Stability of Woodall's theorem and spectral conditions for large cycles

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Abstract

In the 1970s, Erdős asked how many edges are needed in a graph on n vertices, to ensure the existence of a cycle of length exactly $n-k$. In this paper, we consider the spectral analog of Erdős' problem. Indeed, the problem of determining tight spectral radius conditions for a cycle of length ℓ in a graph of order n for each $\ell \in [3, n]$ seems very difficult. We determine tight spectral radius conditions for C_ℓ where ℓ belongs to an interval of the form $[n - \Theta(\sqrt{n}), n]$. As a main tool, we prove a stability result of a theorem due to Woodall, which states that for a graph G of order $n \geqslant 2k+3$ where $k \geqslant 0$ is an integer, if $e(G) > \binom{n-k-1}{2} + \binom{k+2}{2}$ then G contains a C_ℓ for each $\ell \in [3, n-k]$. We prove a tight spectral condition for the circumference of a 2-connected graph with a given minimum degree, of which the main tool is a stability version of a 1976 conjecture of Woodall on the circumference of a 2-connected graph with a given minimum degree proved by Ma and the second author. We also give a brief survey on this area and point out where we are and our predicament.

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

Let $A(G)$ be the adjacency matrix of a graph G and $D(G)$ be the degree matrix of G. The *spectral radius* of G, denoted by $\rho(G)$, is the maximum of the moduli of all eigenvalues of $A(G)$. The *signless Laplacian spectral radius* of G, denoted by $q(G)$, is the largest eigenvalue of the signless Laplacian matrix $Q(G) := A(G) + D(G)$. Throughout this

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paper, the spectral radius condition mainly refers to the sufficient condition in terms of $\rho(G)$ or $q(G)$. We denote by $EX_{spec}(n, H)$ a family of spectral extremal graphs G of order n containing no H which attain the maximal spectral radius.

For two graphs G and G', we denote by $G \subseteq G'$ if G is a subgraph of G'. We denote I_k to be an *isolated set* of k vertices. Let G_1 and G_2 be two graphs. We use $G_1 \cup G_2$ to denote the *(disjoint) union* of G_1 and G_2, i.e., a new graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$, and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$, where $V(G_1) \cap V(G_2) = \emptyset$. We use $G_1 \vee G_2$ to denote the *join* of G_1 and G_2, that is, $G_1 \cup G_2$ together with all new edges xy where $x \in V(G_1)$ and $y \in V(G_2)$.

1.1 A brief survey on previous related results

In this paper, we make a new contribution to the following open problem whose solution seems to be difficult and a big project:

Problem 1. Determine tight spectral radius conditions for the existence of a cycle of length ℓ in a graph of order n for each $\ell \in [3, n]$.

Problem 1 is closely related to a huge of references in spectral graph theory. For small even values of ℓ, the case $\ell = 4$ was originally studied by Babai and Guiduli [1] and Nikiforov [29]. In particular, Nikiforov [29] proved $EX_{spec}(n, C_4) = \{K_1 \vee M_{\frac{n-1}{2}}\}$[1] (when n is odd) and conjectured that $EX_{spec}(n, C_4) = \{K_1 \vee (K_1 \cup M_{\frac{n-2}{2}})\}$ (when n is even) (see [31]). The even case of C_4 was confirmed by Zhai and Wang [43], and the case $\ell = 6$ was solved by Zhai and Lin [41]. In general small even values of ℓ, Nikiforov [32] conjectured that $EX_{spec}(n, C_{2k+2}) = \{S^+_{n,k}\}$ where $k \geqslant 2$ and n is sufficiently large n (related to k), and $S^+_{n,k}$ is obtained from $K_k \vee (n-k)K_1$ by adding a new edge between two isolated vertices in the independent set I_{n-k}. Only very recently, a complete solution to Nikiforov's conjecture was announced by Cioabă, Desai, and Tait [8]. However, in view of the techniques used in their proof [8], we cannot even say Problem 1 has been solved for all even integer ℓ in a range in the form of $[3, \Theta(\sqrt{n})]$. On the other hand, the spectral even cycle problem is quite different from the original even cycle problem in extremal graph theory (see lots of references in [17]). For large cycles, if $\ell = n$, Fiedler and Nikiforov [13] proved $EX_{spec}(n, C_n) = \{K_1 \vee (K_{n-2} \cup K_1)\}$; if $\ell = n - 1$, the solution is obtained by Ge and the author [19], who proved that $EX_{spec}(n, C_{n-1}) = \{K_1 \vee (K_{n-3} \cup K_2)\}$.

In this paper, we contribute to Problem 1 for the range $\ell \in [n - \Theta(\sqrt{n}), n]$. The whole picture and our predicament are depicted in our last section.

It is very natural to compare the above results with the extremal results of large cycles. In the 1970s, Erdős [11] asked how many edges are needed in a graph on n vertices, to ensure the existence of a cycle of length exactly $n-k$. Recall that Woodall [37] proved that for a graph G of order $n \geqslant 2k+3$ where $k \geqslant 0$ is an integer, if $e(G) \geqslant \binom{n-k-1}{2} + \binom{k+2}{2} + 1$, then G contains a C_ℓ for each $\ell \in [3, n-k]$. At almost the same time, some partial result was also obtained by Bondy [4].

[1]Here M_k means a matching of k edges.

For $n \geqslant c \geqslant 2k-1$, we set

$$L_{n,k} = K_1 \vee (K_{n-k-1} \cup K_k) \text{ and } W_{n,k,c} = K_k \vee (K_{c-2k+1} \cup (n-c+k-1)K_1).$$

The graph $L_{n,k+1}$ shows Woodall's theorem is sharp, which means that $ex(n, C_{n-k}) = \binom{n-k-1}{2} + \binom{k+2}{2}$ for $n \geqslant 2k+3$.

If we introduce minimum degree as a new parament, the situation becomes more complicated. For non-hamiltonian graphs, Erdős [10] proved the following result in 1962: For a graph G on n vertices with $\delta(G) \geqslant k$ where $1 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor$, if G is non-hamiltonian then

$$e(G) \leqslant \max\left\{\binom{n-k}{2} + k^2, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\}.$$

Note that the families of graphs $W_{n,k,n-1}$ and $W_{n,\lfloor \frac{n-1}{2} \rfloor, n-1}$ show that the upper bound in Erdős' theorem is sharp.

In 1977, Kopylov [25] determined a sharp edge condition for the circumference of a 2-connected graph, which generalized Erdős' theorem. Indeed, Kopylov [25] proved that: Let G be a 2-connected graph on n vertices. If $2 \leqslant c \leqslant n-1$ and $e(G) \geqslant \max\{e(W_{n,2,c}), e(W_{n,\lfloor \frac{c}{2} \rfloor, c})\}$, then $c(G) \geqslant c+1$, unless $G = W_{n,2,c}$ or $G = W_{n,\lfloor \frac{c}{2} \rfloor, c}$. Woodall [38] proposed a conjecture in 1976, which generalized Kopylov's theorem to a minimum degree version (see Conjecture 12 in Subsection 1.2). Ma and Ning [28] proved a stability version of Woodall's conjecture as follows. To understand Ma and Ning's theorem clearly, we need to introduce some notation (see [16, 28]).

Definition 2. ([16]) For $n > c$ where c is odd, let $\mathcal{X}_{n,c}$ be the family of graphs of order n such that a graph $G \in \mathcal{X}_{n,c}$ if and only if $V(G)$ has a partition $V(G) = A \cup B \cup X$ such that A is a clique, $|A| = \frac{c-1}{2}$, both B and X are independent sets, all vertices in A are adjacent to all vertices in B, and there exist two vertices $a \in A$ and $b \in B$ such that for any $x \in X$, $N_G(x) = \{a, b\}$.

