On the Local and Global Mean Orders of Sub-k-Trees of k-Trees

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Abstract

In this paper we show that for a given k-tree T with a k-clique C, the local mean order of all sub-k-trees of T containing C is not less than the global mean order of all sub-k-trees of T, and the path-type k-trees have the smallest global mean sub-k-tree order among all k-trees of a given order. These two results give solutions to two problems of Stephens and Oellermann [J. Graph Theory 88 (2018), 61-79] concerning the mean order of sub-k-trees of k-trees. Furthermore, the mean sub-k-tree order as a function on k-trees is shown to be monotone with respect to inclusion. This generalizes Jamison's result for the case $k = 1$ [J. Combin. Theory Ser. B 35 (1983), 207-223].

Mathematics Subject Classifications: 05C05, 05C30, 05C35

1 Introduction

In the 1980s Jamison [11, 12] initiated the study of the mean order of the subtrees of a tree. He studied the extremal problem and proved that the path P_n has the smallest mean subtree order, namely $\frac{n+2}{3}$, among all trees of a fixed order n. However, the problem of describing the tree(s) of a given order with the largest mean subtree order remains open, although several other open problems and conjectures posed in [11] and [12] were subsequently solved in [4, 8, 16, 18, 25, 27, 28]. In recent years, some extensions of this

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mean to other connected graphs have been considered, such as the mean order of the subtrees of an arbitrary graph (not necessarily a tree) (see [3, 5, 15, 26]), the mean order of the connected induced subgraphs of a graph (see [1, 9, 10, 13, 22, 23, 24]), and the mean order of the sub-k-trees of a k-tree (see [20]). Note that all of these means are equal to the mean subtree order of a tree if the underlying graph is a tree.

In this paper we further investigate the mean order of the sub-k-trees of a k-tree. As a generalization of trees, the class of k-trees (introduced by Beineke and Pippert in [2]) can be defined recursively as follows.

Definition 1. Fix a positive integer k.

1. The complete graph K_k is a k-tree.
2. If T is a k-tree, then so is the graph obtained from T by adding a vertex adjacent to all vertices of some k-clique of T.

Note that a 1-tree is precisely a tree. Throughout we use T to denote an arbitrary k-tree, and C to denote an arbitrary k-clique of T. The k-tree with just k vertices is *trivial*. All other k-trees are *non-trivial*. It is worth mentioning that non-trivial k-trees are in one-to-one correspondence with "tight $(k+1)$-trees", which are generalisations of trees to $(k+1)$-uniform hypergraphs: each nontrivial k-tree is the underlying graph of a corresponding tight $(k+1)$-tree, and the edge set of each tight $(k+1)$-tree is the set of $(k+1)$-cliques of the corresponding k-tree (see [6], although the "tight" was added in the later usage).

The original k-clique in the recursive construction of a k-tree is called the *base k-clique*. When we refer to a *sub-k-tree* X of a k-tree T, we mean that X is a subgraph of T that is itself a k-tree. A sub-1-tree is also called a *subtree*. We denote by $\mathcal{S}(T)$ the collection of all sub-k-trees of T, and by $\mathcal{S}(T;C)$ the collection of sub-k-trees of T containing the k-clique C. Let $N(T) = |\mathcal{S}(T)|$ and $N(T;C) = |\mathcal{S}(T;C)|$.

For an integer $1 \leqslant r \leqslant k+1$, we use $Q_r(T)$ to denote the number of r-cliques of T. Note that $Q_1(T) = |T|$ is the *order* of T. We denote by $O_r(T)$ the total number of r-cliques over all sub-k-trees of T, that is, $O_r(T) = \sum_{X\in\mathcal{S}(T)} Q_r(X)$. Likewise, let $O_r(T;C)$ denote the number of r-cliques over all sub-k-trees of T containing the k-clique C, i.e., $O_r(T;C) = \sum_{X\in\mathcal{S}(T;C)} Q_r(X)$. For $r = 1$, we use the notations $O(T) = O_1(T)$, $O(T;C) = O_1(T;C)$ and refer to them as the *global order* and *local order* at C, respectively. Then the average number of r-cliques in a sub-k-tree of T is given by

$$\mu_r(T) = \frac{O_r(T)}{N(T)}.$$

The average number of r-cliques in a sub-k-tree of T containing the k-clique C is given by

$$\mu_r(T;C) = \frac{O_r(T;C)}{N(T;C)}.$$

Again we write $\mu(T) = \mu_1(T), \mu(T;C) = \mu_1(T;C)$ and refer to them as the *global mean order* and *local mean order* at C, respectively.

Given a k-clique C, we define the *degree* of C as the number of $(k+1)$-cliques containing it, denoted by $deg_T(C)$, which is consistent with the conventional notion of the degree of a vertex. A k-clique of degree at least 3 will be called a *major k-clique*. We say a vertex is *simplicial* if its neighbours induce a clique. A simplicial vertex of a k-tree of order $n \geqslant k+1$ with degree k is called a *k-leaf*. Thus a 1-leaf is a *leaf*. A k-clique containing a k-leaf is called a *simplicial k-clique*.

Note that adding a vertex into a k-tree induces $\binom{k}{r-1}$ additional r-cliques. Then we have the following formula (see also [7]) for the number of r-cliques in a k-tree.

Proposition 2. *Let T be a k-tree of order n. Then for $1 \leqslant r \leqslant k+1$, the number of r-cliques in T is $\binom{k}{r} + (n-k)\binom{k}{r-1}$. In particular, the number of k-cliques in T is $(n-k)k+1$ and the number of $(k+1)$-cliques in T is $n-k$.*

Some subclasses of k-trees deserve special attention, such as *path-type k-trees, star-type k-trees and aster-type k-trees.* They are generalizations of special subclasses of trees.

