π_i has degree 2 for $i \in \{1, 2\}$: Put

$$f = \frac{1}{q^2+q+1} \sum_{L \in \mathcal{L}_1} x_L - \frac{q+1}{q^2(q^2+q+1)} \sum_{L \in \mathcal{L}_2} x_L.$$

Clearly, f has degree 2 and corresponds to the set Π_1. It remains to see that f is Boolean. All $q^2 + q + 1$ lines in an isotropic plane Π are isotropic, so $f(\Pi) = 1$. A plane Π with a point as radical has $q + 1$ isotropic lines and q^2 nondegenerate lines, so then $f(\Pi) = 0$.

Now assume that n is even and that σ is nondegenerate.

The symmetry group of this example is $\mathrm{Sp}(n, q)$. The number of line orbits equals the number of plane orbits as there are precisely two types of each. Thus, this is an example for equality in Block's lemma, Lemma 17. The group $\mathrm{Sp}(n, q)$ acts transitive on points, hence the example is a 1-design and therefore an equitable bipartition. For $n = 6$, this was already observed in [17]. The quotient matrix is

$$\begin{pmatrix} q[3][n-5] & q^{n-4}[2][3] \\ [2][n-4] & [3]q[n-3] - [2][n-4] \end{pmatrix}.$$

Remark 13. A similar construction works for even degree d and $k = d + 1$.

5.1.2 Quadrics

Let $n \geqslant 6$. Consider a quadratic form Q on $\mathbb{F}_q^n$, for instance $Q(x) = x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n$ for n odd. Let $\mathcal{Q}$ denote the singular points $\langle x \rangle$ (so $Q(x) = 0$). We say that a subspace of $\mathbb{F}_q^n$ is totally singular if all its points are singular.

Let $\mathcal{L}_i$ denote the family of lines which intersect $\mathcal{Q}$ in i points. As Q is quadratic, $\mathcal{L}_i$ is empty unless $i \in \{0, 1, 2, q + 1\}$. A line in one of these sets is called exterior line, tangent, secant, or totally singular line, respectively. There are five types of planes with respect to Q. In bracket we provide the explicit isomorphy type in $\mathbb{F}_q^3$.

Let Π_1 denote the family of totally singular planes (isomorphic to the quadratic form $Q'(x) = 0$).

Let Π_2 denote the family of planes of double line type (isomorphic to $Q'(x) = x_1^2$).

Let Π_3 denote the family of planes with exactly one singular point (isomorphic to $Q'(x) = x_1^2 + \alpha x_1 x_2 + \beta x_2^2$ such that $x_1^2 + \alpha x_1 x_2 + \beta x_2^2$ is irreducible over $\mathbb{F}_q$).

Let Π_4 denote the family of planes with exactly two totally singular lines (isomorphic to $Q'(x) = x_1 x_2$).

Let Π_5 denote the family of conic planes (isomorphic to $Q'(x) = x_1^2 + x_1 x_2$).

Let $A = (a_{ji})$ denote the 5×4 matrix such that a_{ji} denotes the number of lines of $\mathcal{L}_i$ in a plane of Π_j. Then

$$A = \begin{pmatrix} 0 & 0 & 0 & q^2+q+1 \\ 0 & q^2+q & 0 & 1 \\ q^2 & q+1 & 0 & 0 \\ 0 & q-1 & q^2 & 2 \\ \binom{q}{2} & q+1 & \binom{q+1}{2} & 0 \end{pmatrix}.$$

Then

$$A\left(-\frac{q+1}{q^4+q^3+q^2}, \frac{1}{q^2+q+1}, -\frac{q+1}{q^4+q^3+q^2}, \frac{1}{q^2+q+1}\right)^T = (1,1,0,0,0)^T,$$

so $\Pi_1 \cup \Pi_2$ has degree 2. We can write the characteristic function of it as

$$f_1 = -\frac{q+1}{q^4+q^3+q^2} \sum_{L \in \mathcal{L}_0 \cup \mathcal{L}_2} x_L + \frac{1}{q^2+q+1} \sum_{L \in \mathcal{L}_1 \cup \mathcal{L}_{q+1}} x_L.$$

If n and q are even, and Q is of hyperbolic type, then this example is isomorphic to the symplectic example in Section 5.1.1, but not when q is odd. Its quotient matrix is identical to the symplectic example.

