# A bijective proof of a generalization of the non-negative crank–odd mex identity

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**Abstract**

Recent works of Andrews–Newman and Hopkins–Sellers unveil an interesting relation between two partition statistics, the crank and the mex. They state that, for a positive integer $n$, there are as many partitions of $n$ with non-negative crank as partitions of $n$ with odd mex. In this paper, we give a bijective proof of a generalization of this identity provided by Hopkins, Sellers and Stanton. Our method uses an alternative definition of the Durfee decomposition, whose combinatorial link to the crank was recently studied by Hopkins, Sellers, and Yee.

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## 1 Introduction

### 1.1 State of art

An *integer partition* is a finite non-increasing sequence of positive integers. It then has the form $\lambda = (\lambda_1, \ldots, \lambda_s)$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_s \geqslant 1$. The terms $\lambda_i$ are called the parts of $\lambda$, and we denote by $\ell(\lambda) = s$ and $|\lambda| = \lambda_1 + \cdots + \lambda_s$ respectively the length and the weight of the partition $\lambda$. For example, $\ell(\emptyset) = |\emptyset| = 0$. For a non-negative integer $n$, an integer partition with weight $n$ is commonly called a partition of $n$. For example, the partition of 3 are $(3), (2,1)$ and $(1,1,1)$, respectively with length $1, 2$ and 3. Let $\mathcal{P}$ be the set of integer partitions. For $\mathcal{F} \subset \mathcal{P}$, $\overline{\mathcal{F}}$ is the complementary of $\mathcal{F}$, i.e. $\overline{\mathcal{F}} = \{\lambda \in \mathcal{P} : \lambda \notin \mathcal{F}\}$. In the remainder of the paper, the term "partition" stands for an integer partition, and $\sharp A$ denotes the number of elements in the set $A$.

In a 1988 paper [1], Andrews and Garvan formally provide a definition of Dyson's crank, a partition statistic introduced by Dyson in [4] to combinatorially explain a divisibility property of partitions. Let $\lambda$ be a partition. Set $\omega(\lambda) = \sharp\{i \in \{1, \ldots, \ell(\lambda)\} : \lambda_i = 1\}$, the number of occurrences of 1 as part of $\lambda$, and set $\eta(\lambda) = \sharp\{i \in \{1, \ldots, \ell(\lambda)\} : \lambda_i > \omega(\lambda)\}$

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the number of parts greater than $\omega(\lambda)$. The crank of $\lambda$, denoted $\mathrm{crank}(\lambda)$, is defined by the relation

$$\mathrm{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \ \omega(\lambda) = 0\,, \\ \eta(\lambda) - \omega(\lambda) & \text{if } \ \omega(\lambda) > 0\,. \end{cases} \tag{1}$$

In particular, set $\mathrm{crank}(\emptyset) = 0$. One can easily check that $-|\lambda| \leqslant \mathrm{crank}(\lambda) \leqslant |\lambda|$. Given integers $m, n$ with $-n \leqslant m \leqslant n$, $C(m,n)$ denotes the number of partitions of weight $n$ and crank $m$, with the exception that $C(1,1) = -C(0,1) = 1$. In [5], Garvan explicitly provides the generating function for the crank.

**Theorem 1** (Garvan)**.** *We have*

$$(x-1)q + \sum_{\lambda \in \mathcal{P}} x^{\mathrm{crank}(\lambda)} q^{|\lambda|} = \sum_{n \geqslant 0} \sum_{m=-n}^{n} C(m,n) x^m q^n = \frac{(q;q)_\infty}{(qx, qx^{-1};q)_\infty} \tag{2}$$

*where* $(a_1, \ldots, a_t; q)_\infty = \prod_{k \geqslant 0} \prod_{i=1}^{t} (1 - a_i q^k)$.

**Corollary 2.** *Given integers* $n \geqslant m \geqslant 0$*, we have* $C(m,n) = C(-m,n)$.

Recent works involved the use of a new partition statistic, the *minimal excludant* or mex, defined as the smallest positive integer which is not a part of the partition. For $\lambda \in \mathcal{P}$, set $\mathrm{mex}(\lambda)$ to be the mex of $\lambda$. For example, $\mathrm{mex}(\emptyset) = \mathrm{mex}((5,3,2,2)) = 1$. A curious yet interesting connection between the mex and the crank arose from the works of Andrews–Newman [2] and Hopkins–Sellers [6], who independently found the following result.

**Theorem 3.** *At fixed weight, the number of partitions with non-negative crank is equal to the number of partitions with odd mex.*

In [7], Hopkins, Sellers and Stanton provide a broad generalization of Theorem 3 by introducing a notion generalizing the mex. For $j \geqslant 0$, and $\lambda$ a partition containing the part $j$, $\mathrm{mex}_j(\lambda)$ is the smallest integer greater than $j$ which is not a part of $\lambda$.

**Theorem 4.** *For* $j \in \mathbb{Z}_{\geqslant 0}$*, at fixed weight greater than* $1$*, the number of partitions* $\lambda$ *with a part* $j$ *such that* $\mathrm{mex}_j(\lambda) - j$ *is odd is equal to the number of partitions with crank at least equal to* $j$*.*

The case $j = 0$ of the above theorem implies Theorem 3 as $\mathrm{mex}_0(\lambda) = \mathrm{mex}(\lambda)$ and $0$ can be seen as a fictitious part of all partitions.

The aim of this paper is to provide a purely bijective proof Theorem 4. This generalization will derive from a key result related the Durfee decomposition of partitions.

### 1.2 Statement of results

We first extend the generalization of the notion of mex to all partitions.

**Definition 5.** Let $j \in \mathbb{Z}_{\geqslant 0}$. For $\lambda \in \mathcal{P}$, the $j$-mex of $\lambda$, denoted $\operatorname{mex}_j(\lambda)$, is the smallest integer greater than $j$ which is not a part of $\lambda$. For example, we have $\operatorname{mex}_j(\emptyset) = j+1$ for $j \in \mathbb{Z}_{\geqslant 0}$, $\operatorname{mex}_0(5,3,2,2) = 1, \operatorname{mex}_1(5,3,2,2) = \operatorname{mex}_2(5,3,2,2) = \operatorname{mex}_3(5,3,2,2) = 4, \operatorname{mex}_4(5,3,2,2) = 6$, and finally $\operatorname{mex}_j(5,3,2,2) = j+1$ for $j \geqslant 5$. Denote by $\mathcal{M}_j$ the set of partitions with a $j$-mex parity different from $j$. With the previous examples, we have that $\emptyset \in M_j$ for $j \in \mathbb{Z}_{\geqslant 0}$, and

$$(5,3,2,2) \in \mathcal{M}_0 \cap \mathcal{M}_1 \cap \overline{\mathcal{M}}_2 \cap \mathcal{M}_3 \cap \overline{\mathcal{M}}_4 \cap \bigcap_{j \geqslant 5} \mathcal{M}_j\,.$$

In the remainder of the paper, for all $\lambda \in \mathcal{P}$, we set $\lambda_0 = \infty$ and $\lambda_{\ell(\lambda)+1} = 0$, so that the sequence $(\lambda_0, \cdots, \lambda_{\ell(\lambda)+1})$ remains non-increasing. A partition then becomes a non-increasing sequence starting from $\infty$ and ending by 0. For example, the partition $\emptyset$ is associated to the sequence $(\infty, 0)$ with $\emptyset_0 = \infty$ and $\emptyset_1 = 0$. For $j \in \mathbb{Z}_{\geqslant 0}$, denote by $\mathcal{P}_j$ the set of partitions which do not have $j$ as part. Conversely, $\overline{\mathcal{P}}_j$ is the set of partitions with a part $j$. A rephrasing of Theorem 4 is then the following.

**Theorem 6.** *For $j \in \mathbb{Z}_{\geqslant 0}$, at fixed weight greater than 1, the number of partitions in $\mathcal{M}_j \cap \overline{\mathcal{P}}_j$ is equal to the number of partitions with crank at least equal to $j$.*

The bijective proof of Theorem 6 that we provide in this paper, was deeply inspired by the work of Hopkins, Sellers, and Yee, who described in [8] combinatorial relations that link the crank and the Durfee decomposition of a partition. Our work is based on a simple yet subtle definition related the very notion of Durfee decomposition.

**Definition 7.** Let $\lambda \in \mathcal{P}$. The function $i \mapsto \lambda_i - i$ is strictly then decreasing on $\{0, \ldots, \ell(\lambda)+1\}$, with $\lambda_0 - 0 = \infty$ and $\lambda_{\ell(\lambda)+1} - (\ell(\lambda)+1) < 0$. Therefore, for $j \in \mathbb{Z}_{\geqslant 0}$, there exists a unique integer $d_j^\lambda \in \{0, \ldots, \ell(\lambda)\}$ such that $\lambda_{d_j^\lambda} - d_j^\lambda \geqslant j > \lambda_{d_j^\lambda+1} - (d_j^\lambda + 1)$. Formally written,

$$d_j^\lambda = \max\{i \in \{0, \ldots, \ell(\lambda)\} : \lambda_i - i \geqslant j\}\,. \tag{3}$$

For example, $d_j^\emptyset = 0$, and for $\lambda = (5,3,2,2)$,

$$d_0^\lambda = d_1^\lambda = 2,\ d_2^\lambda = d_3^\lambda = d_4^\lambda = 1 \text{ and } d_j^\lambda = 0 \text{ for } j \geqslant 5\,.$$

We also denote by $\mathcal{F}_j$ the set of partitions $\lambda$ such that $j \notin \{\lambda_i - i : i \in \{1, \ldots, \ell(\lambda)\}\}$, which is equivalent to saying that $\lambda_{d_j^\lambda} - d_j^\lambda > j > \lambda_{d_j^\lambda+1} - (d_j^\lambda + 1)$. Conversely, $\overline{\mathcal{F}}_j$ is the set of partitions $\lambda$ satisfying $\lambda_{d_j^\lambda} = d_j^\lambda + j$.

*Remark* 8. For $j \in \mathbb{Z}_{\geqslant 0}$, as $\lambda_{d_{j+1}^\lambda+2} - (d_{j+1}^\lambda + 2) < j < \lambda_{d_{j+1}^\lambda} - d_{j+1}^\lambda$ when $d_{j+1}^k < \ell(\lambda)$, we always have that $d_j^\lambda - d_{j+1}^\lambda \in \{0,1\}$. Therefore, the sequence $(d_j^\lambda)_{j\geqslant 0}$ is non-increasing while the sequence $(d_j^\lambda + j)_{j \geqslant 0}$ is non-decreasing.

