Non-empty intersection of longest paths in H-free graphs

James A. Long Jr. Kevin G. Milans

Department of Mathematics
West Virginia University
USA
jalong@mix.wvu.edu
milans@math.wvu.edu

Andrea Munaro

Department of Mathematical, Physical and Computer Sciences
University of Parma
Italy
andrea.munaro@unipr.it

Submitted: May 24, 2022; Accepted: Dec 15, 2022; Published: Feb 24, 2023

Abstract

We make progress toward a characterization of the graphs H such that every connected H-free graph has a longest path transversal of size 1. In particular, we show that the graphs H on at most 4 vertices satisfying this property are exactly the linear forests. We also show that if the order of a connected graph G is large relative to its connectivity $\kappa(G)$, and its independence number $\alpha(G)$ satisfies $\alpha(G) \leqslant \kappa(G) + 2$, then each vertex of maximum degree forms a longest path transversal of size 1.

Mathematics Subject Classifications: 05C38

1 Introduction

It is a classic result in graph theory that every two longest paths in a connected graph share at least one vertex. Gallai [11] asked whether in fact all longest paths in a connected graph share at least one vertex. This was answered in the negative by Walther [27], who provided a counterexample with 25 vertices. A counterexample with 12 vertices was later

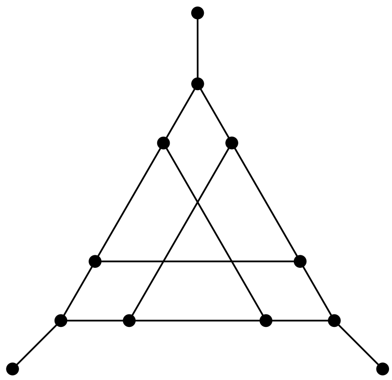

Figure 1: The graph G_0: A 12-vertex graph with no Gallai vertex.

constructed by Walther and Voss [28] and, independently, by Zamfirescu [31] (see Figure 1). Brinkmann and Van Cleemput (see [26]) verified that there is no counterexample with less than 12 vertices.

A *Gallai set* (or *longest path transversal*) in a graph G is a set of vertices S such that every longest path in G has a vertex in S. The *Gallai number* or *longest path transversal number* of G, denoted by $\mathrm{lpt}(G)$, is the minimum size of a Gallai set and a *Gallai family* is a family of graphs $\mathcal{G}$ such that $\mathrm{lpt}(G) = 1$ for each connected graph $G \in \mathcal{G}$. A vertex v in G is a *Gallai vertex* if $\{v\}$ is a Gallai set and a graph is *Gallai* if it has a Gallai vertex.

The counterexamples mentioned above consist of connected graphs G for which $\mathrm{lpt}(G) = 2$. In fact, there are examples of connected graphs G for which $\mathrm{lpt}(G) = 3$ [15, 31] and Walther [27] and Zamfirescu [30] asked if the Gallai number of connected graphs is bounded. In a companion paper [23] we addressed this fifty-year-old question. Improving on [25], we showed that connected graphs admit sublinear longest path transversals. The gap between our upper bound and the constant lower bound 3 remains large.

In this paper we focus on another natural variant of Gallai's question: Which classes of graphs form Gallai families? It is well known that a family of pairwise intersecting subtrees of a tree has non-empty intersection; in particular, trees form a Gallai family. Several other Gallai families have been identified: split graphs and cacti [19], circular-arc graphs [1, 18], series-parallel graphs [6], graphs with matching number at most 3 [5], dually chordal graphs [17], $2K_2$-free graphs [13], P_4-sparse graphs and $(P_5, K_{1,3})$-free graphs [3], bipartite permutation graphs [4], (H_1, H_2)-free graphs such that H_1 and H_2 are connected and every 2-connected (H_1, H_2)-free graph is Hamiltonian (all such pairs are known and each includes $K_{1,3}$) [12].

Let $\mathrm{Free}(H)$ be the class of H-free graphs. A *monogenic class* of graphs has the form $\mathrm{Free}(H)$, for some graph H. In this paper we aim at characterizing monogenic Gallai families. In Section 3, we make progress by showing that if $\mathrm{Free}(H)$ is a Gallai family, then H is a linear forest, and this suffices when $|V(H)| \leqslant 4$. In the spirit of [13], we in fact prove something more general: if H is a linear forest on at most 4 vertices and G is a connected H-free graph, then all maximum degree vertices in G are Gallai. Dichotomies in monogenic classes for structural and algorithmic graph properties have been the subject of several studies. For example, they have been provided for properties such as Hamiltonicity [10, 22], boundedness of clique-width [9], price of connectivity [2, 16], and polynomial-time solvability of various algorithmic problems [14, 20, 21, 24]. In Section 4, we show that if G is a connected graph with independence number $\alpha(G) \leqslant 4$ (i.e., G is $5P_1$-free), then G is Gallai. We then conjecture that the same holds if $\alpha(G) \leqslant 5$.

A celebrated result of Chvátal and Erdős [7] asserts that a graph G has a Hamiltonian cycle when $|V(G)| \geqslant 3$ and $\alpha(G) \leqslant \kappa(G)$, and that G has a Hamiltonian path when $\alpha(G) \leqslant \kappa(G) + 1$. It follows that every vertex in G is Gallai when $\alpha(G) \leqslant \kappa(G) + 1$. In Section 5, we show that if a connected graph G is large relative to its connectivity $\kappa(G)$ and $\alpha(G) \leqslant \kappa(G) + 2$, then each vertex of maximum degree is a Gallai vertex. Moreover, for each $k \geqslant 1$, we provide an infinite family of k-connected graphs G such that $\alpha(G) \leqslant k + 3$ but no maximum degree vertex in G is Gallai (see Example 20). Our result has the following immediate consequence: if a regular graph G is large relative to its connectivity and $\alpha(G) \leqslant \kappa(G) + 2$, then G contains a Hamiltonian path. The condition $\alpha(G) \leqslant \kappa(G)+2$ is best possible up to an additive factor of 2 (this follows from a construction in [8], see Example 22).

2 Preliminaries

In this paper we consider only finite graphs. Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$.

Neighborhoods and degrees. For a vertex $v \in V(G)$, the *neighborhood* $N_G(v)$ is the set of vertices adjacent to v in G. For a set of vertices $S \subseteq V(G)$, the *neighborhood* of S, denoted $N_G(S)$, is $\bigcup_{v \in S} N_G(v)$. We also extend the concept of neighborhood to subgraphs by defining $N_G(H) = N_G(V(H))$ when H is a subgraph of G. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v in G. When G is clear from context, we may write $d(v)$ for $d_G(v)$. A vertex $v \in V(G)$ with $d(v) = 3$ is *cubic*. The *maximum degree* $\Delta(G)$ of G is $\max\{d_G(v) : v \in V\}$. Similarly, the *minimum degree* $\delta(G)$ of G is $\min\{d_G(v) : v \in V\}$.

Paths and cycles. A *path* is a non-empty graph P with $V(P) = \{x_0, x_1, \ldots, x_k\}$ and $E(P) = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$. We may also denote P by listing its vertices in the natural order $x_0x_1 \cdots x_k$. The vertices x_0 and x_k are the *ends* or *endpoints* of P; the other vertices are *interior* vertices of P. The *length* of P is the number of edges in P. We denote the n-vertex path by P_n. A path in a graph G is *Hamiltonian*, or *spanning*, if it contains all vertices of G. A uv-path is a path whose endpoints are u and v. If $P = x_0x_1 \cdots x_k$ is a path and $k \geqslant 2$, the graph with vertex set $V(P)$ and edge set $E(P) \cup x_kx_0$ is a *cycle*. The *length* of a cycle is the number of its edges (or vertices) and the cycle on n vertices is denoted by C_n. A cycle in a graph G is *Hamiltonian*, or *spanning*, if it contains all vertices of G. The *girth* of a graph containing a cycle is the length of a shortest cycle and a graph with no cycle has infinite girth. The *distance* $\mathrm{dist}_G(u, v)$ from a vertex u to a vertex v in a graph G is the length of a shortest path between u and v.

Graph operations. Let G be a graph and let $S \subseteq V(G)$. The graph $G - S$ is obtained from G by deleting all vertices in S and all edges incident to a vertex in S. The subgraph of G induced by a set of vertices S', denoted $G[S']$, is the graph $G - S$, where $S = V(G) - S'$. For $M \subseteq E(G)$, we define $G - M$ analogously. The *union* of simple graphs G and H is denoted $G \cup H$ and has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *disjoint union* of G and H, denoted $G + H$, is the union of a copy of G and a copy of H on disjoint vertex sets. The disjoint union of k copies of G is denoted by kG.

Graph classes and special graphs. If a graph does not contain induced subgraphs isomorphic to graphs in a set Z, it is Z*-free* and the set of all Z-free graphs is denoted by $\mathrm{Free}(Z)$. A *complete graph* is a graph whose vertices are pairwise adjacent and the complete graph on n vertices is denoted by K_n. A *triangle* is the graph K_3. A graph G is r*-partite*, for $r \geqslant 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. An r-partite graph in which every two vertices from distinct parts are adjacent is called *complete* and 2-partite graphs are usually called *bipartite*. An (X, Y)*-bigraph* is a bipartite graph with bipartition $\{X, Y\}$. Given a graph G and $X, Y \subseteq V(G)$, the *induced* (X, Y)*-bigraph* is the bipartite subgraph of G with vertex set $X \cup Y$ and where each edge has one endpoint in X and the other in Y. A *tree* is a connected graph not containing any cycle as a subgraph and the vertices of degree 1 are its *leaves*.