Definition 3 (path-type $\boldsymbol{k}$-trees)**.** Fix a positive integer k.

1. The complete graphs K_k and K_{k+1} are a path-type k-tree.

2. If P is a path-type k-tree, then so is the graph obtained from P by adding a vertex adjacent to all vertices of some simplicial k-clique of P.

Thus every path-type k-tree with more than $k+1$ vertices has precisely two k-leaves, one of which is the most recently added vertex. Moreover, it is easy to see that every path-type k-tree on n vertices has the same number and average order of sub-k-trees, although they are not all isomorphic (see also [20]). The class of path-type k-trees has been previously studied under the name k-path graphs (see [17, 19]).

Definition 4 (star-type $\boldsymbol{k}$-trees)**.** Fix a positive integer k.

1. The complete graph K_k is a star-type k-tree.

2. If S is a star-type k-tree, then so is the graph obtained from S by adding a vertex adjacent to all vertices of the base k-clique of S.

Definition 5 (aster-type $\boldsymbol{k}$-trees)**.** Fix a positive integer k.

1. The complete graph K_k is an aster-type k-tree.

2. If A is an aster-type k-tree, then so is the graph obtained from A by adding a vertex adjacent to all vertices of the base k-clique of A or any simplicial k-clique of A.

For $k = 1$, the above three graphs are exactly the paths, stars, and asters (i.e., trees with at most one vertex of degree greater than 2), respectively. Note that path-type k-trees and star-type k-trees are necessarily aster-type k-trees, but the converse is not true.

Stephens and Oellermann [20] initiated the first study of the global mean and the local mean orders of sub-k-trees of k-trees. Sharp lower bounds on the local mean orders of all sub-k-trees containing a given k-clique and a given sub-k-tree were derived, respectively. A k-tree without major k-cliques is called a *simple-clique k-tree* (or simply, SC k-tree), which forms a wide graph class. For example, SC 2-trees are just maximal outerplanar graphs, and SC 3-trees are just maximal planar chordal graphs (see [17]). And obviously, path-type k-trees are necessarily SC k-trees, but the converse is not true. For the global mean orders of sub-k-trees of a k-tree, Stephens and Oellermann proved that the path-type k-trees have the minimum number of sub-k-trees and the smallest global mean sub-k-tree order among all SC k-trees of a given order. Moreover, several problems were also asked in [20]. In this paper, we mainly consider the following two:

Problem 6 ([20, Problem 1])**.** For a given k-tree T with k-clique C, is the local mean order of all sub-k-trees containing C an upper bound for the global mean order of all sub-k-trees of T?

Problem 7 ([20, Problem 3])**.** Do the path-type k-trees have the smallest global mean sub-k-tree order among all k-trees with a given order?

We start with the notion of the dual of a k-tree in Section 2, which can reduce a k-tree to a block graph. Using this reduction, we show that the path-type k-trees and the star-type k-trees have the minimum and the maximum number of sub-k-trees, respectively, among all k-trees of a given order. This generalizes a known result of Székely and Wang [21] on the number of subtrees of a tree. We also show that for any k-tree T with a k-clique C, the local mean order of all sub-k-trees of T containing C is not less than the global mean order of all sub-k-trees of T. This gives an affirmative answer to Problem 6. In Section 3, we prove that the path-type k-trees have the smallest global mean sub-k-tree order among all k-trees of a given order, thus giving an affirmative answer to Problem 7. It is also shown that for any k-tree, the mean order of its sub-k-trees is asymptotically equal to the mean order of the connected induced subgraphs of its dual. In Section 4, the mean sub-k-tree order as a function on k-trees is shown to be monotone with respect to inclusion. This generalizes Jamison's results [11] for the case $k = 1$. We conclude in Section 5 with an open question.

2 Comparing local and global mean orders

It was shown in [11, Theorem 3.9] that for any tree T and any vertex v in T, the local mean order of subtrees containing v is an upper bound on the global mean order of all subtrees of T, that is, $\mu(T; v) \geqslant \mu(T)$. In this section, we generalize this result by showing that $\mu(T; C) \geqslant \mu(T)$ for any k-tree T and any k-clique C of T, thus answering Problem 6. To do this, we first introduce the notion of the dual of a k-tree T, which is also known as the $(k + 1)$-line graph of T (see [17]). In the case $k = 1$ it is just the normal line graph of the tree. Using this tool, we also obtain an extremal result on the number of sub-k-trees of a k-tree. That is, the path-type k-trees and the star-type k-trees have the minimum and the maximum number of sub-k-trees, respectively, among all k-trees of a given order.

For a k-tree T, we use $\mathsf{CL}_{k+1}(T)$ to denote the set of $(k+1)$-cliques of T.

Definition 8. Let T be a k-tree. The dual of T, denoted by T^*, is the graph defined as follows:

1. If X is a $(k+1)$-clique in T, then X is a vertex in T^*. Hence $V(T^*)$ is the set of $(k+1)$-cliques in T, i.e., $V(T^*) = \mathsf{CL}_{k+1}(T)$.

2. If X and Y are $(k+1)$-cliques in T such that their intersection is a k-clique, then $XY \in E(T^*)$.

It follows from the definition that T^* is a block graph, i.e., graph for which every block (maximal connected subgraph without a cut-vertex) is a clique (see also [20]). And by Proposition 2, T^* has order $|T| - k$. Figure 1 gives an example of a 2-tree T and its dual. The following result was derived in [20, Theorem 26].