*Remark* 9. The Durfee decomposition of a partition $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ is the triplet $(d_0^\lambda, \mu, \nu)$ with $(\mu, \nu) = [(\mu_1, \ldots, \mu_{d_0^\lambda}), (\nu_1, \ldots, \nu_{d_0^\lambda})]$ such that, for all $i \in \{1, \ldots, d_0^\lambda\}$, $\mu_i = \lambda_i - i$ and $\nu_i = \sharp\{u \in \{1, \ldots, \ell(\lambda)\} : \lambda_u \geqslant i\} - i$. Inversely, for any triplet $(t, \mu, \nu)$

such that $\mu$ and $\nu$ are increasing sequences of $t$ non-negative integers, we associate the partition $\lambda$ with length $\nu_1 + 1$ and

$$\begin{cases} \lambda_i = \mu_i + i & \text{for } \ 1 \leqslant i \leqslant t \\ \lambda_i = \sharp\{u : \nu_u + u \geqslant i\} & \text{for } \ t+1 \leqslant i \leqslant \nu_1 + 1 \,. \end{cases}$$

We note $\lambda \equiv (t, \mu, \nu)$, and one can check that $|\lambda| = t + \sum_{u=1}^{t} \mu_u + \nu_u$. In particular, $\emptyset \equiv (0, \emptyset, \emptyset)$.

Observe that, for $j \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{F}_j$ can be equivalently defined as the set of partitions $\lambda \equiv (d_0^\lambda, \mu, \nu)$ such that $j$ is not in $\mu$.

We now provide an intermediate result that plays a fundamental role in the bijective proof of Theorem 4.

**Theorem 10.** *For $j \in \mathbb{Z}_{\geqslant 0}$, at fixed weight, the number of partitions of $\mathcal{M}_j$ is equal to the number of partitions in $\mathcal{F}_j$.*

**Corollary 11.** *For $j \in \mathbb{Z}_{\geqslant 0}$, at fixed weight, the number of partitions of $\overline{\mathcal{M}}_j$ is equal to the number of partitions in $\overline{\mathcal{F}}_j$.*

Using the following reformulation of a result provided by Hopkins, Sellers and Stanton in [7], we derive the generalization of Theorem 3 from Theorem 10 and Corollary 2.

**Theorem 12.** *Let $j \in \mathbb{Z}_{\geqslant 0}$. Then, at fixed weight greater than 1, there are as many partitions in $\mathcal{F}_j \cap \overline{\mathcal{P}}_j$ as partitions with crank at most equal to $-j$.*

By adding a part $j$ to the partitions in $\mathcal{M}_j$ and $\mathcal{F}_j$ when $j > 0$, Theorem 12 implies that, at fixed weight, there are as many partitions in $\mathcal{M}_j \cap \overline{\mathcal{P}}_j$ as partitions in $\mathcal{F}_j \cap \overline{\mathcal{P}}_j$. Then, by Theorem 10, at fixed weight greater than 1, there are as many partitions in $\mathcal{M}_j \cap \overline{\mathcal{P}}_j$ as partitions with crank at most equal to $-j$. We finally obtain Theorem 4 from Corollary 2.

The remainder of the paper is organized as follows. We first provide in Section 2 a simple analytic proof of Theorem 10. Then, in Sections 3 and 4, a bijection for Theorem 10 is given, so as the proof of its well-definedness. After that, in Section 5, we give a direct bijective proof of Corollary 11 in the spirit of the bijection of Section 3. In Section 6, Theorem 12 is proved bijectively. Finally, in Section 8, we provide the full scope of bijective proof of Theorem 4 with a bijection for Corollary 2 given in Section 7.

## 2 Analytic proof of Theorem 10

For $j, k \in \mathbb{Z}_{\geqslant 0}$, set $\Delta_{j,k} = (j+k, \ldots, j+1)$ the partition consisting of $k$ consecutive integers ending by $j+1$, and $\Delta_{j,0} = \emptyset$. Then, $|\Delta_{j,k}| = \frac{k(k+1)}{2} + jk$. Hence, by Definition 5, the set of partitions with $j$-mex equal to $j+k+1$ can be associated to $\{\Delta_{j,k}\} \times \mathcal{P}_{j+k+1}$, i.e. $\operatorname{mex}_j(\lambda) = j+k+1$ if and only if there exists a unique partition, without a part

$j+k+1$, whose parts are exactly those of $\lambda$ except once the parts $j+1,\ldots,j+k$. We then have

$$\sum_{\lambda\in\mathcal{P}} x^{\mathrm{mex}_j(\lambda)}y^{\ell(\lambda)}q^{|\lambda|} = \frac{x^{j+1}}{(qy;q)_\infty}\sum_{k\geqslant 0}(xy)^k(1-yq^{j+k+1})q^{|\Delta_{j,k}|}\,.$$

By Definition 5, $\mathcal{M}_j$ consists of partitions $\lambda$ such that $\mathrm{mex}_j(\lambda)-j\equiv 1 \mod 2$. Hence, using the above relation with $x=y=1$ and running the sum over even $k$, we have

$$\begin{aligned}\sum_{\lambda\in\mathcal{M}_j} q^{|\lambda|} &= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0}(1-q^{j+2k+1})\cdot q^{|\Delta_{j,2k}|}\\ &= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0}(-1)^k q^{|\Delta_{j,k}|}\\ &= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0}(-1)^k q^{\frac{k(k+1)}{2}+jk}\,.\end{aligned}$$

Also, by the Jacobi triple product,

$$(-x;q)_\infty(-x^{-1}q;q)_\infty = \frac{1}{(q;q)_\infty}\cdot\sum_{k\in\mathbb{Z}}(-1)^k x^k q^{\frac{k(k-1)}{2}}\,,$$

and by Remark 9,

$$\begin{aligned}\sum_{\lambda\in\mathcal{F}_j} q^{|\lambda|} &= [x^0]\left[(-x;q)_\infty(-x^{-1}q;q)_\infty\right]\cdot\frac{1}{(1+xq^j)}\\ &= [x^0]\left[\frac{1}{(q;q)_\infty}\cdot\sum_{k\in\mathbb{Z}}(-1)^k x^k q^{\frac{k(k-1)}{2}}\right]\cdot\left(\sum_{k\geqslant 0}(-1)^k x^k q^{jk}\right)\\ &= \frac{1}{(q;q)_\infty}\cdot\sum_{k\geqslant 0}(-1)^k q^{\frac{k(k+1)}{2}+kj}\,.\end{aligned}$$

In conclusion, $\sum_{\lambda\in\mathcal{M}_j} q^{|\lambda|} = \sum_{\lambda\in\mathcal{F}_j} q^{|\lambda|}$.

## 3 Bijection for Theorem 10

Here is a reminder of the key definitions that we use in Sections 3, 4 and 5. For all integers $j\geqslant 1$, $k\geqslant 0$, and for all partitions $\mu=(\mu_1,\ldots,\mu_{\ell(\mu)})$

$$\begin{aligned}&\mathcal{P}_j = \{\lambda\in\mathcal{P} : i\notin\lambda\} \quad\text{and}\quad \overline{\mathcal{P}}_j = \{\lambda\in\mathcal{P} : i\in\lambda\},\\ &\mathcal{M}_j = \{\lambda\in\mathcal{P} : mex_j(\lambda)-j\equiv 1 \mod 2\},\\ &d_j^\mu = \max\{i\in\{0,\ldots,\ell(\mu)\} : \mu_i - i\geqslant j\},\\ &\mathcal{F}_j = \{\lambda\in\mathcal{P} : \lambda_{d_j^\lambda}-d_j^\lambda > j\} \quad\text{and}\quad \overline{\mathcal{F}}_j = \{\lambda\in\mathcal{P} : \lambda_{d_j^\lambda}-d_j^\lambda = j\},\\ &\Delta_{j,k} = (j+k,\ldots,j+1,j).\end{aligned}$$

In this section, we provide two maps inverse of each other for the bijective proof of Theorem 10. In our bijections, we will apply some transformations on pairs of partitions in $\bigsqcup_{k\geqslant 0}\{\Delta_{j,k}\}\times\mathcal{P}$. One has to keep in mind that $\mathcal{P}$ can be trivially associated to $\{\Delta_{j,0}\}\times\mathcal{P}=\{\emptyset\}\times\mathcal{P}$. Finally, we identify $\mathcal{M}_j$ to $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}_{j+2k+1}$ and $\mathcal{F}_j$ to $\{\Delta_{j,0}\}\times\mathcal{F}_j$. Two maps will then be constructed,

$$\Phi_j:\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}_{j+2k+1}\to\{\Delta_{j,0}\}\times\mathcal{F}_j$$

and

$$\Psi_j:\{\Delta_{j,0}\}\times\mathcal{F}_j\to\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}_{j+2k+1}\,,$$

in such a way that the partitions in the pairs have their weight and their length sums conserved during the process. By abuse of notation, if $\lambda\in\mathcal{M}_j$ is identified to the pair $(\Delta_{j,2k},\mu)$, we write $\Phi_j(\lambda)=\nu\in\mathcal{F}_j$ such that $\Phi_j((\Delta_{j,2k},\mu))=(\Delta_{j,0},\nu)$. The same convention stands for $\Psi_j:\mathcal{F}_j\mapsto\mathcal{M}_j$.

### 3.1 From $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}_{j+2k+1}$ to $\{\Delta_{j,0}\}\times\mathcal{F}_j$

Let $\phi_j$ be the map defined on

$$\left(\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}\right)\setminus(\{\Delta_{j,0}\}\times\mathcal{F}_j)=\left(\bigsqcup_{k\geqslant 1}\{\Delta_{j,2k}\}\times\mathcal{F}_{j+2k}\right)\sqcup\left(\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}\right)$$

as follows.

1. For $k\geqslant 0$ and $(\Delta_{j,2k},\lambda)\in\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$, do the transformation

$$\lambda_1,\dots,\lambda_{d^\lambda_{j+2k}}\mapsto\lambda_1+1,\dots,\lambda_{d^\lambda_{j+2k}-1}+1,1+j+2k\,.$$

   This means that, in $\lambda$, we delete the part $\lambda_{d^\lambda_{j+2k}}=d^\lambda_{j+2k}+j+2k$, add 1 to the $d^\lambda_{j+2k}-1$ largest *finite* parts and add a part $1+j+2k+1$. We then obtain a partition $\mu$, and set $\phi_j((\Delta_{j,2k},\lambda))=(\Delta_{j,2k},\mu)$. Observe that $|\mu|=|\lambda|$ and $\ell(\mu)=\ell(\lambda)$, so that the weight and length sums are conserved. Moreover, the transformation does not involved parts less than $1+j+2k$, so that the parts at most equal to $j$ are conserved.

