A characterization of two-dimensional Buchsbaum matching complexes

Bennet Goeckner[*]
Department of Mathematics
University of San Diego
San Diego, California, U.S.A.
bgoeckner@sandiego.edu

Fran Herr
Department of Mathematics
University of Washington
Seattle, Washington, U.S.A.
fanhrr@gmail.com

Legrand Jones II
Department of Mathematics
Indiana University
Bloomington, Indiana, U.S.A.
legjones@iu.edu

Rowan Rowlands[†]
Department of Mathematics
University of Washington
Seattle, Washington, U.S.A.
rowanr@uw.edu

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Abstract

The matching complex $M(G)$ of a graph G is the set of all matchings in G. A Buchsbaum simplicial complex is a generalization of both a homology manifold and a Cohen–Macaulay complex. We give a complete characterization of the graphs G for which $M(G)$ is a two-dimensional Buchsbaum complex. As an intermediate step, we determine which graphs have matching complexes that are themselves connected graphs.

Mathematics Subject Classifications: 05C70, 13F55, 05E45

1 Introduction

Given a graph G, a matching is a collection of edges such that no two share a common endpoint. The matching complex $M(G)$, which is the set of all matchings in G, forms a simplicial complex. Matching complexes and their topology have been studied extensively; see, e.g., [4, 10] for surveys of the field.

Recently, all homology manifolds that arise as matching complexes have been classified [2]. Outside of dimension two, all such complexes are combinatorial (i.e., PL) balls and

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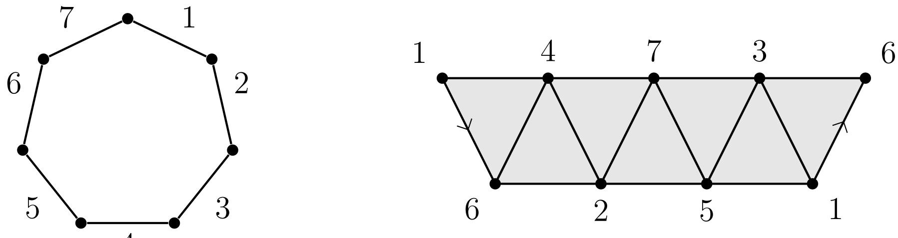


(a) C_7

(b) $M(C_7)$

Figure 1: The matching complex of C_7 is a triangulated Möbius strip. Faces of $M(C_7)$ with the same label are identified.

spheres. In dimension two, more examples appear, including a torus and a Möbius strip. See Figure 1 for one such example.

In this paper, we characterize all graphs G for which $M(G)$ is a two-dimensional Buchsbaum complex, which partially answers a question from [2, Section 6]. Buchsbaum complexes are a generalization of both homology manifolds and Cohen–Macaulay complexes. Though originally defined algebraically, the Buchsbaum condition is in fact a topological property [7]. In dimension two, Buchsbaum complexes can be classified in terms of certain subcomplexes being connected graphs, and this is the notion we will use.

In Section 2, we introduce relevant terminology and background. In Section 3 we classify all graphs G for which $M(G)$ is a connected graph, which allows us to characterize all one-dimensional Buchsbaum and Cohen–Macaulay matching complexes in Theorem 4. Then we consider the local behavior of graphs with two-dimensional Buchsbaum matching complexes. Section 4 gives an explicit description of all graphs G such that $M(G)$ is a two-dimensional Buchsbaum complex in Theorem 11 and then shows that this list is exhaustive. We end with a brief discussion of similar questions in higher dimensions in Section 5.

2 Preliminaries

Our two main objects of study are simple graphs and simplicial complexes. For all terms not defined here, see standard references such as [11] and [8].

A *(simple) graph* $G = (V, E)$ consists of a vertex set $V = V(G)$ and an edge set $E = E(G)$ whose members are two-element subsets of V. If $e = \{a, b\} \in E$, we refer to vertices a and b as the *endpoints* of the edge e; we will often use the notation $e = ab$. Given a graph G, a *matching* is a collection of edges of G such that no two share an endpoint. Unless stated otherwise, we will assume that all graphs mentioned in theorem statements are simple and do not have isolated vertices (i.e., vertices that are not the endpoints of any edges).

A graph is *bipartite* if the vertices can be partitioned into two sets V_1 and V_2 so that every edge has one endpoint in V_1 and one endpoint in V_2, or, equivalently, if the graph contains no odd-sized cycles. We often refer to several common graphs: K_n the complete graph on n vertices, C_n the cycle on n vertices, and S_n the star graph with $n+1$ vertices. We often refer to a path on $n+1$ vertices as a path of length n and denote it as P_{n+1}.

A *simplicial complex* Δ is a collection of sets with the property that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. An element $\sigma \in \Delta$ is called a *face*; throughout, we will use the convention of writing abc in place of $\{a, b, c\}$, etc. for faces of simplicial complexes. The *dimension* of a face σ is $\dim \sigma := |\sigma| - 1$, and the *dimension* of Δ, denoted $\dim \Delta$, is the maximum of the dimensions of its faces. A complex is *pure* if all maximal faces have the same dimension. Faces of dimension 0 and 1 are called vertices and edges respectively.

Note that simple graphs can be thought of as simplicial complexes of dimension one (or less, if the graph has no edges). Throughout, we will often blur the distinctions between graphs and 1-dimensional simplicial complexes and between simplicial complexes and their geometric realizations.

Given a face $\sigma \in \Delta$, its *link*, denoted $\operatorname{link}_\Delta \sigma$ (or simply $\operatorname{link} \sigma$ if Δ is unambiguous), is

$$\operatorname{link}_\Delta \sigma = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\}.$$

For example, the link of vertex 7 in Figure 1(b) is the path with edges $42, 25, 53$ and the link of edge 16 is the pair of isolated vertices 3 and 4. The link of a face captures the local structure of Δ near that face, and many properties of simplicial complexes—including Buchsbaumness—can be defined in terms of links.

The *matching complex* $M(G)$ is the set of all matchings in G. Since any subset of a matching is also a matching, $M(G)$ is a simplicial complex. Note that the vertices of $M(G)$ correspond to the edges of G. For especially small graphs, the matching complex is easy to calculate by hand. Alternatively, we define a Mathematica function “MatchingComplex” in the appendix.

The figures in this paper occasionally include dotted edges. If a pendant edge is dotted, then our arguments relating to that figure allow for arbitrarily many copies of that pendant. Similarly, if a figure includes a dotted path of length 2 that only touches the rest of the graph at the ends of the path, then the graph may include arbitrarily many paths of length 2 attached at the same points.

Many results on matching complexes concern the topological properties of their geometric realizations. Most previous results consider $M(G)$ for a family of graphs (see, e.g., [4] and [10] for a survey of these results), but we will instead specify the properties of $M(G)$ and then determine the structure of G. Motivated by a question from [2, Section 6], we will be primarily interested in complexes that satisfy the following definition.

Definition 1. Let Δ be a two-dimensional simplicial complex. We say that Δ is *Buchsbaum* if for each vertex $v \in \Delta$, $\operatorname{link}_\Delta v$ is a connected graph with at least one edge.

Note that if Δ has any maximal faces of dimension 0 or 1, then Δ cannot satisfy Definition 1, so all maximal faces must have dimension 2, i.e., Δ is pure. For example, the link of every vertex in Figure 1(b) is a path on four vertices, so $M(C_7)$ is Buchsbaum.

Remark 2. In this paper we focus on Buchsbaum complexes in dimension two. In general, a Buchsbaum complex can be defined as a pure complex where the i^{th} reduced homology of $\operatorname{link}\sigma$ is trivial for all $i < \dim\Delta - |\sigma|$ for all nonempty faces $\sigma \in \Delta$. The Buchsbaum condition was first defined in terms of algebraic properties of the complex's associated Stanley–Reisner ring. However, the combinatorial description above is equivalent for two-dimensional complexes, and, moreover, Buchsbaumness is a topological invariant (see [7, 5]).

Remark 3. Buchsbaumness is a generalization of the *Cohen–Macaulay* condition, which additionally requires that $\operatorname{link}_\Delta \varnothing$ (i.e. Δ itself) also has vanishing i^{th} homology for all $i < \dim\Delta$. There is a related and even more restrictive class known as *Gorenstein* complexes. A complete characterization of Gorenstein matching complexes is implicit in [2, Theorem 3.1] (via [8, Chapter II Theorem 5.1]) and is proved independently in [6, Theorem 2.1].

In light of these remarks, we note that if $\dim\Delta = 1$, then Δ is Buchsbaum if and only if Δ is a graph with no isolated vertices, and Δ is Cohen–Macaulay if and only if it is connected. In general, a complex Δ is Buchsbaum if and only if it is pure and $\operatorname{link}_\Delta v$ is Cohen–Macaulay for all vertices v of Δ. We will consider the one-dimensional case in Section 3, and it will be key to developing our results for two-dimensional complexes.

We are not aware of any overt study of Buchsbaum matching complexes in the literature, but there are some results for similar properties for certain families of graphs. For example, [3, Theorem 15] shows that $M(K_{m,n})$ is Cohen–Macaulay if and only if $n \geqslant 2m-1$, and [12, Theorem 2.3] shows that this is in fact equivalent to vertex decomposability for this family of matching complexes.

3 One-dimensional matching complexes and link behavior

The goal of this section is to classify all Cohen–Macaulay and Buchsbaum matching complexes in dimension one. We begin by defining the families $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ and the bowtie graph B in Figure 2. In this figure, dotted edges are optional. For families $\mathcal{G}_1$ and $\mathcal{G}_2$, we may repeatedly introduce filled vertices and connect them to any (nonzero) number of unfilled vertices.

The following is the main result in this section.