For $q = 2$, we also find the following example which corresponds to $\Pi_1 \cup \Pi_3$:

$$f_2 = \frac{15}{64} \sum_{L \in \mathcal{L}_0} x_L - \frac{1}{42} \sum_{L \in \mathcal{L}_1} x_L - \frac{11}{168} \sum_{L \in \mathcal{L}_2} x_L + \frac{1}{7} \sum_{L \in \mathcal{L}_{q+1}} x_L.$$

For $(n, q) = (6, 2)$ with Q of elliptic type $O^-(6, q)$, Π_1 is empty. Hence, for $q = 2$ the sets $\Pi_i = \Pi_1 \cup \Pi_i$ for $i = 2, 3$ have degree 2, and so any Π_i with $i \in \{2, \ldots, 4\}$ has degree 2.

5.2 Examples for Solids

We consider examples on solids, that is $k = 4$. Let $n = 8$. Let Q be a nondegenerate quadratic form of elliptic type $O^-(8, q)$, for instance $Q(x) = x_1^2 + \alpha x_1 x_2 + \beta x_2^2 + x_3^2 + \cdots + x_8^2$ such that $x_1^2 + \alpha x_1 x_2 + \beta x_2^2$ is irreducible over $\mathbb{F}_q$. The terminology is identical to §5.1.2, so $\mathcal{Q}$ is the set of singular points and we partition the lines set into $\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_{q+1}$.

There are the following types of solids. In bracket we provide the explicit isomorphy type in $\mathbb{F}_q^4$.

Let $\mathcal{S}_1$ denote the set of all solids of double plane type (with a quadratic form of type $Q'(x) = x_1^2$).

Let $\mathcal{S}_2$ denote the set of all solids with two totally singular planes (type $Q'(x) = x_1x_2$).

Let $\mathcal{S}_3$ denote the set of all solids with with precisely one totally singular line (type $Q'(x) = x_1^2 + \alpha x_1x_2 + \beta x_2^2$).

Let $\mathcal{S}_4$ denote the set of all solids that intersect $\mathcal{Q}$ in a cone with a point as base over a conic (type $Q'(x) = x_1^2 + x_1x_2$).

Let $\mathcal{S}_5$ denote the set of all nondegenerate solids of hyperbolic type $O^+(4,q)$ (type $Q'(x) = x_1x_2 + x_3x_4$).

Let $\mathcal{S}_6$ denote the set of all nondegenerate solids of elliptic type $O^-(4,q)$ (type $Q'(x) = x_1^2 + \alpha x_1x_2 + \beta x_2^2 + x_3^2$).

The example below can be seen as a generalization of the example by Bruen and Drudge for degree 1 in [8]. Let $A = (a_{ji})$ denote the 6×4 matrix such that a_{ji} denotes the number of lines of $\mathcal{L}_i$ in a solid of Π_j.

$$A = \begin{pmatrix} 0 & q^2(q^2+q+1) & 0 & q^2+q+1 \\ 0 & q(q^2-1) & q^4 & 2q^2+2q+1 \\ q^4 & q(q+1)^2 & 0 & 1 \\ \frac{1}{2}q^3(q-1) & q^3+2q^2 & \frac{1}{2}q^3(q+1) & q+1 \\ \frac{1}{2}q^2(q-1)^2 & (q+1)(q^2-1) & \frac{1}{2}q^2(q+1)^2 & 2(q+1) \\ \frac{1}{2}q^2(q^2+1) & (q+1)(q^2+1) & \frac{1}{2}q^2(q^2+1) & 0 \end{pmatrix}.$$

Then we see that

$$A\left(\frac{q+1}{q^3(q^2+q+1)}, 0, -\frac{q+1}{q^3(q^2+q+1)}, \frac{1}{q^2+q+1}\right)^T = (1,1,1,0,0,0)^T.$$

Hence, $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ is a degree 2 set. The corresponding degree 2 polynomial is

$$f = \frac{q+1}{q^3[3]}\left((\sum_{L \in \mathcal{L}_0} x_L) - (\sum_{L \in \mathcal{L}_2} x_L) \right) + \frac{1}{[3]} \sum_{L \in \mathcal{L}_{q+1}} x_L.$$

Note that $|\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3| = (q^4+1)(q^3+1)(q^2+1)[5]$.