2. For $k\geqslant 1$ and $(\Delta_{j,2k},\lambda)\in\{\Delta_{j,2k}\}\times\mathcal{F}_{j+2k}$, do the transformation

$$\Delta_{j,2k},\lambda_1,\dots,\lambda_{d^\lambda_{j+2k}}\mapsto\Delta_{j,2k-2},\lambda_1-1,\dots,\lambda_{d^\lambda_{j+2k}}-1,d^\lambda_{j+2k}+j+2k,j+2k-1\,.$$

   This means that, in $\Delta_{j,2k}$, we deleted the parts $j+2k,j+2k-1$, and in $\lambda$, we subtract 1 to the $d^\lambda_{j+2k}$ largest finite parts and add the parts $d^\lambda_{j+2k}+j+2k$ and $j+2k-1$ to obtain a partition $\mu$. We finally set $\phi_j((\Delta_{j,2k},\lambda))=(\Delta_{j,2k-2},\mu)$. Note that $|\mu|=|\lambda|+(j+2k)+(j+2k-1)$, and $\ell(\mu)=\ell(\lambda)+2$, so that the weight and length sums are conserved. In addition, The parts at most equal to $j$ are conserved.

For all $(\Delta_{j,2k}, \lambda) \in \{\Delta_{j,2k}\} \times \mathcal{P}_{j+2k+1}$, iterate the map $\phi_j$ as long as it is possible. The iteration stops as soon as we reach a pair in $\{\Delta_{j,0}\} \times \mathcal{F}_j$. We then set $\Phi_j((\Delta_{j,2k}, \lambda)) = \phi_j^u((\Delta_{j,2k}, \lambda)) \in \{\Delta_{j,0}\} \times \mathcal{F}_j$. We finally observe that the transformations occur only on parts greater than $j$. Therefore, if $\Phi_j$ is well-defined, it then induces a map from $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\} \times \left(\mathcal{P}_{j+2k+1} \cap \overline{\mathcal{P}}_j\right)$ to $\{\Delta_{j,0}\} \times \left(\mathcal{F}_j \cap \overline{\mathcal{P}}_j\right)$.

**Example 13.** For $j, k \in \mathbb{Z}_{\geqslant 0}$,

$$\Phi_j((\Delta_{j,2k}, \emptyset)) = (\Delta_{j,0}, \Delta_{j,2k}) \,,$$

as $\phi_j^u((\Delta_{j,2k}, \emptyset)) = (\Delta_{j,2k-2u}, \Delta_{j+2k-2u,2u})$ for $0 \leqslant u \leqslant k$.

**Example 14.** Consider the partition $(11, 8, 7, 7, 5, 5, 4, 3, 2, 2)$. It belongs to $\mathcal{M}_j$ for $j \neq 2, 4, 7, 10$, and the corresponding pairs are respectively

$(\Delta_{0,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)), (\Delta_{1,4}, (11, 8, 7, 7, 5, 2)), (\Delta_{3,2}, (11, 8, 7, 7, 5, 3, 2, 2))$
$(\Delta_{5,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)), (\Delta_{6,2}, (11, 7, 5, 5, 4, 3, 2, 2)),$
and $(\Delta_{j,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2))$ for $10 \neq j \geqslant 8$.

We now represent in tables the different iterations of each $\phi_j$ for $j \in \{0, 1, 3, 5\}$. For $j = 0$,

| Iteration | $k$ | $\lambda$ | $d^\lambda_{2k}$ | case | $\phi_0(\lambda)$ |
|---|---|---|---|---|---|
| 1 | 0 | $(11, 8, 7, 7, 5, 5, 4, 3, 2, 2)$ | 5 | (1) | $(12, 9, 8, 8, 5, 4, 3, 2, 2, 1)$ |
| 2 | 0 | $(12, 9, 8, 8, 5, 4, 3, 2, 2, 1)$ | 5 | (1) | $(13, 10, 9, 9, 4, 3, 2, 2, 1, 1)$ |
| 3 | 0 | $(13, 10, 9, 9, 4, 3, 2, 2, 1, 1)$ | 4 | $-$ | $-$ |

and

$$\begin{aligned}\Phi_0(\Delta_{0,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)) &= \phi_0^2(\Delta_{0,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)) \\ &= (\Delta_{0,0}, (13, 10, 9, 9, 4, 3, 2, 2, 1, 1)).\end{aligned}$$

For $j = 1$,

| Iteration | $k$ | $\lambda$ | $d^\lambda_{1+2k}$ | case | $\phi_1(\lambda)$ |
|---|---|---|---|---|---|
| 1 | 2 | $(11, 8, 7, 7, 5, 2)$ | 2 | (2) | $(10, 7, 7, 7, 7, 5, 4, 2)$ |
| 2 | 1 | $(10, 7, 7, 7, 7, 5, 4, 2)$ | 4 | (1) | $(11, 8, 8, 7, 5, 4, 4, 2)$ |
| 3 | 1 | $(11, 8, 8, 7, 5, 4, 4, 2)$ | 4 | (1) | $(12, 9, 9, 5, 4, 4, 4, 2)$ |
| 4 | 1 | $(12, 9, 9, 5, 4, 4, 4, 2)$ | 3 | (2) | $(11, 8, 8, 6, 5, 4, 4, 4, 2, 2)$ |
| 5 | 0 | $(11, 8, 8, 6, 5, 4, 4, 4, 2, 2)$ | 4 | $-$ | $-$ |

and

$$\begin{aligned}\Phi_1((\Delta_{1,4}, (11, 8, 7, 7, 5, 2))) &= \phi_1^4((\Delta_{1,4}, (11, 8, 7, 7, 5, 2))) \\ \text{““““} &= (\Delta_{1,0}, (11, 8, 8, 6, 5, 4, 4, 4, 2, 2)).\end{aligned}$$

For $j = 3$,

| Iteration | $k$ | $\lambda$ | $d^{\lambda}_{3+2k}$ | case | $\phi_3(\lambda)$ |
|---|---|---|---|---|---|
| 1 | 1 | $(11, 8, 7, 7, 5, 3, 2, 2)$ | 2 | (2) | $(10, 7, 7, 7, 7, 5, 4, 3, 2, 2)$ |
| 2 | 0 | $(10, 7, 7, 7, 7, 5, 4, 3, 2, 2)$ | 4 | (1) | $(11, 8, 8, 7, 5, 4, 4, 3, 2, 2)$ |
| 3 | 0 | $(11, 8, 8, 7, 5, 4, 4, 3, 2, 2)$ | 4 | (1) | $(12, 9, 9, 5, 4, 4, 4, 3, 2, 2)$ |
| 4 | 0 | $(12, 9, 9, 5, 4, 4, 4, 3, 2, 2)$ | 3 | $-$ | $-$ |

and

$$\begin{aligned}\Phi_3((\Delta_{3,2}, (11, 8, 7, 7, 5, 3, 2, 2))) &= \phi_3^3((\Delta_{3,2}, (11, 8, 7, 7, 5, 3, 2, 2))) \\ &= (\Delta_{3,0}, (12, 9, 9, 5, 4, 4, 4, 3, 2, 2)).\end{aligned}$$

For $j = 5$,

| Iteration | $k$ | $\lambda$ | $d^{\lambda}_{5+2k}$ | case | $\phi_5(\lambda)$ |
|---|---|---|---|---|---|
| 1 | 0 | $(11, 8, 7, 7, 5, 5, 4, 3, 2, 2)$ | (2) | $-$ | $-$ |

and

$$\begin{aligned}\Phi_5(\Delta_{5,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)) &= \phi_5^0(\Delta_{5,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)) \\ &= (\Delta_{5,0}, (11, 8, 7, 7, 5, 5, 4, 3, 2, 2)).\end{aligned}$$

### 3.2 From $\{\Delta_{j,0}\} \times \mathcal{F}_j$ to $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\} \times \mathcal{P}_{j+2k+1}$

Let $\psi_j$ be the map defined on

$$\left(\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\} \times \mathcal{P}\right) \setminus \left(\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\} \times \mathcal{P}_{j+2k+1}\right) = \bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\} \times \overline{\mathcal{P}}_{j+2k+1}$$

as follows. Let $k \geqslant 0$.

1. For $(\Delta_{j,2k}, \mu) \in \{\Delta_{j,2k}\} \times \left(\overline{\mathcal{F}}_{j+2k+1} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$, do the transformation

   $$\Delta_{j,2k}, \mu_1, \dots, \mu_{d^{\mu}_{1+j+2k}}, 1 + j + 2k \mapsto \Delta_{j,2k+2}, \mu_1 + 1, \dots, \mu_{d^{\mu}_{1+j+2k}-1} + 1\,.$$

   This means that, in $\Delta_{j,2k}$, we add the parts $1+j+2k, j+2k+2$, and in $\mu$, we add 1 to the $d^{\mu}_{1+j+2k}-1$ largest finite parts and delete the parts $\mu_{d^{\mu}_{1+j+2k}} = d^{\mu}_{1+j+2k}+1+j+2k$ and $1 + j + 2k$ to obtain a partition $\lambda$. We finally set $\psi_j((\Delta_{j,2k}, \mu)) = (\Delta_{j,2k+2}, \lambda)$. Note that $|\lambda| = |\mu| - (j + 2k + 2) - (j + 2k + 1)$, and $\ell(\lambda) = \ell(\mu) - 2$.

2. For $(\Delta_{j,2k}, \mu) \in \{\Delta_{j,2k}\} \times \left(\mathcal{F}_{j+2k+1} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$, do the transformation

   $$\mu_1, \dots, \mu_{d^{\mu}_{1+j+2k}}, 1 + j + 2k \mapsto \mu_1 - 1, \dots, \mu_{d^{\mu}_{1+j+2k}} - 1, d^{\mu}_{1+j+2k} + 1 + j + 2k\,.$$

   This means that, in $\mu$, we deleted the part $1 + j + 2k$, subtract 1 to the $d^{\mu}_{1+j+2k}$ largest finite parts and add the part $d^{\mu}_{1+j+2k} + 1 + j + 2k$ to obtain a partition $\lambda$. We finally set $\psi_j((\Delta_{j,2k}, \mu)) = (\Delta_{j,2k}, \lambda)$. Observe that $|\lambda| = |\mu|$ and $\ell(\lambda) = \ell(\mu)$.

For all $(\Delta_{j,0},\mu)\in\{\Delta_{j,0}\}\times\mathcal{F}_j$, iterate the map $\psi_j$ as long as it is possible. The iteration stops as soon as we reach a pair in $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}_{j+2k+1}$. We then set $\Psi_j((\Delta_{j,0},\mu)) = \psi_j^u((\Delta_{j,0},\mu))\in\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\mathcal{P}_{j+2k+1}$. Similarly to $\Phi_j$, if $\Psi_j$ is well-defined, it then induces a map from $\{\Delta_{j,0}\}\times\left(\mathcal{F}_j\cap\overline{\mathcal{P}}_j\right)$ to $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\}\times\left(\mathcal{P}_{j+2k+1}\cap\overline{\mathcal{P}}_j\right)$.