Definition 3. ([16]) For $n > c$ where c is odd, let $\mathcal{Y}_{n,c}$ be the family of graphs of order n such that a graph $G \in \mathcal{Y}_{n,c}$ if and only if $V(G)$ has a partition $V(G) = A \cup B \cup Y$ such that A is a clique, $|A| = \frac{c-1}{2}$, B is an independent set, and every component of $G[Y]$ is a star $K_{1,r}$ with $r \geqslant 1$, all vertices in A are adjacent to all vertices in B, and there exist two vertices $a_1, a_2 \in A$ such that for every component H of $G[Y]$, $N_G(H) = \{a_1, a_2\}$ and if $|H| \geqslant 3$, then there exists a_i, $i = 1, 2$, such that $N_G(y) \cap A = \{a_i\}$ for all leaves y of H.

Definition 4. ([28]) For $n > c > 2k$ with $k-1 | n-c$, we set

$$Z_{n,k,c} = K_2 \vee \left(K_{c-k+1} \cup \left(\frac{n-c}{k-1} + 1\right) K_{k-1}\right).$$

Moreover, we set $\mathcal{Z}_{n,k,c} = \mathcal{X}_{n,c} \cup \mathcal{Y}_{n,c}$ if $k = 2$ and c is odd; $\mathcal{Z}_{n,k,c} = \{Z_{n,k,c}\}$ if $k \geqslant 3$ and $k-1 | n-c$, and $\mathcal{Z}_{n,k,c} = \emptyset$ otherwise.

The stability version of Woodall's conjecture is as follows, which generalized results mentioned in several papers [26, 14, 15]. Throughout this paper, we use $c(G)$ to denote the length of a longest cycle in a graph G, i.e., the circumference of G.

Theorem 5 (Ma and Ning [28])**.** *Let G be a 2-connected graph on n vertices with $\delta(G) \geqslant k$ and circumference $c(G) \leqslant c$, where $10 \leqslant c \leqslant n-1$. If*

$$e(G) > \max\left\{e(W_{n,k+1,c}), e(W_{n,\lfloor\frac{c}{2}\rfloor-1,c)}\right\},$$

then one of the following holds:
(a) $G \subseteq W_{n,k,c}$, or $G \subseteq W_{n,\lfloor\frac{c}{2}\rfloor,c}$;
(b) $k=2$, c is odd, and $G \subseteq H$ for some graph $H \in \mathcal{X}_{n,c} \cup \mathcal{Y}_{n,c}$; and
(c) $k \geqslant 3$, $k-1|n-c$, and $G \subseteq Z_{n,k,c}$.

Ma and Ning's theorem is a main tool for proving one of our theorems (i.e., Theorem 13).

1.2 Main results

An open problem on large cycles was given in [19] as follows:

Problem 6 ([19])**.** Let G be a connected graph of order n and $k \geqslant 1$ be an integer, where n is sufficiently large compared to k.
(a) Suppose that $\rho(G) > \rho(L_{n,k})$. Does G contain a C_{n-k+1}?
(b) Suppose that $q(G) > q(L_{n,k})$. Does G contain a C_{n-k+1}?

In this paper, we first give a positive answer (in stronger form) to Problem 6. When $k=2$, it implies one main theorem in [19].

Theorem 7. *Let $k \geqslant 1$ be an integer. Let G be a graph of order n. If either*
(a) $\rho(G) \geqslant \rho(L_{n,k})$ where $n \geqslant \max\{6k+11, \frac{(k+3)(k+4)}{2}\}$ or,
(b) $q(G) \geqslant q(L_{n,k})$ where $n \geqslant \max\{6k+11, k^2+2k+3\}$,
then G contains a C_ℓ for each integer $\ell \in [3, n-k+1]$ unless $G = L_{n,k}$.

In [37], Woodall determined Turán numbers of large cycles as follows.

Theorem 8 (Woodall [37])**.** *For a graph G of order $n \geqslant 2k+3$ where $k \geqslant 0$ is an integer, if $e(G) \geqslant \binom{n-k-1}{2} + \binom{k+2}{2} + 1$, then G contains a C_ℓ for each $\ell \in [3, n-k]$.*

Let us introduce some notation.

Definition 9. Let k and $n \geqslant 2k+1$ be integers. We define $\mathcal{L}_{n,k}$ to be a family of graphs, such that a graph $G \in \mathcal{L}_{n,k}$ if and only if G is a graph of order n in which there is a subgraph $K \cong K_{n-k}$, and for each component H of $G - V(K)$, $V(H)$ is a clique and all vertices in H are adjacent to a same vertex in K. (Notice that for distinct cliques of $G - V(K)$, the vertices sharing with the clique K may be different.) Specially, the graph $L_{n,k}$ is the one in $\mathcal{L}_{n,k}$ with the maximum number of edges.

As a main tool for proving Theorem 7, we shall prove a stability result of Theorem 8.

Theorem 10. *Let G be a graph of order $n \geqslant \max\{6k+17, \frac{(k+4)(k+5)}{2}\}$ where $k \geqslant 0$. If*

$$e(G) \geqslant e(L_{n,k+2}) = \binom{n-k-2}{2} + \binom{k+3}{2},$$

then G contains a cycle C_ℓ for each $\ell \in [3, c(G)]$. Suppose that G contains no C_{n-k}. Then one of the following holds:
(a) $G \subseteq L$ for some $L \in \mathcal{L}_{n,k+1}$;
(b) $G = L_{n,k+2} \cong K_1 \vee (K_{n-k-3} \cup K_{k+2})$;
(c) $k = 0$ and $G \subseteq W_{n,2,n-1} = K_2 \vee (K_{n-4} \cup 2K_1)$;
(d) $k = 1$ and $G = W_{n,2,n-2} = K_2 \vee (K_{n-5} \cup 3K_1)$.

Let G be a graph that is not a forest. The *girth* (*circumference*) of G, denoted by $g(G)$ ($c(G)$), is the length of a shortest (longest) cycle in G. We say that a graph is *weakly pancyclic* if it contains all cycles of lengths from $g(G)$ to $c(G)$. We also refine Woodall's Theorem (by determining the unique extremal graph).

Theorem 11. *Let G be a graph of order $n \geqslant \max\{6k+11, \frac{(k+3)(k+4)}{2}\}$ where $k \geqslant 0$. If*

$$e(G) \geqslant e(L_{n,k+1}) = \binom{n-k-1}{2} + \binom{k+2}{2},$$

then G is weakly pancyclic with girth 3. Furthermore, one of the following is true:
(a) G contains a C_ℓ for each $\ell \in [3, n-k]$;
(b) $G = L_{n,k+1} \cong K_1 \vee (K_{n-k-2} \cup K_{k+1})$.

In 1976, Woodall [38] proposed the following conjecture in the setting of 2-connected graphs with a given minimum degree.

Conjecture 12 (Woodall [38])**.** Let G be a 2-connected graph on n vertices with minimum degree $\delta(G) \geqslant k \geqslant 2$. If $2 \leqslant c \leqslant n-1$ and

$$e(G) \geqslant \max\{e(W_{n,k,c}), e(W_{n,\lfloor \frac{c}{2} \rfloor,c})\},$$

then $c(G) \geqslant c+1$, unless $G = W_{n,k,c}$ or $G = W_{n,\lfloor \frac{c}{2} \rfloor,c}$.

Lastly, we prove a sharp spectral condition for the circumference of 2-connected graphs, which is a partial spectral analog of Woodall's conjecture (i.e., Conjecture 12).

Theorem 13. *Let G be a 2-connected graph on n vertices with minimum degree $\delta(G) \geqslant k \geqslant 2$. If either*
(a) $\rho(G) \geqslant \rho(W_{n,k,c})$ where $n > c \geqslant \max\{\frac{5n+6k+5}{6}, n - \sqrt{2n} + \frac{3k}{4} + 3\}$, or
(b) $q(G) \geqslant q(W_{n,k,c})$ where $n > c \geqslant \max\{\frac{5n+6k+5}{6}, n - \frac{2}{3}\sqrt{3n} + \frac{2(2k+3)}{3}\}$,
then $c(G) \geqslant c+1$ unless $G = W_{n,k,c}$.