This example for $J_q(8,4)$ seems to depend on almost all coordinates which is in contrast to $J(8,4)$ where we only obtained an example depending on 5 coordinates, see Section 3. Formally, we can show the following:

Proposition 14. *The example $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ depends on at least $q^4 - q^3 + q^2 - q + 3$ variables of type x_P and x_H for P a 1-space and H an $(n-1)$-space.*

Proof. Put $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. On average, a point of $\mathbb{F}_q^8$ lies in $(q^3+1)(q^2+1)[5]$ elements of $\mathcal{S}$. A singular point lies in $(q+1)(q^2+1)^2(q^3+1)$ elements of $\mathcal{S}$. Hence, a non-singular point lies on

$$\frac{[8] \cdot (q^3+1)(q^2+1)[5] - (q^4+1)(q^2+q+1) \cdot (q+1)(q^2+1)^2(q^3+1)}{[8] - (q^4+1)(q^2+q+1)}$$
$$= (q^3+1)[4]^2 =: \alpha$$

elements of $\mathcal{S}$. Dually, a hyperplane of $\mathbb{F}_q^8$ contains at most α elements of $\mathcal{S}$. Hence, we need at least

$$\left\lceil \frac{|\mathcal{S}|}{\alpha} \right\rceil = q^4 - q^3 + q^2 - q + 3$$

points and hyperplanes to cover all elements of $\mathcal{S}$. Now suppose that $\mathcal{S}$ only depends on a set $\mathcal{R}$ of less than $q^4 - q^3 + q^2 - q + 3$ points and hyperplanes. Then there exist 4-spaces $S \in \mathcal{S}$ and $T \notin \mathcal{S}$ which are non-incident with all elements of $\mathcal{R}$. Hence, we cannot distinguish between S and T based on $\mathcal{R}$ which contradicts that $\mathcal{S}$ only depends on $\mathcal{R}$. □

By Lemma 8, the preceding result is tight up to a constant factor (as $q \to \infty$). Hence, there exists a degree 2 function on $J_q(8,4)$ which depends, up to a constant factor, the maximum number of coordinates.

6 Other Examples

6.1 Local Degree 2 Examples for Planes

Here we provide an (incomplete) selection of examples which are of degree 2 and which stabilize a partial flag of subspaces, but are not trivial.

6.1.1 A Line and a Complementary Spread

For $n = 6$, let L be a line and let $\mathcal{C}$ be a set of q^2+1 colines through L which pairwise meet in L. Note that $\mathcal{C}$ exists because $\mathbb{F}_q^4$ possesses line spreads. Then the set $\{\Pi \text{ a plane} : \dim(\Pi \cap L) = 0 \text{ and } \exists C \in \mathcal{C} : \Pi \subseteq C\}$ has degree 2 and size $(q^2+1) \cdot q^2(q+1)$. We can write its characteristic function f as

$$f = \sum_{C \in \mathcal{C}} (x_C - x_L).$$

The example is a $(q+1)(q^2+2)$-junta: we can decide if an element is in the set by testing inclusion for each of the $q+1$ hyperplanes through the q^2+1 colines through L, together with testing the inclusion for each of the $q+1$ points of L.