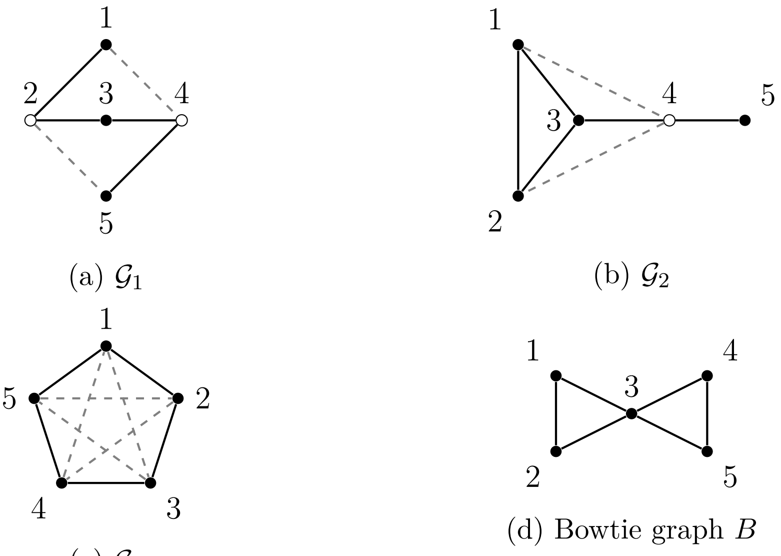
Theorem 4. *Let G be a graph and assume* $\dim M(G) = 1$.

(a) $M(G)$ is Cohen–Macaulay if and only if either

- *(i) G has two components which are each either a K_3 or a star graph, or*
- *(ii) G is in one of the families $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, or G is the bowtie graph B, all of which are defined in Figure 2.*

(b) $M(G)$ is Buchsbaum if and only if G is a graph described in (a) or $G = K_4$ or $G = C_4$.

We will spend the rest of this section proving Theorem 4. We first introduce a tool that will be used throughout the remainder of the paper.



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$ (c) $\mathcal{G}_3$ (d) Bowtie graph B

Figure 2: The families of connected graphs whose matching complexes are themselves connected graphs. Dotted edges are optional. For families $\mathcal{G}_1$ and $\mathcal{G}_2$, we may repeatedly introduce filled vertices and connect them to any (nonzero) number of unfilled vertices.

Definition 5. For a simple graph G and an edge $e \in E(G)$, the *non-adjacent subgraph* of e, denoted N_e, is the subgraph induced by all edges of G that do not share any endpoints with e.

Observe that N_e will never have any isolated vertices. Furthermore, the link of the vertex e in $M(G)$ is $\operatorname{link}_{M(G)} e = M(N_e)$. This allows us to use non-adjacent subgraphs to translate the two-dimensional Buchsbaum condition for the matching complex $M(G)$ into conditions for the graph G.

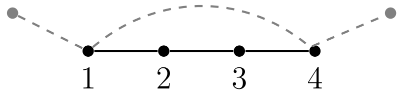
Lemma 6. *Given a graph G, $M(G)$ is a two-dimensional Buchsbaum complex if and only if $M(N_e)$ is a connected graph with at least one edge for all $e \in E(G)$.*

Proof. A matching complex $M(G)$ is two-dimensional if and only if the largest size of a matching in G is three. This is equivalent to the largest dimension of the link of a vertex in $M(G)$ being one (i.e., the link is a graph with at least one edge).

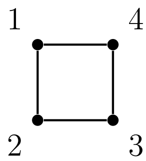
Let $M(G)$ be a two-dimensional complex. Then $M(G)$ is Buchsbaum if and only if $\operatorname{link}_{M(G)} e$ is a connected graph with at least one edge for each vertex e of $M(G)$. Since $\operatorname{link}_{M(G)} e = M(N_e)$, this completes the proof. □

We will use the above result throughout as our main tool for characterizing two-dimensional Buchsbaum matching complexes.

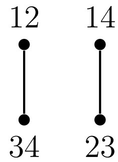
We will first consider graphs G for which $M(G)$ is a connected *graph*, i.e., a 1-dimensional simplicial complex. These graphs will be instrumental in Section 4, and they answer the question for 1-dimensional Cohen–Macaulay and Buchsbaum matching complexes. We first turn our attention to matching complexes of disconnected graphs.



(a) Edges that can be added to the path $\{12, 23, 34\}$ without touching 23.



(b) C_4



(c) $M(C_4)$

Figure 3: Graphs appearing in the proof of Lemma 8

Lemma 7. *Let G be a disconnected graph. Then $M(G)$ is a connected graph if and only if G has two components which are each either a K_3 or star graph.*

Proof. By direct computation we easily see that the matching complexes of $K_3 \sqcup K_3$, $K_3 \sqcup S_n$, and $S_n \sqcup S_m$ are all connected graphs.

Suppose that $M(G)$ is a connected graph and G is disconnected. Observe that if G had more than two components or if any component contained two non-adjacent edges, then G would contain a 3-matching. Thus G has exactly two components and all edges of each component are adjacent to each other, so the only possibility for each component of G is K_3 or a star graph. $\square$

We now will focus on *connected* graphs whose matching complexes are also connected graphs. We start with the following lemma.

Lemma 8. *Suppose G is a connected graph with at least two edges. If $M(G)$ is also a connected graph, then G contains a path of length four and no paths of length five or more.*

Proof. If G has a path of length five or more, we get a 3-matching by taking the first, third, and fifth edges of the path, so $M(G)$ is not 1-dimensional.

Assume that edges $e = uv$ and $e' = u'v'$ form a matching in G. Since G is connected, there must be a path connecting u to u'. Let P be the shortest such path (which may contain e or e'). Since the vertices of e and e' are distinct, P must contain at least one edge besides e and e', so $P \cup \{e, e'\}$ is a path containing at least three edges.

We now only need to show that G has a path of length four in particular. Suppose G contains a path of length three, say $\{12, 23, 34\}$. Now consider the edge 23 (see Figure 3(a)). Since $M(G)$ is a connected graph, there must be some other edge in G which does not share endpoints with 23, otherwise 23 becomes an isolated vertex in $M(G)$. Because G is connected, the only way to do this without having a path of length four is to add the edge 14, i.e., to have C_4 as a subgraph of our graph G. However, $M(C_4)$ is not a connected graph (see Figures 3(b) and 3(c)), so this subgraph cannot contain all the edges of G. Furthermore, the matching complex of any graph on four vertices that contains a 4-cycle is also disconnected. Thus we must have some edge in G which (without loss of generality) shares no endpoints with both 12 and 23 while keeping G a connected graph, which gives a path of length four. $\square$

We now turn our attention again to Figure 2, and note that each graph depicted in this figure contains a path of length four.

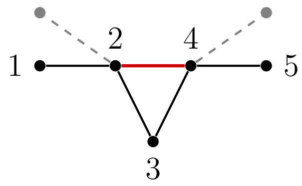


Figure 4: Graph appearing in Case 2(b) in the proof of Lemma 9.

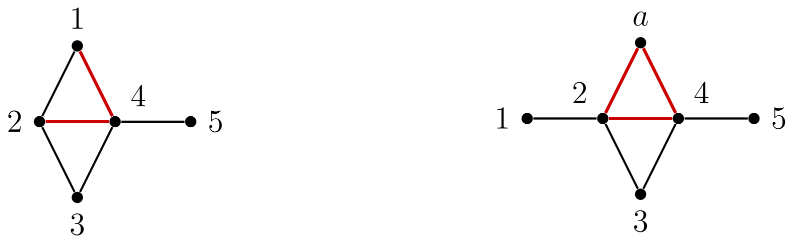


Figure 5: Graphs appearing in Case 4(b) in the proof of Lemma 9.

Lemma 9. *Suppose G is a connected graph with at least two edges. Then $M(G)$ is a connected graph if and only if $G \in \mathcal{G}_1$, $\mathcal{G}_2$, or $\mathcal{G}_3$, or G is the bowtie graph B.*

Proof. It is straightforward to check that the matching complex of any graph in these families is indeed a connected graph; we omit the details of these calculations.

For the other direction of the proof, assume $M(G)$ is a connected graph. By Lemma 8, G contains a path P of length four, say $P = \{12, 23, 34, 45\}$. If G is simply this path, then $G \in \mathcal{G}_1$. If this path is not all of G, let us consider what we can add. Note that we cannot add any edges which produce a 3-matching in the graph, so in particular we cannot have any edges of the form $1a$, $3a$, or $5a$ where a is a new vertex, and we cannot add any edges that do not share a vertex with this path of length four. We will break the remainder of the proof into cases.

Case 1: Suppose G has C_5 as a subgraph. We note that $M(C_5)$ is itself a connected graph (namely, $M(C_5) = C_5$). If G contains any edge whose endpoints are not both contained in this C_5, then G contains a 3-matching. However, $M(G)$ remains connected and 1-dimensional if we add any number of edges between vertices in the C_5. Thus G is in the family $\mathcal{G}_3$.

Case 2: Suppose G has C_3 as a subgraph but not C_4 or C_5. Without loss of generality, this can only occur if we add edge 13 or 24 to our path of length four because otherwise we immediately get a 3-matching. We will consider these as Cases 2(a) and 2(b), respectively.

In Case 2(a), we add the edge 13. Adding any edge to vertices 1 or 2 will either create a 4-cycle (if no new vertices are introduced) or 3-matching (if there are new vertices) and thus isn't allowed. Adding any edge to vertex 5 except the edge 35 will again create a 4-cycle or 3-matching. Thus the only allowed options are to add any number of pendant edges off vertex 4 or to instead add the edge 35. Observe that doing both of these would create a 3-matching. The first of these options puts G in $\mathcal{G}_2$, the second shows that G is the bowtie graph B.

In Case 2(b), we add the edge 24 as depicted in Figure 4. Observe that any additional edge with 1, 3, or 5 as an endpoint will create either 4-cycle or 3-matching and thus is not allowed. The only possible additional edges in this case have either 2 or 4 as an endpoint. However, any graph in this family has a disconnected matching complex—in particular, the edge 24 is not in a matching with any other edge. Thus this case is impossible.