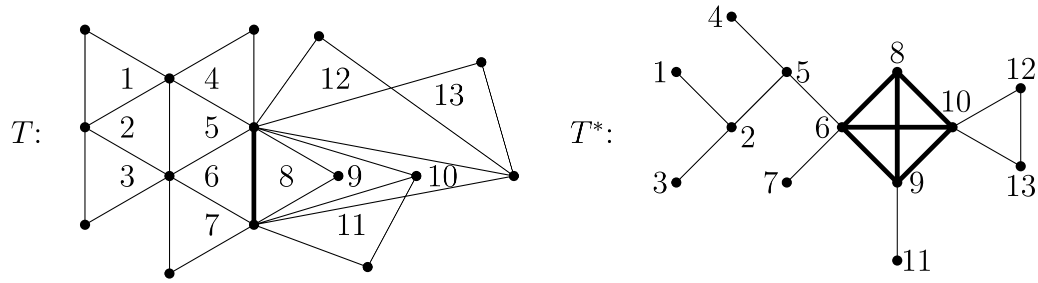

Figure 1: A 2-tree T (left) with its 3-cliques labeled and the dual T^* (right). Bolded in T is a major 2-clique of degree 4, which corresponds to the bolded 4-clique in T^*.

Theorem 9 ([20, Theorem 26]). *For any k-tree T, there is a one-to-one correspondence between non-trivial sub-k-trees of T and connected induced subgraphs of the dual of T.*

We need more helpful concepts before presenting the main results of this section. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is $|G|$. Denote by $\mathcal{C}(G)$ the collection of all connected induced subgraphs of G. For $U \subseteq V(G)$, denote by $\mathcal{C}(G;U)$ (resp., $\mathcal{C}^\star(G;U)$) the collection of all connected induced subgraphs of G containing all (resp., at least one) of the vertices of U. Let $\overline{N}(G) = |\mathcal{C}(G)|, \overline{N}(G;U) = |\mathcal{C}(G;U)|, \overline{N}^\star(G;U) = |\mathcal{C}^\star(G;U)|$, $\overline{O}(G) = \sum_{X\in\mathcal{C}(G)} |X|, \overline{O}(G;U) = \sum_{X\in\mathcal{C}(G;U)} |X|$, and $\overline{O}^\star(G;U) = \sum_{X\in\mathcal{C}^\star(G;U)} |X|$. Then

$$\overline{\mu}(G) = \frac{\overline{O}(G)}{\overline{N}(G)}, \quad \overline{\mu}(G;U) = \frac{\overline{O}(G;U)}{\overline{N}(G;U)} \quad \text{and} \quad \overline{\mu}^\star(G;U) = \frac{\overline{O}^\star(G;U)}{\overline{N}^\star(G;U)}$$

denote, respectively, the mean order of all connected induced subgraphs of G, the mean order of all connected induced subgraphs of G containing every vertex of U and the mean order of all connected induced subgraphs of G containing at least one vertex of U. If U contains only one single vertex v, then $\overline{\mu}(G;U) = \overline{\mu}^\star(G;U) = \overline{\mu}(G;v)$.

The following result established in [1] is useful for our subsequent proofs.

Lemma 10 ([1, Theorem 4.7]). *Let G be a connected block graph.*

(i) If v is a vertex of G, then $\overline{\mu}(G;v) \geqslant \overline{\mu}(G)$.

(ii) If U is the vertex set of a block of G, then $\overline{\mu}^\star(G;U) \geqslant \overline{\mu}(G)$.

Recall that $Q_k(T)$ is the number of k-cliques of a k-tree T. Then the following result is obvious from Theorem 9.

Lemma 11. *For any k-tree T with dual T^*, we have*

$$N(T) = \overline{N}(T^*) + Q_k(T).$$

Székely and Wang [21] studied the extremal problem concerning the number of subtrees of a tree. They proved the following:

Theorem 12 ([21, Theorem 3.1]). *The path P_n has $\binom{n+1}{2}$ subtrees, fewer than any other tree of order n. The star $K_{1,n-1}$ has $2^{n-1}+n-1$ subtrees, more than any other tree of order n.*

Note that every tree is a connected block graph. The extremal problem concerning the number of subtrees of a block graph was recently considered in [14]. We also note that a subtree of a tree T is a connected induced subgraph of T. Below we consider the extremal problem concerning the number of connected induced subgraphs of a block graph.

Lemma 13. *For any connected block graph G of order n, we have*

$$\binom{n+1}{2} \leqslant \overline{N}(G) \leqslant 2^n - 1$$

with left equality if and only if $G \cong P_n$ and right equality if and only if $G \cong K_n$.

Proof. The right inequality $\overline{N}(G) \leqslant \overline{N}(K_n) = 2^n - 1$ clearly holds because each nonempty subset of vertices in a complete graph induces a connected subgraph, and clearly equality holds if and only if $G \cong K_n$. Then we focus on the left inequality. Since the connected induced subgraphs of a tree are precisely the subtrees of that tree, by Theorem 12, we have $\overline{N}(P_n) = \binom{n+1}{2}$. Let X be a spanning tree of G. It is easy to see that $\overline{N}(G) \geqslant \overline{N}(X)$ with equality if and only if $G \cong X$. Combining it with Theorem 12, we conclude that $\overline{N}(G) \geqslant \overline{N}(X) \geqslant \overline{N}(P_n)$ with equality if and only if $G \cong P_n$. □

It is clear from the proof that Lemma 13 holds for general connected graphs, not just for block graphs. The following extremal result on the number of sub-k-trees of a k-tree is a generalization of Theorem 12.

Theorem 14. *For any k-tree T of order n, we have*

$$\binom{n-k+1}{2} + (n-k)k + 1 \leqslant N(T) \leqslant 2^{n-k} + (n-k)k$$

with left equality if and only if T is a path-type k-tree and right equality if and only if T is a star-type k-tree.