1.3 Organization

The paper is organized as follows. In Section 2, we give proofs of our main results. In Subsection 2.1, we firstly prove Theorem 10 and Theorem 11. After that, we prove Theorem 7 in Subsection 2.2. As a byproduct, we answer Problem 1 completely. In Subsection 2.3, we prove Theorem 13. In the last section, we summarize all the results related to Problem 1. We mention some unsolved problems related to cycles of given lengths or longest cycles in 2-connected graphs for further study.

2 Proofs

2.1 Proofs for structural results

The proof of Theorem 11 needs the following three lemmas. The *n-closure* $cl_n(G)$ of a graph G on n vertices is defined to be its one supergraph with order n, which is obtained by recursively joining any pair of nonadjacent vertices with degree sum at least n till there is no such pair. A graph G is *closed* if $G = cl_n(G)$ (i.e., every two nonadjacent vertices of G have a degree sum less than n).

Lemma 14 (Bondy and Chvátal [5]). *Let G be a graph of order n. Then $c(G) = c(cl_n(G))$.*

Lemma 15 (Bondy [3]). *Let G be a graph of order n. If $c(G) = c$ and $e(G) > \frac{c(2n-c)}{4}$, then G is weakly pancyclic with girth 3.*

For the third lemma, its original form in [26] needs the condition "$k \geqslant 1$". Here we prove the small case that $k = 0$. This lemma is the key tool for our proof. Denote by $\omega(G)$ the clique number of a graph G. Clearly $c(G) \geqslant \omega(G)$.

Lemma 16. *Let G be a closed graph of order $n \geqslant 6k + 5$ where $k \geqslant 0$. If*

$$e(G) > e(W_{n,k+1,n-1}) = \binom{n-k-1}{2} + (k+1)^2$$

then $\omega(G) \geqslant n - k$.

Proof. Recall that the case of $k \geqslant 1$ was proved in [26]. Let $k = 0$. Suppose that there exist two non-adjacent vertices $x, y \in V(G)$ such that $d(x) + d(y) \leqslant n - 1$. Let $H := G - \{x, y\}$. Then $e(G) \leqslant e(H) + d(x) + d(y) \leqslant \binom{n-2}{2} + n - 1 = \binom{n-1}{2} + 1$, a contradiction. Thus, for any two nonadjacent vertices, the degree sum of them is at least n. By the definition of n-closure, $G = K_n$ and so $\omega(G) = n$. □

We are now in stand for proving Theorem 11.

Proof of Theorem 11. Suppose that G is a graph satisfying the condition. We first show that G is weakly pancyclic with girth 3. Let $c := c(G)$. By Lemma 15, we only need to show that $\binom{n-k-1}{2} + \binom{k+2}{2} > \frac{c(2n-c)}{4}$. If not, then we have $\frac{nc}{2} - \frac{c^2}{4} \geqslant \frac{n^2-(2k+3)n}{2} + (k+1)(k+2)$, which implies $c^2 - 2nc + 2(n^2 - (2k+3)n) + 4(k+2)(k+1) \leqslant 0$. However, the discriminant of the quadratic form $\Delta = (2n)^2 - 4\left(2(n^2 - (2k+3)n) + 4(k+1)(k+2)\right) < 0$ for $n \geqslant 2k+5$, a contradiction. This proves the first part of the theorem.

Now let $G' = cl_n(G)$. Since

$$e(G') \geqslant e(G) \geqslant \binom{n-k-1}{2} + \binom{k+2}{2} \geqslant \binom{n-k-2}{2} + (k+2)^2 + 1$$

for $n \geqslant \frac{(k+3)(k+4)}{2}$, by Lemma 16, $\omega(G') \geqslant n-k-1$ when $n \geqslant \max\{6k+11, \frac{(k+3)(k+4)}{2}\}$. This implies that $c(G') \geqslant n-k-1$. If $c(G') \geqslant n-k$, then $c(G) = c(G') \geqslant n-k$ by Lemma 14. Recall that G is weakly pancyclic, implying (a) holds. So assume $c(G') \leqslant n-k-1$, implying that $c(G') = \omega(G') = n-k-1$.

Let S be a clique of G' with $|S| = n-k-1$, $K = G'[S]$ and $H = G' - S$. Thus K is complete. Let H_1 be a component of H. For any vertex $v \in V(H_1)$, if $|N_{G'}(v) \cap S| \geqslant 2$ then $c(G') \geqslant n-k$, a contradiction. Thus, $|N_{G'}(v) \cap S| \leqslant 1$ holds for every vertex $v \in V(H)$. Now

$$\begin{aligned} e(G) \leqslant e(G') &= e(K) + e(H) + e_{G'}(S, V(H)) \\ &\leqslant \binom{n-k-1}{2} + \binom{k+1}{2} + (k+1) \\ &= \binom{n-k-1}{2} + \binom{k+2}{2} \leqslant e(G). \end{aligned}$$

(Recall that $e(G) \geqslant \binom{n-k-1}{2} + \binom{k+2}{2}$.) We infer that every inequality above becomes equality. This implies that $G = G'$, H is complete, and $|N_G(v) \cap S| = 1$ for every $v \in V(H)$. Recall that $|N(H) \cap S| = 1$. All vertices in H have a common neighbor in S. Thus $G = L_{n,k+1}$, and so (b) holds. The proof is complete. ∎

We further prove a stability version of Woodall's theorem.

Proof of Theorem 10. The argument used here is similar to Theorem 11. However, more details are needed. We first claim that G is weakly pancyclic with girth 3. We showed above that if $e(G) \geqslant e(L_{n,k+1})$ and $n \geqslant 2k+5$, then G is weakly pancyclic with girth 3. Similarly, we can prove: if $e(G) \geqslant e(L_{n,k+2})$ and $n \geqslant 2k+7$, then G is weakly pancyclic with girth 3 as well.

Let $G' := cl_n(G)$. If $c(G') \geqslant n-k$, then by Lemma 14, $c(G) = c(G') \geqslant n-k$. Recall that G is weakly pancyclic, implying that G contains a C_{n-k}. So we assume that $c(G') \leqslant n-k-1$. Since

$$e(G') \geqslant e(G) \geqslant \binom{n-k-2}{2} + \binom{k+3}{2} \geqslant \binom{n-k-3}{2} + (k+3)^2 + 1$$

for $n \geqslant \frac{(k+4)(k+5)}{2}$, by Lemma 16, $\omega(G') \geqslant n-k-2$ when $n \geqslant 6k+17$. If $\omega(G') \geqslant n-k$. then $c(G') \geqslant \omega(G') \geqslant n-k$, a contradiction. Now we assume that $\omega(G') = n-k-2$ or $\omega(G') = n-k-1$. Let S be a clique of G' with $|S| = \omega(G')$, $K = G'[S]$ and $H = G' - S$.

Case A. $\omega(G') = n-k-1$. Let H_1 be a component of H. If $|N_{G'}(H_1) \cap S| \geqslant 2$, then $c(G') \geqslant n-k$ (recall that S is a clique of G'), a contradiction. Thus, every component

H_1 of $G' - S$ satisfies $|N_{G'}(H_1) \cap S| \leqslant 1$. It follows that $G \subseteq G' \subseteq L$ for some $L \in \mathcal{L}_{n,k+1}$, and (a) holds.

Case B. $\omega(G') = n-k-2$. Set $T = \{v \in V(H) : |N_{G'}(v) \cap S| \geqslant 2\}$. We distinguish the following subcases.

Case B.1. $|T| = 0$. In this case, $|N_{G'}(v) \cap S| \leqslant 1$ for every vertex $v \in V(H)$. Now

$$\begin{aligned} e(G) \leqslant e(G') &= e(K) + e(H) + e_{G'}(S, V(H)) \\ &\leqslant \binom{n-k-2}{2} + \binom{k+2}{2} + (k+2) \\ &= \binom{n-k-2}{2} + \binom{k+3}{2} \leqslant e(G). \end{aligned}$$

(Recall that $e(G) \geqslant \binom{n-k-2}{2} + \binom{k+3}{2}$.) We infer that each inequality becomes equality. This implies that $G = G'$, H is complete, and $|N_G(v) \cap S| = 1$ for every $v \in V(H)$. Recall that $|N(H) \cap S| = 1$. All vertices in H have a common neighbor in S. Thus $G = L_{n,k+1}$, and so (b) holds.