Case 3: Suppose G has C_4 as a subgraph but not C_3 or C_5. Without loss of generality, the only way this can occur without introducing a 3-matching is to add edge 14 to the P_4 subgraph. Observe that any additional edge with 1 or 3 as an endpoint will either create a disallowed cycle or 3-matching. This is the same for any edge with 5 as an endpoint except the edge 25, which is allowed. The edge 24 would create a 3-cycle and thus is not allowed. However, any number of pendants off 2 and 4 and any paths of length two connecting vertices 2 and 4 are allowed. Therefore $G \in \mathcal{G}_1$.

Case 4: Suppose G has both C_3 and C_4 as a subgraph but not C_5. There are two ways to introduce the C_3 (without loss of generality): As in Case 2, we can add the edge 13 or 24. We will call these Cases 4(a) and 4(b), respectively.

In Case 4(a), we add the edge 13. The only edges involving 1 and 2 that we can add without introducing a 3-matching or 5-cycle are 14 and 24. We must add at least one of these to create a 4-cycle in G. Once we do so, we may add the other edge and any number of pendants off 4. Any edge with 5 as an endpoint will create a 3-matching or 5-cycle. Thus $G \in \mathcal{G}_2$.

In Case 4(b), we add the edge 24. As before, pendant edges off 1, 3, or 5 create 3-matchings, and adding the edge 15 creates a 5-cycle. Without loss of generality, the only way to create a 4-cycle without introducing new vertices is to add the edge 14. The only way to prevent 24 from being an isolated vertex in the matching complex is to add the edge 13. Now adding pendants to 2 produces a 3 matching, thus $G \in \mathcal{G}_2$.

If instead we create a 4-cycle with a new vertex, the only possible way without creating 3-matchings or disallowed cycles is to add $2a$ and $4a$ for a new vertex a as in Figure 5. However, as in Case 2(b), we see that 24 must be an isolated vertex in the matching complex. Thus this case is impossible.

Case 5: Suppose G contains no cycles. In this case the only edges we can add without getting a 3-matching are pendants off vertices 2 and 4. Thus $G \in \mathcal{G}_1$. $\square$

We immediately get the following corollary, combining the above results of this section with the definition of a two-dimensional Buchsbaum complex.

Corollary 10. *Let G be a graph. Then $M(G)$ is a two-dimensional Buchsbaum complex if and only if for each edge e of G, either*

(a) N_e has two components which are each either a K_3 or a star graph, or

(b) N_e is in one of the families $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, or N_e is the bowtie graph B.

(That is, N_e is one of the graphs described in Theorem 4(a).)

We are now able to complete the proof of Theorem 4, characterizing graphs whose matching complex is one-dimensional and either Cohen–Macaulay or Buchsbaum.

Proof of Theorem 4. We recall that a 1-dimensional complex is Cohen–Macaulay if and only if it is connected, and it is Buchsbaum if and only if it has no isolated vertices.

Therefore Theorem 4(a) follows immediately. For Theorem 4(b), the only graphs that have disconnected matching complexes with no isolated vertices are K_4 and C_4 by [2, Theorem 2.9]. □

4 Buchsbaum graph families

The goal of this section is to provide an explicit description of all graphs G for which $M(G)$ is two-dimensional and Buchsbaum. The following is our main result.

Theorem 11. *Let G be a graph. Then $M(G)$ is a two-dimensional Buchsbaum complex if and only if G is one of the following graphs (which are defined below and depicted in Figure 6):*

(a) a member of one of the families $\mathcal{B}_{C_7}$, $\mathcal{B}_P$, or $\mathcal{B}_i$ for some $i \in \{1, \dots, 9\}$,

(b) one of the two exceptional graphs E_1 and E_2, or

(c) one of the disconnected graphs described in Proposition 12, i.e.,

  (i) G has three components, each of which is either K_3 or a star graph, or

  (ii) G has two components, one of which is K_3 or a star graph and the other is either the bowtie graph B or a graph in one of the families $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$.

The rest of this section is devoted to proving Theorem 11, and we will now provide a brief outline of the proof. We first briefly consider when G is disconnected in Proposition 12. Next, we collect a variety of results on graphs containing cycles of certain sizes, including bipartite graphs. We then split up our remaining casework using the notion of a "link connected" graph, which we introduce in Definition 18 (a graph is link connected if all non-adjacent subgraphs N_e are connected). In Section 4.1, we examine the graphs that are link connected: We consider which cycles can appear in these graphs, and use this analysis to deduce which families these graphs belong to. This leaves the non-link-connected graphs, which we deal with in Section 4.2. These graphs by definition have some non-adjacent subgraph N_e that is not connected. By Corollary 10, there are only a few possibilities for what this N_e can be, and we examine each possibility one by one.

First, let us handle the case where G is disconnected.

Proposition 12. *Suppose G is a disconnected graph. Then $M(G)$ is two-dimensional and Buchsbaum if and only if either*

(a) G has three components, each of which is either K_3 or a star graph, or

(b) G has two components, one of which is K_3 or a star graph and the other is either the bowtie graph B or a graph in one of the families $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$.

Proof. First, if G is a graph satisfying (a) or (b), it is straightforward to check that $M(G)$ is indeed two-dimensional and Buchsbaum.

Conversely, suppose that G is a disconnected graph and $M(G)$ is two-dimensional and Buchsbaum. If G has four or more connected components, then it is guaranteed to have a 4-matching and $M(G)$ is thus not two-dimensional. If G has three components, then each must be either K_3 or a star graph because each component must have all edges adjacent to each other; otherwise we could find a 4-matching in G.

Next, suppose that G has exactly two connected components and $G = G_1 \sqcup G_2$. At least one component must not contain a 2-matching: Say this component is G_1. Then G_1 is K_3 or a star graph. If we take any edge $e \in G_1$, we see that N_e is precisely G_2. Thus by Lemma 9, G_2 must be in $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ or G_2 is the bowtie graph B. $\square$

We now turn our attention to Figure 6, which depicts several families of graphs $\mathcal{B}_i$ and two exceptional graphs E_1 and E_2 which all have two-dimensional Buchsbaum matching complexes. The families are defined as follows: Solid edges are necessary and dotted edges are optional. Observe the filled and unfilled vertices in Figure 6: For each graph, we may add any number of new filled vertices and attach each to a non-zero number of the unfilled vertices. We note that the families $\mathcal{B}_1$ through $\mathcal{B}_6$ have nonempty intersection. Otherwise these families are mutually disjoint.

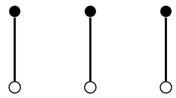
We now describe two additional families that are not depicted in Figure 6. The first is $\mathcal{B}_{C_7}$, which is defined to be the family of all graphs containing C_7 as a subgraph that have two-dimensional Buchsbaum matching complexes. We discuss this family in more detail later in this section, in particular in Lemmas 16 and 17, Table 1, and Figure 11.

The last family is $\mathcal{B}_P$, which we call petal graphs. These are formed by taking three graphs—each either a K_3 or star graph with at least two edges—and then gluing these graphs together at a single vertex. For the star graphs, the gluing vertex must be a non-central vertex of the star. The resulting graph will have one main central vertex and three 'petals,' each of which is a K_3 or star graph. For an example of a petal graph, see Figure 7.

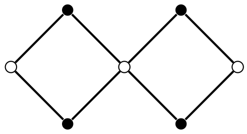
We can straightforwardly verify that any graph in Figure 6 or the families $\mathcal{B}_{C_7}$ and $\mathcal{B}_P$ has a two-dimensional Buchsbaum matching complex: For petal graphs and the families in Figure 6, we observe that—regardless of whether any dotted edges are included—the non-adjacent subgraph of any edge is a graph from Theorem 4. Thus we only need to consider added edges in Figure 6. The non-adjacent subgraphs of these edges correspond precisely to removing an unfilled vertex from the graph in question; again, we see that all such graphs appear in Theorem 4.

We will spend the remainder of this section showing that any connected graph with a two-dimensional Buchsbaum matching complex must be in one of the families in Figure 6 or the $\mathcal{B}_{C_7}$ or $\mathcal{B}_P$ families. Doing so will complete the proof of Theorem 11.

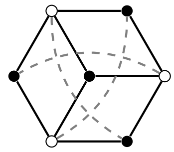
We recall that the only bipartite graphs in Lemmas 7 and 9 are either the disjoint union of two star graphs or in graph family $\mathcal{G}_1$. Considering Figure 6, we make precise the following observation.



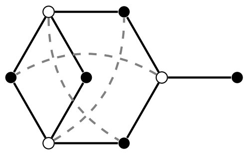
(a) $\mathcal{B}_1$



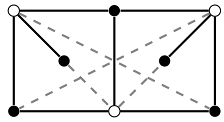
(b) $\mathcal{B}_2$



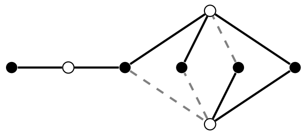
(c) $\mathcal{B}_3$



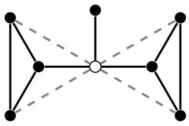
(d) $\mathcal{B}_4$



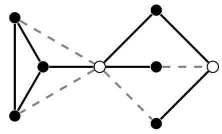
(e) $\mathcal{B}_5$



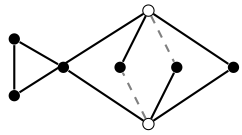
(f) $\mathcal{B}_6$



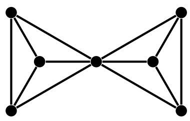
(g) $\mathcal{B}_7$



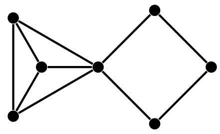
(h) $\mathcal{B}_8$



(i) $\mathcal{B}_9$



(j) Exceptional graph E_1



(k) Exceptional graph E_2

Figure 6: Graphs whose matching complexes are two-dimensional and Buchsbaum. As before, solid edges are required and dotted edges are optional. For each graph, any number of filled vertices may be introduced and each new vertex attached to some nonzero number of unfilled vertices.