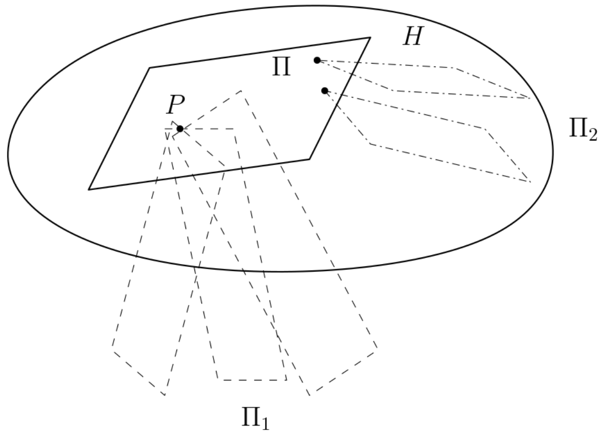

Figure 1: The point-plane-hyperplane example from §6.1.2. The planes of Π_1 and Π_2 correspond to the planes with dashed border.

6.1.2 Incident Point-Plane-Hyperplane

Let $n = 6$. Pick a point P, a plane Π, and a hyperplane H such that $P \subseteq \Pi \subseteq H$.
Let Π_1 be the set of all planes not in H which meet Π in a line through P.
Let Π_2 the set of all planes in H whose meet with Π is a point different from P.

Then $\Pi_1 \cup \Pi_2$ corresponds to a degree 2 function and has size $q^3(q+1) + (q^2+q)q^4 = (q^2+1) \cdot q^3(q+1)$. To see that it has degree 2, consider the types of lines:
Let $\mathcal{L}_1$ be the set of all lines in Π through P.
Let $\mathcal{L}_2$ be the set of all lines in Π not through P.
Let $\mathcal{L}_3$ be the set of all lines in H whose meet with Π is P.
Let $\mathcal{L}_4$ be the set of all lines in H whose meet with Π is a point, but not P.
Let $\mathcal{L}_5$ be the set of all lines in H which are skew to Π.
Let $\mathcal{L}_6$ be the set of all lines whose meet with H is P.
Let $\mathcal{L}_7$ be the set of all lines whose meet with H is a point in Π, but not P.
Let $\mathcal{L}_8$ be the set of all lines whose meet with H is a point not in Π.

Then we can write the characteristic function f of $\Pi_1 \cup \Pi_2$ as

$$\begin{aligned} f =& \frac{q^3}{[3][2]} \sum_{L \in \mathcal{L}_1} x_L + \frac{-q}{[3]} \sum_{\mathcal{L}_2 \cup \mathcal{L}_3} x_L \\ &+ \frac{1}{[3][2]} \sum_{\mathcal{L}_4 \cup \mathcal{L}_7} x_L + \frac{q+1}{q[3]} \sum_{\mathcal{L}_5 \cup \mathcal{L}_6} x_L + \frac{-1}{q^2[3]} \sum_{\mathcal{L}_8} x_L. \end{aligned}$$

See Figure 1 for an illustration. The example is a (q^2+q+2)-junta: we can decide if an element is in $\Pi_1 \cup \Pi_2$ by testing inclusion for the q^2+q+1 points in Π and H.

6.2 Some Sporadic Examples

Here are some sporadic example for $(n, q) = (6, 2)$. Despite our best efforts, we did not manage to generalize them. The reader can find some more sporadic examples, which are also equitable bipartitions, in [37].

6.2.1 Incident Line-Solid

For $n = 6$ and $q = 2$, let M be a line and let C be a coline with $M \subseteq C$. There are $q + 1$ hyperplanes through C and $q + 1$ points on M. For each of the $q + 1$ points $P \subseteq M$, choose a distinct hyperplane H_P through C.

Let Π_1 denote the set of all planes π with $M \subseteq \pi \subseteq C$.

Let Π_2 denote the set of all planes π not in C which meet M in some point P and satisfy $\pi \subseteq H_p$.

Let Π_3 denote the set of all planes which meet M in some point p and and C in a line. Note that $\Pi_2 \subseteq \Pi_3$.

The set $\Pi_1 \cup \Pi_3$ has degree 2 and size $(q + 1) + q^3(q + 1)^2 + q^3(q^2 + 1)(q + 1)$. It is a $2(q + 1)$-junta: test all points on L and all hyperplanes through C.