Proof. Let P and S be a path-type k-tree and a star-type k-tree, respectively, of order n. Then P^* is a path of order $n-k$ and S^* is a complete graph of order $n-k$. Recall that T^* is a block graph of order $n-k$. By Lemma 13, we have $\overline{N}(P^*) \leqslant \overline{N}(T^*) \leqslant \overline{N}(S^*)$. Moreover, it follows from Proposition 2 that $Q_k(P) = Q_k(S) = Q_k(T) = (n-k)k+1$. Thus

$$\overline{N}(P^*) + Q_k(P) \leqslant \overline{N}(T^*) + Q_k(T) \leqslant \overline{N}(S^*) + Q_k(S).$$

Therefore, by Lemmas 11 and 13, we have

$$\binom{n-k+1}{2} + (n-k)k + 1 \leqslant N(T) \leqslant 2^{n-k} + (n-k)k.$$

Note that the dual T^* of a k-tree T is a path (resp., a complete graph) if and only if T is a path-type k-tree (resp., a star-type k-tree). Hence the left equality holds if and only if T is a path-type k-tree and the right equality holds if and only if T is a star-type k-tree. □

Next we compare the global and the local mean orders, which needs the following three lemmas.

Lemma 15. *For any k-tree T of order n with dual T^*, we have*

$$\mu(T) = \frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1} + k.$$

Proof. By Proposition 2, the number of $(k+1)$-cliques in a k-tree of order n is $n-k$. Note that each term differs by k and so does the average. Hence we have $\mu_{k+1}(T) = \mu(T) - k$. Moreover, by Lemma 11 and Proposition 2, we have $N(T) = \overline{N}(T^*) + Q_k(T) = \overline{N}(T^*) + (n-k)k + 1$. Hence

$$\begin{aligned}
\mu_{k+1}(T) &= \frac{O_{k+1}(T)}{N(T)} = \frac{1}{N(T)} \sum_{X \in \mathcal{S}(T)} Q_{k+1}(X) \\
&= \frac{1}{N(T)} \sum_{X \in \mathcal{S}(T)} |X^*| = \frac{1}{N(T)} \sum_{Y \in \mathcal{C}(T^*)} |Y| \\
&= \frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1}.
\end{aligned}$$

Combining these two equalities, we obtain the desired result. □

For a k-tree T with a k-clique C, we denote by $\mathsf{CL}_{k+1}(T;C)$ the set of $(k+1)$-cliques of T containing C.

Lemma 16. *For any k-tree T with a k-clique C, let $B \subseteq V(T^*)$ such that $B = \mathsf{CL}_{k+1}(T;C)$. Then*

$$\mu(T;C) = \frac{\overline{O}^{\star}(T^*;B)}{\overline{N}^{\star}(T^*;B) + 1} + k.$$

Proof. Recall that $\mathcal{C}^\star(T^*;B)$ is a set of connected induced subgraphs of T^* containing at least one vertex of B. We define a function $f : \mathcal{S}(T;C) \setminus \{C\} \mapsto \mathcal{C}^\star(T^*;B)$ by $f(X) = X^*$ for all $X \in \mathcal{S}(T;C) \setminus \{C\}$. Note that for any $X \in \mathcal{S}(T;C) \setminus \{C\}$, X contains at least one $(k+1)$-clique of $\mathsf{CL}_{k+1}(T;C)$, which implies X^* contains at least one vertex of B, i.e., $X^* \in \mathcal{C}^\star(T^*;B)$. Thus f is well-defined. Then the argument used in [20, Theorem 26] can be used to show that f is a bijection, from which it follows that $N(T;C) = \overline{N}^\star(T^*;B)+1$. Then

$$\begin{aligned}\mu_{k+1}(T;C) &= \frac{O_{k+1}(T;C)}{N(T;C)} = \frac{1}{N(T;C)} \sum_{X\in\mathcal{S}(T;C)} Q_{k+1}(X)\\ &= \frac{1}{N(T;C)} \sum_{X\in\mathcal{S}(T;C)} |X^*| = \frac{1}{N(T;C)} \sum_{Y\in\mathcal{C}^\star(T^*;B)} |Y|\\ &= \frac{\overline{O}^\star(T^*;B)}{\overline{N}^\star(T^*;B)+1}.\end{aligned}$$

Moreover, the same argument as in Lemma 15 shows that

$$\mu_{k+1}(T;C) = \mu(T;C) - k.$$

Therefore

$$\mu(T;C) = \frac{\overline{O}^\star(T^*;B)}{\overline{N}^\star(T^*;B)+1} + k. \quad \square$$

Lemma 17. *For a connected graph G of order n with $v \in V(G)$, we have*

$$n\overline{N}(G;v) \geqslant \overline{N}(G)$$

with equality if and only if $G \cong K_1$.

Proof. We proceed by induction on n. If $n = 1$, then $G \cong K_1$ and the result follows trivially. Now let $n \geqslant 2$ and suppose that the statement holds for all connected graphs of order less than n. Let v be a vertex of G.

If v is not a cut-vertex of G, then $G - v$ is a connected graph of order $n-1$. Let u be a neighbor of v. By induction hypothesis, we have $(n-1)\overline{N}(G-v;u) \geqslant \overline{N}(G-v)$. Note that $\overline{N}(G;v) > \overline{N}(G-v;u)$ because each connected induced subgraph in $\mathcal{C}(G-v;u)$ together with v is a connected induced subgraph of $\mathcal{C}(G;v)$ and there is an additional singleton $\{v\}$ in $\mathcal{C}(G;v)$. Hence

$$(n-1)\overline{N}(G;v) > (n-1)\overline{N}(G-v;u) \geqslant \overline{N}(G-v),$$

Thus we have $n\overline{N}(G;v) > \overline{N}(G-v) + \overline{N}(G;v) = \overline{N}(G)$.