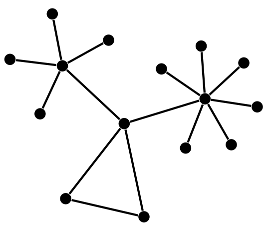


Figure 7: Example of a petal graph in $\mathcal{B}_P$. Each petal is either a K_3 or star graph with at least two edges.

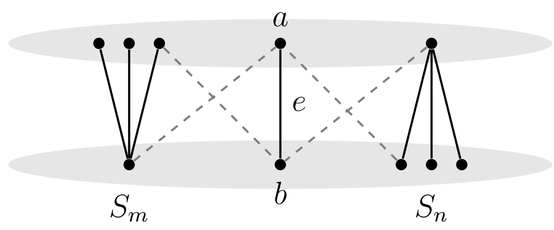


Figure 8: The case where N_e is a disjoint union of two stars, bipartitioned badly, in Proposition 13.

Proposition 13. *If G is a connected, bipartite graph and $M(G)$ is a two-dimensional Buchsbaum complex, then one side of the bipartition has exactly three vertices.*

Proof. First, since $M(G)$ is two-dimensional, there is some matching of three edges in G. Each of these edges must have one vertex in each side of the bipartition, so each side must contain at least three vertices.

Now, let $e = ab$ be any edge of G. We know that the matching complex of N_e is a connected graph and that N_e itself must be bipartite, so by Lemmas 7 and 9, N_e is either a graph in $\mathcal{G}_1$ or a disjoint union of two star graphs.

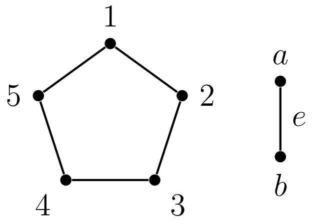
Assume $N_e \in \mathcal{G}_1$. For every graph in $\mathcal{G}_1$, any bipartition has one side with only two vertices, namely the two vertices labeled 2 and 4 in Figure 2. The edge e itself contributes one more vertex to each side, so this side has exactly 3 vertices in G.

This only leaves the case where N_e is a disjoint union of two stars, i.e., $N_e = S_m \sqcup S_n$. Since this subgraph is disconnected, there is more than one way to bipartition it—the centers of the stars can be on either the same side of the bipartition or opposite sides. If the centers of the stars are on the same side (or either of i and j is 1), the same argument as above works, since e again contributes one more vertex to this side. However, in the bipartition of N_e which puts the centers on opposite sides (see Figure 8), each side may be arbitrarily large: We must argue that this is not allowed in G.

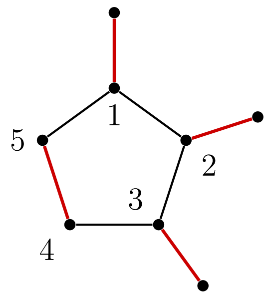
Since G is connected, there must be at least one edge in G connecting e to each star. We claim that such an edge cannot connect to the center of a star: Assume that the edge x connects vertex a to the center of S_m. Given the constraints on G, we see that N_x cannot contain a path of length four, so N_x must be disconnected. Thus there cannot be an edge between b and the center of S_n. Therefore we must have an edge between a and a non-central vertex of S_n; call this edge y. Returning to N_x, we see that there must be an edge containing vertex b that does not connect to S_n; call this edge z. Observe that this creates a 4-matching in G: Take the edges y and z, any edge in S_m that is not adjacent to z, and any edge in S_n that is not adjacent to y.

Similarly, if there are only edges between e and non-central vertices of S_m and S_n then G will also contain a 4-matching: Take two such non-adjacent edges, then for each star we can always find another non-adjacent edge.

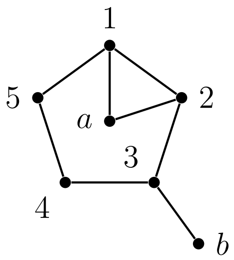
Thus this case is impossible, so one side of the bipartition of G must have exactly 3 vertices. □



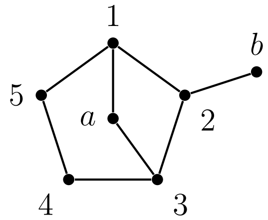
(a) All other edges of G must be adjacent to e or contained in the depicted 5-cycle.



(b) A graph that contains a 4-matching.



(c) The additional edges off 1 and 2 share a vertex that is distinct from the endpoint of the additional edge off 3.



(d) The additional edges off 1 and 3 share a vertex that is distinct from the endpoint of the additional edge off 2.

Figure 9: Graphs appearing in the proof of Proposition 14.

There are several relevant observations about non-bipartite graphs that we can make as well.

Proposition 14. *Assume G is connected and $M(G)$ is a two-dimensional Buchsbaum complex. If G contains C_5, then G contains C_7.*

Proof. Suppose first that G contains a 5-cycle and an edge $e = ab$ that is disjoint from this cycle, as depicted in Figure 9(a).

Now, N_e must be in $\mathcal{G}_3$, so all other edges of G must be either adjacent to e or have both vertices in this 5-cycle. Since G is connected, we assume the edge $x = 1a$ exists without loss of generality. Considering N_x, we see that there must be an edge between b and some vertex of C_5 other than vertex 1. If either of the edges $2b$ or $5b$ exist, then G will contain a 7-cycle. Without loss of generality, we assume that $y = 3b$ exists. We see that $N_{45} \in \mathcal{G}_3$, thus G cannot contain any additional vertices.

Observe that adding $4a$, $5b$, or any additional edge with vertex 2 as an endpoint will create a 7-cycle. Considering N_x again, we see that it is impossible for this subgraph to contain a path of length four without creating a 7-cycle in G. Therefore, whenever G has an edge disjoint from the 5-cycle, then G contains a 7-cycle.

We now instead assume that all other edges of G share at least one vertex with this 5-cycle. Observe, for example, that the edges 23 and 45 form a matching. Since $M(G)$ is two-dimensional and Buchsbaum, these two edges must be part of a 3-matching with some additional edge whose endpoints are vertex 1 and some vertex u_1 outside of the 5-cycle. By applying the same argument for each pair of non-adjacent edges in this 5-cycle, we conclude that each of the five vertices in this 5-cycle is an endpoint of some edge whose other endpoint is a vertex outside the 5-cycle. Furthermore, we must be able to choose five such edges—one for each vertex in the 5-cycle—such that not all share the same new vertex as an endpoint (otherwise any non-adjacent subgraph would be connected but have only four vertices). Thus G has at least two vertices outside this 5-cycle.

Consider the aforementioned edges for each of the five vertices in the 5-cycle. If we take three vertices of the 5-cycle in a row, then at least two of these edges must have a shared endpoint, otherwise G would contain a 4-matching as depicted in Figure 9(b).

Without loss of generality, there are two options for the additional edges off vertices 1, 2, and 3, as depicted in Figures 9(c) and 9(d).

Assume that the edges $1a$, $2a$, and $3b$ exist as in Figure 9(c). Then, considering the vertices $2, 3, 4$ in a row with the same logic as above, either $4a$ or $4b$ exists. If $4b$ exists, then G contains a 7-cycle. If $4a$ exists, we apply the same logic to $3, 4, 5$ to see that either $5a$ or $5b$ exists. If $5b$ exists, then G contains a 7-cycle. If $5a$ exists, observe that $N_{3b} \in \mathcal{G}_3$, which implies that G cannot have any additional vertices. Furthermore, N_{23} needs a path of length four. The only way to create such a path is to have some edge between one of the vertices in N_{23} and b, which creates a 7-cycle.

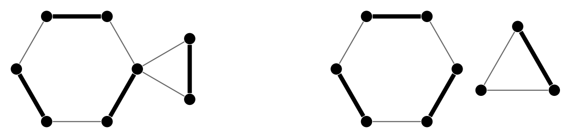
Assume instead that the edges $1a$ and $3a$ exist as in Figure 9(d). We perform the same analysis as in the previous paragraph—either one of the edges $4b$ or $5b$ exists which creates a 7-cycle, or both of the edges $4a$ and $5a$ exist. In the latter case, we again see that we need some edge between one of the vertices in N_{23} and b, which creates a 7-cycle. $\square$

We know by Proposition 14 that the existence of a 5-cycle will force the existence of a 7-cycle. We now consider non-bipartite graphs containing 6-cycles.

Lemma 15. *Let G be a graph whose matching complex is a two-dimensional Buchsbaum complex. If G has both C_6 and C_3 as subgraphs then it also has C_7 as a subgraph.*

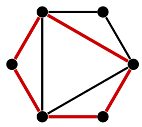
Proof. There are a number of ways that the C_3 and C_6 could interact. By Proposition 14, any time we deduce that G must have a C_5 it must also contain a C_7.

- If the C_3 subgraph shares at most one vertex with the C_6, then we have a 4-matching—take alternating edges of the C_6 together with an edge of the C_3 that does not touch the C_6. Hence this case cannot appear.

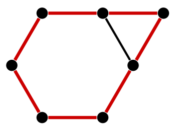


- Suppose our C_3 subgraph shares two vertices with the C_6 and no edges. These two vertices are either distance 2 or distance 3 apart in the C_6, and in both cases our graph contains C_5 as a subgraph, so Proposition 14 gives us a C_7.

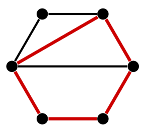
- Suppose that our C_3 subgraph shares three vertices with the C_6 and no edges. There is only one way to do this, without loss of generality, and this way gives us a C_5 as a subgraph, so again Proposition 14 says that G contains C_7 as a subgraph.



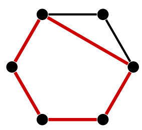
- If our C_3 subgraph shares two vertices and one edge with the C_6, we immediately get C_7 as a subgraph.



- If the C_3 shares three vertices and one edge with the C_6, we again get C_5 as a subgraph, so again we must also have C_7.



- Finally, if the C_3 shares all three vertices and two edges with the C_6, we obtain a C_5 and thus also a C_7.



This covers all possibilities, so G must always contain C_7 as a subgraph. □

Once a graph contains a C_7, it must have a very constrained structure as described in the lemma below.