Graph parameters. A set of vertices or edges of a graph is *maximum* with respect to the property $\mathcal{P}$ if it has maximum size among all subsets having property $\mathcal{P}$. An *independent set* of a graph is a set of pairwise non-adjacent vertices and the *independence number* $\alpha(G)$ is the size of a maximum independent set of G. A *clique* of a graph is a set of pairwise adjacent vertices. A *matching* in G is a set of edges with distinct endpoints. A matching M *saturates* a set of vertices S if each vertex in S is the endpoint of an edge in M. A graph G is k*-connected* if $|V(G)| > k$ and $G - S$ is connected for each $S \subseteq V(G)$ with $|S| < k$. The *connectivity* of G, denoted $\kappa(G)$, is the maximum k such that G is k-connected.

3 Monogenic Gallai families

In this section we make progress toward a classification of monogenic Gallai families. We first show that a necessary condition for a monogenic family $\mathrm{Free}(H)$ to be Gallai is that H is a linear forest on at most 9 vertices, where a *linear forest* is a forest in which every component is a path. Let G_0 be the graph in Figure 1 with $\mathrm{lpt}(G_0) = 2$ [28, 31]. We obtain necessary conditions on monogenic Gallai families by subdividing edges or replacing cubic vertices with triangles in G_0 to obtain new counterexamples with arbitrarily large girth or no induced claw, respectively.

In the following, we say that a graph H is a *fixer* if $\mathrm{Free}(H)$ is a Gallai family; that is, forbidding H "fixes" the answer to Gallai's question.

Proposition 1. *If H is a fixer, then H is a linear forest on at most 9 vertices.*

Proof. Let H be a fixer. By definition, if G is a graph with $\operatorname{lpt}(G) > 1$, then H is an induced subgraph of G.

Note that G_0 is obtained from the Petersen graph by splitting an arbitrary vertex into a set R of three vertices, each of degree 1 (see Figure 1). Clearly, G_0 is triangle-free and every path in G_0 avoids at least one vertex in R. Since the Petersen graph has no Hamiltonian cycle [29], every path in G_0 omits at least 2 vertices. Moreover, since the Petersen graph is vertex-transitive [29] and has a 9-cycle, it follows that for each vertex $x \in V(G_0) - R$, there is a longest path in G_0 with both ends in R that omits only x and the other vertex in R.

Let M be the set of 3 edges incident to the vertices in R. Let G_1 be the graph obtained from G_0 by replacing each edge in M with a path of length q and replacing each edge outside M with a path of length p, where $p > |V(H)|$. Provided that $q > |E(G_0)| \cdot p$, the longest paths in G_1 are in bijective correspondence with the longest paths in G_0 that have both ends in R. Recalling that, for each $x \in V(G_0) - R$, there is a longest path in G_0 with both ends in R that omits x, we have $\operatorname{lpt}(G_1) > 1$. Since G_1 has girth larger than $|V(H)|$ and H is an induced subgraph of G_1, it follows that H is acyclic.

Let S be the set of cubic vertices in G_1. We obtain G_2 from G_1 by replacing each vertex $w \in S$ with a triangle T_w such that the three edges incident to w in G_1 are incident to distinct vertices of T_w in G_2. Clearly, G_2 is claw-free. Let P be a longest path in G_2. Again, provided that q is sufficiently large, P has its ends in R. When P visits a vertex in some T_w, it must visit all vertices in T_w before leaving. It follows that the longest paths in G_2 are in bijective correspondence with the longest paths in G_1 and $\operatorname{lpt}(G_2) > 1$.

Since H is an induced subgraph of G_1 and G_2, it follows that H is triangle-free and claw-free, and so $\Delta(H) \leqslant 2$. Recalling that H is acyclic, we have that H is a linear forest. But H is also an induced subgraph of G_0 and to obtain an induced linear forest as a subgraph of G_0, a vertex must be deleted from the closed neighborhood of each cubic vertex of G_0. Let R' be the set of neighbors of vertices in R. Since the vertices in R' are cubic and have disjoint closed neighborhoods, each induced linear forest has at most $|V(G_0)| - |R'|$ vertices, and so $|V(H)| \leqslant |V(G_0)| - |R'| = 12 - 3 = 9$. □

Remark 2. Gao and Shan [12, Problem 6] asked whether all longest paths in a connected claw-free graph have a non-empty intersection. Proposition 1 answers this question in the negative.

For $|V(H)| \leqslant 4$, we show that H is a fixer if and only if H is a linear forest. Necessity follows from Proposition 1. For sufficiency, we show that every 4-vertex linear forest is a fixer. The linear forests of order 4 are P_4, $P_3 + P_1$, $2P_2$, $P_2 + 2P_1$, and $4P_1$ (see Figure 2). Cerioli and Lima [3] showed that P_4-sparse graphs, a superclass of P_4-free graphs, form a Gallai family, whereas Golan and Shan [13] showed that $2P_2$-free graphs form a Gallai family. In other words, P_4 and $2P_2$ are fixers. In the following, we address the remaining cases: $P_3 + P_1$, $P_2 + 2P_1$, and $4P_1$.

We begin with some basic but useful observations. Given vertices $x, y \in V(G)$, an *xy-fiber* is a longest path among all the xy-paths. Similarly, an *x-fiber* is a longest path among all the paths having x as an endpoint, and a *fiber* is a longest path in G. Note

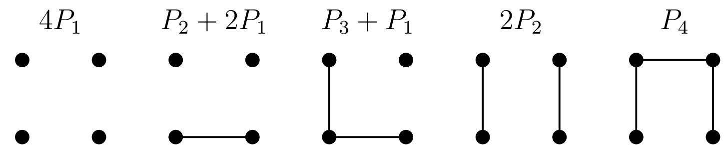

Figure 2: The linear forests on 4 vertices. These are exactly the graphs H on 4 vertices such that Free(H) is a Gallai family.

that every fiber is an x-fiber for some vertex x, and every x-fiber is an xy-fiber for some vertex y.

The following two basic lemmas are used repeatedly, sometimes implicitly. Similar ideas are key to the results in [7]. The first basic lemma treats single neighbors of fibers.

Lemma 3. *Let P be an xy-path in a graph G, where $P = v_0 \cdots v_\ell$ with $x = v_0$ and $y = v_\ell$. Let H be a component of $G - V(P)$ with a neighbor v_i on P. If P is an x-fiber, then $i < \ell$. Moreover, if $0 < i$, then $v_\ell v_{i-1} \notin E(G)$. Similarly, if P is a y-fiber, then $0 < i$, and if $i < \ell$, then $v_0 v_{i+1} \notin E(G)$.*

Proof. Suppose P is an x-fiber. No vertex in H is adjacent to y, or else P extends to a longer x-fiber, a contradiction. Therefore, $i < \ell$. Also, if $i > 0$ and $v_{i-1}v_\ell \in E(G)$, then following P from v_0 to v_{i-1}, traversing $v_{i-1}v_\ell$, following P backward from v_ℓ to v_i, and traveling to H produces a longer x-fiber. The case that P is a y-fiber is symmetric. □

In many of our arguments, we show that a path P in G has some desired property or else we obtain a longer path. We now formalize two common ways to obtain longer paths. Given two lists of objects a and b, a *splice* of a with b is a sequence obtained from a by (1) replacing a non-empty interval of a with b, or (2) inserting b between consecutive elements in a, or (3) prepending or appending b to a. Given a *host path* P and a *patching path* Q, a *splice* of P with Q is a path whose vertices are ordered according to a splice of the ordered list of vertices in P with the ordered list of vertices in Q. A splice of P that has the same endpoints as P is an *interior splice*; otherwise, the splice is *exterior*.

A *detour* of an xy-path P is a path obtained from P by using two patching paths Q_1 and Q_2 as follows. Suppose that Q_i is a u_iw_i-path for $i \in \{1,2\}$ and u_1, u_2, w_1, w_2 are distinct vertices appearing in order along P. We follow P from x to u_1, traverse Q_1, follow P backward from w_1 to u_2, traverse Q_2, and finally follow P from w_2 to y.

Note that our definitions of a splice and detour require the resulting object to be a path and therefore implicitly impose certain disjointness conditions on segments of the host and the patching paths. Also, note that interior splices and detours of P have the same endpoints as P. A splice or detour of P is *augmenting* if it is longer than P.

Let P be a path in G and let H be a component of $G - V(P)$. A vertex $s \in V(P)$ with a neighbor in H is an *attachment point* of H. Our next lemma concerns pairs of attachment points.

Lemma 4. *Let P be an xy-path in a graph G and let H be a component of $G - V(P)$ with attachment points s and s', where s appears before s' when traversing P from x to y. The following hold:*

1. *If s and s' are consecutive on P, then there is an augmenting interior splice of P.*

2. *If s and s' are not consecutive along P, w and w' immediately follow s and s' respectively, and $ww' \in E(G)$, then there is an augmenting detour of P.*

3. *If s and s' are not consecutive along P, w and w' immediately precede s and s' respectively, and $ww' \in E(G)$, then there is an augmenting detour of P.*

Proof. For part 1, since s and s' are consecutive attachment points on P, we obtain an augmenting interior splice by inserting an appropriate path in H between s and s'. For part 2, let Q_1 be an ss'-path with interior vertices in H and let Q_2 be the path ww'. There is an augmenting detour of P using patching paths Q_1 and Q_2. The case in part 3 is symmetric. □

When P is a kind of fiber and a component H of $G - V(P)$ has many attachment points, our next lemma obtains a large independent set contained in P consisting of non-attachment points.