The reader can easily check that, by applying the corresponding $\Psi_j$ to the pairs obtained in Example 14, we retrieve the pairs corresponding to $(11,8,7,7,5,5,4,3,2,2)$ by the exact inverse process.

## 4 Proof of the well-definedness of the bijection

The maps $\Phi_j$ and $\Psi_j$ preserve the weight and length sums of the pair of partitions, as they result from iterations of $\phi_j$ and $\psi_j$ which have this property. To prove that maps $\Phi_j$ and $\Psi_j$ are inverse of one another, we first prove that $\phi_j$ and $\psi_j$ are inverse of each other, then prove the well-definedness of $\Phi_j$ and $\Psi_j$ and conclude.

### 4.1 The maps $\phi_j$ and $\psi_j$ are inverse of each other

We first prove that $\phi_j$ and $\psi_j$ are inverse of each other with the following result.

**Proposition 15.** *For all $k\geqslant 0$, $\phi_j$ describes a bijection*

1. *from $\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$ to $\{\Delta_{j,2k}\}\times\left(\mathcal{F}_{1+j+2k}\cap\overline{\mathcal{P}}_{j+2k+1}\right)$,*
2. *from $\{\Delta_{j,2k+2}\}\times\mathcal{F}_{j+2k+2}$ to $\{\Delta_{j,2k}\}\times\left(\overline{\mathcal{F}}_{1+j+2k}\cap\overline{\mathcal{P}}_{j+2k+1}\right)$,*

*and $\psi_j$ is the inverse of $\phi_j$.*

*Proof.* Let $k,j\geqslant 0$.

1. The map $\phi_j$ describes a bijection from $\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$ to $\{\Delta_{j,2k}\}\times\left(\mathcal{F}_{1+j+2k}\cap\overline{\mathcal{P}}_{j+2k+1}\right)$, and $\psi_j=\phi_j^{-1}$.

   (a) For $(\Delta_{j,2k},\lambda)\in\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$, set $\ell(\lambda)\geqslant t\geqslant d^\lambda_{j+2k}$ such that $\lambda_t\geqslant 1+j+2k>\lambda_{t+1}$. By the first case of Section 3.1, $\mu$ consists of the parts

$$\begin{cases}\mu_i=\lambda_i+1 & \text{for}\quad 1,\leqslant i<d^\lambda_{j+2k}\,,\\ \mu_i=\lambda_{i+1} & \text{for}\quad d^\lambda_{j+2k}\leqslant i<t\,,\\ \mu_i=\lambda_i & \text{for}\quad t<i\leqslant\ell(\lambda)\,,\\ \mu_t=1+j+2k\,. & \end{cases}$$

   Since $1+j+2k$ is a part of $\mu$, we have that $\mu\in\overline{\mathcal{P}}_{1+j+2k}$. Moreover, $\lambda_{d^\lambda_{j+2k}}\geqslant\mu_{d^\lambda_{j+2k}}$, and

$$\mu_{d^\lambda_{j+2k}-1}-(d^\lambda_{j+2k}-1)=\lambda_{d^\lambda_{j+2k}-1}-(d^\lambda_{j+2k}-1)+1$$

$$
\begin{aligned}
&> \lambda_{d^\lambda_{j+2k}} - d^\lambda_{j+2k} + 1 \\
&= 1 + j + 2k \\
&> \mu_{d^\lambda_{j+2k}} - d^\lambda_{j+2k} \,,
\end{aligned}
$$

so that $d^\mu_{1+j+2k} = d^\lambda_{j+2k} - 1$ and $\mu \in \mathcal{F}_{1+j+2k}$. Thus,

$$\phi_j((\Delta_{j,2k}, \lambda)) \in \{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right) .$$

As $d^\mu_{1+j+2k} = d^\lambda_{j+2k} - 1$, by applying the second case of Section 3.2 on $(\Delta_{j,2k}, \mu)$, it is straightforward that $\psi_j(\phi_j((\Delta_{j,2k}, \lambda))) = (\Delta_{j,2k}, \lambda)$.

(b) Let $(\Delta_{j,2k}, \mu) \in \{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$. Similarly to the previous case, as

$$\mu_{d^\mu_{1+j+2k}+1} \leqslant 1 + d^\mu_{1+j+2k} + j + 2k \leqslant \mu_{d^\mu_{1+j+2k}} - 1 \,,$$

we observe that the part $\lambda_{d^\mu_{1+j+2k}+1} = 1 + d^\mu_{1+j+2k} + j + 2k$ so that $d^\lambda_{j+2k} = d^\mu_{1+j+2k} + 1$ and $\lambda \in \mathcal{F}_{j+2k}$. Then,

$$\psi_j((\Delta_{j,2k}, \mu)) \in \{\Delta_{j,2k}\} \times \overline{\mathcal{F}}_{j+2k}.$$

Using the first case of Section 3.1 on $(\Delta_{j,2k}, \lambda)$ with $d^\lambda_{j+2k} = d^\mu_{1+j+2k} + 1$, it is straightforward that $\phi_j(\psi_j((\Delta_{j,2k}, \mu))) = (\Delta_{j,2k}, \mu)$.

2. The map $\phi_j$ describes a bijection from $\{\Delta_{j,2k+2}\} \times \mathcal{F}_{j+2k+2}$ to $\{\Delta_{j,2k}\} \times \left(\overline{\mathcal{F}}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$, and $\psi_j = \phi_j^{-1}$.

(a) Let $(\Delta_{j,2k+2}, \lambda) \in \{\Delta_{j,2k+2}\} \times \mathcal{F}_{j+2k+2}$. Note that

$$\lambda_{d^\lambda_{j+2k+2}} - d^\lambda_{j+2k+2} > j + 2k + 2 > \lambda_{d^\lambda_{j+2k+2}+1} - (d^\lambda_{j+2k+2} + 1) \,.$$

Then, the partition $\mu$ consists of the parts $\lambda_1 - 1, \ldots, \lambda_{d^\lambda_{j+2k+2}} - 1, \lambda_{d^\lambda_{j+2k+2}}, \ldots, \lambda_\ell$ and the parts $d^\lambda_{j+2k+2} + j + 2k + 2, 1 + j + 2k$. Since $1 + j + 2k$ is a part $\mu$, $\mu \in \overline{\mathcal{P}}_{1+j+2k}$. Moreover, $\lambda_{d^\lambda_{j+2k+2}} - 1 \geqslant d^\lambda_{j+2k+2} + j + 2k + 2 \geqslant \lambda_{d^\lambda_{j+2k+2}}$ and $d^\lambda_{j+2k+2} + j + 2k + 2 > 1 + j + 2k$, so that $\mu_{d^\lambda_{j+2k+2}+1} = d^\lambda_{j+2k+2} + j + 2k + 2$. Hence, $d^\mu_{1+j+2k} = d^\lambda_{j+2k+2} + 1$ and $\mu \in \overline{\mathcal{F}}_{1+j+2k}$, and

$$\phi_j((\Delta_{j,2k+2}, \lambda)) \in \{\Delta_{j,2k}\} \times \left(\overline{\mathcal{F}}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right) .$$

Finally, by using the first case of Section 3.2 on $(\Delta_{j,2k}, \mu)$ with $d^\mu_{1+j+2k} = d^\lambda_{j+2k+2} + 1$, we retrieve the fact that $\psi_j(\phi_j((\Delta_{j,2k+2}, \lambda))) = (\Delta_{j,2k+2}, \lambda)$.

(b) Let $(\Delta_{j,2k}, \mu) \in \{\Delta_{j,2k}\} \times \left(\overline{\mathcal{F}}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$. Using the first case of Section 3.2, we have that $\lambda$ consists of the parts $\mu_1 + 1, \ldots, \mu_{d^\mu_{1+j+2k}-1} + 1$, and the parts $\mu_{d^\mu_{1+j+2k}+1}, \ldots, \mu_{\ell(\mu)}$ except $1 + j + 2k$. Moreover, $\mu_{d^\mu_{1+j+2k}-1} + 1 - (d^\mu_{1+j+2k} - 1) > \mu_{d^\mu_{1+j+2k}} - d^\mu_{1+j+2k} + 1 = 2 + j + 2k > \mu_{d^\mu_{1+j+2k}+1} - d^\mu_{1+j+2k}$, so that $d^\lambda_{j+2k+2} = d^\mu_{1+j+2k} - 1$ and $\lambda \in \mathcal{F}_{j+2k+2}$. Hence,

$$\psi_j((\Delta_{j,2k}, \mu)) \in \{\Delta_{j,2k+2}\} \times \mathcal{F}_{j+2k} \,.$$

We prove similarly to the previous case that $\phi_j(\psi_j((\Delta_{j,2k}, \mu))) = (\Delta_{j,2k}, \mu)$. $\square$

### 4.2 Well-definedness of $\Phi_j$

By Proposition 15, $\phi_j$ is injective, and this implies that a pair is not fixed by $\phi_j$ if and only if its iterations are not fixed. Hence, it suffices to check that $(\Delta_{j,2k},\lambda)\in\{\Delta_{j,2k}\}\times\mathcal{P}_{1+j+2k}$ is not fixed by $\phi_j$, and that we reach $\{\Delta_{j,0}\}\times\mathcal{F}_j$ after a finite number of iterations of $\phi_j$. In this regard, we state the following proposition.

**Proposition 16.** *Let* $(\Delta_{j,2k},\lambda)\in\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$. *Then,* $\lambda_{d^\lambda_{j+2k}}=d^\lambda_{j+2k}+j+2k$, *and* $\lambda_1\geqslant 1+j+2k$.

1. *If* $\lambda_1=1+j+2k$, *then* $d^\lambda_{j+2k}=1$ *and* $(\Delta_{j,2k},\lambda)$ *is fixed by* $\phi_j$. *Inversely, all the pairs fixed by* $\phi_j$ *have the form* $(\Delta_{j,2k},\lambda)$ *with* $\lambda_1=1+j+2k$.
2. *If* $\lambda_1>1+j+2k$, *then* $d^\lambda_{j+2k}\geqslant 2$ *and, by setting* $u=\sharp\{i\geqslant d^\lambda_{j+2k}:\lambda_i=d^\lambda_{j+2k}+j+2k\}$, *we have that*
$$\phi_j^0((\Delta_{j,2k},\lambda)),\dots,\phi_j^{u-1}((\Delta_{j,2k},\lambda))\in\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}\,,$$
*and* $\phi_j^u((\Delta_{j,2k},\lambda))\in\{\Delta_{j,2k}\}\times\mathcal{F}_{j+2k}$.