Lemma 16. *If G is a graph (with no isolated vertices) that contains C_7 as a subgraph and $M(G)$ is two-dimensional, then G has exactly 7 vertices.*

Proof. See Figure 10. □

Figure 10: Any graph containing C_7 and any additional non-isolated vertices will always contain a 4-matching.

# edges added to C_7	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	total
# graphs up to isomorphism	1	2	10	30	58	77	73	56	37	20	10	5	2	1	1	383
# graphs where $M(G)$ is 2D Buchsbaum, up to isomorphism	1	1	3	7	11	18	19	20	18	12	7	4	2	1	1	125

Table 1: Data on graphs containing C_7.

We note that the graphs in Lemma 16 are all Hamiltonian, i.e., they each contain a cycle that uses all vertices of the graph.

A consequence of Lemma 16 is that there are only finitely many possibilities to check to find all graphs containing C_7 whose matching complex is two-dimensional and Buchsbaum: Simply take a C_7 and add every subset of the $\binom{7}{2} - 7 = 14$ edges that could be added, giving $2^{14} = 16384$ possibilities. While this would be impractical to check by hand, a computer can search these possibilities without difficulty. We have included this code in an appendix. Out of the 383 isomorphism classes of graphs on 7 vertices containing a C_7, 125 of them have a matching complex that is two-dimensional and Buchsbaum. See Table 1 for more refined data.

In particular, C_7 itself has a two-dimensional Buchsbaum matching complex, as does K_7. Deleting any one or two edges from K_7 gives a two-dimensional Buchsbaum matching complex, and so does one of the two ways of adding a single edge to C_7 (up to isomorphism).

Lemma 16 also puts restrictions on what N_e can be for any edge e in these graphs: There must be exactly five vertices in N_e, and there are only a few possibilities allowed by Lemmas 7 and 9 with only 5 vertices.

Lemma 17. *If G is a graph containing C_7 whose matching complex is two-dimensional and Buchsbaum, and e is an edge of G, then N_e must be one of the graphs in Figure 11.*

4.1 Link connected graphs

We split up our remaining casework using the following definition.

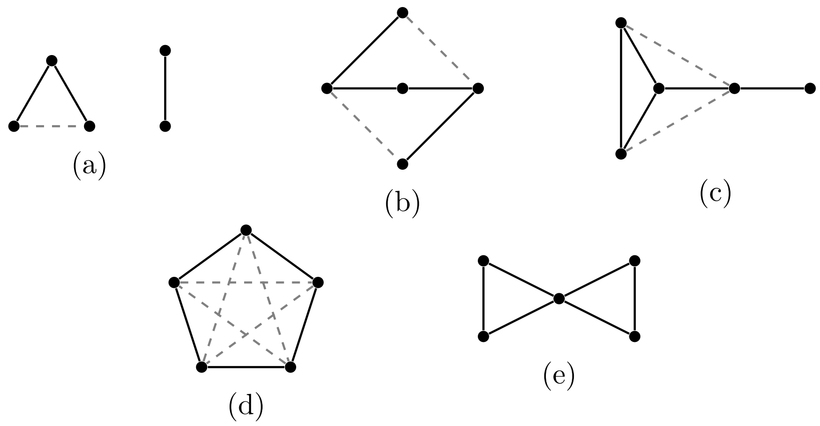


Figure 11: Graphs with 5 vertices whose matching complex is a connected graph.

Definition 18. Let G be a connected graph. We call G *link connected* if N_e is a connected graph for every edge $e \in E(G)$.

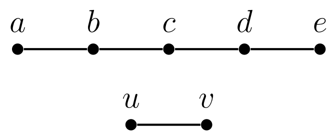
Remark 19. The above definition is similar to the definition of a 3-vertex-connected graph. In particular a 3-vertex-connected graph (or a "3-connected graph") is a graph such that the removal of any two vertices cannot disconnect the graph. The difference here is that we call G link connected if it is a graph such that the removal of any two *adjacent* vertices cannot disconnect the graph. Furthermore, 3-connected graphs should not have any isolated vertices after removing the two specified vertices, but our definition of N_e omits isolated vertices by construction.

In this section we will describe all link connected graphs whose matching complexes are two-dimensional Buchsbaum complexes. We start with some tools that will assist us with this.

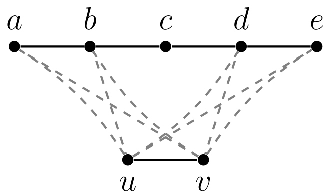
Lemma 20. *Suppose G is a link connected graph. If $M(G)$ is a two-dimensional Buchsbaum complex, then G has C_k as a subgraph for some $k \in \{4, 5, 6, 7\}$.*

Proof. First note that for $k \geqslant 8$, G cannot have C_k as a subgraph, since such cycles all contain a 4-matching.

Take any edge $uv \in E(G)$ and consider N_{uv}. By Lemma 8 we know that we have the following as a subgraph of G:



Since G is a link connected graph, N_{bc} must be connected, so there must be a path in G between either u or v and either d or e. If this path contains a, then it creates a k-cycle for $k \geqslant 4$ without using either of u or v, so we assume this does not occur. Similarly, N_{cd} must be connected, so there is a path in G from either u or v to either a or b. Similarly, we may assume this path does not contain e.

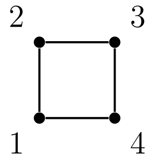


Patching these paths together, we get a cycle whose length is at least 4. □

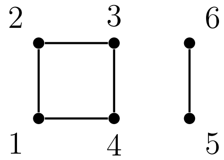
We will be using Lemma 20 to break up our casework in this section.

Lemma 21. *Let G be a link connected graph. If $M(G)$ is a two-dimensional Buchsbaum complex and G has C_4 as a subgraph, then either G has C_k as a subgraph for some $k \in \{5, 6, 7\}$ or $G \in \mathcal{B}_1$.*

Proof. By assumption, G has C_4 as a subgraph. Label the vertices of this subgraph as follows:

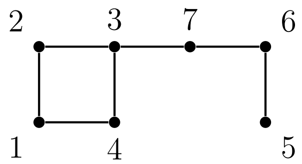


Because 12 and 34 are in a 2-matching together, they must be part of a 3-matching. Therefore, we must have at least one more edge 56 which is disjoint from the 4-cycle:

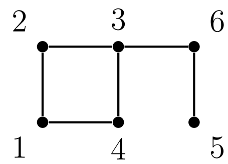


Since G is link connected, N_{12} must be a connected graph, so without loss of generality there must be a path between vertices 3 and 6. This path cannot have length three or more, as that would introduce a 4-matching, so we must have one of these two cases:

Case 1:



Case 2:

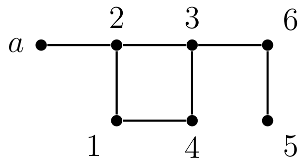


First, let us consider Case 1. Since N_{37} must be connected, there must be a path from 1, 2, or 4 to either 5 or 6, but every way to do this produces a cycle C_k with $k \geqslant 5$.

This leaves us with Case 2. The edges 14 and 36 form a 2-matching together, so they must be part of a 3-matching with some other edge of G. This other edge cannot be disjoint from the picture above, since that would give us a 4-matching made of edges 12, 34, 56 and the new edge. So the possibilities for the new edge are:

- 25, in which case G has C_6 as a subgraph;

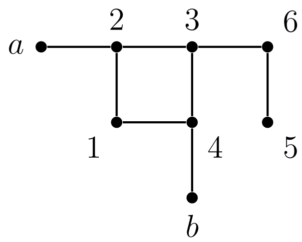
- $5a$ where a is a vertex not previously in our subgraph, which puts us into Case 1, which we have already dealt with;
- or $2a$ where a is a vertex not previously in our subgraph.



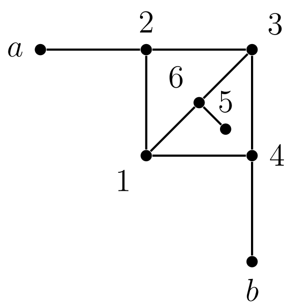
In this case, now, 12 and 36 are in a 2-matching together, so once again they must be part of a 3-matching. The possibilities for the third edge in this 3-matching are:

- 45, which gives us a C_6;
- $4a$, which results in some C_k for $k \in \{5, 6, 7\}$ after noticing that N_{23} must also be connected;
- $4b$ where b is a new vertex, which we will come back to momentarily;
- $5a$, which gives a C_5;
- $5b$, which contains Case 1 and is thus already dealt with;
- or ab, which gives a 4-matching.

Now we consider the case with edge $4b$.



Applying similar logic again, N_{34} must be a connected graph, so we can deduce that either G has C_k with $k \in \{5, 6, 7\}$ or has the following as a subgraph:



This graph is in $\mathcal{B}_1$ (with vertices 2, 4, and 6 being the unfilled vertices and vertices 1 and 3 introduced and attached to all three of the unfilled vertices). If this is a proper subgraph of G, then the only other edges we can add without introducing a 4-matching or a larger cycle are pendants attached to vertices 2, 4, and 6, which keep G in $\mathcal{B}_1$. □

As a consequence, we can split link connected graphs in the following way.

Corollary 22. *Suppose G is link connected and $M(G)$ is a two-dimensional Buchsbaum complex. Then either G has exactly 7 vertices and has C_7 as a subgraph, or G is bipartite.*

Proof. First, G cannot contain any cycles on 8 or more vertices without including a 4-matching.

Lemma 20 tells us that G must contain C_k, with $4 \leqslant k \leqslant 7$. If G includes C_7, we are done immediately by Lemma 16; and if G includes a C_5, then Proposition 14 gives us a C_7 and we are done.

This only leaves the case where G contains C_4 or C_6 but not C_5 or C_7. If G contains C_4 but not C_5 or C_7, then Lemma 21 implies that either G is in $\mathcal{B}_1$ and thus bipartite, or G contains C_6. And if G contains C_6 but not C_5 or C_7, then Lemma 15 implies that G cannot contain C_3 either, so G has no odd cycles and is thus bipartite. This covers all cases. □

We now consider link connected graphs that contain a 6-cycle, which will complete our discussion of link connected graphs.