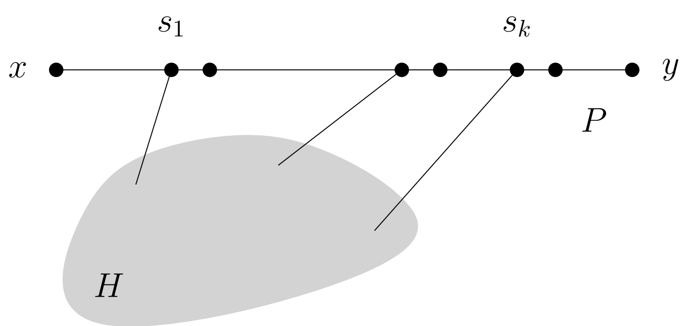

Figure 3: Construction of A in the proof of Lemma 5.

Lemma 5. *Let P be an xy-path in a graph G, let H be a component of $G - V(P)$ and let k be the number of attachment points of H. There is an independent set A of G such that $A \subseteq V(P)$, no edge joins a vertex in A and a vertex in $V(H)$, and the following hold:*

1. *If P is an xy-fiber, then $A \subseteq V(P) - \{x, y\}$ and $|A| \geqslant k - 1$.*

2. *If P is an x-fiber, then $A \subseteq V(P) - \{x\}$ and $|A| \geqslant k$.*

3. *If P is a fiber, then $A \subseteq V(P)$ and $|A| \geqslant k + 1$.*

Proof. Let $s_1, \dots, s_k$ be the attachment points of H, with indices increasing from x to y along P, and let $S = \{s_1, \dots, s_k\}$ (see Figure 3).

For part 1, let A be the set of vertices in P that immediately follow some s_i with $1 \leqslant i < k$. Since P is an xy-fiber, Lemma 4 implies that s_i and s_{i+1} are not consecutive along P. Therefore, S and A are disjoint and so no vertex in A has a neighbor in H. By Lemma 4, it follows that A is an independent set.

For part 2, suppose in addition that P is an x-fiber. By Lemma 3, $s_k \neq y$, and we may take A to be the set of vertices that immediately follow some s_i with $1 \leqslant i \leqslant k$.

For part 3, suppose in addition that P is a fiber. By Lemma 3, we have $s_1 \neq x$. Let A be the set of vertices that immediately follow an attachment point together with x. Note that since P is also a y-fiber, it follows from Lemma 3 that x has no neighbor in A, and so A is an independent set of size $k + 1$. □

We can finally show in the following sections that $P_3 + P_1$, $P_2 + 2P_1$, and $4P_1$ are all fixers.

3.1 $P_3 + P_1$ is a fixer

Theorem 6. *If G is a connected $(P_3 + P_1)$-free graph, then every vertex of degree at least $\Delta(G) - 1$ is a Gallai vertex.*

Proof. Let P be a longest path in G, where $P = v_0 \cdots v_\ell$ with $x = v_0$ and $y = v_\ell$. Suppose for a contradiction that there is a vertex u with $d(u) \geqslant \Delta(G) - 1$ but $u \notin V(P)$. Let H be the component of $G - V(P)$ containing u. Let $T = V(H)$, let S be the set of attachment points of H on P, let $k = |S|$, and let $t = |T|$.

Note that H is a complete graph, or else an induced copy of P_3 in H together with an endpoint of P would induce a copy of $P_3 + P_1$ in G. We now claim that $xv_i \in E(G)$ for each $v_i \in S$. Otherwise, by Lemma 3, given a neighbor z of v_i in H, $\{z, v_i, v_{i+1}, x\}$ would induce a copy of $P_3 + P_1$.

Next we claim that $v_{i-1}v_{i+1} \notin E(G)$ when $v_i \in S$. Otherwise, we obtain a longer path by starting with a neighbor z of v_i in H, walking along zv_ix, following P from x to v_{i-1}, traversing $v_{i-1}v_{i+1}$, and following P from v_{i+1} to y. Therefore $zv_i \in E(G)$ for each $z \in T$ and $v_i \in S$, otherwise $\{z, v_{i-1}, v_i, v_{i+1}\}$ would induce a copy of $P_3 + P_1$. It follows that $N(z) = (T - \{z\}) \cup S$ for each $z \in T$. In particular, $d(u) = (t - 1) + k$.

Next we claim that, if $v_i, v_j \in S$ with $i \neq j$, then $v_iv_{j+1} \in E(G)$. Otherwise, given a neighbor z of v_i in H, the set $\{z, v_i, v_{i+1}, v_{j+1}\}$ would induce a copy of $P_3 + P_1$ since $v_{i+1}v_{j+1} \notin E(G)$ by Lemma 4. This implies that, if $v_i \in S$, then the neighborhood of v_i contains x, T, and $\{v_{j+1} \colon v_j \in S\}$, and so $d(v_i) \geqslant 1 + t + k$. Therefore $\Delta(G) \geqslant d(v_i) \geqslant d(u) + 2$, a contradiction. □

The degree assumption in Theorem 6 is best possible. Indeed, the complete bipartite graph $K_{t,t+2}$ is $(P_3 + P_1)$-free, has maximum degree $t + 2$, and the vertices of degree t are not Gallai.

3.2 $P_2 + 2P_1$ is a fixer

Proposition 7. *If G is a connected $(P_2 + 2P_1)$-free graph, then every vertex of maximum degree is a Gallai vertex.*

Proof. Let G be a connected $(P_2 + 2P_1)$-free graph and let $P = v_0 \cdots v_\ell$ be a longest path in G with ends $x = v_0$ and $y = v_\ell$. Suppose for a contradiction that u is a vertex of maximum degree and $u \notin V(P)$. Let $k = d(u) = \Delta(G)$, and let H be the component of

$G - V(P)$ containing u. Note that $xy \notin E(G)$, or else we obtain a longer path by starting at a vertex in H with a neighbor on P and traveling around the cycle $P + xy$. Also, $V(G) - V(P)$ is an independent set, or else, by Lemma 3, an adjacent pair of vertices in $V(G) - V(P)$ together with x and y would induce a copy of $P_2 + 2P_1$.

Let S be the set of attachment points of H. Since H has one vertex, we have $|S| = k$. Applying Lemma 5 where H is the graph with the single vertex u, there is an independent set $A \subseteq V(P)$ such that $|A| = k + 1$ and $A \cap S = \varnothing$.

If some vertex $s \in S$ has two non-neighbors $w_1, w_2 \in A$, then $\{u, s, w_1, w_2\}$ induces a copy of $P_2 + 2P_1$. Hence every vertex in S has at least k neighbors in A. Counting u, every vertex in S has degree at least $k + 1$, contradicting that $\Delta(G) = k$. □

Vertices of degree $\Delta(G) - 1$ in a $(P_2 + 2P_1)$-free graph G need not be Gallai. Indeed, consider the graph G obtained from $K_{t,t+2}$ by removing a matching saturating the part of size t. G is $(P_2 + 2P_1)$-free and $\Delta(G) = t + 1$. The longest paths in G omit one vertex, and the Gallai vertices are those in the smaller part. Two of the non-Gallai vertices in the larger part have degree t, which equals $\Delta(G) - 1$.

3.3 $4P_1$ is a fixer

For a path P in a graph G containing the vertices x and y, the *closed subpath of P with boundary points x and y*, denoted $P[x, y]$, is the subpath of P with endpoints x and y. The *open subpath of P with boundary points x and y*, denoted $P(x, y)$, is $P[x, y] - \{x, y\}$. Additionally, we define the *semi-open* subpaths $P[x, y)$ and $P(x, y]$ analogously.

Let $x, y \in V(G)$, let P be an xy-path in G, and let H be a component of $G - V(P)$. For each non-attachment point $w \in V(P)$, we define the *rank* of w, denoted $\mathrm{rank}(w)$, to be the maximum length of a subpath of $P[x, w]$ containing w but no attachment points. Note that if $s_1, \ldots, s_k$ are the attachment points with indices increasing from x to y, then the rank of a non-attachment point $w \in V(P(s_i, s_{i+1}))$ is $\mathrm{dist}_P(s_i, w) - 1$.

Lemma 8. *Let P be an xy-path in a graph G and let H be a complete component of $G - V(P)$. Let S be the set of attachment points of H on P, where $S = \{s_1, \ldots, s_k\}$, with indices increasing from x to y, and suppose that the induced $(S, V(H))$-bigraph has a matching saturating S_0 when $S_0 \subseteq S$ and $|S_0| \leqslant |V(H)|$. The following hold.*

1. *If $s_1 = x$, then P has an augmenting splice with endpoint y. If $s_k = y$, then P has an augmenting splice with endpoint x. If s_i and s_{i+1} are consecutive on P, then P has an augmenting interior splice.*

2. *If some component P_0 of $P - S$ has fewer than $|V(H)|$ vertices, then P has an augmenting splice replacing P_0.*

3. *If w and w' are in distinct components of $P - S - V(P[x, s_1])$, $\mathrm{rank}(w) + \mathrm{rank}(w') < |V(H)|$, and $ww' \in E(G)$, then P has an augmenting detour.*

4. *If w and w' are in distinct components of $P - S$, $\mathrm{rank}(w) + \mathrm{rank}(w') < |V(H)|$, and $ww' \in E(G)$, then G has a path with endpoint y that is longer than P.*

Proof. For part 1, if $s_1 = x$ or $s_k = y$, then we obtain an augmenting splice of P by prepending or appending a Hamiltonian path of H. If s_i and s_{i+1} are consecutive along P, then it follows from Lemma 4 that P has an augmenting interior splice.