Then set $\Pi_1 \cup \Pi_2$ has degree 2 and size $(q+1)+q^3(q+1)^2$. The example is a $2(q+1)$-junta (as before). Its characteristic function can be written as

$$f = \sum_{P \in \left[{M \atop 1}\right]} (x_P - x_P x_{H_P}),$$

where f is the characteristic function of $\Pi_1 \cup \Pi_2 \cup \Pi_3$ and also has degree 2.

6.2.2 Incident Point-Line-Plane-Hyperplane

For $n = 6$ and $q = 2$, let P be a 1-space, M a 2-space, Π a 3-space, and H a 5-space such that $P \subseteq M \subseteq \Pi \subseteq H$.

Let Π_1 denote all planes in H which contain M.

Let Π_2 denote all planes not in H which meet Π in a line through P different from M.

Let Π_3 denote all planes in H which meet Π in a point on M different from P.

The set $\Pi_1 \cup \Pi_2 \cup \Pi_3$ has degree 2 and size $7 + 16 + 32 = 55$. The example is a $(3q + 1)$-junta, that is a 7-junta: we can decide if an element is in $\Pi_1 \cup \Pi_2$ by testing inclusion for H and the points in a triangle which includes M.

6.3 Unexplained Computer Examples

We found some examples by computer which we could not derive from any of the other examples. We present these and their symmetries in the following table. In the structure description, we denote the the cyclic group of order m by C_m, the symmetric group of order $m!$ by S_m, and the dihedral group of order m by D_m. We write a^b if an orbit of length a occurs b times. For plane orbits, we only provide those that constitute the degree 2 example.

size	stab size	point orbits	line orbits	plane orbits	structure
80	61440	$32, 20, 10, 1$	$320, 160, 60, 40^2, 16, 10, 5$	40^2	$C_2^5 \rtimes (C_2^4 \rtimes S_5)$
85	86016	$32, 14^2, 2, 1$	$224^2, 84, 32, 28^2, 16, 7^2, 1$	$56, 21, 8$	G, see below
177	64	$32, 4^7, 2^4, 1^7$	$32^8, 16^{15}, 8^8, 4^8, 2^{24}, 1^{11}$	$16^4, 8^{10}, 2^{13}, 1^7$	$C_2^3 \times D_8$
420	126	$42, 21$	$126^4, 63, 42, 21, 14, 7$	$126^3, 42$	$S_3 \times (C_7 \rtimes C_3)$

Here $G = (C_4 \times C_2^3) \rtimes (C_2 \times (C_2^3 \rtimes PSL(3,2)))$.

7 Concluding Remarks

(1) In case of the Johnson graph, let us remark that a classification of Boolean degree 2 functions in $J(n,k)$ appears feasible, but goes beyond the scope of the present work.

(2) For $d = 1$ and $k = 2$, Bamberg and Penttila [1] classified all subgroups of $\mathrm{P\Gamma L}(n,q)$ which have the same number of orbits on points and lines. This answered a conjecture by Cameron and Liebler in [9]. In light of §A, the following generalization is natural:

Problem 15. Classify all subgroups of $\mathrm{P\Gamma L}(n,q)$ with the same number of orbits on d-spaces and k-spaces, that is all Boolean degree d functions on $J_q(n,k)$ for which equality holds in Lemma 17.

(3) In [13] it was shown for several domains that one can write a Boolean degree d function as a constant depth decision tree. In case of the Grassmann graph, the natural queries are "Is a point contained in a subspace?" and "Is a subspace in a hyperplane?". In light of our collection of examples, one is tempted to make the following conjecture:

Conjecture 16. For every given q and d, there exists a k_0 such that if $k, n-k \geqslant k_0$, then every Boolean degree d function on $J_q(n,k)$ is a constant depth decision tree.

One might also conjecture that the depth only depends polynomially on q. A related, but less specific conjecture can be found in [27].