Proposition 23. *Suppose G is a link connected graph. If $M(G)$ is a two-dimensional Buchsbaum complex and G has C_6 as a subgraph, then $G \in \mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, $\mathcal{B}_5$, or $\mathcal{B}_{C_7}$.*

Proof. Note that if G also contains C_3 or C_5, then $G \in \mathcal{B}_{C_7}$ by Corollary 22. Thus we will limit our consideration to bipartite graphs, which implies that $N_e \in \mathcal{G}_1$ for all edges e of G. Furthermore, G must contain some edges that have a vertex outside of the 6-cycle (since each N_e must have at least 5 vertices), and G cannot have an edge that is disjoint from this 6-cycle (since this would create a 4-matching). Therefore we assume G contains the subgraph in Figure 12. Observe that the filled and unfilled vertices form a bipartition for the graph, so by Proposition 13, there cannot be any more unfilled vertices. (Note that we do not assume that the filled and unfilled vertices follow the convention in Figure 6; however, these vertices will turn out to follow this convention.)

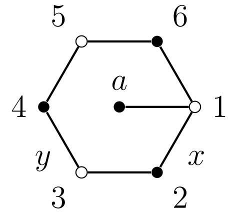


Figure 12: A subgraph appearing in the proof of Proposition 23.

Considering N_x, we see that G must contain an additional edge e with vertex 3 as an endpoint. Similarly, considering N_y, we see that G must contain an additional edge e' with vertex 5 as an endpoint.

Case 1: The other endpoints of e and e' are not part of the C_6.

First we note that if the edges $1a$, e, and e' are all pendants, then $G \in \mathcal{B}_1$.

Observe that if $e = 3a$ and $e' = 5a$, then $G \in \mathcal{B}_3$. Assume instead $e = 3a$ and $e' = 5b$ where b is a vertex not shown in Figure 12. (Observe that this case is equivalent to if $e = 3b$ and $e' = 5a$ or if $e = 3b$ and $e' = 5b$.) If neither of the edges $1b$ and $3b$ exist, then $G \in \mathcal{B}_4$.

If both edges $1b$ and $3b$ exist, then again $G \in \mathcal{B}_3$. Assume without loss of generality that the edge $3b$ exists but $1b$ does not. We claim that the family that G is in depends on whether any additional edges between the vertices of the C_6 in question exist. If no such edges exist or only the edge 36 exists, then $G \in \mathcal{B}_2$. If either or both of the other possible edges (i.e., 14 and 25) exist, then $G \in \mathcal{B}_3$.

Case 2: At least one of e and e' has both vertices in this C_6.

Assume without loss of generality that $e = 36$ and consider N_{36}, which we recall must be a member of $\mathcal{G}_1$. Thus G must contain the edge 14 or one of the edges 25 and $5a$.

Assume G does not contain either of the edges 25 or $5a$. Thus G must contain 14 and $5b$ where b is a vertex not shown in Figure 12. Therefore $G \in \mathcal{B}_5$.

If instead G contains $5a$, then $G \in \mathcal{B}_3$. Finally, we consider the case where G contains 25 but not $5a$. Considering N_x, we see that G must either contain the edge $3a$ (in which case $G \in \mathcal{B}_3$) or either $3b$ or $5b$ where where b is a vertex not shown in Figure 12. In either of these latter cases, $G \in \mathcal{B}_5$. $\square$

4.2 Non-link connected graphs

We now consider connected graphs which are not link connected. By definition, any such graph must have at least one edge e for which the non-adjacent subgraph N_e is not a connected graph. By Lemma 7, every such connected graph G with a two-dimensional Buchsbaum matching complex must contain an edge e such that N_e is a graph in Figure 13. Note for example that under our convention for pendants, each component of the graph in Figure 13(f) represents any star graph with at least two edges. Though P_2 (the path on two vertices) is also a star graph, it does not have a single center vertex and thus behaves somewhat differently from other star graphs, so we consider it separately.

The following lemma shows that any two edges with disconnected non-adjacent subgraphs must share a vertex.

Lemma 24. *Let G be a connected graph such that $M(G)$ is two-dimensional. If there exist edges $e, e' \in E$ such that both N_e and $N_{e'}$ are disconnected, then e and e' must share a vertex.*

Proof. Assume that N_e is disconnected with components G_1 and G_2. All edges of G that are not in one of these components must share a vertex with edge e. Without loss of generality, let e' be an edge in G_1.

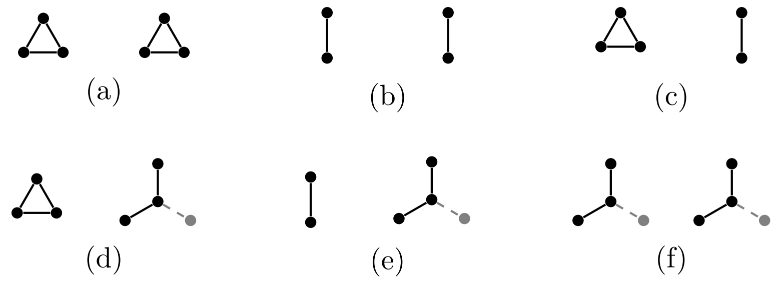
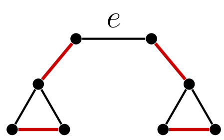
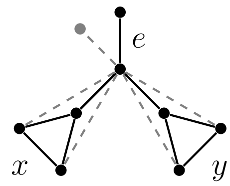


Figure 13: The possibilities for N_e disconnected.



(a) A subgraph of G in Proposition 25.



(b) A subgraph of G in Proposition 25.

Figure 14: Graphs appearing in Proposition 25.

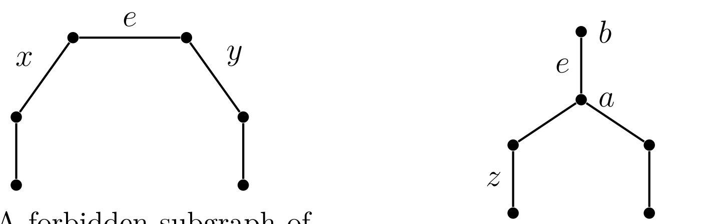
Since G is connected, there must be an edge connecting e and G_2. Thus e and G_2 form a connected subgraph of G, so the only way for $N_{e'}$ to be disconnected is for it to contain some edge $x \in G_1$ that is disjoint from e'. But this shows that $\{e, e', x, y\}$ is a 4-matching for any edge $y \in G_2$, which contradicts that $M(G)$ is two-dimensional. $\square$

We now consider each of the cases from Figure 13 in turn. As a reminder, G will always be a connected graph in the remainder of this section, and all additional edges of G must be adjacent to the edge e.

Proposition 25. *Assume G is connected but not link connected and $M(G)$ is a two-dimensional Buchsbaum matching complex. If G contains an edge e such that $N_e = K_3 \sqcup K_3$ (i.e., N_e is the graph in Figure 13(a)), then $G \in \mathcal{B}_7$.*

Proof. Since G is connected and all additional edges must touch e, each K_3 must be connected to e with an edge. If the two copies of K_3 are connected to e via different endpoints of e, as in Figure 14(a), then G will contain a 4-matching. Since all remaining edges of G must share a vertex with e, we therefore must have the graph depicted in Figure 14(b), and the other endpoint of e cannot connect to either K_3. Considering N_x and N_y, we see that each must be a member of $\mathcal{G}_2$. This completely determines G, so we see that $G \in \mathcal{B}_7$. $\square$

Proposition 26. *Assume G is connected but not link connected and $M(G)$ is a two-dimensional Buchsbaum matching complex. If G contains an edge e such that $N_e = P_2 \sqcup P_2$ (i.e., N_e is the graph in Figure 13(b)), then $G \in \mathcal{B}_P$.*



(a) A forbidden subgraph of G.

(b) A subgraph of G.

Figure 15: Graphs appearing in the proof of Proposition 26

Proof. Assume that the two copies of P_2 are attached to e via different endpoints of e. Call these two new edges x and y as in Figure 15(a). Since all additional edges of G are adjacent to e, there is no 3-matching containing x and y, thus $M(G)$ is not Buchsbaum. Thus, only one endpoint of e can be connected to the edges in N_e.

This gives the graph depicted in Figure 15(b). Since N_z must contain a path of length four, we must add an edge off vertex b. Pendants off vertex a would form a 4-matching and are thus not allowed. We may add any number of pendants off b, or, instead, we can add a single edge between a and the pendant edge connected to b to form a K_3. Similarly, we may connect up a to either edge in N_e. Thus $G \in \mathcal{B}_P$. $\square$

Proposition 27. *Assume G is connected but not link connected and $M(G)$ is a two-dimensional Buchsbaum matching complex. If G contains an edge e such that $N_e = K_3 \sqcup P_2$ (i.e., N_e is the graph in Figure 13(c)), then $G \in \mathcal{B}_7, \mathcal{B}_8, \mathcal{B}_{C_7}$, or G is one of the two exceptional graphs E_1 and E_2.*

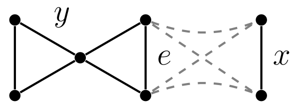
Proof. Let x be the isolated edge of N_e. By Lemma 24, N_x is connected and thus either $N_x = B$ or $N_x \in \mathcal{G}_2$ or $\mathcal{G}_3$. If $N_x \in \mathcal{G}_3$, then $G \in \mathcal{B}_{C_7}$ by Proposition 14.

If $N_x = B$, then the graph in Figure 16(a) is a subgraph of G and all other edges of G are adjacent to both e and x. By Lemma 24, N_y must be connected and thus have a path of length four. But this is impossible, so this case cannot happen.