For part 2, let P_0 be a component of $P - S$ with $1 \leqslant |V(P_0)| < |V(H)|$. Note that P_0 is $P[x, s_1)$, or $P(s_k, y]$, or $P(s_i, s_{i+1})$ for some i. Suppose that $P_0 = P(s_i, s_{i+1})$. Hence there is a matching $\{s_i z, s_{i+1} z'\}$ joining s_i and s_{i+1} to distinct vertices $z, z' \in V(H)$. Since H is complete, H contains a spanning zz'-path Q. Since $|V(P_0)| < |V(H)|$, we obtain an augmenting interior splice by replacing P_0 with Q. The cases $P_0 = P[x, s_1)$ and $P_0 = P(s_k, y]$ are similar, except that we obtain an augmenting external splice.

For part 3, we may assume that w appears before w' when traversing P from x to y (see Figure 4). Let i and j be indices such that $w \in V(P(s_i, s_{i+1}))$ and $w' \in V(P(s_j, s_{j+1}))$ except that we set $j = k$ if $w' \in V(P(s_k, y])$. Since w and w' are in distinct components of $P - S$, we have $i < j$. If $|V(H)| = 1$, then $\mathrm{rank}(w) + \mathrm{rank}(w') < |V(H)|$ implies that w immediately follows s_i and w' immediately follows s_j. By Lemma 4 part (2), we have that P has an augmenting detour. Otherwise, $|V(H)| \geqslant 2$ and there is a matching $\{s_i z, s_j z'\}$ joining s_i and s_j to distinct vertices $z, z' \in V(H)$. Let Q_1 be an $s_i s_j$-path whose interior vertices form a spanning zz'-path in H, and let Q_2 be the path ww'. The detour of P with patching paths Q_1 and Q_2 adds the vertices in $V(H)$ but omits the $\mathrm{rank}(w)$ vertices in $P(s_i, w)$ and the $\mathrm{rank}(w')$ vertices in $P(s_j, w')$. Since $\mathrm{rank}(w) + \mathrm{rank}(w') < |V(H)|$, the detour is augmenting.

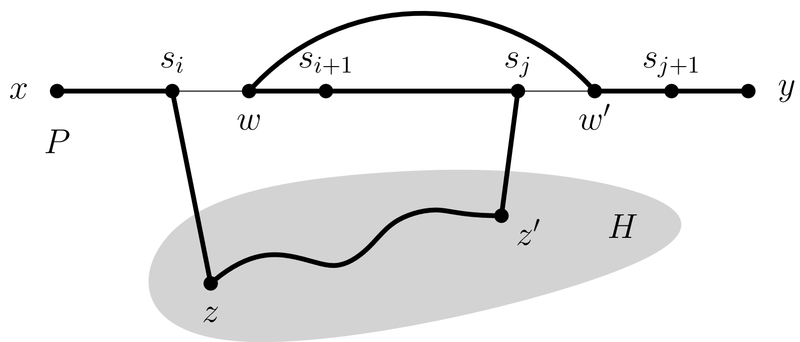

Figure 4: Part 3 in Lemma 8.

For part 4, we may apply the argument for part 3 unless $w \in V(P[x, s_1])$. As before, let j be the index such that $w' \in V(P(s_j, s_{j+1}))$, except that we set $j = k$ if $w' \in V(P(s_k, y])$. We obtain a new path P' by following P backward from y to w', traversing $w'w$, following P forward from w to s_j, traversing an edge joining s_j and a vertex in H, and finishing with a Hamiltonian path in H. The path P' includes all of $V(H)$ but omits the $\mathrm{rank}(w)$ vertices in $P[x, w)$ and the $\mathrm{rank}(w')$ vertices in $P(s_j, w')$. Since $\mathrm{rank}(w) + \mathrm{rank}(w') < |V(H)|$, the path P' is longer than P. □

Our next lemma provides additional structure when G is k-connected and $\alpha(G) \leqslant k+2$.

Lemma 9. *Let P be a longest path in a graph G with endpoints x and y, and let H be a component of $G - V(P)$. Suppose that G is k-connected and $\alpha(G) \leqslant k+2$. The following hold.*

1. *The set S of attachment points of H on P has size k.*

2. *The subgraph H is complete.*

3. *The graph $P - S$ has $k+1$ components, and each has at least $|V(H)|$ vertices.*

4. *If w and w' are in distinct components of $P - S$ and $\mathrm{rank}(w) + \mathrm{rank}(w') < |V(H)|$, then $ww' \notin E(G)$.*

5. *The vertices in each component of $P - S$ of rank less than $|V(H)|$ form a clique.*

Proof. Let $S = \{s_1, \ldots, s_r\}$, with indices increasing from x to y. Since G is k-connected and H is a component of $G - V(P)$, it follows that $r \geqslant k$, or else S separates $V(H)$ from x and y. Since P is a fiber, it follows from Lemma 5 that G contains an independent set A with $|A| = r+1$ such that $A \subseteq V(P)$ and no edge joins A and $V(H)$. Since $1 + (k+1) \leqslant \alpha(H) + (r+1) = \alpha(H) + |A| \leqslant \alpha(G) \leqslant k+2$, it follows that $\alpha(H) = 1$ and $r = k$. Hence, there are exactly k attachment points and H is complete.

Let $S_0 \subseteq S$ with $|S_0| \leqslant |V(H)|$ and let B be the induced $(S_0, V(H))$-bigraph. If B has no matching saturating S_0, then Hall's Theorem [29] implies that there exists $S_1 \subseteq S_0$ such that $|N_B(S_1)| < |S_1|$. Since $|N_B(S_1)| < |S_1| \leqslant |S_0| \leqslant |V(H)|$, it follows that $N_B(S_1) \cup (S - S_1)$ is a cutset of size less than k, contradicting that G is k-connected. Therefore Lemma 8 applies, and since P is a longest path, parts 3 and 4 follow.

It remains to establish part 5. Suppose for a contradiction that w and w' are distinct vertices in the same component W of $P - S$ such that $\mathrm{rank}(w), \mathrm{rank}(w') < |V(H)|$ and $ww' \notin E(G)$. Let A be the set of non-attachment points in P with rank 0, and obtain A' from A by deleting the vertex in $W \cap A$ and adding w and w'. Note that, with the possible exception of $\{w, w'\}$, each pair of vertices in A' has rank sum less than $|V(H)|$ and intersects two components of $P - S$. It follows from part 4 that A' is an independent set. Since $|A'| = k+2$ and A consists of non-attachment points, we may add any vertex in H to obtain an independent set of size $k+3$, a contradiction. □

Theorem 10. *Let $k \in \{1, 2\}$. If G is k-connected and $\alpha(G) \leqslant k+2$, then every longest path in G contains every vertex of degree at least $\Delta(G) - (2-k)$.*

Proof. Let P be a longest path in G with endpoints x and y, and suppose for a contradiction that there exists $u \notin V(P)$ with $d(u) \geqslant \Delta(G) - (2-k)$. Let H be the component of $G - V(P)$ containing u, and let $t = |V(H)|$. Let $s_1, \ldots, s_k$ be the attachment points of H on P, indexed in order from x to y, and let $S = \{s_1, \ldots, s_k\}$. Note that $\Delta(G) \leqslant d(u) + (2-k) \leqslant ((t-1)+k) + (2-k) = t+1$.

For each component W of $P - S$, let $f(W)$ be the set of vertices w in W with $\mathrm{rank}(w) < t$. We claim that $N(s_1)$ either contains $V(H)$ or $f(W)$, for some component W of $P - S$. If not, then let A be the set of vertices consisting of the lowest-ranked non-neighbor of

s_1 in each component of $P - S$. Note that if $\{w, w'\}$ is a pair of vertices in A, then $\text{rank}(w) + \text{rank}(w') < t$, or else s_1 has a set B of at least t neighbors in the components of $P - S$ containing w and w'. Let z be the vertex in $P[x, s_1]$ that preceeds s_1. Note that $z \notin B$, since some non-neighbor of s_1 separates z and the initial segment of $P[x, s_1)$ consisting of vertices belonging to B. Counting B together with z, it follows that $d(s_1) \geqslant t + 2$, contradicting that $\Delta(G) \leqslant t + 1$. Hence $\text{rank}(w) + \text{rank}(w') < t$ and it follows from Lemma 9 part (4) that A is an independent set. But A together with s_1 and a non-neighbor of s_1 in H forms an independent set of size $k+3$, contradicting that $\alpha(G) \leqslant k+2$. Therefore $N(s_1)$ either contains $V(H)$ or $f(W)$ for some component W of $P - S$.

Note that $|V(H)| = t$ and $|f(W)| = t$ for each component W of $P - S$. Let v and v' be the immediate neighbors of s_1 along P, and let v'' be a neighbor of s_1 in H. Noting that $V(H)$ and each $f(W)$ intersect $\{v, v', v''\}$ in at most one vertex, it follows that $d(s_1) \geqslant t + 3 - 1$, contradicting that $\Delta(G) \leqslant t + 1$. □

We note two consequences.