Case B.2. $|T| = 1$. Let v_1 be the unique vertex in T. Let H_1 be a component of $H - v_1$. If $v_1 \in N_{G'}(H_1)$, then $N_{G'}(H_1) \cap S = \emptyset$; for otherwise $c(G') \geqslant n-k$. Furthermore, If $|N_{G'}(H_1) \cap S| \geqslant 2$, then there are two independent edges between S and $V(H_1)$ (notice that in G', every vertex in H_1 has at most one neighbor in S), implying that $c(G') \geqslant n-k$, a contradiction. Thus, $|N_{G'}(H_1) \cap (S \cup \{v_1\})| \leqslant 1$ for every component H_1 of $G' - (S \cup \{v_1\})$. This implies that $G \subseteq G' \subseteq L$ for some $L \in \mathcal{L}_{n,k+1}$, and (a) holds.

Case B.3. $|T| \geqslant 2$. Let v_1 be a vertex in T and u_1, u_2 be two vertices in $N_{G'}(v_1) \cap S$. For any other vertex $v_2 \in T$, we have that $N_{G'}(v_2) \cap S = \{u_1, u_2\}$, for otherwise $c(G') \geqslant n-k$. Furthermore, $N_{G'}(v_1) \cap S = \{u_1, u_2\}$. In brief, we have $N_{G'}(T) \cap S = \{u_1, u_2\}$. If there are two vertices in T which are adjacent in G', then $c(G') \geqslant n-k$, a contradiction. Hence T is an independent set in G'. For any vertex $v \in V(G) \backslash (S \cup T)$, we claim that $|N_{G'}(v) \cap (S \cup T)| \leqslant 1$. Indeed, as $v \notin T$, v cannot have two neighbors in S. If $N_{G'}(v)$ contains two vertices in T or contains one vertex in T and one vertex in S, then $c(G') \geqslant n-k$, a contradiction. Set $H_1 = H - T$ and $t = |T|$. Notice that $2 \leqslant t \leqslant k+2$. Now

$$\begin{aligned} e(G) \leqslant e(G') &= e(K) + e_{G'}(S, T) + e(H_1) + e_{G'}(S \cup T, V(H_1)) \\ &\leqslant \binom{n-k-2}{2} + 2t + \binom{k+2-t}{2} + (k+2-t) \\ &= \binom{n-k-2}{2} + \binom{k+3}{2} + \frac{t^2 - (2k+1)t}{2} \\ &\leqslant e(G) + \frac{t(t-2k-1)}{2}. \end{aligned}$$

This implies that $t \geqslant 2k+1$. As $2 \leqslant t \leqslant k+2$, we only can have $k = 0$ and $t = 2$, or $k = 1$ and $t = 3$. For each case, $V(G) = S \cup T$. For the first case, $G \subseteq G' = W_{n,2,n-1}$, and (c) holds. For the second case $G' = W_{n,2,n-2}$. Moreover, equality holds in the above inequalities, implying that $G = G'$ and (d) holds. ∎

2.2 Proofs for spectral results

In the following, we prove the corresponding spectral part of this section. Let G be a graph and $u, v \in V(G)$. We use $G[u \to v]$ to denote a new graph obtained from G, by replacing all edges uw by vw, where $w \in N_G(u) \backslash (N_G(v) \cup \{v\})$. Following Brouwer et al.'s book [6], we call this "Kelmans operation".

We need some results on the spectral properties of graphs under Kelmans operations. These theorems will play important roles in our answer to Problem 1.

Theorem 17 (Csikvári [9])**.** *Let G be a graph. Let $u, v \in V(G)$ and $G' := G[u \to v]$. Then $\rho(G') \geqslant \rho(G)$.*

Theorem 18 (Li and Ning [26])**.** *Let G be a graph. Let $u, v \in V(G)$ and $G' := G[u \to v]$. Then $q(G') \geqslant q(G)$.*

For two variants of Theorems 17 and 18, see [39, 24].

The following theorem is very basic and famous.

Theorem 19. *Let G be a connected graph and G' be a proper subgraph of G. Then $\rho(G') < \rho(G)$ and $q(G') < q(G)$.*

The following spectral inequalities help us to invert our problems into ones in extremal style.

Theorem 20 (Hong [22])**.** *Let G be a graph on n vertices and m edges. If $\delta(G) \geqslant 1$ then $\rho(G) \leqslant \sqrt{2m - n + 1}$.*

Theorem 21 (Feng and Yu [12])**.** *Let G be a graph on n vertices and m edges. Then $q(G) \leqslant \frac{2m}{n-1} + n - 2$.*

The following two lemmas will be used to determine the extremal graphs.

Lemma 22. *Suppose that G is a subgraph of a member in $\mathcal{L}_{n,k}$ where $n \geqslant 2k + 1$.*
(a) If $\rho(G) \geqslant \rho(L_{n,k})$ then $G = L_{n,k}$.
(b) If $q(G) \geqslant q(L_{n,k})$ then $G = L_{n,k}$.

Proof. (a) Let $L \in \mathcal{L}_{n,k}$ with $G \subseteq L$. Since $G \subseteq L$, $\rho(G) \leqslant \rho(L)$ by Theorem 19, with equality if and only if $G = L$ (recall that L is connected). Let K be the clique of L with $|K| = n - k$. Let $H_1, H_2, \ldots, H_t$ be the components of $L - K$, and let v_i, $i \in [1, t]$, be the unique vertex in $N(H_i) \cap K$. By a series of Kelmans operations from v_i to v_1 for all $v_i \neq v_1$, we get a graph L' which is a subgraph of $L_{n,k}$. (In fact, the graph L' consists of some cliques sharing one common vertex.) By Theorem 17,

$$\rho(G) \leqslant \rho(L) \leqslant \rho(L') \leqslant \rho(L_{n,k}).$$

Notice that L' is always a subgraph of $L_{n,k}$. If L' is a proper subgraph of $L_{n,k}$, then $\rho(G) < \rho(L_{n,k})$. Thus, $L' = L_{n,k}$, specially we have $t = 1$ and $L = L'$. Furthermore, if $G \neq L$ then we also have $\rho(G) < \rho(L)$, a contradiction. Thus, $G = L = L_{n,k}$. This proves statement (a).

(b) The proof is almost the same as the one of (a). We just use Theorem 18 instead of Theorem 17 in the whole proof. We omit the details. $\square$

Lemma 23. *Let n, k be integers where $k \geqslant 1$. Then*
(a) $\rho(L_{n,k}) > \rho(L_{n,k+1})$ and $q(L_{n,k}) > q(L_{n,k+1})$, for all $n \geqslant 2k+3$;
(b) $\rho(L_{n,1}) > \rho(W_{n,2,n-1})$ and $q(L_{n,1}) > q(W_{n,2,n-1})$ for all $n \geqslant 6$;
(c) $\rho(L_{n,2}) > \rho(W_{n,2,n-2})$ for all $n \geqslant 7$ and $q(L_{n,2}) > q(W_{n,2,n-2})$ for all $n \geqslant 8$.

Proof. (a) Let $V(L_{n,k+1}) = X \cup Y \cup \{z\}$, where $X \cup \{z\}$ is the $(k+2)$-clique in $L_{n,k+1}$ and $Y \cup \{z\}$ is the $(n-k-1)$-clique in $L_{n,k+1}$. Choose $x \in X$. $L_{n,k}$ can be obtained from $L_{n,k+1}$ by deleting all edges xx' for $x' \in X$ and adding all edges xy' for $y' \in Y$.