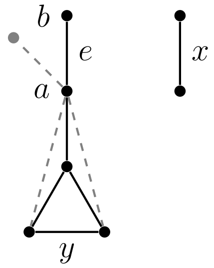
If instead $N_x \in \mathcal{G}_2$, then the graph in Figure 16(b) is a subgraph of G and all other edges of G are adjacent to both e and x or are of the form of the dotted edges in Figure 16(b). Again, N_y must be connected and have a path of length four, which forces the edge connecting vertex b and an endpoint of x to exist. Calling this edge z and considering N_z, at least one of the edges labeled p and q in Figure 16(c) must appear.

Assume the edge labeled p exists. If the edge between b and the other endpoint of x exists, then $G \in \mathcal{B}_7$. In not, the only possibility is to have $G \in \mathcal{B}_8$.

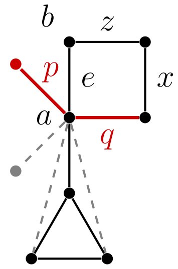
If the edge labeled p does not exist, then q must exist. Considering the other edges of the K_3 in N_z in turn, we see that the K_4 containing vertex a and this K_3 must be completed because each edge of this K_3 needs a path of length four in its non-adjacent subgraph. Furthermore, the other two edges between e and x can either both exist or neither exist. Thus G is either E_1 or E_2, the exceptional graphs in Figure 6. $\square$



(a) The case when $N_x = B$ in Proposition 27. We further consider N_y, which leads to a contradiction.



(b) The case when $N_x \in \mathcal{G}_2$ in Proposition 27. Considering N_y, we see that there must be an edge connecting b and an endpoint of x.



(c) The case when $N_x \in \mathcal{G}_2$ in Proposition 27. Considering N_z, least one of the red edges p and q must exist.

Figure 16: Graphs appearing in the proof of Proposition 27.

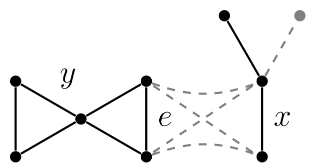
Proposition 28. *Assume G is connected but not link connected and $M(G)$ is a two-dimensional Buchsbaum matching complex. If G contains an edge e such that $N_e = K_3 \sqcup S_n$ with $n \geqslant 2$ (i.e., N_e is the graph in Figure 13(d)), then $G \in \mathcal{B}_8$.*

Proof. Let x be an edge of the star graph in N_e and consider N_x. Observe that $N_x \in \mathcal{G}_3$ is impossible by Proposition 14 and Lemma 16. Since N_x contains a K_3, either $N_x = B$ or $N_x \in \mathcal{G}_2$.

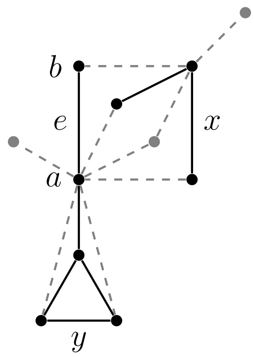
Assume $N_x = B$ and we will show that this is impossible. We know that G contains the graph in Figure 17(a). Observe that an edge connecting e to a non-central vertex of the star containing x produces a 4-matching and thus is forbidden. Instead if there is an edge connecting e and the central vertex of the star, we consider N_y, where y is the edge labeled in Figure 17(a). Observe that N_y is connected but cannot have a path of length four, so this situation is impossible.

Instead assume $N_x \in \mathcal{G}_2$. We claim that $G \in \mathcal{B}_8$. By assumption, G contains the graph Figure 17(b) as a subgraph, and we claim that any additional edge of G is of the form of a dotted edge in this figure. Observe that we cannot have an edge that connects b to a non-central vertex of the star containing x since this would give a 4-matching in G. Let y be the edge of K_3 labeled in Figure 17(b). Considering N_y, we see that there cannot be an edge that connects a to the central vertex of the star, since this would render N_y connected without a path of length four.

If G has an edge connecting a to a non-central vertex of the star containing x, then $G \in \mathcal{B}_8$. If G does not have such an edge, then G must have an edge connecting b to the



(a) The case when $N_x = B$ in Proposition 28.



(b) The case when $N_x \in \mathcal{G}_2$ in Proposition 28.

Figure 17: Graphs appearing in the proof of Proposition 28

center of this star. Call this new edge z. Considering N_z, we see that there must be a pendant off vertex a, so again we can conclude that $G \in \mathcal{B}_8$. $\square$

Proposition 29. *Assume G is connected but not link connected and $M(G)$ is a two-dimensional Buchsbaum matching complex. If G contains an edge e such that $N_e = P_2 \sqcup S_n$ with $n \geqslant 2$ (i.e., N_e is the graph in Figure 13(e)), then $G \in \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8, \mathcal{B}_9, \mathcal{B}_{C_7}$, or $\mathcal{B}_P$.*

Proof. Let x be the isolated edge in N_e, let c be the center of the S_n in N_e, and consider N_x. Since all edges of N_x besides e and the edges in the S_n are adjacent to e, $N_x = B$ is impossible, and if $N_x \in \mathcal{G}_3$, then $G \in \mathcal{B}_{C_7}$ by Proposition 14.

First assume that $N_x \in \mathcal{G}_2$. In this case, G contains the graph in Figure 18(a) as a subgraph, and e is one of the edges of the solid K_3 in this figure. All additional edges of G are of the form of one of the dotted edges or are between e and x.

If e is the vertical edge in the K_3 in Figure 18(a), we let y be the edge labeled in this figure and consider N_y. By Lemma 24, N_y is connected and thus must have a path of length four. However, this is impossible given the restrictions on G; thus we conclude that e is one of the other edges of the solid triangle in this figure. We must add either or both edges between vertex a and edge x to ensure that G is connected, and we may also add pendants off the center of the star S_n; no other edges are allowed. In either case, $G \in \mathcal{B}_P$ with one or two petals being K_3.

The last case to consider is when $N_x \in \mathcal{G}_1$. We will consider two cases: In the first e connects to the star in N_e via a non-central vertex (Figure 18(b)); in the second no such connections are allowed (Figure 18(c)). In each case, all remaining edges in N_x are of the form of a dotted edge in the respective figure. All other edges of G must be adjacent to both e and x.

Case 1: We have the case in Figure 18(b); i.e., the star containing c connects to edge e via at least one non-central vertex.

Consider N_y where y is the edge indicated in this figure. Note that N_y must be connected and thus have a path of length four. Thus G must have an edge connecting edge x and the vertex a (call this edge z) and also at least one of the red edges in Figure 18(d).

If $N_y \in \mathcal{G}_1$, then the only possible addition to G—apart from the edges in Figure 18(d)—is the edge which connects vertex b to the endpoint of x that is not an endpoint of z. Assume this edge exists and call it w. In this case G cannot contain the edge ac, since N_{ac} would be connected without a path of length four. If the red edge connecting b and a non-central vertex of the star exists then $G \in \mathcal{B}_2$, and if this edge does not exist then $G \in \mathcal{B}_6$. If instead G does not contain the edge w, then $G \in \mathcal{B}_6$. (To see this, consider N_z in Figure 18(d) to see we must have an additional edge off b or c.)

If instead $N_y \in \mathcal{G}_2$, then G must contain the edge that creates a triangle with edges x and z. Thus G contains the graph in Figure 18(e) and all other edges of G are of the form of the dotted or red edges in this figure. Assume the edge between a and c exists. Considering the non-adjacent subgraph for this edge, we see that no additional edges between e and x can exist. Considering N_z, we see that either b or c must have a pendant and thus $G \in \mathcal{B}_9$.

Assume instead that the edge between a and c does not exist. If the other two edges between e and x both exist, then either G is the exceptional graph E_2 or $G \in \mathcal{B}_8$. If G contains only one of the other possible edges between e and x we assume, without loss of generality, that it is the edge adjacent to z. Considering N_z shows G must have another edge off b or c (that doesn't attach to a). Thus $G \in \mathcal{B}_8$.

Case 2: We have the case in Figure 18(c); i.e., the star containing c is to connected to edge e only via the edge ac.

Consider N_y where y is the edge labeled in this figure. For the same reasons as the previous case, there must be an edge between x and vertex a, and all other edges of G are either between e and x or of the form of a dotted edge in this figure.

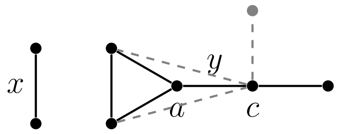
Assume $N_y \in \mathcal{G}_1$. The only way for N_y to contain a cycle in this case is to have edge w as in Figure 18(f). However, we see that N_w cannot contain a path of length four, so this edge cannot exist. Thus in this case G is the graph in Figure 18(c) with an additional edge from x to vertex a. So $G \in \mathcal{B}_P$ with all three petals being star graphs.

If instead $N_y \in \mathcal{G}_2$, then the edge between vertex a and the other endpoint of x must exist. Furthermore, neither of the other two edges connecting e and x can exist. Thus $G \in \mathcal{B}_P$ with one petal being a K_3. This completes the proof. □

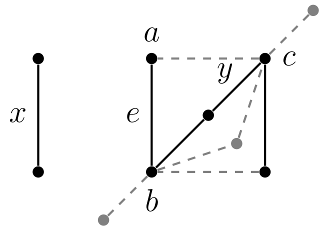
Proposition 30. *Assume G is connected but not link connected and $M(G)$ is a two-dimensional Buchsbaum matching complex. If G contains an edge e such that $N_e = S_m \sqcup S_n$ with $m, n \geqslant 2$ (i.e., N_e is the graph in Figure 13(f)), then $G \in \mathcal{B}_2, \mathcal{B}_5, \mathcal{B}_6$, or $\mathcal{B}_P$.*

Proof. Let c be the center vertex of S_m and d be the center vertex of S_n. Let x be any edge in S_m and consider N_x. Since all remaining edges of N_x are adjacent to e, $N_x = B$ is impossible. Similarly $N_x \in \mathcal{G}_3$ is impossible by Proposition 14 and Lemma 16 since G has at least eight vertices.