Now assume that v is a cut-vertex of G. Let $H_1, H_2, \ldots, H_k$ be the components of $G - v$. For $i = 1, 2, \ldots, k$, denote by G_i the subgraph of G induced by the vertices

$V(H_i) \cup \{v\}$, and let n_i be their respective orders. By induction hypothesis, we have $n_i\overline{N}(G_i;v) \geqslant \overline{N}(G_i)$. Hence $(n_i-1)\overline{N}(G_i;v) \geqslant \overline{N}(G_i) - \overline{N}(G_i;v) = \overline{N}(H_i)$. Note that

$$\overline{N}(G) = \overline{N}(G-v) + \overline{N}(G;v) = \sum_{i=1}^{k} \overline{N}(H_i) + \prod_{i=1}^{k} \overline{N}(G_i;v).$$

Thus

$$\begin{aligned}
\overline{N}(G) &\leqslant \sum_{i=1}^{k}[(n_i-1)\overline{N}(G_i;v)] + \prod_{i=1}^{k} \overline{N}(G_i;v) \\
&< (n-1)\sum_{i=1}^{k} \overline{N}(G_i;v) + \prod_{i=1}^{k} \overline{N}(G_i;v) \\
&\leqslant (n-1)\prod_{i=1}^{k} \overline{N}(G_i;v) + \prod_{i=1}^{k} \overline{N}(G_i;v) \\
&= n\overline{N}(G;v).
\end{aligned}$$

Here, the inequality $\sum_{i=1}^{k} \overline{N}(G_i;v) \leqslant \prod_{i=1}^{k} \overline{N}(G_i;v)$ holds because $\overline{N}(G_i;v) \geqslant 2$ for every $1 \leqslant i \leqslant k$. This completes the induction. □

Now we establish an inequality between the global and the local mean orders, which provides an affirmative answer to Problem 6.

Theorem 18. *For any k-tree T of order n with a k-clique C, we have $\mu(T;C) \geqslant \mu(T)$ with equality if and only if $T \cong K_k$.*

Proof. If $T \cong K_k$, then $\mu(T;C) = \mu(T) = k$. So we may suppose that $|T| > k$. Let $B \subseteq V(T^*)$ such that $B = \mathsf{CL}_{k+1}(T;C)$. Note that $|\mathsf{CL}_{k+1}(T;C)| = deg_T(C)$. Clearly, if $deg_T(C) = 1$, then B is a single vertex of T^*, and if $deg_T(C) > 1$, then B is the vertex set of a block of T^* which is a clique of size $deg_T(C)$. By Lemmas 15 and 16, it suffices to show that

$$\frac{\overline{O}^{\star}(T^*;B)}{\overline{N}^{\star}(T^*;B)+1} > \frac{\overline{O}(T^*)}{\overline{N}(T^*)+(n-k)k+1},$$

that is,

$$\overline{O}^{\star}(T^*;B)\overline{N}(T^*) + [k(n-k)+1]\overline{O}^{\star}(T^*;B) > \overline{O}(T^*)\overline{N}^{\star}(T^*;B) + \overline{O}(T^*).$$

Since T^* is a block graph, and B is either a block or a single vertex, it follows from Lemma 10 that $\overline{\mu}^{\star}(T^*;B) \geqslant \overline{\mu}(T^*)$, which is equivalent to

$$\frac{\overline{O}^{\star}(T^*;B)}{\overline{N}^{\star}(T^*;B)} \geqslant \frac{\overline{O}(T^*)}{\overline{N}(T^*)}, \tag{1}$$

that is,

$$\overline{O}^{\star}(T^*;B)\overline{N}(T^*) \geqslant \overline{O}(T^*)\overline{N}^{\star}(T^*;B).$$

Then we just need to prove that

$$[k(n-k)+1]\overline{O}^{\star}(T^*;B) > \overline{O}(T^*). \tag{2}$$

Consider the order of B. Note that $|T^*| = n-k$. If $|B| = deg_T(C) = 1$, by Lemma 17, we have

$$\overline{N}(T^*) \leqslant (n-k)\overline{N}^{\star}(T^*;B) < [k(n-k)+1]\overline{N}^{\star}(T^*;B).$$

Multiplying this inequality with (1), we can deduce that (2) holds.

If $|B| = deg_T(C) \geqslant 2$, let G be a graph obtained from T^* by contracting B to a new vertex u. Then $|G| = n-k-|B|+1$. By Lemma 17, we have $(n-k-|B|+1)\overline{N}(G;u) \geqslant \overline{N}(G)$, that is, $(n-k-|B|)\overline{N}(G;u) \geqslant \overline{N}(G-u)$. Further, we note that $\overline{N}^{\star}(T^*;B) \geqslant \overline{N}(T^*;B) = \overline{N}(G;u)$ and $T^* - B = G - u$. Then we have $\overline{N}(G-u) = \overline{N}(T^*-B) = \overline{N}(T^*) - \overline{N}^{\star}(T^*;B)$. Hence

$$[k(n-k)]\overline{N}^{\star}(T^*;B) > (n-k-|B|)\overline{N}(G;u) \geqslant \overline{N}(G-u) = \overline{N}(T^*) - \overline{N}^{\star}(T^*;B).$$

Thus we have

$$[k(n-k)+1]\overline{N}^{\star}(T^*;B) > \overline{N}(T^*).$$

Again, this together with (1) yields (2), and we are done. □

3 *k*-trees with extremal global mean orders

In this section, we consider the extremal problems regarding the global mean order. We prove that the path-type-k-trees have the smallest global mean sub-k-tree orders among all k-trees, answering Problem 7. We also show that the star-type-k-trees have the largest global mean sub-k-tree orders among all aster-type-k-trees. These two results generalize the results of Jamison [11] for the case $k=1$. Moreover, it is shown that the global mean order of the sub-k-trees of T of sufficiently large order is asymptotically equal to the mean order of all connected induced subgraphs of the dual T^*.