Let $\boldsymbol{\alpha}$ be the Perron vector concerning $\rho_1 = \rho(L_{n,k+1})$, where $\alpha_x, \alpha_y, \alpha_z$ correspond to the eigencomponents of vertices in X, vertices in Y and the vertex z, respectively. By eigenequation, we have $\rho_1\alpha_x = k\alpha_x + \alpha_z$ and $\rho_1\alpha_y = (n-k-3)\alpha_y + \alpha_z$. It follows that $(\rho_1 - k)\alpha_x = (\rho_1 - (n-k-3))\alpha_y$. Since $n \geqslant 2k+3$, we have $\alpha_y \geqslant \alpha_x > 0$. By Rayleigh quotient, we have

$$\rho(L_{n,k}) - \rho(L_{n,k+1}) \geqslant 2(n-k-2)\alpha_x\alpha_y - 2k\alpha_x^2 = 2\alpha_x((n-k-2)\alpha_y - k\alpha_x) > 0.$$

This proves $\rho(L_{n,k}) > \rho(L_{n,k+1})$ for $n \geqslant 2k+3$.

Let $\boldsymbol{\beta}$ be the Perron vector with respect to $q_1 = q(L_{n,k+1})$, where $\beta_x, \beta_y, \beta_z$ correspond to the eigencomponents of vertices in X, vertices in Y and the vertex z, respectively. By eigenequation, we have $q_1 x = (2k+1)\beta_x + \beta_z$ and $q_1\beta_y = (2n-2k-5)\beta_y + \beta_z$. It follows that $(q_1 - (2k+1))\beta_x = (q_1 - (2n-2k-5))\beta_y$. If $n \geqslant 2k+3$, then $\beta_y \geqslant \beta_x > 0$. By Rayleigh quotient, we have

$$q(L_{n,k}) - q(L_{n,k+1}) \geqslant (n-k-2)(\beta_x + \beta_y)^2 - k(\beta_x + \beta_x)^2 > 0.$$

This proves $q(L_{n,k}) > q(L_{n,k+1})$ for $n \geqslant 2k+3$.

(b) By Theorems 20 and 21, for $n \geqslant 6$,

$$\rho(W_{n,2,n-1}) \leqslant \sqrt{2e(W_{n,2,n-1}) - n + 1} = \sqrt{n^2 - 6n + 15} < n - 2 < \rho(L_{n,1}),$$
$$q(W_{n,2,n-1}) \leqslant \frac{2e(W_{n,2,n-1})}{n-1} + n - 2 = \frac{2n^2 - 8n + 16}{n-1} \leqslant 2(n-2) < q(L_{n,1}).$$

(c) If $n \geqslant 8$, then by Theorem 20,

$$\rho(W_{n,2,n-2}) \leqslant \sqrt{2e(W_{n,2,n-2}) - n + 1} = \sqrt{n^2 - 8n + 25} \leqslant n - 3 < \rho(L_{n,2}).$$

If $n \geqslant 10$, then by Theorem 21,

$$q(W_{n,2,n-2}) \leqslant \frac{2e(W_{n,2,n-2})}{n-1} + n - 2 = \frac{2n^2 - 10n + 26}{n-1} \leqslant 2(n-3) < q(L_{n,2}).$$

For the remainder case that n is small, one can solve it by computer (see Table 1). ☐

Proof of Theorem 7. If G is disconnected, then we can add some edges between different components recursively, and get a connected graph G' with $\rho(G') > \rho(G)$ and

Table 1: Special radii of $W_{n,2,n-2}$ and $L_{n,2}$ with $n \in [6,9]$

Graphs	$W_{6,2,4}$	$L_{6,2}$	$W_{7,2,5}$	$L_{7,2}$	$W_{8,2,6}$	$L_{8,2}$	$W_{9,2,7}$	$L_{9,2}$
$\rho(\cdot)$	3.3723	3.2618	3.9095	4.1413	4.6056	5.0874	5.4172	6.0593
$q(\cdot)$	7.4641	7.0000	8.7355	8.7016	10.5311	12.4244	11.7492	12.4389

$q(G') > q(G)$. Since the added edges are not contained in any cycle, if G' contains some cycles, then so does G. Thus we need only deal with the case that G is connected.

Suppose that (a) holds. Furthermore, suppose that G does not contain a C_ℓ for every $\ell \in [3, n-k+1]$. We shall show that $G = L_{n,k}$.

By Theorem 20, we have

$$\sqrt{2e(G) - n + 1} \geqslant \rho(G) \geqslant \rho(L_{n,k}) \geqslant n - k - 1.$$

It follows that

$$e(G) \geqslant \frac{(n-k-1)^2 + n - 1}{2} \geqslant \binom{n-k-1}{2} + \binom{k+2}{2}$$

for $n \geqslant \frac{(k+2)^2}{2}$. By Theorem 10, G is weakly pancyclic with girth 3 for $n \geqslant \max\{6k + 11, \frac{(k+3)(k+4)}{2}\}$. Furthermore, if G does not contain a C_{n-k+1}, then one of the following is true: (1) $G \subseteq L$ for some $L \in \mathcal{L}_{n,k}$; (2) $G = L_{n,k+1}$; (3) $k = 1$ and $G \subseteq W_{n,2,n-1}$, or $k = 2$ and $G \subseteq W_{n,2,n-2}$. By Lemma 22 and Lemma 23, $G = L_{n,k}$.

Suppose that (b) holds. By Theorem 21, we obtain

$$\frac{2e(G)}{n-1} + n - 2 \geqslant q(G) \geqslant q(L_{n,k}) \geqslant 2(n-k-1),$$

which implies that

$$e(G) \geqslant \frac{(2(n-k-1) - n + 2)(n-1)}{2} \geqslant \binom{n-k-1}{2} + \binom{k+2}{2}$$

for $n \geqslant k^2 + 2k + 2$. By Theorem 10, G is weakly pancyclic with girth 3 for $n \geqslant \max\{6k+11, \frac{(k+3)(k+4)}{2}\}$. Notice that $\frac{(k+3)(k+4)}{2} \leqslant \max\{6k+11, k^2+2k+2\}$. Furthermore, if G does not contain a C_{n-k+1}, then one of the following is true: (1) $G \subseteq L$ for some $L \in \mathcal{L}_{n,k}$; (2) $G = L_{n,k+1}$; (3) $k = 1$ and $G \subseteq W_{n,2,n-1}$, or $k = 2$ and $G \subseteq W_{n,2,n-2}$. By Lemma 22 and Lemma 23, $G = L_{n,k}$. The proof is complete. ∎

2.3 Proof of Theorem 13

The goal of this section is to prove Theorem 13. One useful tool is a classical spectral inequality by Hong, Shu, and Fang [23].

Theorem 24 (Hong, Shu and Fang [23]). *Let G be a connected graph on n vertices and m edges. If $\delta(G) \geqslant k \geqslant 1$, then*

$$\rho(G) \leqslant \frac{k-1}{2} + \sqrt{2m - kn + \frac{(k+1)^2}{4}}.$$

The next spectral inequality was originally proposed by Guo, Wang, and Li [21] as a conjecture and proved by Sun and Das [36].

Theorem 25 (Sun and Das [36]). *Let G be a graph with minimum degree $\delta(G) \geqslant 1$. For any $v \in V(G)$, we have $\rho^2(G-v) \geqslant \rho^2(G) - 2d(v) + 1$.*

Lemma 26. *Let G be a connected graph and H be an induced subgraph of G. Suppose that $|V(G)\backslash V(H)| = k$ and $e(G) - e(H) = m$. Then $\rho^2(H) \geqslant \rho^2(G) - 2m + k$.*

Proof. We remark that in Theorem 25, the condition $\delta(G) \geqslant 1$ can be replaced by that $d_G(v) \geqslant 1$ and then also ensures $\rho^2(G-v) \geqslant \rho^2(G) - 2d(v) + 1$. If G has some isolated vertices, then taking $G' = G - V_0$, where V_0 is the set of isolated vertices of G, and we have $\rho(G) = \rho(G')$ and $\rho(G-v) = \rho(G'-v)$.

Now consider the connected graph G and an induced subgraph H. Let $H_i : 1 \leqslant i \leqslant t$ be the components of $G-H$, u_i be a vertex in $N(H) \cap V(H_i)$, and T_i be a spanning tree of H_i rooted at u_i. Now H can be obtained from G by removing vertices one by one, such that each time we remove a leaf of some tree $T_i' \subseteq T_i$, which is never isolated. Now applying the variant of Theorem 25 we discussed above, we have shown $\rho^2(H) \geqslant \rho^2(G) - 2m + k$. $\square$

Next, we present a lemma involving pure computation, which is used in proofs of Lemma 28 and Theorem 13.