Corollary 11. *If G is a connected graph with $\alpha(G) \leqslant 3$ and $\Delta(G) - \delta(G) \leqslant 1$, or if G is a 2-connected regular graph with $\alpha(G) \leqslant 4$, then G has a Hamiltonian path.*

Corollary 12. *The graph $4P_1$ is a fixer.*

4 A 5-vertex fixer

In this section, we show that $5P_1$ is a fixer. Although $5P_1$ is a fixer, there are connected $5P_1$-free graphs in which no vertex of maximum degree is Gallai (see Example 20). By contrast, for each fixer F of order at most 4, the vertices of maximum degree in a connected F-free graph are all Gallai: Golan and Shan [13] show this for $F = 2P_2$, our results in Section 3 show this for $F \in \{P_3 + P_1, P_2 + 2P_1, 4P_1\}$, and we leave the case $F = P_4$ as an exercise.

The statement that $5P_1$ is a fixer is equivalent to the statement that if G is a connected graph with $\alpha(G) \leqslant 4$, then G has a Gallai vertex. In the case that G is 2-connected, the result already follows from Theorem 10. When G has cut-vertices, we exploit the block-cutpoint structure of G. We need the following two variants of Theorem 10 in the case that P is an x-fiber or an xy-fiber for distinguished vertices $x, y \in V(G)$.

Lemma 13. *Let G be a 2-connected graph with a distinguished vertex x. If $\alpha(G-x) \leqslant 3$, then every x-fiber contains every vertex in G of maximum degree.*

Proof. Let P be an x-fiber with other endpoint y, and suppose for a contradiction that u is a vertex of maximum degree not on P. Let H be the component of $G - V(P)$ containing u, and let r be the number of attachment points of H on P. Note that $r \geqslant 2$, or else there is at most one attachment point separating y and H, contradicting that G is 2-connected. Moreover, by Lemma 5 part (2), we have that $r + \alpha(H) \leqslant \alpha(G - x) \leqslant 3$. Since $r \geqslant 2$ and $\alpha(H) \geqslant 1$, it follows that $r = 2$ and $\alpha(H) = 1$. Therefore H is a complete graph. Let

$\{s_1, s_2\}$ be the set of attachment points of H on P, with indices increasing from x to y, and let $S = \{s_1, s_2\}$.

Since G is 2-connected, there is a matching in the induced $(S, V(H))$-bigraph saturating S or $|V(H)| = 1$. Let $t = |V(H)|$ and note that $d(u) \leqslant (t-1) + 2 = t + 1$. Since P is an x-fiber, it follows from Lemma 8 that both $P(s_1, s_2)$ and $P(s_2, y]$ are non-empty (part (1)) and have at least t vertices (part (2)). If s_2 has at least t neighbors in some set in $\{V(H), V(P(s_1, s_2)), V(P(s_2, y])\}$, then $d(s_2) \geqslant t + 2 > d(u)$, contradicting that u has maximum degree. Hence s_2 has fewer than t neighbors in each of $V(H)$, $V(P(s_1, s_2))$, and $V(P(s_2, y])$. Let w_1 and w_2 be the non-neighbors of s_2 of minimum rank in $P(s_1, s_2)$ and $P(s_2, y]$, respectively, and let z be a non-neighbor of s_2 in H.

We claim that $\{s_2, z, w_1, w_2\}$ is an independent set, contradicting $\alpha(G - x) \leqslant 3$. By construction, s_2 has no neighbor in $\{z, w_1, w_2\}$. Since w_1 and w_2 are not attachment points, z has no neighbor in $\{w_1, w_2\}$. If $w_1 w_2 \in E(G)$, then Lemma 8 part (3) and the fact that P is an x-fiber imply that $\text{rank}(w_1) + \text{rank}(w_2) \geqslant t$. Hence s_2 is adjacent to all vertices in $P(s_1, w_1)$ and $P(s_2, w_2)$, and there are at least t of them. Together with the vertex preceding s_2 in P and a neighbor of s_2 in H, we have $d(s_2) \geqslant t + 2$, contradicting that u has maximum degree. □

Lemma 14. *Let G be a 2-connected graph and let x and y be distinct vertices of G. If $\alpha(G - \{x, y\}) \leqslant 2$, then every xy-fiber contains every vertex in G of maximum degree or $G - \{x, y\}$ is the disjoint union of two complete graphs.*

Proof. Let P be an xy-fiber, let u be a vertex of maximum degree not on P, and let H be the component of $G - V(P)$ containing u. Let $\{s_1, \ldots, s_r\}$ be the set of attachment points of H, with indices increasing from x to y, and let $S = \{s_1, \ldots, s_r\}$. Since G is 2-connected, we have $r \geqslant 2$, or else deleting S separates H from $V(P) - S$ (which is non-empty since $x \neq y$). By Lemma 5, there is an independent set $A \subseteq V(P - \{x, y\})$ such that $|A| = r - 1$ and there are no edges joining A and $V(H)$. Therefore $1 + 1 \leqslant (r-1) + \alpha(H) \leqslant \alpha(G - \{x, y\}) \leqslant 2$. It follows that $r = 2$ and $\alpha(H) = 1$.

Let $t = |V(H)|$. Note that H is complete and, since G is 2-connected, there is a matching in the induced $(S, V(H))$-bigraph saturating S or $|V(H)| = 1$. By Lemma 8, we have $|V(P(s_1, s_2))| \geqslant t$ or else there is an augmenting interior splice of P replacing $P(s_1, s_2)$, contradicting that P is an xy-fiber.

Let $W = V(P(s_1, s_2))$. Note that W is a clique, or else a non-adjacent pair of vertices in W together with a vertex in H gives an independent set of size 3, contradicting $\alpha(G - \{x, y\}) \leqslant 2$.

If $(x, y) = (s_1, s_2)$, then $G - \{x, y\}$ is the disjoint union of the complete graph H and the complete graph on W. Otherwise, if $x \neq s_1$, then s_1 has a non-neighbor in H and a non-neighbor in W, or else $d(s_1) \geqslant t + 2 > d(u)$. So s_1 together with a non-neighbor in W and a non-neighbor in H form an independent set of size 3 in $G - \{x, y\}$, a contradiction. The case that $y \neq s_2$ is similar. □

The *block-cutpoint graph* of a graph G is a bipartite graph H in which one part consists of the cut-vertices of G and the other has a vertex b_i for each block B_i of G. Moreover,

vb_i is an edge of H if and only if $v \in B_i$. When G is connected, its block-cutpoint graph is a tree whose leaves are the blocks of G (see, e.g., [29]). We say that a block B of a graph G is *special* if every longest path in G contains an edge in B.

Lemma 15. *If no cut-vertex in a connected graph G is Gallai, then G has a special block.*

Proof. Let G be a connected graph such that no cut-vertex is Gallai. Suppose for a contradiction that no block of G is special. Let T be the block-cutpoint tree of G. We construct a digraph D on $V(T)$ in which each vertex has out-degree 1. Let B be a block in G. We identify a particular cut-vertex $x \in V(B)$ and we include the directed edge Bx in D. Since B is not special, some longest path of G is contained in some component H of $G - E(B)$. Note that H and B have exactly one vertex in common, and we take x to be this cut-vertex.

Let x be a cut-vertex in G. We specify a particular block B that contains x and we include the directed edge xB in D. Since x is not Gallai, some component H of $G - x$ contains a longest path in G. Let B be the block containing x such that $B - x \subseteq H$. We add the directed edge xB to $E(D)$.

Since $|E(D)| = |V(T)| > |E(T)|$, it follows that there is a block B and a cut-vertex x such that both Bx and xB are edges in D. This implies that G has vertex-disjoint longest paths, contradicting the fact that every two longest paths in a connected graph share at least one vertex. □

Lemma 16. *If G is a connected graph, $\alpha(G) \leqslant 4$, and G has a special block, then G has a Gallai vertex.*

Proof. Let G be a connected graph with $\alpha(G) \leqslant 4$ and with a special block B. Let S be the set of cut-vertices in B, with $S = \{x_1, \ldots, x_k\}$. Since $\alpha(G) \leqslant 4$, we have $k \leqslant 4$.

Case $k = 0$. In this case, $G = B$ and so G is 2-connected. It follows from Theorem 10 that G has a Gallai vertex.

Case $k = 1$. Let $u \in V(B)$ with $d_B(u) = \Delta(B)$. We claim that u is a Gallai vertex in G. Let P be a longest path in G. If P is contained in B, then $u \in V(P)$ by Theorem 10. If P leaves B through the cut-vertex x_1, then $P \cap B$ is an x_1-fiber in B and it follows that $u \in V(P)$ by Lemma 13.

Case $k = 2$. Suppose first that $B - S$ is not the disjoint union of two complete graphs. Let $u \in V(B)$ with $d_B(u) = \Delta(B)$. We claim that u is a Gallai vertex. Let P be a longest path in G. Since B is special, it follows that $P \cap B$ is a nontrivial subpath of P. Note that, as a subgraph of B, the path $P \cap B$ is either a fiber, an x_1-fiber or an x_2-fiber, or an x_1x_2-fiber, depending on whether P has two, one, or zero endpoints in B, respectively. It follows from Theorem 10, Lemma 13, or Lemma 14 that $u \in V(P \cap B)$, respectively.