We first prove the well-definedness of $\Phi_j$ assuming that the above proposition is true. Let $\lambda\in\mathcal{P}_{1+j+2k}$. Since $1+j+2k$ is not a part of $\lambda$, $\lambda_1\neq 1+j+2k$ and by Proposition 16, $(\Delta_{j,2k},\lambda)$ is not fixed by $\Phi_j$. Hence, its iterations are not fixed by $\phi_j$. We can then use the second case of Proposition 16 and deduce the existence of non-negative integers $u_l$ that counts the numbers of iterations of $(\Delta_{j,2k},\lambda)$ respectively in $\{\Delta_{j,2l}\}\times\overline{\mathcal{F}}_{j+2l}$. More precisely, we have for all $l\in\{0,\dots,k\}$,

$$\phi_j^{k-l+u+\sum_{t=l+1}^k u_t}((\Delta_{j,2k},\lambda))\in\{\Delta_{j,2l}\}\times\overline{\mathcal{F}}_{j+2l}\text{ for }0\leqslant u<u_l$$
$$\text{and }\phi_j^{k-l+\sum_{t=l}^k u_t}((\Delta_{j,2k},\lambda))\in\{\Delta_{j,2l}\}\times\mathcal{F}_{j+2l}\,.$$

Since these iterations are not fixed, by setting $n=|\lambda|+|\Delta_{j,2k}|$, we have by definition of $u_l$ in Proposition 16 that $u_l+1\leqslant\frac{n}{2+j+2l}\leqslant\frac{n}{2}\cdot\frac{1}{l+1}$. Therefore, there is at most $\frac{n}{2}(1+\log(k+2))$ iterations, hence finite. Moreover,

$$\Phi_j((\Delta_{j,2k},\lambda))=\phi_j^{k+\sum_{t=0}^k u_t}((\Delta_{j,2k},\lambda))\in\{\Delta_{j,0}\}\times\mathcal{F}_j$$

so that $\Phi_j((\Delta_{j,2k},\lambda))$ is well-defined.

*Proof of Proposition 16.* Let $(\Delta_{j,2k},\lambda)\in\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$. Recall that $\lambda_{d^\lambda_{j+2k}}=d^\lambda_{j+2k}+j+2k$ and set $\phi_j((\Delta_{j,2k},\lambda))=(\Delta_{j,2k},\mu)$.

1. As $\lambda_1\geqslant 1+j+2k$, the fact that $d^\lambda_{j+2k}$ is unique implies that $\lambda_1=1+j+2k$ if and only if $d^\lambda_{j+2k}=1$. The fact that $\phi_j(\Delta_{j,2k},\lambda)=(\Delta_{j,2k},\lambda)$ is trivial, as $\mu$ is obtained by deleting $\lambda_1$, adding $1+j+2k$, and not adding 1 to any other part. Inversely, only the pairs of $\{\Delta_{j,2k}\}\times\overline{\mathcal{F}}_{j+2k}$ can be fixed by $\phi_j$, and when $d^\lambda_{j+2k}\geqslant 2$, $\mu_1=\lambda_1+1>\lambda_1$ so that $\phi_j(\Delta_{j,2k},\lambda)\neq(\Delta_{j,2k},\lambda)$. Hence, the only pairs fixed by $\phi_j$ have the form $(\Delta_{j,2k},\lambda)$ with $\lambda_1=1+j+2k$.

2. When $\lambda_1 > 1+j+2k$, $d^\lambda_{j+2k} \geqslant 2$. Since $\mu_{d^\lambda_{j+2k}-1} > \lambda_{d^\lambda_{j+2k}} \geqslant \mu_{d^\lambda_{j+2k}}$, we then have that $\mu_{d^\lambda_{j+2k}-1} - (d^\lambda_{j+2k}-1) > j+2k \geqslant \mu_{d^\lambda_{j+2k}} - d^\lambda_{j+2k}$. Hence, $\mu \in \overline{\mathcal{F}}_{j+2k}$ if and only if $d^\lambda_{j+2k} = d^\mu_{j+2k}$ and $\mu_{d^\lambda_{j+2k}} = d^\lambda_{j+2k} + j + 2k > 1+j+2k$. This occurs only if $\mu_{d^\lambda_{j+2k}} = \lambda_{d^\lambda_{j+2k}+1} = d^\lambda_{j+2k} + j + 2k$. Thus, by formally setting $(\Delta_{j,2k}, \lambda^{(w)}) = \phi^w_j((\Delta_{j,2k}, \lambda))$ for $w \in \{0,\dots,u\}$ and $\ell(\lambda) \geqslant t \geqslant d^\lambda_{j+2k} + u - 1$ such that $\lambda_t \geqslant 1+j+2k > \lambda_{t+1}$, we have that
$$\begin{cases} \lambda^{(w)}_i = \lambda_i + w & \text{if} \quad 1 \leqslant i \leqslant d^\lambda_{j+2k} - 1\,, \\ \lambda^{(w)}_i = \lambda_{i+w} & \text{if} \quad d^\lambda_{j+2k} \leqslant i \leqslant t - w\,, \\ \lambda^{(w)}_i = 1+j+2k & \text{if} \quad t-w+1 \leqslant i \leqslant t\,, \\ \lambda^{(w)}_i = \lambda_i & \text{if} \quad t+1 \leqslant i \leqslant \ell(\lambda)\,, \end{cases}$$
as we recursively obtain that $d^{\lambda^{(w)}}_{j+2k} = d^\lambda_{j+2k}$ and $\lambda^{(w)}_{d^\lambda_{j+2k}} = \lambda_{d^\lambda_{j+2k}+w} = d^\lambda_{j+2k} + j + 2k$ for all $w \in \{0,\dots,u-1\}$. Moreover, $\lambda^{(u)}_{d^\lambda_{j+2k}}$ is either $\lambda_{d^\lambda_{j+2k}+u}$ when $t \geqslant d^\lambda_{j+2k} + u$, or $1+j+2k$ when $t = d^\lambda_{j+2k} + u - 1$. In all cases, $\lambda^{(u)}_{d^\lambda_{j+2k}} - d^\lambda_{j+2k} < j+2k < \lambda^{(u)}_{d^\lambda_{j+2k}-1} - (d^\lambda_{j+2k}-1) - u$, so that $\lambda^{(u)} \in \mathcal{F}_{j+2k}$.

□

### 4.3 Well-definedness of $\Psi_j$

As before, we only need to check that $(\Delta_{j,0}, \lambda) \in \{\Delta_{j,0}\} \times \mathcal{F}_j$ is not fixed by $\psi_j$, and that we reach $\bigsqcup_{k\geqslant 0}\{\Delta_{j,2k}\} \times \mathcal{P}_{j+2k+1}$ after a finite number of iterations of $\psi_j$. The following propositions helps in that purpose.

**Proposition 17.** *For $(\Delta_{j,2k}, \mu) \in \{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$, we have $\mu_{d^\mu_{1+j+2k}} - d^\mu_{1+j+2k} > 1+j+2k > \mu_{d^\mu_{1+j+2k}+1} - (d^\mu_{1+j+2k}+1)$.*

1. *If $d^\mu_{1+j+2k} = 0$, then $\mu_1 = 1+j+2k$ and $(\Delta_{j,2k}, \mu)$ is fixed by $\psi_j$. Inversely, all the pairs fixed by $\psi_j$ have the form $(\Delta_{j,2k}, \mu)$ with $\mu_1 = 1+j+2k$.*

2. *If $d^\mu_{1+j+2k} \geqslant 1$, set $u = \mu_{d^\mu_{1+j+2k}} - d^\mu_{1+j+2k} - (1+j+2k)$, and $v$ the number occurrences of $1+j+2k$ in $\mu$.*

   (a) *If $v > u$, then*
   $$\psi^0_j((\Delta_{j,2k}, \mu)), \dots, \psi^{u-1}_j((\Delta_{j,2k}, \mu)) \in \{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)\,,$$
   *and $\psi^u_j((\Delta_{j,2k}, \mu)) \in \{\Delta_{j,2k}\} \times \left(\overline{\mathcal{F}}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$.*

   (b) *If $v \leqslant u$, then*
   $$\psi^0_j((\Delta_{j,2k}, \mu)), \dots, \psi^{v-1}_j((\Delta_{j,2k}, \mu)) \in \{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)\,,$$
   *and $\psi^v_j((\Delta_{j,2k}, \mu)) \in \{\Delta_{j,2k}\} \times \mathcal{P}_{j+2k+1}$.*

To prove the well-definedness of $\Psi_j$, we first observe that, for $\mu \in \mathcal{F}_j$, $\mu_1 \neq j+1$, and by Proposition 17, $(\Delta_{j,0}, \mu)$ is not fixed by $\psi_j$. Thus, as $\psi_j$ is injective, its iterations are not fixed by $\psi_j$. Moreover, for $l$ such that $|\Delta_{j,2l}| = l(1+2j+2l)) > |\mu|$, these iterations do not reach $\{\Delta_{j,2l}\} \times \mathcal{P}$ since $\psi_j$ conserves the weight sum of the pair. Let then $k$ be the largest $l$ such that $\{\Delta_{j,2l}\} \times \mathcal{P}$ is reached by the iterations. By Proposition 17, for $0 \leqslant l \leqslant k$, $w_l$ the number of iterations of $(\Delta_{j,0}, \mu)$ by $\psi_j$ in $\{\Delta_{j,2l}\} \times \mathcal{P}$ is at most equal to one plus the number of occurrences of $1+j+2l$, hence at most equal to $\frac{|\mu|}{1+j+2l}$. We then have in total at most $\frac{|\mu|(2+\log(k+1))}{2}$ iterations. Finally, at $k$,

$$\Psi_j((\Delta_{j,0}, \mu)) = \psi_j^{k+\sum_{l=0}^{k} w_l}((\Delta_{j,0}, \mu))$$

is necessarily in $\{\Delta_{j,2k}\} \times \mathcal{P}_{j+2k+1}$ as the iterations stop. Hence, $\Psi_j$ is well-defined.

*Proof of Proposition 17.* As $1+j+2k$ is a part of $\mu$, we then have $\mu_1 \geqslant 1+j+2k$. Set $\psi_j((\Delta_{j,2k}, \mu)) = (\Delta_{j,2k}, \lambda)$.