Lemma 27. *(1) If $n \geqslant 2k+1 \geqslant 1$ and $p \geqslant \frac{n}{3} + k - \frac{1}{2}$, then*

$$\frac{p-1}{2} + \sqrt{pn - \frac{3p^2+2p-1}{4}} \leqslant 2p - k.$$

(2) If $n \geqslant 2k+1 \geqslant 1$ and $p \geqslant \frac{4n+k-5}{10}$, then

$$\frac{n}{2} + p - 1 + \sqrt{\left(\frac{n}{2} + p - 1\right)^2 - 2(p^2 - p)} \leqslant 4p - 2k.$$

(3) If $0 \leqslant n - c \leqslant \sqrt{2n} - \frac{3k}{4} - 3$, then

$$\frac{1}{2}\left(kn + \left(c - \frac{3k-1}{2}\right)^2 - \frac{(k+1)^2}{4}\right) \geqslant \binom{c-k-1}{2} + (n-c+k+1)^2.$$

(4) If $n \geqslant 12$ and $0 \leqslant n - c \leqslant \frac{2}{3}\sqrt{3n} - k - 1$, then

$$\frac{(2(c-k)-(n-2))(n-1)}{2} \geqslant \binom{c-k-1}{2} + (n-c+k+1)^2.$$

Proof. (1) The inequality is equal to

$$f(p) := 3p^2 - (n+3k-2)p + k(k-1) \geqslant 0.$$

One can compute that $f(\frac{n}{3}+k-\frac{1}{2}) = \frac{n}{6} - \frac{k}{2} + k^2 - \frac{1}{4} \geqslant 0$, and $f(p)$ is increasing when $p \geqslant \frac{n}{3}+k-\frac{1}{2}$. This proves the inequality.

(2) The inequality is equal to

$$f(p) := 5p^2 - (2n+6k-3)p + k(n+2k-2) \geqslant 0.$$

One can compute that $f(\frac{4n+k-5}{10}) = \frac{n}{5} + \frac{7k^2}{4} - \frac{7k}{10} - \frac{1}{4} \geqslant 0$, and $f(p)$ is increasing when $p \geqslant \frac{4n+k-5}{10}$. This proves the inequality.

(3) The inequality is equal to

$$f(n-c) := (n-c)^2 + \left(\frac{3k}{2}+4\right)(n-c) - \left(2n - \frac{k^2+9k+4}{2}\right) \leqslant 0.$$

When $n \geqslant \frac{k^2+9k+4}{4}$, the quadratic equation $f(n-c)=0$ for $n-c$ has two real roots, one of which is non-positive. Moreover, one can obtain that $f(\sqrt{2n} - \frac{3k}{4} - 3) = -2\sqrt{2n} - \frac{k^2}{16} + \frac{3k}{2} - 1 \leqslant 0$. Thus $f(n-c) \leqslant 0$ when $0 \leqslant n-c \leqslant \sqrt{2n} - \frac{3k}{4} - 3$. Notice that $\sqrt{2n} - \frac{3k}{4} - 3 \geqslant 0$ implies $n \geqslant \frac{k^2+9k+4}{4}$. This proves the inequality.

(4) The inequality is equal to

$$f(n-c) := (n-c)^2 + \left(2k+\frac{5}{3}\right)(n-c) - \left(\frac{4n}{3} - \frac{3k^2+5k+6}{3}\right) \leqslant 0.$$

When $n \geqslant \frac{3k^2+5k+6}{4}$, the quadratic equation $f(n-c)=0$ for $n-c$ has two real roots, one of which is non-positive. Moreover, one can compute that $f(\frac{2}{3}\sqrt{3n}-k-1) = -\frac{2}{9}\sqrt{3n}+\frac{4}{3} \leqslant 0$ when $n \geqslant 12$. Thus $f(n-c) \leqslant 0$ when $0 \leqslant n-c \leqslant \frac{2}{3}\sqrt{3n} - k - 1$. Notice that $\frac{2}{3}\sqrt{3n} - k - 1 \geqslant 0$ implies $n \geqslant \frac{3k^2+5k+6}{4}$. This proves the inequality. $\square$

Lemma 28. *Let $n > c \geqslant 2k+1$, where n, c, k are positive integers.*
(a) If $c \geqslant \frac{2n}{3}+k$, then $\rho(W_{n,\lfloor \frac{c}{2} \rfloor,c}) < \rho(W_{n,k,c})$; if $c \geqslant \frac{4n+k}{5}$, then $q(W_{n,\lfloor \frac{c}{2} \rfloor,c}) < q(W_{n,k,c})$.
(b) For any graph $G \in \mathcal{X}_{n,c}$, we have $\rho(G) < \rho(W_{n,\lfloor \frac{c}{2} \rfloor,c})$ and $q(G) < q(W_{n,\lfloor \frac{c}{2} \rfloor,c})$.
(c) For any graph $G \in \mathcal{Y}_{n,c}$, we have $\rho(G) < \rho(W_{n,\lfloor \frac{c}{2} \rfloor,c})$ and $q(G) < q(W_{n,\lfloor \frac{c}{2} \rfloor,c})$.
(d) We have $\rho(Z_{n,k,c}) < \rho(W_{n,k,c})$ and $q(Z_{n,k,c}) < q(W_{n,k,c})$.

Proof. (a) Set $p = \lfloor \frac{c}{2} \rfloor$. If c is even, then $W_{n,\lfloor \frac{c}{2} \rfloor,c} = S_{n,p}$. By simple algebraic operations,

$$\rho(W_{n,\lfloor \frac{c}{2} \rfloor,c}) = \rho(S_{n,p}) = \frac{p-1}{2} + \sqrt{pn - \frac{3p^2+2p-1}{4}}.$$

On the other hand, $\rho(W_{n,k,c}) > c-k = 2p-k$. To prove that $\rho(W_{n,\lfloor \frac{c}{2} \rfloor,c}) < \rho(W_{n,k,c})$, we only need to prove that

$$\frac{p-1}{2} + \sqrt{pn - \frac{3p^2+2p-1}{4}} \leqslant 2p-k.$$

If c is odd, then $W_{n,\lfloor\frac{c}{2}\rfloor,c} = S^+_{n,p}$, and

$$\rho(W_{n,\lfloor\frac{c}{2}\rfloor,c}) \leqslant \rho(S_{n,p}) + 1 \leqslant \frac{p+1}{2} + \sqrt{pn - \frac{3p^2+2p-1}{4}}.$$

On the other hand, $\rho(W_{n,k,c}) > c - k = 2p - k + 1$. To prove that $\rho(W_{n,\lfloor\frac{c}{2}\rfloor,c}) < \rho(W_{n,k,c})$, we only need to prove that

$$\frac{p+1}{2} + \sqrt{pn - \frac{3p^2+2p-1}{4}} \leqslant 2p - k + 1.$$

Both of the inequalities hold when $p \geqslant \frac{n}{3} + k - \frac{1}{2}$ (see Lemma 27). Thus if $c \geqslant \frac{2n}{3} + 2k$, then $\rho(W_{n,\lfloor\frac{c}{2}\rfloor,c}) < \rho(W_{n,k,c})$.