Otherwise, suppose that $B - S$ is the disjoint union of two complete graphs W_1 and W_2 (see Figure 5). Since B is 2-connected, for $i \in \{1, 2\}$, there is a matching in the induced $(S, V(W_i))$-bigraph saturating S or $|V(W_i)| = 1$. Also, since S is a minimum cut in B, each vertex in S has neighbors in $V(W_1)$ and $V(W_2)$. It follows that B has a Hamiltonian cycle. We claim that x_2 is a Gallai vertex. Let P be a longest path in G, and suppose for a contradiction that $x_2 \notin V(P)$. Since B is special, P has at least one endpoint in

B. Replacing the subpath of P inside B with an appropriate Hamiltonian path gives a longer path in G.

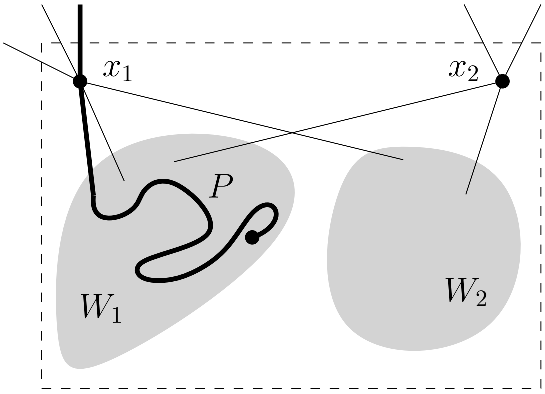

Figure 5: Case $k = 2$ in the proof of Lemma 16.

Case $k = 3$. Note that $B - S$ is a complete graph W_1 or else $\alpha(G) > 4$. Suppose there is a pair of cut-vertices, say $\{x_1, x_3\}$, such that $B - \{x_1, x_3\}$ is the disjoint union of two complete graphs. These are necessarily W_1 and the 1-vertex subgraph consisting of x_2; let W_2 be this 1-vertex subgraph. As in the case $k = 2$, it follows that B has a Hamiltonian cycle containing $x_1x_2x_3$ as a subpath. We claim that x_3 is a Gallai vertex. Let P be a longest path in G and suppose for a contradiction that $x_3 \notin V(P)$. Note that P cannot have an endpoint in B, or else replacing $P \cap B$ with an appropriate Hamiltonian path gives a longer path in G. Therefore, as a subgraph of B, the path $P \cap B$ is an x_1x_2-fiber. But B has a spanning x_1x_2-path, contradicting $x_3 \notin V(P)$.

Otherwise, there is no pair of cut-vertices whose removal from B results in the disjoint union of two complete graphs. Let $u \in V(B)$ with $d_B(u) = \Delta(B)$. We claim that u is a Gallai vertex. Let P be a longest path in G. It follows that, as a subgraph of B, the path $P \cap B$ is a fiber, an x_i-fiber for some $x_i \in S$, or an x_ix_j-fiber for some $x_i, x_j \in S$, depending on whether P has two, one, or zero endpoints in B, respectively. It follows from Theorem 10, Lemma 13, or Lemma 14 that $u \in V(P \cap B)$, respectively.

Case $k = 4$. The condition $\alpha(G) \leqslant 4$ requires that $|V(B)| = 4$ and 2-connectivity requires that B contains a 4-cycle C. Let x_i be a cut-vertex in B which maximizes the length of an x_i-fiber in $G - E(B)$. We claim that x_i is a Gallai vertex. Let P be a longest path in G, and suppose for a contradiction that $x_i \notin V(P)$. The path P decomposes into three subpaths P_1, P_2, and P_3, where $P_2 = P \cap B$. Let x_j be the vertex in $V(P_1) \cap V(P_2)$, and let x_k be the vertex in $V(P_2) \cap V(P_3)$. Since $|V(B)| = 4$, it follows that x_j or x_k is a neighbor of x_i in C. If $x_kx_i \in E(C)$, then we find a longer path in G by keeping P_1, extending P_2 by the edge x_kx_i to obtain P_2', and replacing P_3 with an x_i-fiber P_3' in $G - E(B)$. Since P_2' is longer than P_2 and P_3' is at least as long as P_3 by our choice of x_i, the path obtained by combining P_1, P_2', and P_3' is longer than P. The case $x_jx_i \in E(C)$ is symmetric. □

Applying our lemmas gives the following.

Theorem 17. *Let G be a connected graph. If $\alpha(G) \leqslant 4$, then G has a Gallai vertex. Equivalently, $5P_1$ is a fixer.*

Proof. If some cut-vertex in G is Gallai, then the claim follows. Otherwise, we have that G has a special block by Lemma 15, and hence G has a Gallai vertex by Lemma 16. □

The graph G_0 from Figure 1 shows that there is a connected graph G such that G has no Gallai vertex and $\alpha(G) = 6$. The case $\alpha(G) \leqslant 5$ remains open.

Conjecture 18. If $\alpha(G) \leqslant 5$ and G is connected, then G has a Gallai vertex.

When G is 3-connected, $\alpha(G) \leqslant 5$, and G is sufficiently large, Theorem 19 shows that G has a Gallai vertex. Outside of a finite number of cases when $\kappa(G) \geqslant 3$, resolving Conjecture 18 reduces to the cases that $\kappa(G) = 1$ and $\kappa(G) = 2$. Although it is reasonable to expect that the case $\kappa(G) = 1$ may be treated by analyzing the block structure of G, it is less clear how to handle the case $\kappa(G) = 2$.

5 A Chvátal–Erdős type result

A celebrated result of Chvátal and Erdős [7] states that if $\alpha(G) \leqslant \kappa(G)$, then G has a Hamiltonian cycle, and the same technique shows that G has a Hamiltonian path when $\alpha(G) \leqslant \kappa(G) + 1$. Clearly, when G has a Hamiltonian path, every vertex in G is Gallai. We show that if $\alpha(G) \leqslant \kappa(G) + 2$ and G is sufficiently large in terms of $\kappa(G)$, then the maximum degree vertices in G are Gallai.

Theorem 19. *For each positive integer k, there exists an integer n_0 such that if G is an n-vertex k-connected graph with $\alpha(G) \leqslant k + 2$ and $n \geqslant n_0$, then each vertex of maximum degree is Gallai.*

Proof. We take $n_0 = k(k+2)(2k+3)+1$. Let P be a longest path in G with endpoints x and y, and suppose for a contradiction that $u \in V(G) - V(P)$ and $d(u) = \Delta(G)$. Let H be the component of $G - V(P)$ containing u, and let $t = |V(H)|$. From Lemma 9, it follows that H is complete and H has a set S of k attachment points on P. Let $S = \{s_1, \ldots, s_k\}$ with indices increasing from x to y. For $1 \leqslant i < k$, let $W_i = V(P(s_i, s_{i+1}))$; we also define $W_0 = V(P[x, s_1))$ and $W_k = V(P(s_k, y])$. By Lemma 9, we have that $|W_i| \geqslant t$ for $0 \leqslant i \leqslant k$. Since $u \in V(H)$, we have that $N(u) \subseteq (V(H) - \{u\}) \cup S$ and therefore $\Delta(G) = d(u) \leqslant (t-1) + k$. If $t \leqslant 2k(k+1)$, then $\Delta(G) \leqslant k(2k+3) - 1$ and so $\alpha(G) \geqslant n/(\Delta(G)+1) \geqslant n/[k(2k+3)] > k+2$, since $n \geqslant n_0$. Therefore we may assume that $t > 2k(k+1)$.

We claim that H is the only component of $G - V(P)$. If $G - V(P)$ contains a second component H', then let S' be the set of attachment points of H' on P. By Lemma 9, it follows that $|S'| = k$. For each i, choose $a_i \in W_i$ among the vertices with ranks in $\{0, \ldots, k\}$ so that $a_i \notin S'$. Let $A = \{a_0, \ldots, a_k\}$. Since $t > 2k(k+1) > 2k$, it follows from

Lemma 9 that A is an independent set of size $k+1$. Since A is disjoint from $S \cup S'$, we may extend A to an independent set of size $k+3$ by adding a vertex in H and a vertex in H'. Since $\alpha(G) \leqslant k+2$, we obtain a contradiction, and so H is the only component of $G - V(P)$.

Next, we claim that each vertex $w \in W_i$ has at most k neighbors outside W_i. Let A be the subset of $V(P) - S$ consisting of the vertices w such that $\mathrm{rank}(w) = 0$. By Lemma 9, we have that A is an independent set with $|A| = k+1$. Note that each vertex $w \in V(P) - (S \cup A)$ has at least one neighbor in A, or else w together with A and a vertex in H would give an independent set of size $k+3$. Since $|A| = k+1$ and $\Delta(G) \leqslant t+k-1$, it follows that $|V(P) - (S \cup A)| \leqslant (k+1)(t+k-1)$ and hence $|V(P) - S| \leqslant (k+1)(t+k) = t(k+1) + k(k+1)$. Since $V(P) - S = \bigcup_{i=0}^{k} W_i$ and $|W_i| \geqslant t$ for each i, it follows that $t \leqslant |W_i| \leqslant t + k(k+1)$. By Lemma 9, in each W_i, the t vertices of smallest rank form a clique. By symmetry, in each W_i, the t vertices of largest rank also form a clique. Since $|W_i| \leqslant t + k(k+1) < 2t$, it follows that each vertex in W_i is among the t vertices with smallest rank or the t vertices with largest rank. In particular, each vertex in W_i has at least $t-1$ neighbors in W_i and hence at most k neighbors outside W_i.