Our proof requires the following result established in [1].

Theorem 19 ([1, Theorem 3.1])**.** *If G is a connected block graph of order n, then $\overline{\mu}(G) \geqslant \frac{n+2}{3}$ with equality if and only if $G \cong P_n$.*

In [11] Jamison proved that $\mu(T) \geqslant \frac{n+2}{3}$ for every tree T of order n with equality only for P_n. A tree is a special block graph and a subtree of a tree T is a connected induced subgraph of T, thus Theorem 19 extends Jamison's lower bound from trees to block graphs. Vince [22] and Haslegrave [9] later extend this lower bound to all connected graphs. They independently and almost simultaneously proved that the path P_n uniquely minimizes the mean order of the connected induced subgraphs among all connected graphs of order n (the two preprints were submitted only one day apart, first by Vince [22] and second by Haslegrave [9]). In the following, we extend this lower bound from trees to k-trees.

Theorem 20. *For any k-tree T of order n, we have*

$$\mu(T) \geqslant \frac{\binom{n-k+2}{3}}{\binom{n-k+1}{2} + (n-k)k + 1} + k$$

with equality if and only if T is a path-type-k-tree.

Proof. Let P be a path-type k-tree of order n, then P^* is a path of order $n-k$. One easily computes that $\overline{N}(P^*) = \binom{n-k+1}{2}$ and $\overline{O}(P^*) = \binom{n-k+2}{3}$ (see also [11]). It follows from Lemma 15 that

$$\mu(P) = \frac{\binom{n-k+2}{3}}{\binom{n-k+1}{2} + (n-k)k + 1} + k.$$

Now assume that T is not a path-type k-tree. It suffices to show that $\mu(T) > \mu(P)$. By Lemma 15, we have

$$\mu(T) = \frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1} + k,$$

$$\mu(P) = \frac{\overline{O}(P^*)}{\overline{N}(P^*) + (n-k)k + 1} + k.$$

Then $\mu(T) > \mu(P)$ is equivalent to

$$\frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1} > \frac{\overline{O}(P^*)}{\overline{N}(P^*) + (n-k)k + 1},$$

that is,

$$\overline{O}(T^*)\overline{N}(P^*) + \overline{O}(T^*)[(n-k)k+1] > \overline{O}(P^*)\overline{N}(T^*) + \overline{O}(P^*)[(n-k)k+1]. \tag{3}$$

Note that T^* is a block graph of order $n-k$ which is not a path. It follows from Theorem 19 that $\overline{\mu}(T^*) > \overline{\mu}(P^*)$, which is equivalent to

$$\frac{\overline{O}(T^*)}{\overline{N}(T^*)} > \frac{\overline{O}(P^*)}{\overline{N}(P^*)}, \tag{4}$$

that is,

$$\overline{O}(T^*)\overline{N}(P^*) > \overline{O}(P^*)\overline{N}(T^*).$$

To obtain (3), we just need to prove that

$$\overline{O}(T^*) > \overline{O}(P^*).$$

By Lemma 13, we have $\overline{N}(T^*) > \overline{N}(P^*)$. Multiplying this with inequality (4) yields $\overline{O}(T^*) > \overline{O}(P^*)$, completing the proof. □

In [11] Jamison proved the following maximum property of stars among all asters of a fixed order.

Theorem 21 ([11, Theorem 5.12]). *Among all asters on n vertices, the star $K_{1,n-1}$ uniquely achieves the largest global mean subtree order.*

We show that this result can be generalized as follows.

Theorem 22. *For any aster-type k-tree T of order n, we have*

$$\mu(T) \leqslant \frac{(n-k)2^{n-k-1}}{2^{n-k} + (n-k)k} + k$$

with equality if and only if T is a star-type k-tree.

Proof. Let S be a star-type k-tree of order n, then S^* is a complete graph of order $n-k$. One easily computes that $\overline{N}(S^*) = 2^{n-k} - 1$ and $\overline{O}(S^*) = (n-k)2^{n-k-1}$. It follows from Lemma 15 that

$$\mu(S) = \frac{\overline{O}(K_{n-k})}{\overline{N}(K_{n-k}) + (n-k)k + 1} + k = \frac{(n-k)2^{n-k-1}}{2^{n-k} + (n-k)k} + k.$$

If T is a path-type k-tree, then the result holds from Theorem 20. Now let T be an aster-type k-tree that is neither path-type nor star-type. It suffices to show that $\mu(T) < \mu(S)$.

Let C be the k-clique of maximum degree in T. Clearly T^* is a block graph of order $n-k$, which is obtained from a complete graph of order $deg_T(C)$ by attaching to each vertex at most one pendant path (there is at least one vertex of T^* with a pendant path attached to it since T is not star-type while being aster-type). It follows from the structure of T^* that there exists an aster-type 1-tree A of order $n-k+1$ whose dual is also T^*. Then, by Theorem 21, $\mu(A) < \mu(K_{1,n-k})$. Note that the dual of $K_{1,n-k}$ is the complete graph K_{n-k}. Applying Lemma 15, we obtain

$$\frac{\overline{O}(T^*)}{\overline{N}(T^*) + n - k + 1} < \frac{\overline{O}(K_{n-k})}{\overline{N}(K_{n-k}) + n - k + 1},$$

that is,

$$\overline{O}(T^*)\overline{N}(K_{n-k}) + (n-k+1)\overline{O}(T^*) < \overline{O}(K_{n-k})\overline{N}(T^*) + (n-k+1)\overline{O}(K_{n-k}).$$

Moreover, we note that $n-k+1 \leqslant (n-k)k+1$ and $\overline{O}(T^*) < \overline{O}(K_{n-k})$. It follows that

$$\overline{O}(T^*)\overline{N}(K_{n-k}) + [(n-k)k+1]\overline{O}(T^*) < \overline{O}(K_{n-k})\overline{N}(T^*) + [(n-k)k+1]\overline{O}(K_{n-k}),$$

which is equivalent to

$$\frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1} + k < \frac{\overline{O}(K_{n-k})}{\overline{N}(K_{n-k}) + (n-k)k + 1} + k,$$

that is, $\mu(T) < \mu(S)$, according to Lemma 15. □

Lastly, we show that the global mean order of the sub-k-trees of T is asymptotically equal to the mean order of all connected induced subgraphs of the dual T^* if the order of T is sufficiently large.