Now we show that $q(W_{n,\lfloor\frac{c}{2}\rfloor,c}) < q(W_{n,k,c})$. If c is even, then by Lemma 27,

$$q(W_{n,\lfloor\frac{c}{2}\rfloor,c}) = q(S_{n,p}) = \frac{n}{2} + p - 1 + \sqrt{\left(\frac{n}{2} + p - 1\right)^2 - 2(p^2 - p)}.$$

On the other hand, $q(W_{n,k,c}) > 2c - 2k = 4p - 2k$. To prove that $q(W_{n,\lfloor\frac{c}{2}\rfloor,c}) < q(W_{n,k,c})$, we only need to prove that

$$\frac{n}{2} + p - 1 + \sqrt{\left(\frac{n}{2} + p - 1\right)^2 - 2(p^2 - p)} \leqslant 4p - 2k.$$

If c is odd, then $W_{n,\lfloor\frac{c}{2}\rfloor,c} = S^+_{n,p}$, and

$$q(W_{n,\lfloor\frac{c}{2}\rfloor,c}) \leqslant q(S_{n,p}) + 2 \leqslant \frac{n}{2} + p + 1 + \sqrt{\left(\frac{n}{2} + p - 1\right)^2 - 2(p^2 - p)}.$$

On the other hand, $q(W_{n,k,c}) > 2c-2k = 4p-2k+2$. To prove that $q(W_{n,\lfloor\frac{c}{2}\rfloor,c}) < q(W_{n,k,c})$, we only need to prove that

$$\frac{n}{2} + p + 1 + \sqrt{\left(\frac{n}{2} + p - 1\right)^2 - 2(p^2 - p)} \leqslant 4p - 2k + 2.$$

Both of inequalities hold when $p \geqslant \frac{4n+k-5}{10}$ (see Lemma 27). Thus if $c \geqslant \frac{4n+k}{5}$, then $q(W_{n,\lfloor\frac{c}{2}\rfloor,c}) < q(W_{n,k,c})$.

(b) Let $A, B, X \subseteq V(G)$, and $a \in A, b \in B$, as given in Definition 2 (see Subsection 1.1). Let $a' \in A\backslash\{a\}$, and let $G' = G[b \to a']$. By Theorems 17 and 18, $\rho(G') \geqslant \rho(G)$ and $q(G') \geqslant q(G)$. Observe that G' is a proper subgraph of $W_{n,\lfloor\frac{c}{2}\rfloor,c}$. Hence $\rho(G) < \rho(W_{n,\lfloor\frac{c}{2}\rfloor,c})$ and $q(G) < q(W_{n,\lfloor\frac{c}{2}\rfloor,c})$. The proof of (b) is complete.

(c) Let $A, B, Y \subseteq V(G)$, and $a_1, a_2 \in A$, as given in Definition 3 (see Subsection 1.1). Let $a' \in A\backslash\{a_1, a_2\}$. Recall that each component of $G[Y]$ is a star. Let $y_1, \ldots, y_r$ be the centers of these stars (for a K_2-component, take any one of its two vertices as a center). Let G' be the graph obtained from G by Kelmans operations from y_i to a', $i = 1, \ldots, r$.

Observe that G' is a proper subgraph of $W_{n,\lfloor\frac{c}{2}\rfloor,c}$. By the same analysis as (b), we can prove (c).

(d) Set $G \cong Z_{n,k,c}$. Let A be the $(c-k+1)$-clique, and $Z_1, Z_2, \ldots, Z_r$, where $r = \frac{n-(c-k+1)}{k-1}$, be the $(k+1)$-cliques of G as given in Definition 4 (see Subsection 1.1). Let x, x' be the two common vertices of these cliques. Choose $A' = \{a_1, a_2, \ldots, a_{k-2}\} \subseteq A \backslash \{x, x'\}$, and for each $j = 1, \ldots, r$, $Z'_j = \{z_1^j, z_2^j, \ldots, z_{k-2}^j\} \subseteq Z_j \backslash \{x, x'\}$. Let G' be the graph obtained from G by Kelmans operations from z_i^j to a_i, $i = 1, \ldots, k-2$, $j = 1, \ldots, r$. Observe that G' is a proper subgraph of $W_{n,\lfloor\frac{c}{2}\rfloor,c}$. By the same analysis as (b), we can prove (d). □

Proof of Theorem 13. Let $G' = cl_n(G)$. We will first show that under the condition, $\omega(G') \geqslant c-k$. Suppose that (a) holds. By Theorem 24, we have

$$\frac{k-1}{2} + \sqrt{2e(G') - kn + \frac{(k+1)^2}{4}} \geqslant \rho(G') \geqslant \rho(G) \geqslant \rho(W_{n,k,c}) > c-k.$$

We infer that

$$e(G') > \frac{1}{2}\left(\left(c - \frac{3k-1}{2}\right)^2 - \frac{(k+1)^2}{4} + kn\right) \geqslant \binom{c-k-1}{2} + (n-c+k+1)^2,$$

for $0 \leqslant n-c \leqslant \sqrt{2n} - \frac{3k}{4} - 3$ (see Lemma 27 (3)). By Lemma 16, $\omega(G') \geqslant c-k$ when $n \geqslant 6(n-c+k)+5$.

Suppose now that (b) holds. By Theorem 21, we have

$$\frac{2e(G')}{n-1} + n - 2 \geqslant q(G') \geqslant q(G) \geqslant q(W_{n,k,c}) > 2(c-k).$$

We infer that

$$e(G') > \frac{(2(c-k)-(n-2))(n-1)}{2} \geqslant \binom{c-k-1}{2} + (n-c+k+1)^2,$$

for $n \geqslant 12$ and $0 \leqslant n-c \leqslant \frac{2}{3}\sqrt{3n} - k - 1$ (see Lemma 27 (4)). By Lemma 16, $\omega(G') \geqslant c-k$ when $n \geqslant 6(n-c+k)+5$. Notice that $n \geqslant 6(n-c+k)+5$ implies $n \geqslant 12$. In each case, we have $\omega(G') \geqslant c-k$, as claimed.

We next show that $e(G') > e(W_{n,k+1,c})$. Let S be a maximum clique of G' and $H = G'[S]$. Suppose first that $\omega(G') \geqslant c-k+1$. Since $\delta(G') \geqslant \delta(G) \geqslant k$, we have

$$\begin{aligned} e(G') &> e(H) + \frac{1}{2}\sum_{v \in V(G')\backslash S} d_{G'}(v) \\ &\geqslant \binom{c-k+1}{2} + \frac{k(n-c+k-1)}{2} \\ &\geqslant \binom{c-k}{2} + (k+1)(n-c+k) = e(W_{n,k+1,c}), \end{aligned}$$

for $n-c \leqslant \frac{2n-(k^2+5k)}{k+4}$. Recall that there holds $0 \leqslant n-c \leqslant \frac{2}{3}\sqrt{3n}-k-1$. We only need to prove that $\frac{2}{3}\sqrt{3n}-k-1 \leqslant \frac{2n-(k^2+5k)}{k+4}$, which is equivalent to

$$3(n+2)^2 \geqslant n(k+4)^2. \tag{1}$$

Since $n > n-\frac{2}{3}\sqrt{3n}+k+1$ (i.e., the condition for the case (b)), we have $\frac{2}{3}\sqrt{3n} > k+1$, that is, $n \geqslant \frac{3(k+1)^2+1}{4}$. If we have $3n^2 \geqslant n(k+4)^2$ (i.e., $n \geqslant \frac{(k+4)^2}{3}$), then (1) holds. Notice that if $k \geqslant 5$, then $\frac{3(k+1)^2+1}{4} \geqslant \frac{(k+4)^2}{3}$, which proves (1) for the case $k \geqslant 5$. Next, we shall deal with the tiny cases that $k = 2, 3, 4$. When $k = 2$, (1) is equivalent to $n^2-8n+4 \geqslant 0$, which obviously holds (recall that $n \geqslant 12$); when $k = 3$, (1) is equivalent to $n(3n-37)+12 \geqslant 0$, which holds for $n \geqslant 12$; when $k = 4$, (1) is equivalent to $3n^2-52n+12 \geqslant 0$. Observe that $n > n-\frac{2}{3}\sqrt{3n}+k+1$ implies that $n \geqslant \frac{3(k+1)^2}{4} = \frac{75}{4}$, and it follows that (1) holds.