It now follows that each W_i is a clique. Indeed, if $w_i, w_i' \in W_i$ but $w_i w_i' \notin E(G)$, then we obtain an independent set A with $A \subseteq V(P) - S$ and $|A| = k+2$ as follows. Starting with $A = \{w_i, w_i'\}$, we add a vertex to A from each W_j with $j \neq i$. Since $|W_j| \geqslant t > k(k+1)$ and each of the vertices already in A have at most k neighbors in W_j, some vertex in W_j can be added to A. The set A together with a vertex in H gives an independent set of size $k+3$, a contradiction. Hence each W_i is a clique.

A vertex z *dominates* a set of vertices B if z is adjacent to each vertex in B. Next, we claim that each $s_i \in S$ dominates some set in $\{W_0, \ldots, W_k, V(H)\}$. If some attachment point s_i has more than k^2 non-neighbors in each W_j and a non-neighbor v in H, then we may obtain an independent set of size $k+3$ by starting with $\{s_i, v\}$ and adding one vertex from each W_j. It follows that each s_i has at least $t - k^2$ neighbors in some set in $\{W_0, \ldots, W_k, V(H)\}$. Let $W_{k+1} = V(H)$, let s_i be an attachment vertex, and choose j such that $0 \leqslant j \leqslant k+1$ and s_i has at least $t - k^2$ neighbors in W_j. We claim that s_i dominates W_j. Indeed, if $w \in W_j$ but $s_i w \notin E(G)$, then we obtain an independent set A of size $k+3$ starting with $A = \{s_i, w\}$ and adding one vertex from each W_ℓ with $0 \leqslant \ell \leqslant k+1$ and $\ell \neq j$. Since s_i has at most $(t+k-1) - (t-k^2)$ neighbors in W_ℓ, each of the other vertices already in A has at most k neighbors in W_ℓ, and $|W_\ell| \geqslant t > (k(k+1)-1) + (k+1)k$, it follows that W_ℓ contains a vertex that can be added to A. Since $\alpha(G) \leqslant k+2$, we obtain a contradiction, and so s_i dominates W_j.

Let $1 \leqslant i < k$. Since W_i is a clique and $W_i = V(P(s_i, s_{i+1}))$, we obtain a path P' with $V(P) = V(P')$ and the same set of attachment points by reordering the vertices in W_i arbitrarily, so long as the first vertex is adjacent to s_i and the last vertex is adjacent to s_{i+1}. Similarly, we may reorder W_0 provided that the last vertex in W_0 is adjacent to s_1 and we may reorder W_k provided that the first vertex in W_k is adjacent to s_k. Let R be the set of neighbors of S in P. Note that for each $w \in W_i - R$ and each q with $1 \leqslant q \leqslant |W_i| - 2$, we may obtain a path P' with $V(P) = V(P')$ and the same attachment points in which $\operatorname{rank}(w) = q$ by an appropriate reordering of W_i. It follows that if $ww' \in E(G)$, for some $w \in W_i$ and $w' \in W_j$, with i and j distinct in $\{0, \dots, k\}$, then $w, w' \in R$. Otherwise, we may reorder W_i and W_j to obtain a new path P' in which either $\operatorname{rank}(w) \leqslant 1$ and $\operatorname{rank}(w') \leqslant 1$, or $\operatorname{rank}(w) \geqslant |W_i| - 2$ and $\operatorname{rank}(w') \geqslant |W_j| - 2$. In the latter case, reversing P' gives a path P'' in which $\operatorname{rank}(w) \leqslant 1$ and $\operatorname{rank}(w') \leqslant 1$. This contradicts Lemma 9 with respect to P' or P'' since $\operatorname{rank}(w) + \operatorname{rank}(w') \leqslant 2$ but $|V(H)| = t > 2k(k+1) \geqslant 4$.

We obtain a final contradiction by showing that some attachment point has degree exceeding $\Delta(G)$. Let $D = \sum_{i=1}^{k} d(s_i)$ and note that $D \leqslant k(t + k - 1)$. We give a lower bound on D using three sets of edges. First, for each s_i, let T_i be a set of 3 edges incident to s_i consisting of the edges joining s_i to its two neighbors in R and a third edge joining s_i and a vertex in H. Second, for $0 \leqslant i \leqslant k$, there is a matching M_i of size k joining vertices in W_i and $V(G) - W_i$, or else the Kőnig-Egerváry Theorem [29] implies that the induced $(W_i, V(G) - W_i)$-bigraph has a vertex cover of size less than k, which is also a vertex cut since $|W_i|, |V(G) - W_i| \geqslant t > k$. Obtain M'_i from M_i by discarding edges incident to vertices in $W_i \cap R$. Note that $|M'_i| \geqslant |M_i| - 2 \geqslant k - 2$ always, but for $i \in \{0, k\}$ we have $|M'_i| \geqslant |M_i| - 1 \geqslant k - 1$. Suppose that $e \in M'_i$, let w be the endpoint of e in W_i, and let v be the other endpoint of e in $V(G) - W_i$. Since w is not an attachment point, we have $v \notin V(H)$, and since H is the only component of $G - V(P)$, it follows that $v \in V(P) - W_i$. Since $w \notin R$, it follows that v must be an attachment point. Hence each edge in M'_i joins a vertex in $W_i - R$ and a vertex in S. Moreover, M'_i and T_j are disjoint, as each edge in T_j has an endpoint in $R \cup V(H)$ and no edge in M'_i has such an endpoint. With $Z = \bigcup_{i=0}^{k} M'_i \cup \bigcup_{j=1}^{k} T_j$, we have $|Z| \geqslant [(k-1)(k-2) + 2(k-1)] + 3k = k(k+2)$. Third, for $1 \leqslant i \leqslant k$, let F_i be the set of edges joining s_i and a set in $\{W_0, \dots, W_k, V(H)\}$ dominated by s_i. Note that $|F_i \cap Z| \leqslant 2$, since F_i contains at most one edge in $\bigcup_{i=0}^{k} M'_i$ and at most one edge in $\bigcup_{j=1}^{k} T_j$. Let $F = \bigcup_{j=1}^{k} F_i$, and note that $|F| \geqslant tk$ and $|F \cap Z| \leqslant 2k$.

We compute $D \geqslant |F \cup Z| = |F| + |Z| - |F \cap Z| \geqslant tk + k(k+2) - 2k = tk + k^2 = k(t+k)$. Since $D \leqslant k(t + k - 1)$, it follows that $k(t+k) \leqslant D \leqslant k(t + k - 1)$, contradicting that k is positive. □

Example 20. The assumption $\alpha(G) \leqslant \kappa(G) + 2$ in Theorem 19 is best possible. Let G be the graph obtained from the star $K_{1,k+2}$ with leaves $\{x_1, \dots, x_{k+2}\}$ by replacing the center vertex with a k-clique S and replacing each leaf vertex x_i with a t-clique X_i containing a set of k distinguished vertices Y_i that are joined to S. Since $V(G)$ can be covered by $k+3$ cliques, we have $\alpha(G) \leqslant k+3$. Also, we have $\kappa(G) = k$ since S is a cutset of size k and when $R \subseteq V(G)$ and $|R| < k$, the graph $G - R$ contains at least one vertex in each of $S, Y_1, \dots, Y_{k+2}$, implying that $G - R$ is connected.

We claim that the set of Gallai vertices in G is S. Since $|S| = k$ and $G - S$ is the disjoint union of $k+2$ copies of K_t, it follows that every path in G has at most $|V(G)| - t$ vertices. Paths in G that achieve this bound contain S and all but one of $X_1, \dots, X_{k+2}$, implying that $u \in V(G)$ is Gallai if and only if $u \in S$. By construction, each vertex in S has degree $k(k+2) + (k-1)$. Hence, when t is sufficiently large, the set of vertices in G of maximum degree is $Y_1 \cup \dots \cup Y_{k+2}$, and none of these is Gallai.

Although maximum degree vertices are not Gallai, our construction still has Gallai vertices. It is natural to ask whether every graph with sufficiently high connectivity has a Gallai vertex [30, 32]. As noted in Section 1, there are k-connected graphs having no Gallai vertices when $k \leqslant 3$. The question remains open for $k \geqslant 4$.

The complete bipartite graphs $K_{s,s+2}$ show that the condition $\alpha(G) \leqslant \kappa(G)+1$ cannot in general be relaxed to $\alpha(G) \leqslant \kappa(G)+2$ while still guaranteeing existence of Hamiltonian paths [7]. However, Theorem 19 immediately implies that this is possible for sufficiently large regular graphs.

Corollary 21. *For each positive integer k, there exists n_0 such that every k-connected regular graph G with $\alpha(G) \leqslant k+2$ and $n \geqslant n_0$ vertices has a Hamiltonian path.*

We do not know whether the condition $\alpha(G) \leqslant k+2$ in Corollary 21 is best possible. The following construction from [8] shows that it cannot be relaxed to $\alpha(G) \leqslant k+5$.

Example 22. Let $k \geqslant 6$ be even. Let G_1 be K_{k+1} minus an edge and let G_2 be K_{k+1} minus a matching on $k-4$ vertices. Let G be the graph obtained from two copies of G_1 and one copy of G_2 by adding a new vertex adjacent to all k vertices of degree $k-1$. We have that G is a 1-connected regular graph with $\alpha(G) = 6$ and no Hamiltonian path.