Theorem 23. *If $\{T_n\}$ is a sequence of k-trees with $|T_n| = n$, then*

$$\lim_{n\to\infty} \frac{\mu(T_n)}{\overline{\mu}(T_n^*)} = 1.$$

Proof. By Lemma 15, we have

$$\mu(T_n) = \frac{\overline{O}(T_n^*)}{\overline{N}(T_n^*) + (n-k)k + 1} + k.$$

Then

$$\frac{\mu(T_n)}{\overline{\mu}(T_n^*)} = \left[\frac{\overline{O}(T_n^*)}{\overline{N}(T_n^*) + (n-k)k + 1} + k\right] \frac{\overline{N}(T_n^*)}{\overline{O}(T_n^*)} = \frac{\overline{N}(T_n^*)}{\overline{N}(T_n^*) + (n-k)k + 1} + \frac{k}{\overline{\mu}(T_n^*)}.$$

Let $\{P_n\}$ be a sequence of path-type k-trees with $|P_n| = n$. It follows from Lemma 13 that $\overline{N}(T_n^*) \geqslant \overline{N}(P_n^*) = \binom{n-k+1}{2}$. Then

$$1 < \frac{\overline{N}(T_n^*) + (n-k)k + 1}{\overline{N}(T_n^*)} \leqslant 1 + \frac{(n-k)k + 1}{\binom{n-k+1}{2}}.$$

As $n \to \infty$, by the squeeze theorem, we have

$$\lim_{n\to\infty} \frac{\overline{N}(T_n^*)}{\overline{N}(T_n^*) + (n-k)k + 1} = 1.$$

Moreover, by Theorem 19, $\overline{\mu}(T_n^*) \geqslant \frac{n-k+2}{3}$. Thus we have $\lim\limits_{n\to\infty} \frac{k}{\overline{\mu}(T_n^*)} = 0$. Therefore

$$\lim_{n\to\infty} \frac{\mu(T_n)}{\overline{\mu}(T_n^*)} = \lim_{n\to\infty} \frac{\overline{N}(T_n^*)}{\overline{N}(T_n^*) + (n-k)k + 1} + \lim_{n\to\infty} \frac{k}{\overline{\mu}(T_n^*)} = 1. \qquad \square$$

4 Inclusion monotonicity

In [11], the following monotonicity results on the mean subtree order of trees were established.

Theorem 24 ([11, Theorem 4.8])**.** *If S is a proper subtree of a tree T, then $\mu(S) < \mu(T)$.*

Theorem 25 ([11, Theorem 4.5])**.** *If $R \subset S$ are subtrees of a tree T, then*

$$\mu(T; R) < \mu(T; S) \leqslant \mu(T; R) + \frac{|S| - |R|}{2}.$$

Theorem 26 ([11, Theorem 4.7])**.** *For any subtree R of a proper subtree S of a tree T, we have $\mu(S;R) < \mu(T;R)$.*

In this section, we extend the above results from trees to k-trees. A useful tool is the simple fact that if a population is divided into subpopulations, then its mean is a convex combination (or weighted average) of the means over the subpopulations. As an example, for a k-tree T with a vertex $v \in V(T)$, we denote by $\mu(T;v)$ the mean order of the sub-k-trees of T containing v and $\mu(T-v)$ the mean order of the sub-k-trees of T not containing v. Since every sub-k-tree of T either contains v or not, we can write

$$\mu(T) = \lambda_1 \mu(T;v) + \lambda_2 \mu(T-v),$$

where $\lambda_1 = \frac{N(T;v)}{N(T)}$ and $\lambda_2 = \frac{N(T-v)}{N(T)}$ satisfying $\lambda_1 + \lambda_2 = 1$. It follows that $\mu(T)$ is a convex combination of $\mu(T;v)$ and $\mu(T-v)$, which implies

$$\min\{\mu(T;v), \mu(T-v)\} \leqslant \mu(T) \leqslant \max\{\mu(T;v), \mu(T-v)\}.$$

Theorem 27. *If S is a proper sub-k-tree of a k-tree T, then $\mu(S) < \mu(T)$.*

Proof. Since any sub-k-tree of T can be obtained from T by a sequence of k-leaf deletions, we may suppose that $S = T - v$ for some k-leaf v of T, the general case following from this by induction. Let $\mu(T;v)$ be the mean order of the sub-k-trees of T containing v. Then $\mu(T)$ is a convex combination of $\mu(T;v)$ and $\mu(S)$. To obtain $\mu(S) < \mu(T)$, we only need to show that $\mu(T;v) \geqslant \mu(T)$.