1. If $d^{\mu}_{1+j+2k} = 0$, then $\mu_1 \leqslant 1+j+2k$, so that $\mu_1 = 1+j+2k$. One can easily check that such pair $(\Delta_{j,2k}, \mu)$ is fixed by $\psi_j$. Inversely, the only pairs fixed by $\psi_j$ are in $\{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$ for some $k \geqslant 0$. By Proposition 15, the pairs of $\{\Delta_{j,2k}\} \times \left(\mathcal{F}_{1+j+2k} \cap \overline{\mathcal{P}}_{j+2k+1}\right)$ fixed by $\psi_j$ are exactly the pairs of $\{\Delta_{j,2k}\} \times \mathcal{F}_{j+2k}$ fixed by $\phi_j$. Finally, Proposition 16 gives us the form of the fixed pairs, which is $(\Delta_{j,2k}, \mu)$ with $\mu_1 = 1+j+2k$.

2. If $d^{\mu}_{1+j+2k} \geqslant 1$, then $\mu_1 \geqslant \mu_{d^{\mu}_{1+j+2k}} > 1+j+2k+d^{\mu}_{1+j+2k}$. By the second part of Section 3.2,
$$\mu_{d^{\mu}_{1+j+2k}} - 1 \geqslant 1+j+2k+d^{\mu}_{1+j+2k} \geqslant \mu_{d^{\mu}_{1+j+2k}+1}\,,$$
so that $\lambda_{d^{\mu}_{1+j+2k}+1} = 1+j+2k+d^{\mu}_{1+j+2k}$. Hence,
$$\lambda_{d^{\mu}_{1+j+2k}} - d^{\mu}_{1+j+2k} \geqslant 1+j+2k > \lambda_{d^{\mu}_{1+j+2k}+1} - (d^{\mu}_{1+j+2k}+1)\,,$$
and $d^{\lambda}_{1+j+2k} = d^{\mu}_{1+j+2k}$. Therefore, both $u = \mu_{d^{\mu}_{1+j+2k}} - (1+j+2k+d^{\mu}_{1+j+2k})$ and $v$ the number of occurrences of $1+j+2k$ decrease by 1 after $\psi_j$ is applied once, while $d^{\mu}_{1+j+2k}$ is conserved. Let $\ell(m) \geqslant t \geqslant d^{\mu}_{1+j+2k}+v$ such that $\mu_t = 1+j+2k > \mu_{t+1}$.

   (a) If $v > u$, by formally setting $(\Delta_{j,2k}, \mu^{(w)}) = \psi_j^w((\Delta_{j,2k}, \mu))$ for $w \in \{0, \dots, u\}$, we have that
$$\begin{cases} \mu^{(w)}_i = \mu_i - w & \text{if} \quad 1 \leqslant i \leqslant d^{\mu}_{1+j+2k}\,, \\ \mu^{(w)}_i = 1+d^{\mu}_{1+j+2k}+j+2k & \text{if} \quad d^{\mu}_{1+j+2k}+1 \leqslant i \leqslant d^{\mu}_{1+j+2k}+w\,, \\ \mu^{(w)}_i = \mu_{i-w} & \text{if} \quad d^{\mu}_{1+j+2k}+w+1 \leqslant i \leqslant t\,, \\ \mu^{(w)}_i = \mu_i & \text{if} \quad t+1 \leqslant i \leqslant \ell(\mu)\,, \end{cases}$$

as we recursively obtain that $d^{\mu^{(w)}}_{1+j+2k} = d^{\mu}_{1+j+2k}$,

$$\mu^{(w)}_{d^{\mu}_{1+j+2k}} = \mu_{d^{\mu}_{1+j+2k}} - w > d^{\mu}_{1+j+2k} + 1 + j + 2k$$

and $\mu^{(w)}_t = 1 + j + 2k$ for all $w \in \{0, \dots, u-1\}$. Finally, we have $\mu^{(u)}_{d^{\mu}_{1+j+2k}} = d^{\mu}_{1+j+2k} + 1 + j + 2k$ and $\mu^{(u)}_t = 1 + j + 2k$ so that $\mu^{(u)} \in \overline{\mathcal{F}}_{1+j+2k} \cap \overline{\mathcal{P}}_{1+j+2k}$.

(b) If $v \leqslant u$, by formally setting $(\Delta_{j,2k}, \mu^{(w)}) = \psi^w_j((\Delta_{j,2k}, \mu))$ for $w \in \{0, \dots, v\}$, we have that

$$\begin{cases} \mu^{(w)}_i = \mu_i - w & \text{if} \quad 1 \leqslant i \leqslant d^{\mu}_{1+j+2k}\,, \\ \mu^{(w)}_i = 1 + d^{\mu}_{1+j+2k} + j + 2k & \text{if} \quad d^{\mu}_{1+j+2k} + 1 \leqslant i \leqslant d^{\mu}_{1+j+2k} + w\,, \\ \mu^{(w)}_i = \mu_{i-w} & \text{if} \quad d^{\mu}_{1+j+2k} + w + 1 \leqslant i \leqslant t\,, \\ \mu^{(w)}_i = \mu_i & \text{if} \quad t + 1 \leqslant i \leqslant \ell(\mu)\,, \end{cases}$$

as we recursively obtain that $d^{\mu^{(w)}}_{1+j+2k} = d^{\mu}_{1+j+2k}$,

$$\mu^{(w)}_{d^{\mu}_{1+j+2k}} = \mu_{d^{\mu}_{1+j+2k}} - w > d^{\mu}_{1+j+2k} + 1 + j + 2k$$

and $\mu_t = 1 + j + 2k$ for all $w \in \{0, \dots, v-1\}$. Finally, we have $d^{\mu^{(w)}}_{1+j+2k} = u - v + d^{\mu}_{1+j+2k} + 1 + j + 2k$, and $\mu^{(u)}_t$ is either $1 + d^{\mu}_{1+j+2k} + j + 2k$ when $t = d^{\mu}_{1+j+2k} + v$, or $\mu_{t-v} > 1 + j + 2k$ when $t > d^{\mu}_{1+j+2k} + v$, so that $\mu^{(u)} \in \overline{\mathcal{P}}_{1+j+2k}$.

□

### 4.4 The maps $\Phi_j$ and $\Psi_j$ are inverse of each other

For $(\Delta_{j,2k}, \lambda) \in \{\Delta_{j,2k}\} \times \mathcal{P}_{1+j+2k}$, there exists a unique finite non-negative integer $u$ such that $\Phi_j((\Delta_{j,2k}, \lambda)) = \phi^u_j((\Delta_{j,2k}, \lambda)) \in \{\Delta_{j,2k}\} \times \mathcal{F}_j$. Then, by Proposition 15, $\psi^u_j(\Psi_j((\Delta_{j,2k}, \lambda))) = (\Delta_{j,2k}, \lambda)$, and as it belongs to $\{\Delta_{j,2k}\} \times \mathcal{P}_{1+j+2k}$, it is by definition $\Psi_j(\Phi_j((\Delta_{j,2k}, \lambda)))$. Similarly, we prove that, for $\mu \in \mathcal{F}_j$, $\Phi_j(\Psi_j((\Delta_{j,0}, \mu))) = (\Delta_{j,0}, \mu)$.

Finally, since the bijections preserve the part less or equal to $j$, $\Phi_j$ then induces a bijection from $\mathcal{M}_j \cap \overline{\mathcal{P}}_j$ to $\mathcal{F}_j \cap \overline{\mathcal{P}}_j$ and $\Psi_j = \Phi_j^{-1}$.

## 5 Bijection for Corollary 11

We here provide a bijection of Corollary 11 in the spirit of Section 3. First, observe the following correspondence.

**Lemma 18.** *There is a weight-preserving bijection between $\overline{\mathcal{F}}_j$ and $\{\Delta_{j,1}\} \times \mathcal{F}_{j+1}$.*

*Proof.* For all $\lambda$ in $\overline{\mathcal{F}}_j$, we have $\lambda_{d_j^\lambda} = j + d_j^\lambda$. Hence, set $\psi'_j(\lambda)$ to be the pair $((j+1),\mu)$, where $\mu$ consists of the parts $\lambda_1 + 1, \dots, \lambda_{d_j^\lambda - 1} + 1$ and $\lambda_i$ for $i > d_j^\lambda$. Inversely, for $((j+1),\mu) \in \{\Delta_{j,1}\} \times \mathcal{F}_{j+1}$, we set $\phi'_j((j+1),\mu)) = \lambda$ whose parts are $\mu_1 - 1, \dots, \mu_{d_{j+1}^\mu} - 1$, $j + 1 + d_{j+1}^\mu$and $\mu_i$ for $i > d_{j+1}^\mu$. The proof that $\phi'_j$ and $\psi' j$ are inverse of each other is similar to the proof of Proposition 15, as $d_j^\lambda = d_{j+1}^\mu + 1$. $\square$

To bijectively prove Corollary 11, we build two maps,

$$\Phi'_j : \bigsqcup_{k \geqslant 0} \{\Delta_{j,2k+1}\} \times \mathcal{P}_{j+2k+2} \to \{\Delta_{j,1}\} \times \mathcal{F}_{j+1}$$

and

$$\Psi'_j : \{\Delta_{j,1}\} \times \mathcal{F}_{j+1} \to \bigsqcup_{k \geqslant 0} \{\Delta_{j,2k+1}\} \times \mathcal{P}_{j+2k+2}.$$

The map $\Phi'_j$ is simply obtained by going through the process of $\Phi_j$, except that we replace all the occurrences of "$2k$" by "$2k+1$". Similarly, $\Phi'_j$ is obtained by $\Psi_j$ by replacing "$2k$" by "$2k+1$". The proof of the well-definedness of the bijection is the same as the proof provided in Section 4.

## 6 Bijective proof of Theorem 12

Before constructing the bijection for Theorem 12, we first state the key result given by Hopkins, Sellers and Yee in [8], and that provides a combinatorial link between the crank and the Durfee decomposition. Recall that, for all partitions $\lambda$,

$$\begin{aligned}
\omega(\lambda) &= \sharp\{i \in \{1,\dots,\ell(\lambda)\} : \lambda_i = 1\},\\
\eta(\lambda) &= \sharp\{i \in \{1,\dots,\ell(\lambda)\} : \lambda_i > \omega(\lambda)\},\\
\mathrm{crank}(\lambda) &= \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0\,,\\ \eta(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0\,. \end{cases}\quad.
\end{aligned}$$

**Lemma 19** (Hopkins-Sellers-Yee)**.** *Let $j \in \mathbb{Z}_{\geqslant 0}$ and $\lambda \in \mathcal{P}$. Then,*

$$\mathrm{crank}(\lambda) \leqslant -j \textit{ if and only if } \omega(\lambda) \geqslant d_j^\lambda + j\,\cdot$$

*Remark* 20. For $j \in \mathbb{Z}_{\geqslant 0}$ and $\lambda \in \mathcal{P}$, Lemma 19 implies that $\mathrm{crank}(\lambda) = -j$ if and only if $d_{j+1}^\lambda + j + 1 > \omega(\lambda) \geqslant d_j^\lambda + j$. By Remark 8, it equivalently means that $\omega(\lambda) = d_j^\lambda + j, \eta(\lambda) = d_j^\lambda$ and $d_{j+1}^\lambda = d_j^\lambda$.