6 Concluding remarks and open problems

In this paper we aimed at characterizing monogenic Gallai families. Let $\mathcal{H}$ be the set of fixers, and recall that $H \in \mathcal{H}$ if and only if $\mathrm{Free}(H)$ is a Gallai family. We showed that $\mathcal{H}$ contains $5P_1$ (Theorem 17) and all linear forests on at most 4 vertices (Section 3). Also, $\mathcal{H}$ is contained in the family of linear forests that are induced subgraphs of G_0 (Proposition 1). It remains open to decide if $H \in \mathcal{H}$ in finitely many cases:

Question 23. Let H be a linear forest induced subgraph of G_0 such that $5 \leqslant |V(H)| \leqslant 9$ and $H \neq 5P_1$. Is $\mathrm{Free}(H)$ a Gallai family?

We believe that $\mathrm{Free}(6P_1)$ provides an affirmative answer (Conjecture 18). It turns out that $3P_3$ and P_7+2P_1 are the only linear forest induced subgraphs of G_0 on 9 vertices, and hence the only candidates for 9-vertex fixers, as shown in the following.

Remark 24. The graphs $3P_3$ and P_7+2P_1 are the only 9-vertex linear forest induced subgraphs of G_0. The argument is as follows. Let H be an induced linear forest of G_0 on 9 vertices and let $P = v_1 \cdots v_i$ be a longest path in H.

Suppose first that P contains two vertices of degree 1 in G_0. Since the first 3 vertices and the last 3 vertices of P determine P, it is easy to see that, up to symmetry, P is one of the bold paths depicted in Figure 6. It follows that H is a copy of P_7+2P_1.

Suppose finally that P contains at most one vertex of degree 1 in G_0. We claim that $i \leqslant 3$. Indeed, if $i \geqslant 4$, then P contains at least $i-1 \geqslant 3$ vertices of degree 3 in G_0, say without loss of generality v_1, v_2, v_3. Note that v_1 has two neighbors in $V(G_0)-V(H)$ and both v_2 and v_3 have one neighbor in $V(G_0)-V(H)$. Since G_0 has girth 5, these

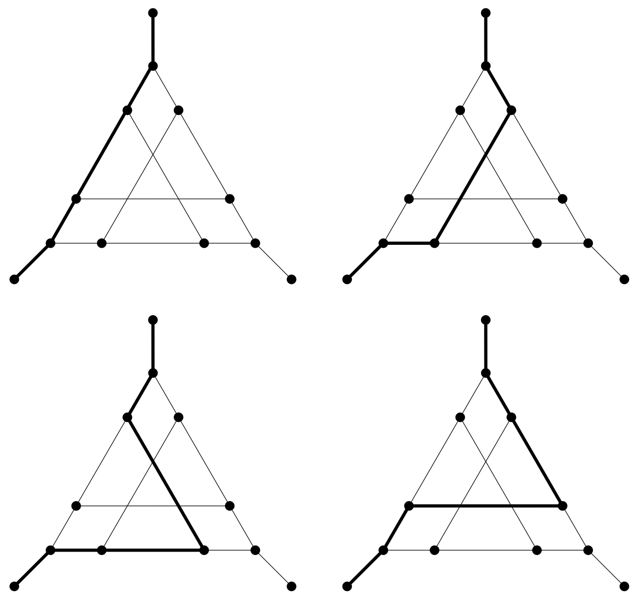

Figure 6: The induced paths in G_0 containing two degree-1 vertices of G_0.

neighbors are distinct and so $|V(H)| \leqslant 12-4=8$, a contradiction. Suppose now H has k components. Note that H has $9-k$ edges and $G_0 - E(H)$ has $6+k$ edges, each of which has an endpoint in $V(G_0)-V(H)$. Since G_0 is subcubic and $|V(G_0)-V(H)| = 12-9 = 3$, it follows that $6+k \leqslant 3 \cdot 3$, and so $k \leqslant 3$. Hence $H = 3P_3$.

In Corollary 21, we observed the following Chvátal–Erdős type result: for a regular graph G, if $\alpha(G) \leqslant \kappa(G)+2$ and G is sufficiently large in terms of $\kappa(G)$, then G contains a Hamiltonian path. We also observed that we cannot relax $\alpha(G) \leqslant \kappa(G)+2$ to $\alpha(G) \leqslant \kappa(G)+5$ and we conclude by asking to determine the best possible condition.

References

[1] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp. Longest paths in circular arc graphs. *Combinatorics, Probability and Computing*, 13(3):311–317, 2004.

[2] R. Belmonte, P. van 't Hof, M. Kamiński, and D. Paulusma. The price of connectivity for feedback vertex set. *Discrete Applied Mathematics*, 217:132–143, 2017.

[3] M. R. Cerioli and P. T. Lima. Intersection of longest paths in graph classes. *Discrete Applied Mathematics*, 281:96–105, 2020.

[4] M. R. Cerioli, C. G. Fernandes, R. Gómez, J. Gutiérrez, and P. T. Lima. Transversals of longest paths. *Discrete Mathematics*, 343(3):111717, 2020.

[5] F. Chen. Nonempty intersection of longest paths in a graph with a small matching number. *Czechoslovak Mathematical Journal*, 65(2):545–553, 2015.

[6] G. Chen, J. Ehrenmüller, C. G. Fernandes, C. G. Heise, S. Shan, P. Yang, and A. N. Yates. Nonempty intersection of longest paths in series–parallel graphs. *Discrete Mathematics*, 340(3):287–304, 2017.

[7] V. Chvátal and P. Erdős. A note on Hamiltonian circuits. *Discrete Mathematics*, 2 (2):111–113, 1972.

[8] D. W. Cranston and S. O. Hamiltonicity in connected regular graphs. *Information Processing Letters*, 113(22):858–860, 2013.

[9] K. K. Dabrowski and D. Paulusma. Clique-width of graph classes defined by two forbidden induced subgraphs. *The Computer Journal*, 59(5):650–666, 2016.

[10] R. J. Faudree and R. J. Gould. Characterizing forbidden pairs for hamiltonian properties. *Discrete Mathematics*, 173(1):45–60, 1997.

[11] T. Gallai. Problem 4. In P. Erdős and G. Katona, editors, *Theory of Graphs, Proceedings of the Colloquium Held at Tihany, Hungary, September 1966*, page 362. Academic Press, New York, 1968.

[12] Y. Gao and S. Shan. Nonempty intersection of longest paths in graphs without forbidden pairs. *Discrete Applied Mathematics*, 304:76–83, 2021.

[13] G. Golan and S. Shan. Nonempty intersection of longest paths in $2K_2$-free graphs. *Electronic Journal of Combinatorics*, 25(2), 2018.

[14] P. A. Golovach, D. Paulusma, and J. Song. Closing complexity gaps for coloring problems on H-free graphs. *Information and Computation*, 237:204–214, 2014.

[15] B. Grünbaum. Vertices missed by longest paths or circuits. *Journal of Combinatorial Theory, Series A*, 17(1):31–38, 1974.

[16] T. R. Hartinger, M. Johnson, M. Milanič, and D. Paulusma. The price of connectivity for cycle transversals. *European Journal of Combinatorics*, 58:203–224, 2016.

[17] A. S. Jobson, A. E. Kézdy, J. Lehel, and S. C. White. Detour trees. *Discrete Applied Mathematics*, 206:73–80, 2016.

[18] F. Joos. A note on longest paths in circular arc graphs. *Discussiones Mathematicae Graph Theory*, 35:419–426, 2015.

[19] S. Klavžar and M. Petkovšek. Graphs with nonempty intersection of longest paths. *Ars Combinatoria*, 29:43–52, 1990.

[20] D. V. Korobitsin. On the complexity of domination number determination in monogenic classes of graphs. *Discrete Mathematics and Applications*, 2(2):191–200, 1992.

[21] D. Král', J. Kratochvíl, Z. Tuza, and G. J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. In A. Brandstädt and V. B. Le, editors, *Graph-Theoretic Concepts in Computer Science*, volume 2204 of *Lecture Notes in Computer Science*, pages 254–262. Springer, 2001.

[22] B. Li and P. Vrána. Forbidden pairs of disconnected graphs implying hamiltonicity. *Journal of Graph Theory*, 84(3):249–261, 2017.

[23] J. A. Long Jr., K. G. Milans, and A. Munaro. Sublinear longest path transversals. *SIAM Journal on Discrete Mathematics*, 35(3):1673–1677, 2021.

[24] A. Munaro. Boundary classes for graph problems involving non-local properties. *Theoretical Computer Science*, 692:46–71, 2017.

[25] D. Rautenbach and J.-S. Sereni. Transversals of longest paths and cycles. *SIAM Journal on Discrete Mathematics*, 28(1):335–341, 2014.

[26] A. Shabbir, C. T. Zamfirescu, and T. I. Zamfirescu. Intersecting longest paths and longest cycles: A survey. *Electronic Journal of Graph Theory and Applications*, 1(1): 56–76, 2013.

[27] H. Walther. Über die nichtexistenz eines knotenpunktes, durch den alle längsten wege eines graphen gehen. *Journal of Combinatorial Theory*, 6(1):1–6, 1969.

[28] H. Walther and H.-J. Voss. *Über Kreise in Graphen*. Deutscher Verlag der Wissenschaften, 1974.

[29] D. B. West. *Introduction to Graph Theory*. Prentice Hall, 2nd edition, 2001.

[30] T. Zamfirescu. A two-connected planar graph without concurrent longest paths. *Journal of Combinatorial Theory, Series B*, 13(2):116–121, 1972.

[31] T. Zamfirescu. On longest paths and circuits in graphs. *Mathematica Scandinavica*, 38:211–239, 1976.

[32] T. Zamfirescu. Intersecting longest paths or cycles: A short survey. *Analele Univ. Craiova Ser. Mat. Inform.*, 28:1–9, 2001.