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A Permutation Groups and Block's Lemma

In 1982 Cameron and Liebler investigated subgroups of $\mathrm{P\Gamma L}(n,q)$ and their orbits on points and lines of the vector space of the vector space $\mathbb{F}_q^n$, see [9]. An application of Block's lemma [3] shows that any subgroup of $\mathrm{P\Gamma L}(n,q)$ has at least as many orbits on lines as it has on points. In case of equality the orbits of lines have, in our terminology here, degree 1.

In this section we concisely discuss a generalization of this application of Block's lemma to degree d, i.e. we show that a subgroup of $\mathrm{P\Gamma L}(n,q)$ has at least as many orbits on k-spaces of $\mathbb{F}_q^n$ as on d-spaces if $d \leqslant k \leqslant n/2$. Again, if equality occurs, then the orbits on k-spaces have degree d in the Grassmann graph $J_q(n,k)$.

Lemma 17 ([3])**.** *Let G be a group acting on two finite sets X and X', with respective sizes n and m. Let $O_1,\dots,O_s$, respectively $O'_1,\dots,O'_t$ be the orbits of the action on X, respectively X'. Suppose that $R \subseteq X \times X'$ is a G-invariant relation and call $A=(a_{ij})$ the $n\times m$ matrix of this relation, i.e. $a_{ij}=1$ if and only if $x_i R x'_j$ and $a_{ij}=0$ otherwise, after having ordered the elements of X and X' arbitrarily. Let χ_S denote the characteristic vector of a set S.*

(i) The vectors $A^T\chi_{O_i}$, $i=1,\dots,s$, are linear combinations of the vectors $\chi_{O'_j}$.

(ii) If A has full row rank, then $s \leqslant t$. If $s=t$, then all vectors $\chi_{O'_j}$ are linear combinations of the vectors $A^T\chi_{O_i}$, hence $\chi_{O'_j} \in \mathrm{Im}(A^T)$.

Let $G \leqslant \mathrm{P\Gamma L}(n,q)$, $n \geqslant 4$, let $2 \leqslant k < n$, $d \leqslant k$ and $d \leqslant n/2$, and let X, respectively X' be the set of d-spaces, respectively k-spaces of $\mathbb{F}_q^n$. The incidence, i.e. the symmetrised set theoretic containment, between an element of X and X' is G-invariant. Furthermore, the incidence matrix is the incidence matrix of the k-space design of $\mathbb{F}_q^n$, and this matrix has full row rank by [5]. If $s=t$, i.e. if G has equally many orbits on the d-spaces as on the k-spaces, then the characteristic vector of each of the orbits of k-spaces lies in $\mathrm{Im}(A^T)$.

In [9], Cameron and Liebler studied collineation groups having equally many point as line orbits, that is $(d,k)=(1,2)$. Conjecture 5 translates in this context that such a group is line transitive, or fixes a hyperplane and acts transitively on the lines of the hyperplanes, or, dually, fixes a point and acts transitively on the lines through the fixed point.

We call two k-spaces of $\mathbb{F}_q^n$ *skew* if and only if they only share the zero vector. A partition of the points of $\mathbb{F}_q^n$ in k-spaces is called a *spread of $\mathbb{F}_q^n$ in k-spaces.* It is well known that such a spread exists if and only if $k \mid n$, e.g. when $n=4$ and $k=2$, there are spreads of lines in $\mathbb{F}_q^n$. By Proposition 3.1 of [9], for any set L of lines of $\mathbb{F}_q^4$, $\chi_L \in \mathrm{Im}(A^T) \iff |L \cap S| = x$ (a natural number, only depending on L), and for any line spread S of $\mathbb{F}_q^4$. Often, a Cameron-Liebler line class of $\mathbb{F}_q^4$ is defined using its characterization with relation to line spreads of $\mathbb{F}_q^4$. When $k \nmid n$, a set K of k-spaces of $\mathbb{F}_q^n$ is called a Cameron-Liebler set k-spaces of $\mathbb{F}_q^n$ if and only if $\chi_K \in \mathrm{Im}(A^T)$. This is the case $d=1$ and $k \geqslant 1$ found in a geometrical context in [4].