*Proof of Lemma 19.* We have that $\mathrm{crank}(\emptyset) = 0$ and $d_j^\emptyset = 0$. The equivalence then stands for $\lambda = \emptyset$. Now suppose that $\lambda \neq \emptyset$, which equivalently means that $\lambda_1 > 0$ and $d_0^\lambda > 0$.

1. If $\omega(\lambda) \geqslant d_j^\lambda + j$, then, by Remark 8, $\omega(\lambda) \geqslant d_0^\lambda > 0$, and $\eta(\lambda) \leqslant d_j^\lambda$ as $\lambda_{d_j^\lambda+1} \leqslant d_j^\lambda + j \leqslant \omega(\lambda)$. Therefore, $\mathrm{crank}(\lambda) = \eta(\lambda) - \omega(\lambda) \leqslant d_j^\lambda - (d_j^\lambda + j) = -j$.

2. Otherwise, if $0 < \omega(\lambda) < d_j^\lambda + j$, then $\lambda_{d_j^\lambda} > \omega(\lambda)$ so that $\eta(\lambda) \geqslant d_j^\lambda$. Hence, $\operatorname{crank}(\lambda) = \eta(\lambda) - \omega(\lambda) > d_j^\lambda - (d_j^\lambda + j) = -j$. Finally, if $\omega(\lambda) = 0$, $\lambda_1 > 0$ so that $\operatorname{crank}(\lambda) > 0 \geqslant -j$. $\square$

For an integer $n$, set $\mathcal{C}_{\leqslant n} = \{\lambda \in \mathcal{P} : \operatorname{crank}(\lambda) \leqslant n\}$ and $\mathcal{C}_{\geqslant n} = \{\lambda \in \mathcal{P} : \operatorname{crank}(\lambda) \geqslant n\}$. Theorem 12 is then equivalent to saying that, for $j \in \mathbb{Z}_{\geqslant 0}$, there exists a weight-preserving bijection $\Gamma_j$ between $\mathcal{F}_j \cap \overline{\mathcal{P}}_j \setminus \{\emptyset, (1)\}$ and $\mathcal{C}_{\leqslant -j} \setminus \{\emptyset, (1)\}$. Also, for $\lambda \neq \emptyset$, we then always have $d_j^\lambda + j > 0$. The construction of $\Gamma_j$ is the following.

1. For $\lambda \in \mathcal{F}_j \cap \overline{\mathcal{P}}_j$ with $|\lambda| > 1$, recall that $\lambda_{d_j^\lambda} - d_j^\lambda > j > \lambda_{d_j^\lambda+1} - (d_j^\lambda + 1)$. Hence $\lambda_{d_j^\lambda} > j$. Let $\ell(\lambda) \geqslant t^\lambda \geqslant d_j$ such that $\lambda_{t^\lambda} > j = \lambda_{t^\lambda+1}$. The map $\Gamma_j$ consists in sustracting 1 to the $d_j^\lambda$ largest finite parts, deleting the part $\lambda_{t^\lambda+1} = j$ and adding $d_j^\lambda + j$ parts equal to 1, so that it is obviously weight preserving. Formally, $\Gamma_j(\lambda) = \mu$ with

$$\begin{cases} \mu_i = \lambda_i - 1 & \text{if } \; 1 \leqslant i \leqslant d_j^\lambda \,, \\ \mu_i = \lambda_i & \text{if } \; d_j^\lambda < i \leqslant t^\lambda \,, \\ \mu_i = \lambda_{i+1} & \text{if } \; t^\lambda < i \leqslant \ell(\lambda) - \chi(j \geqslant 1) \,, \\ \mu_i = 1 & \text{if } \; \ell(\lambda) + \chi(j = 0) \leqslant i \leqslant \ell(\lambda) + d_j^\lambda + j - \chi(j \geqslant 1) \,. \end{cases}$$

   Here $\chi(A)$ equals 1 if $A$ is true and 0 if not. Observe that $\mu_{d_j^\lambda} - d_j^\lambda \geqslant j > \mu_{d_j^\lambda+1} - (d_j^\lambda + 1)$, as $\mu_{d_j^\lambda+1} = \lambda_{d_j^\lambda+1}$ if $d_j^\lambda < t^\lambda$, or $\lambda_{d_j^\lambda+2}$ if $d_j^\lambda = t^\lambda < \ell(\lambda) - \chi(j \geqslant 1)$, or 1 if $d_j^\lambda = t^\lambda = \ell(\lambda) - \chi(j \geqslant 1)$. Therefore, $d_j^\mu = d_j^\lambda$ and $\mu \in \mathcal{C}_{\leqslant -j}$. Finally, note that $\ell(\mu) = \ell(\lambda) + d_j^\lambda + j - \chi(j \geqslant 1) \geqslant 2d_j^\mu + j - \chi(j \geqslant 1)$.

2. Inversely, let $\emptyset \neq \mu \in \mathcal{C}_{-j}$ with $|\lambda| > 1$. If $\mu_{d_j^\mu} = 1$, then $d_j^\mu = 1$ and $j = 0$. In that case, as $\mu \neq (1)$, $\ell(\mu) \geqslant 2$, so that $\ell(\mu) \geqslant 2d_j^\mu$. If $\mu_{d_j^\mu} > 1 = \mu_{\ell(\mu) - d_j^\mu + j + 1}$, then $\ell(\mu) \geqslant 2d_j^\mu + j$. We thus always have $\ell(\mu) \geqslant 2d_j^\mu + j$. For $j \geqslant 1$, let $d_j^\mu \leqslant t^\mu \leqslant \ell(\mu) - d_j^\mu - j$ such that $\mu_t > j \geqslant \mu_{t+1}$, and for $j = 0$, set $t^\mu = \ell(\mu) - d_0^\mu$. The map $\Gamma_j^{-1}$ then consists in deleting the $d_j^\mu + j$ smallest parts equal to 1, adding 1 to the $d_j^\mu$ largest finite parts and a part $j$. Formally, $\Gamma_j^{-1}(\mu) = \lambda$ with

$$\begin{cases} \lambda_i = \mu_i + 1 & \text{if } \; 1 \leqslant i \leqslant d_j^\mu \,, \\ \lambda_i = \mu_i & \text{if } \; d_j^\mu < i \leqslant t^\mu \,, \\ \lambda_i = \mu_{i-1} & \text{if } \; t^\mu + 1 < i \leqslant \ell(\mu) - d_j^\mu - j + 1 \,, \\ \lambda_{t^\mu+1} = j \,. \end{cases}$$

   As $\mu_{d_j^\mu} - d_j^\mu \geqslant j > \mu_{d_j^\mu+1} - (d_j^\mu + 1)$ and $\mu_{d_j^\mu+1} \geqslant \lambda_{d_j^\mu+1}$, we have that $\lambda_{d_j^\mu} - d_j^\mu > j > \lambda_{d_j^\mu+1} - (d_j^\mu + 1)$. Therefore, $d_j^\lambda = d_j^\mu$ and $\lambda \in \mathcal{F}_j \cap \overline{\mathcal{P}}_j$. Note that $\ell(\lambda) = \ell(\mu) - d_j^\mu - j + \chi(j \geqslant 1)$.

The map $\Gamma_j$ is well-defined as $\Gamma_j((\mathcal{F}_j \cap \overline{\mathcal{P}}_j) \setminus \{\emptyset, (1)\}) \subset \mathcal{C}_{\leqslant -j}$, and since $\mu = \Gamma_j(\lambda)$ satisfies $d_j^\mu = d_j^\lambda$, it is straightforward that $\Gamma_j^{-1}(\Gamma_j(\lambda)) = \lambda$ as the corresponding $t^\mu$ equals $t^\lambda$. Inversely, $\Gamma_j^{-1}(\mu) \subset \mathcal{F}_j \cap \overline{\mathcal{P}}_j$ and $\Gamma_j(\Gamma_j^{-1}(\mu)) = \mu$ for all $\mu \in \mathcal{C}_{\leqslant -j}$ with $|\lambda| > 1$.

**Example 21.** We have the following table:

| $j$ | $\lambda$ | $d_j^\lambda$ | $\Gamma_j(\lambda)$ | $\mathrm{crank}(\Gamma_j(\lambda))$ |
|---|---|---|---|---|
| 1 | $(1)$ | 0 | $(1)$ | $-1$ |
| 0 | $\Delta_{0,2k}$ | $k$ | $(\underbrace{2k-1,\dots,k}_{k \text{ consecutive}}, \underbrace{k,\dots,1}_{k \text{ consecutive}}, \underbrace{1,\dots,1}_{k})$ | $-3\chi(k \geqslant 1)$ |
| $\geqslant 1$ | $\Delta_{j-1,2k}$ | $k$ | $(\underbrace{2k-1+j,\dots,k+j}_{k \text{ consecutive}}, \underbrace{k+j,\dots,1+j}_{k \text{ consecutive}}, \underbrace{1,\dots,1}_{k+j})$ | $-j-\chi(k \geqslant 1)$ |
| 0 | $(13,10,9,9,4,3,2,2,1,1)$ | 4 | $(12,9,8,8,4,3,2,2,1,1,1,1,1,1)$ | $-2$ |
| 3 | $(12,9,9,5,4,4,4,3,2,2)$ | 3 | $(11,8,8,5,4,4,4,2,2,1,1,1,1,1,1)$ | $-3$ |
| 5 | $(11,8,7,7,5,5,4,3,2,2)$ | 2 | $(10,7,7,7,5,4,3,2,2,1,1,1,1,1,1,1)$ | $-6$ |

.

# 7 Bijective proof of Corollary 2

We here present a crank-sign reversing involution provided by Berkovich and Garvan in [3]. The involution $\Lambda$ on $\mathcal{P} \setminus \{\emptyset, (1)\}$ is such that $\mathrm{crank}(\Lambda(\lambda)) = -\mathrm{crank}(\lambda)$. Let $\lambda \in \mathcal{P}$ with $|\lambda| > 1$, and construct $\nu = \Lambda(\lambda)$ as follows.