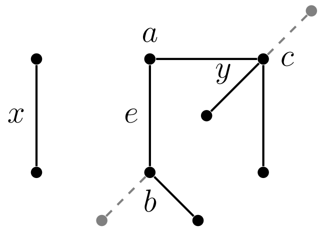
Assume that $N_x \in \mathcal{G}_2$. In this case, G contains the graph in Figure 19(a) as a subgraph, and e is one of the edges of the solid K_3 in this figure. All remaining edges of G are



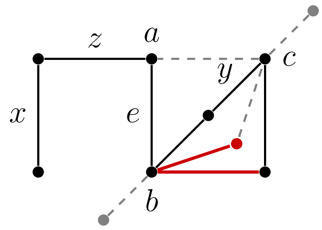
(a) The case where $N_x \in \mathcal{G}_2$.



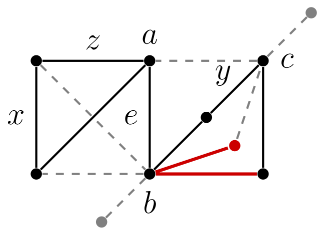
(b) The edge e connects to the star via a non-central vertex.



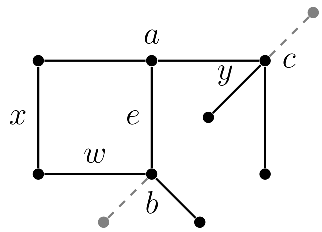
(c) The edge e connects to the star only via the central vertex.



(d) The graph G must contain at least one of the red edges.

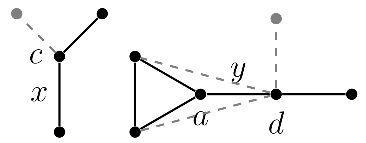


(e) All remaining edges are of the form of the dotted or red edges.

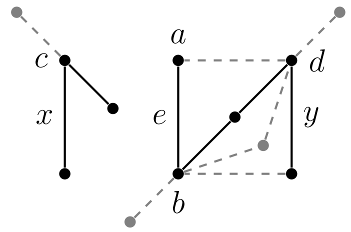


(f) The non-adjacent subgraph N_w leads to a contradiction.

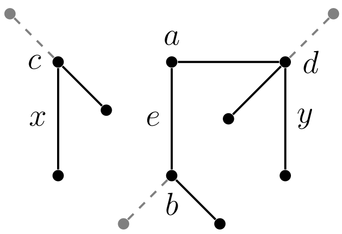
Figure 18: Graphs appearing in the proof of Proposition 29.



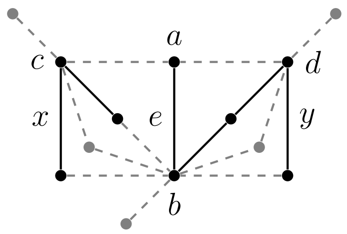
(a) The case where $N_x \in \mathcal{G}_2$.



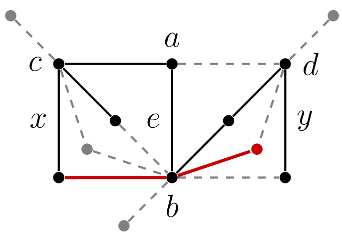
(b) The edge e connects to the star via a non-central vertex.



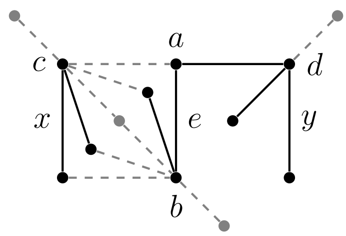
(c) The edge e connects to the star only via the central vertex.



(d) The only possible additional edges of G are dotted in this figure.



(e) The graph G must contain at least one of the red edges. The only possible additional edges of G are dotted in this figure.



(f) The only possible additional edges of G are dotted in this figure.

Figure 19: Graphs appearing in the proof of Proposition 30.

either of the form of the dotted edges in this figure or adjacent to both e and x. As in Proposition 29, we see that e cannot be the vertical edge in Figure 19(a) by considering N_y. Similarly, if e is a different edge of the triangle, then G must have an edge between vertices a and c. Furthermore, the only edges we can add to G are more pendants off these stars (and not the dotted edges completing the K_4 in Figure 19(a)), so $G \in \mathcal{B}_P$ with two pendants being stars and the other a K_3.

The only remaining possibility is to have $N_x \in \mathcal{G}_1$. Observe that e can connect to the star with center vertex d via a non-central vertex as in Figure 19(b) (and possibly also d) or only via the central vertex d as in Figure 19(c). All remaining edges in N_x have to be of the form of one of the dotted edges in these figures. All other edges of G must be adjacent to both e and x or to the vertex c.

Case 1: We have the case in Figure 19(b); i.e., the star containing d connects to edge e via at least one non-central vertex.

Consider N_y where y is the edge indicated in this figure. Note that N_y must be connected and thus have a path of length four. Observe that connecting b to c or connecting a to a non-central vertex of the star containing c both create a contradiction for N_y. Thus the only remaining edges in G must be of the form of the dotted edges in Figure 19(d).

There must be some edge connecting e and the star containing x. Assume the edge ac does not exist. Without loss of generality, this implies that the edge connecting b to x exists. If, for each star, there is at least one more edge connecting e to the star, then $G \in \mathcal{B}_2$. If not, then $G \in \mathcal{B}_6$.

Assume instead that the edge ac does exist. Considering N_{ac}, this forces the existence of an additional edge containing b that is not adjacent to edge y. These are indicated in red in Figure 19(e). If there are no edges besides ac connecting e and the star containing c, then $G \in \mathcal{B}_6$. If instead there exists an additional edge connecting b to the star containing c, we consider a few options. If the edge ad exists, then $G \in \mathcal{B}_5$. If ad does not exist but the edge between b and y exists, then $G \in \mathcal{B}_2$. If neither ad nor the edge between b and y exist, then $G \in \mathcal{B}_6$.

Case 2: We have the case in Figure 19(c); i.e., the star containing d is to connected to edge e only via the edge ad.

Let y be the edge indicated in this Figure 19(c). Considering N_y, we see that adding either the edge from b to c or the edge from a to a non-central vertex of the star containing c would create a contradiction.

Thus all additional edges in G must be of the form of the dotted edges in Figure 19(f). There must be some edge connecting e and the star containing x. If the edge between c and the end of the pendant off vertex b in Figure 19(f) exists, then an additional edge off b must exist, which shows that $G \in \mathcal{B}_6$. Similarly, if a different edge exists between b and a non-central vertex of this star, then $G \in \mathcal{B}_6$. If instead the only edge connecting e and the star containing x is the edge between vertices a and c, then $G \in \mathcal{B}_P$ (with all three petals being stars). This completes the proof. □

This completes our characterization of graphs with two-dimensional Buchsbaum matching complexes.

5 Concluding remarks

Outside of dimension two, all homology manifolds that arise as matching complexes are combinatorial spheres and balls [2]. However, in the Buchsbaum case, we do not expect higher dimensions to be as well behaved. Though a complete characterization is perhaps currently infeasible in general, it may be possible when restricted to certain families of graphs. For example, we can give an answer for complete bipartite graphs: If $G = K_{m,n}$ is a complete bipartite graph with $m \leqslant n$, then $M(G)$ is Buchsbaum if and only if $n \geqslant 2m-2$. This follows from [3, Theorem 15], which says that $M(K_{m,n})$ is Cohen–Macaulay if and only if $n \geqslant 2m - 1$.

We further note that [12, Theorem 2.3] shows that $M(K_{m,n})$ is vertex decomposable if and only if $n \geqslant 2m - 1$, so vertex decomposability, shellability, and Cohen–Macaulayness are equivalent for matching complexes of complete bipartite graphs. For other families of graphs, however, none of these properties hold in general. For example, if $n \geqslant 8$, then $M(K_n)$ is not Cohen–Macaulay (and thus not shellable or vertex decomposable). However, work has been done on the shellability and vertex decomposability of skeleta of these matching complexes [1, 9]. Thus it may be interesting to study Buchsbaumness of skeleta of matching complexes.

Lastly, we note that some other classes of simplicial complexes can be easily described in the context of matching complexes. Recall that a complex is a matroid indepedence complex (or, simply, a matroid) if every induced subcomplex is pure. Using this definition, it is straightforward to see that $M(G)$ is a matroid if and only if G does not contain a path of length three. Thus $M(G)$ is a matroid if and only if G is the disjoint union of star graphs and copies of K_3.

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Appendix

The following is Mathematica code for computing matching complexes and checking whether a matching complex is two-dimensional and Buchsbaum. The code also checks whether a Hamiltonian graph on 7 vertices has a Buchsbaum matching complex, which was used to generate the data in Table 1.

```
MatchingComplex[g_] := RelationGraph[DisjointQ, EdgeList@g]
(* Computes the 1-skeleton of the matching complex. (The full
   matching complex is the clique complex of this graph.) *)

VertexLink[g_,v_] := VertexDelete[NeighborhoodGraph[g,v], v]
```

```
(* Computes the link of v by finding its neighborhood and
   deleting v *)

TwoDBuchsbaumQ[g_] := (Length@First@FindClique@g == 3) &&
   AllTrue[VertexList@g, (ConnectedGraphQ@# && EdgeCount@#>0)&
   @ VertexLink[g,#]&]
(* Checks for 2D Buchsbaumness by checking that the largest
   clique has size 3 and all vertex links are connected graphs
   with at least one edge *)

c7andedges = Table[EdgeAdd[CycleGraph@7,#]& /@
   Subsets[EdgeList@GraphComplement@CycleGraph@7, {i}] //
   DeleteDuplicates[#,IsomorphicGraphQ]&, {i,0,14}];
(* Takes all subsets of edges in the complement of C7, grouped
   by size of the subset, and adds them to C7, then throws out
   all but one graph from each isomorphism class *)

c7buchsbaums = Select[#, TwoDBuchsbaumQ@*MatchingComplex]& /@
   c7andedges;
(* Picks out the graphs whose matching complex is 2D and
   Buchsbaum *)
```

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