Note that the statements of Block's lemma are only unidirectional, i.e. an orbit of k-spaces under a collineation group with equally many point orbits as orbits on k-spaces is a Cameron-Liebler set of k-spaces, but the converse is not true, the union of all k-spaces through a fixed point P and contained in a fixed hyperplane not through P is a Cameron-Liebler set of k-spaces which is not the orbit under a collineation group with equally many point as k-space orbits.

We do not know if a classification of such subgroups of $\mathrm{P\Gamma L}(n, q)$ is feasible, but we will see in §5.1.1 that the symplectic group $\mathrm{Sp}(n, q)$ provides us with some examples when n and d are even.

B Divisibility Conditions

In the following we summarize known divisibility conditions based on the survey by Braun, Kiermaier, Wassermann [6].

B.1 Small Parameters

For $(n, k) = (6, 3)$, a 2-$(6, 3, c(q + 1))$ design has size $c(q^3 + 1)[5]$. Existence is known for $(q, c) = (2, 1), (3, 3), (4, 2), (5, 13)$. Hence, for $q = 2, 3, 4, 5$ we obtain that $|\mathcal{F}|$ needs to be divisible by $5, 10, 17, 2$, respectively.

For $(n, k) = (7, 3)$, a 2-$(7, 3, \lambda)$ design has size $\lambda(q^2 - q + 1)[7]$. Existence is known for $(q, \lambda) = (2, 3), (3, 5), (4, 21), (5, 31)$. Hence, for $q = 2, 3, 4, 5$ we obtain that $|\mathcal{F}|$ needs to be divisible by $[5] = \frac{q^5-1}{q-1}$.

For $(n, k) = (8, 4)$, a 2-$(8, 4, c(q^2 + q + 1))$ design has size $c(q^4 + 1)[7]$. Existence is known for $(q, c) = (2, 7), (3, 455), (4, 5733), (5, 20181)$. Hence, for $q = 2, 3, 4, 5$ we obtain that $|\mathcal{F}|$ needs to be divisible by $93, 121, 341, 781$, respectively.

B.2 Suzuki's construction

Let q be a prime and $n \geqslant 7$ be an integer satisfying $\gcd(n, 4!) = 1$. Then there exists a 2-$(n, 3, q^2 + q + 1)$ design. See [6, Theorem 11]. Hence,

Lemma 18. *Let $\mathcal{F}$ be a degree 2 family of 3-spaces in $\mathbb{F}_q^n$. Then $(q^3 - 1)|\mathcal{F}|$ is divisible by $q^{n-2} - 1$.*

For instance, for $n = 11$ and $q = 2, 3, 4, 5$ or 7: $(q^3 - 1)|\mathcal{F}|$ is divisible by $511, 19\,682, 262\,143, 1\,953\,124$ or $40\,353\,606$, respectively.

B.3 More Conditions in the Binary Case

Lemma 19. *Let $m \geqslant 3$. Suppose that $\mathcal{F}$ is a set of 3-spaces in $\mathbb{F}_2^n$ of degree 2, then the following holds:*

(a) If $n = 8m$, then $C|\mathcal{F}|$ is divisible by $2^{8m-2} - 1$, where $C \in \{42, 312\}$.

(b) If $n = 9m$*, then* $42 \cdot |\mathcal{F}|$ *is divisible by* $2^{9m-2} - 1$.

(c) If $n = 10m$*, then* $210 \cdot |\mathcal{F}|$ *is divisible by* $2^{10m-2} - 1$.

(d) If $n = 13m$*, then* $42 \cdot |\mathcal{F}|$ *is divisible by* $2^{13m-2} - 1$.

Proof. By [6, Section 5.2], there exist (a) 2-$(8m, 3, C)_2$ designs for $C \in \{42, 312\}$, (b) 2-$(9m, 3, 42)_2$ designs for $m \geqslant 3$, (c) 2-$(10m, 3, 210)_2$ designs for $m \geqslant 3$, and (d) 2-$(13m, 3, 42)_2$ designs for $m \geqslant 3$. □