Suppose now that $\omega(G') = c-k$. By Lemma 26, $\rho^2(H) \geqslant \rho^2(G') - 2(e(G')-e(H)) + |V(G)\backslash S|$. Thus we have

$$\begin{aligned}
e(G') &\geqslant \frac{1}{2}(\rho^2(G') - \rho^2(H) + 2e(H) + |V(G)\backslash S|) \\
&> \frac{(c-k)^2 - (c-k-1)^2 + (c-k)(c-k-1) + (n-c+k)}{2} \\
&= \frac{(c-k)^2+n-1}{2} \\
&\geqslant \binom{c-k}{2} + (k+1)(n-c+k) = e(W_{n,k+1,c})
\end{aligned}$$

for $n-c \leqslant \frac{2n-(2k^2+3k+1)}{2k+3}$ (which holds when $0 \leqslant n-c \leqslant \frac{2}{3}\sqrt{3n} - \frac{2(2k+3)}{3}\}$).

If $c(G) \leqslant c$, then $c(G') = c(G) \leqslant c$. Since $f(n,k,c) = \binom{c-k+1}{2} + k(n-c+k-1)$, we have $\frac{df}{dk} = n-2c+3k-\frac{3}{2}$. As $c \geqslant \frac{5n+6k+5}{6}$, we have $f'(k) \leqslant n - \frac{5n+6k+5}{3} + 3k - \frac{3}{2} = -\frac{2n}{3} + k - \frac{19}{6}$. Furthermore, by condition, we obtain $n \geqslant \frac{(2k+3)^2}{3}$, and hence $f'(k) \leqslant -\frac{2(2k+3)^2}{9} + k - \frac{19}{6} \leqslant 0$. This implies that $e(W_{n,k+1,c}) \geqslant e(W_{n,\lfloor\frac{c}{2}-1\rfloor,n})$.

By Theorem 5, G' (and G) is a subgraph of some graph in $\{W_{n,k,c}, W_{n,\lfloor\frac{c}{2}\rfloor,n}\} \cup \mathcal{Z}_{n,k,c}$. By Lemma 28, G can only be $W_{n,k,c}$. The proof is complete. ∎

3 Concluding remarks

Recall Nikiforov [32] conjectured that (a) every graph on sufficiently large order n contains a C_{2k+1} or a C_{2k+2} if $\rho(G) \geqslant \rho(S_{n,k})$, unless $G = S_{n,k}$ where $S_{n,k} := K_k \vee (n-k)K_1$, and that (b) every graph on sufficiently large order n contains a C_{2k+2} if $\rho(G) \geqslant \rho(S^+_{n,k})$, unless $G = S^+_{n,k}$ where $S^+_{n,k}$ is obtained from $S_{n,k}$ by adding an edge in the $n-k$ isolated vertices. One can easily compute that $\rho(S_{n,k}) = \Theta(\sqrt{n})$ and $\rho(S^+_{n,k}) = \Theta(\sqrt{n})$. The following refined version of Nikiforov's conjecture is helpful to Problem 1.

Problem 29 (A refined version of Nikiforov's conjecture). For any integer $k \geqslant 3$, determine the infimum $\alpha := \alpha(k)$ such that every graph of order $n = \Omega(k^\alpha)$ (where $\Omega(k^\alpha)$ means there exists some constant c which is not related to k and n, such that $n \geqslant ck^\alpha$) satisfying $\lambda(G) > \lambda(S^+_{n,k})$ contains a C_{2k+2}.

If these conjectures turned out to be true (in a clear form), for example, if these conjectures will be confirmed for $n = \Omega(k^2)$, then we maybe obtain a tight spectral condition for an even cycle C_ℓ where $\ell \in [3, \Theta(\sqrt{n})] \cup [n - \Theta(\sqrt{n}), n]$. It is still mysterious to determine tight spectral conditions for C_ℓ, where $\ell = cn$, $0 < c < 1$, such as $C_{\frac{n}{2}}$ and etc.

A classical result in Bollobás' textbook [2, Corrolary 5.4] states a graph G contains all cycles C_ℓ for each $\ell \in [3, \lfloor \frac{n+3}{2} \rfloor]$ if $e(G) > \lfloor \frac{n^2}{4} \rfloor$. Nikiforov [30] proposed a corresponding spectral analogous open problem:

Problem 30 (Nikiforov [30]). What is the maximum C such that for all positive $\varepsilon < C$ and sufficiently large n, every graph G of order n with $\rho(G) > \sqrt{\lfloor \frac{n^2}{4} \rfloor}$ contains a cycle of length ℓ for every integer $\ell \leqslant (C - \varepsilon)n$.

Different from the original edge extremal case, Nikiforov [30] constructed the class of graphs $G = K_s \vee (n-s)K_1$ where $s = \left\lfloor \frac{(3-\sqrt{5})n}{4} \right\rfloor$ (see [30]) from which one can find $C \leqslant \frac{(3-\sqrt{5})}{2}$. On the other hand, Nikiforov [30] proved that $C \geqslant \frac{1}{320}$ and later it was slightly improved by the second author and Peng [33] to $C \geqslant \frac{1}{160}$. Recently, Zhai and Lin [42] proved that every graph G on n vertices contains cycles of length $\ell \in [3, \frac{n}{7}]$ if $\rho(G) > \sqrt{\lfloor \frac{n^2}{4} \rfloor}$. Notice that they do not need the assumption that n is sufficiently large. Only very recently, the current authors [27] showed $C \geqslant \frac{1}{4}$ by completely different methods.

Theorem 31 (Li and Ning [27]). *Let ε be real with $0 < \varepsilon < \frac{1}{4}$. Then there exists an integer $N := N(\varepsilon)$, such that if G is a graph on n vertices with $n \geqslant N$ and $\rho(G) > \sqrt{\lfloor \frac{n^2}{4} \rfloor}$, then G contains all cycles C_ℓ with $\ell \in [3, (\frac{1}{4} - \varepsilon)n]$.*

From Theorem 31, we can see Problem 1 is known for the case $\ell < (\frac{1}{4} - o(1))n$ where ℓ is odd and n is large enough. Now we can summarize all results related to Problem 1 (see Subsection 1.1 and the last section here) as follows. For other related references, we refer the reader to [40, 18, 7, 42].

Remark 32. Problem 1 is solved for each even integer $\ell \in [4, c] \cup [n - \lfloor \frac{-1+\sqrt{8n+1}}{2} \rfloor + 1, n]$ where c is any fixed even integer and n is sufficiently large related to c, and each odd integer $\ell \in [3, \frac{n}{4} - K(\frac{1}{8})] \cup [n - \lfloor \frac{-1+\sqrt{8n+1}}{2} \rfloor + 1, n]$ where n is sufficiently large.[2]

Note that each of the extremal graphs in Theorem 7 has a cut-vertex. This motivates us to conclude this paper with the following open problem (and also a more general one).

[2]Here $K(\frac{1}{8})$ is a constant only related to $\frac{1}{8}$. The detailed definition can be found in [20]. In fact, the current authors proved a slightly stronger result than the statement mentioned in Theorem 31 (see [27]).

Problem 33. Determine tight spectral radius conditions for a cycle of length ℓ in a 2-connected graph of order n for each $\ell \in [\lfloor \frac{n+3}{2} \rfloor, n]$.

Problem 34. Determine tight spectral radius conditions for a cycle of length ℓ in a 2-connected graph of order n with minimum degree $\delta \geqslant k \geqslant 2$ for each $\ell \in [3, n]$.

One may find that under the condition (a) of Theorem 13, we need

$$c \geqslant \max\left\{\frac{5n+6k+5}{6}, n-\sqrt{2n}+\frac{3k}{4}+3\right\}.$$

If there is a real $\alpha < 1$ such that $n^\alpha \geqslant k$, then we know $\frac{3k}{4} \leqslant \frac{3n^\alpha}{4}$. It means the part (a) of Theorem 13 also holds when $n \geqslant k^{\frac{1}{\alpha}}$ and $c \geqslant \max\{\frac{5n}{6}+n^\alpha+\frac{5}{6}, n-\sqrt{2n}+\frac{3}{4}n^\alpha+3\}$. Thus, for large graphs, it is an open problem to study the case when $2k+1 \leqslant c < n-\sqrt{2n}+\frac{3}{4}n^\alpha+3$ in Theorem 13. As the first step, can we prove the case $c \leqslant l_1 n$, where $l_1 < 1$ is a constant independent of k and n?

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