1. (a) If $\omega(\lambda) = 0$, then $\lambda_1 \geqslant \lambda_{\ell(\lambda)} > 1$. Set

$$\begin{cases} \nu_i = \lambda_{i+1} & \text{if } \ 1 \leqslant i \leqslant \ell(\lambda) - 1\,, \\ \nu_i = 1 & \text{if } \ \ell(\lambda) \leqslant i \leqslant \lambda_1 + \ell(\lambda) - 1\,. \end{cases}$$

Hence, $|\nu| = |\lambda|$, $\omega(\nu) = \lambda_1 > 0$ and $\eta(\nu) = 0$ so that $\mathrm{crank}(\nu) = -\lambda_1 = -\mathrm{crank}(\lambda)$.

(b) If $\omega(\lambda) > 0$ and $\eta(\lambda) = 0$, then set

$$\begin{cases} \nu_1 = \omega(\lambda) \\ \nu_i = \lambda_{i-1} & \text{if } \ 2 \leqslant i \leqslant \ell(\lambda) - \omega(\lambda) + 1\,. \end{cases}$$

Hence, $|\nu| = |\lambda|$, $\omega(\nu) = 0$, and $\mathrm{crank}(\nu) = \omega(\lambda) = -\mathrm{crank}(\lambda)$.

One can easily check that these two cases are inverse of one another.

2. If $\omega(\lambda), \eta(\lambda) > 0$, let $\rho(\lambda) = \max\{\omega(\lambda), \lambda_2 - 1\}$ and let $\lambda^*$ be the conjugate of $\lambda$, which is defined by $\lambda_i^* = \sharp\{u : \lambda_u \geqslant i\}$ for all $i \in \{1, \dots, \lambda_1\}$. Then, $\ell(\lambda^*) = \lambda_1$, $\eta(\lambda) = \lambda^*_{\omega(\lambda)+1}$ and $\omega(\lambda) = \lambda_1^* - \lambda_2^*$. We thus set

$$\begin{cases} \nu_1 = \lambda_2^* + \lambda_1 - \rho(\lambda)\,, \\ \nu_i = 1 + \lambda_i^* & \text{if } \ 2 \leqslant i \leqslant \omega(\lambda)\,, \\ \nu_i = \lambda_{i+1}^* & \text{if } \ \omega(\lambda) < i \leqslant \rho(\lambda)\,, \\ \nu_i = 1 & \text{if } \ \rho(\lambda) < i \leqslant \rho(\lambda) + \eta(\lambda)\,. \end{cases}$$

For all $\omega(\lambda) < i \leqslant \rho(\lambda)$, $2 = \lambda^*_{\lambda_2} \leqslant \nu_i \leqslant \lambda^*_{\omega(\lambda)+2} \leqslant \eta(\lambda)$. Moreover, $\lambda^*_2 \geqslant \lambda^*_{\omega(\lambda)+1} = \eta(\lambda)$, $\lambda_1 - \rho(\lambda) \geqslant 1$ as $\lambda_1 - \omega(\lambda) \geqslant 1$ and $\lambda_1 - \lambda_2 + 1 \geqslant 1$, and for all $2 \leqslant i \leqslant \omega(\lambda)$, $\nu_i \geqslant 1 + \lambda^*_{\omega(\lambda)} \geqslant 1 + \eta(\lambda)$. Therefore, $\omega(\nu) = \eta(\lambda)$ and $\eta(\nu) = \omega(\lambda)$, and $\mathrm{crank}(\nu) = -\mathrm{crank}(\mu)$. Furthermore, since $\rho(\lambda)+1 \geqslant \lambda_2$, $\lambda^*_i = 1$ for all $\rho(\lambda)+1 < i \leqslant \lambda_1$, and

$$\begin{aligned}|\nu| &= \underbrace{\omega(\lambda) + \lambda^*_2}_{\lambda^*_1} + (\lambda_1 - \rho(\lambda) - 1) + \sum_{i=2}^{\omega(\lambda)} \lambda^*_i + \sum_{\omega(\lambda)+2}^{\rho(\lambda)+1} \lambda^*_i + \underbrace{\eta(\lambda)}_{\lambda^*_{\omega(\lambda)+1}} \\ &= \sum_{i=1}^{\lambda_1} \lambda^*_i = |\lambda^*|\end{aligned}$$

so that $|\nu| = |\lambda|$. In addition,

$$\begin{cases} \nu^*_i = \lambda_i - 1 & \text{if } \ 2 \leqslant i \leqslant \eta(\lambda)\,, \\ \nu^*_i = \lambda_{i-1} & \text{if } \ \eta(\lambda)+1 < i \leqslant \lambda^*_2 + 1\,, \\ \nu^*_{\eta(\lambda)+1} = \omega(\lambda)\,, \end{cases}$$

and $\rho(\nu) = \lambda^*_2$ if $\omega(\lambda) > 1$ and $\rho(\nu) = \eta(\lambda) = \lambda^*_2$ if $\omega(\lambda) = 1$ so that $\rho(\nu) = \lambda^*_2$. Finally, $\nu^*_2 = \rho(\lambda)$ if $\rho(\lambda) > \omega(\lambda)$, and $\nu^*_2 = \omega(\lambda) = \rho(\lambda)$ if $\rho(\lambda) > \omega(\lambda)$ as $\nu_{\omega(\lambda)} \geqslant \omega(\lambda) + 1 \geqslant 2$. Hence, $\nu^*_2 = \rho(\lambda)$, and for $\Lambda(\nu) = \kappa$, we have

$$\begin{cases} \kappa_1 = \rho(\lambda) + \nu_1 - \lambda^*_2 = \lambda_1\,, \\ \kappa_i = 1 + \nu^*_i = \lambda_i & \text{if } \ 2 \leqslant i \leqslant \eta(\lambda)\,, \\ \kappa_i = \nu^*_{i+1} = \lambda_i & \text{if } \ \eta(\lambda) < i \leqslant \lambda^*_2\,, \\ \kappa_i = 1 & \text{if } \ \lambda^*_2 < i \leqslant \lambda^*_2 + \omega(\lambda) = \lambda^*_1\,. \end{cases}$$

We then conclude that $\Lambda(\Lambda(\lambda)) = \lambda$.

**Example 22.** We have the following table:

| $\lambda$ | $\omega(\lambda)$ | $\eta(\lambda)$ | $\rho(\lambda)$ | $\Lambda(\lambda)$ |
|---|---|---|---|---|
| $(12,9,8,8,4,3,2,2,1,1,1,1,1,1)$ | 6 | 4 | 8 | $(12,9,7,6,5,5,4,2,1,1,1,1)$ |
| $(11,8,8,5,4,4,4,2,2,1,1,1,1,1,1)$ | 6 | 3 | 7 | $(13,10,8,8,5,4,3,1,1,1)$ |
| $(10,7,7,7,5,4,3,2,2,1,1,1,1,1,1,1)$ | 7 | 1 | 7 | $(12,10,8,7,6,5,5,1)$ |

.

# 8 Conclusion

As we construct the different intermediate bijections in Sections 3, 6 and 7, we now present the full scope of the bijection for Theorem 4. For $j \in \mathbb{Z}_{\geqslant 0}$, the bijection between $\mathcal{M}_j \cap \overline{\mathcal{P}}_j$ and the set of partitions with crank at least equal to $j$ is given by $\Lambda \circ \Gamma_j \circ \Phi_j$, and its inverse is $\Psi_j \circ \Gamma_j^{-1} \circ \Lambda$.

**Example 23.** Using Examples 14, 21 and 22, the images of $(11, 8, 7, 7, 5, 5, 4, 3, 2, 2)$ in the cases $j = 0, 3, 5$ are respectively

$$(12, 9, 7, 6, 5, 5, 4, 2, 1, 1, 1, 1), (13, 10, 8, 8, 5, 4, 3, 1, 1, 1) \text{ and } (12, 10, 8, 7, 6, 5, 5, 1)\,.$$

In particular, in Theorem 3, the partition $(11, 8, 7, 7, 5, 5, 4, 3, 2, 2)$ with odd mex 1 can be associated to the partition $(12, 9, 7, 6, 5, 5, 4, 2, 1, 1, 1, 1)$ with non-negative crank 2.

**Example 24.** Here is a list of all the partitions of 9 in $\mathcal{M}_0$ and their successive images by applying $\Phi_0$, $\Gamma_0$ and $\Lambda$.

| $\lambda \in \mathcal{M}_0$ | $\Phi_0(\lambda) \in \mathcal{F}_0$ | $\Gamma_0(\Phi_0(\lambda)) \in \mathcal{C}_{\leqslant 0}$ | $\Lambda(\Gamma_0(\Phi_0(\lambda))) \in \mathcal{C}_{\geqslant 0}$ |
|---|---|---|---|
| $(9)$ | $(9)$ | $(8,1)$ | $(8,1)$ |
| $(7,2)$ | $(8,1)$ | $(7,1,1)$ | $(6,2,1)$ |
| $(6,3)$ | $(6,3)$ | $(5,2,1,1)$ | $(5,3,1)$ |
| $(5,4)$ | $(5,4)$ | $(4,3,1,1)$ | $(4,3,1,1)$ |
| $(5,2,2)$ | $(7,1,1)$ | $(6,1,1,1)$ | $(4,2,2,1)$ |
| $(4,3,2)$ | $(4,3,2)$ | $(3,2,2,1,1)$ | $(4,4,1)$ |
| $(3,3,3)$ | $(4,4,1)$ | $(3,3,1,1,1)$ | $(3,3,3)$ |
| $(3,2,2,2)$ | $(6,1,1,1)$ | $(5,1,1,1,1)$ | $(2,2,2,2,1)$ |
| $(6,2,1)$ | $(5,3,1)$ | $(4,2,1,1,1)$ | $(3,3,2,1)$ |
| $(5,2,1,1)$ | $(4,3,1,1)$ | $(3,2,1,1,1,1)$ | $(4,3,2)$ |
| $(4,2,2,1)$ | $(3,3,2,1)$ | $(2,2,2,1,1,1)$ | $(3,2,2,2)$ |
| $(4,2,1,1,1)$ | $(3,3,1,1,1)$ | $(2,2,1,1,1,1,1)$ | $(5,2,2)$ |
| $(2,2,2,2,1)$ | $(5,1,1,1,1)$ | $(4,1,1,1,1,1)$ | $(5,4)$ |
| $(2,2,2,1,1,1)$ | $(4,1,1,1,1,1)$ | $(3,1,1,1,1,1,1)$ | $(6,3)$ |
| $(2,2,1,1,1,1,1)$ | $(3,1,1,1,1,1,1)$ | $(2,1,1,1,1,1,1,1)$ | $(7,2)$ |
| $(2,1,1,1,1,1,1,1)$ | $(2,1,1,1,1,1,1,1)$ | $(1,1,1,1,1,1,1,1,1)$ | $(9)$ |

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