Let C be a k-clique containing v. Clearly $deg_T(C) = 1$. Thus there exist k k-cliques containing v, one of which is C. Note that each non-trivial sub-k-tree in T that contains v must contain C. It follows that $N(T;v) = N(T;C) + k - 1$ and $O(T;v) = O(T;C) + k(k-1)$. Recall that $\mathsf{CL}_{k+1}(T;C)$ is the set of $(k+1)$-cliques of T containing C. Let $B \subseteq V(T^*)$ such that $B = \mathsf{CL}_{k+1}(T;C)$. In terms of Lemma 16, we have $O(T;C) = \overline{O}^{\star}(T^*;B) + k\left(\overline{N}^{\star}(T^*;B) + 1\right)$ and $N(T;C) = \overline{N}^{\star}(T^*;B) + 1$. It follows that

$$\begin{aligned}
\mu(T;v) = \frac{O(T;v)}{N(T;v)} &= \frac{O(T;C) + k(k-1)}{N(T;C) + k - 1} \\
&= \frac{\overline{O}^{\star}(T^*;B) + k\left(\overline{N}^{\star}(T^*;B) + 1\right) + k(k-1)}{\overline{N}^{\star}(T^*;B) + 1 + k - 1} \\
&= \frac{\overline{O}^{\star}(T^*;B)}{\overline{N}^{\star}(T^*;B) + k} + k.
\end{aligned}$$

By Lemma 15,

$$\mu(T) = \frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1} + k.$$

To obtain $\mu(T;v) \geqslant \mu(T)$, it suffices to show that

$$\frac{\overline{O}^{\star}(T^*;B)}{\overline{N}^{\star}(T^*;B) + k} > \frac{\overline{O}(T^*)}{\overline{N}(T^*) + (n-k)k + 1},$$

that is,

$$\overline{O}^{\star}(T^*;B)\overline{N}(T^*)+[(n-k)k+1]\,\overline{O}^{\star}(T^*;B)>\overline{O}(T^*)\overline{N}^{\star}(T^*;B)+k\overline{O}(T^*).$$

Note that B is a single vertex of T^* since $deg_T(C)=1$. Then, by Lemma 10, $\overline{\mu}^{\star}(T^*;B)\geqslant\overline{\mu}(T^*)$, which is equivalent to

$$\frac{\overline{O}^{\star}(T^*;B)}{\overline{N}^{\star}(T^*;B)}\geqslant\frac{\overline{O}(T^*)}{\overline{N}(T^*)},\tag{5}$$

that is,

$$\overline{O}^{\star}(T^*;B)\overline{N}(T^*)\geqslant\overline{O}(T^*)\overline{N}^{\star}(T^*;B).\tag{6}$$

Moreover, using Lemma 17, we have $(n-k)\overline{N}^{\star}(T^*;B)>\overline{N}(T^*)$. This together with (5) yields $(n-k)\overline{O}^{\star}(T^*;B)>\overline{O}(T^*)$, which implies that $[(n-k)k+1]\overline{O}^{\star}(T^*;B)>k\overline{O}(T^*)$. Adding this inequality and (6) gives us the desired result. $\square$

Theorem 27 is a generalization of Theorem 24. Now we generalize Theorems 25 and 26 from trees to k-trees. This requires a useful method for reducing a k-tree to a 1-tree.

It is well-known that every non-trivial k-tree has at least two k-leaves. If v is a k-leaf of a k-tree T of order $n\geqslant k+1$, then $T-v$ is a k-tree. Then for any non-trivial k-tree T with a k-clique C, there exists a sequence of vertices $(v_1,v_2,\ldots,v_p)$ such that (i) $\{v_1,v_2,\ldots,v_p\}\cup V(C)=V(T)$, (ii) v_1 is a k-leaf in T, (iii) v_i is a k-leaf in $T-\{v_1,v_2,\ldots,v_{i-1}\}$ for all $i\geqslant 2$. Such a sequence is called a *perfect elimination ordering of T down to C*. The following result established in [20] is a generalization to k-trees of the fact that any two vertices of a tree have a unique path between them.

Lemma 28 ([20, Lemma 7])**.** *For any k-tree T with a k-clique C, let v be any vertex of T that is not in C. Then there exists a unique sequence $A_T(C,v)=(C,w_1,w_2,\ldots,w_s,v)$, where all terms except C are vertices of T, such that*

1. *the graph induced by $V(C)\cup\{w_1,w_2,\ldots,w_{s-1},w_s,v\}$, denoted by $P_T(C,v)$, is a path-type k-tree, and C is simplicial in $P_T(C,v)$.*

2. *the sequence $(v,w_s,w_{s-1},\ldots,w_2,w_1)$ is a perfect elimination ordering of $P_T(C,v)$ down to C.*

We use $L(T)$ to denote the set of k-leaves of T. The next lemma shows that each k-tree has a *path-type representation* consisting of path-type k-trees starting at C and ending at a k-leaf.

Lemma 29 ([20, Theorem 8])**.** *For any k-tree T with a k-clique C, we have*

$$T=\left(\bigcup_{v\in L(T)}V\left(P_T(C,v)\right),\ \bigcup_{v\in L(T)}E\left(P_T(C,v)\right)\right).$$

For each $v \in V(T) - V(C)$, we define the graph $P'_T(C, v)$ with

$$V\left(P'_T(C, v)\right) = \{C, w_1, w_2, \ldots, w_{s-1}, w_s, v\},$$

and

$$E\left(P'_T(C, v)\right) = \{Cw_1, w_1w_2, w_2w_3, \ldots, w_{s-1}w_s, w_sv\}.$$

Thus the graph $P'(C, v)$ is a path of order $s + 2$.

Definition 30. Let T be a k-tree with k-clique C. The characteristic 1-tree of T with respect to C, is the graph T'_C defined as follows:

$$T'_C = \left(\bigcup_{v \in L(T)} V\left(P'_T(C, v)\right), \bigcup_{v \in L(T)} E\left(P'_T(C, v)\right) \right).$$

It follows from Lemma 28 that T'_C is a tree, and is the unique tree for which $A_{T'_C}(C, v) = A_T(C, v)$ for all $v \in L(T)$. Note that the characteristic 1-tree of a path-type k-tree with respect to an arbitrary k-clique is a path. Figure 2 gives an example of a 2-tree and its two characteristic 1-trees. The following result is rather intuitive.

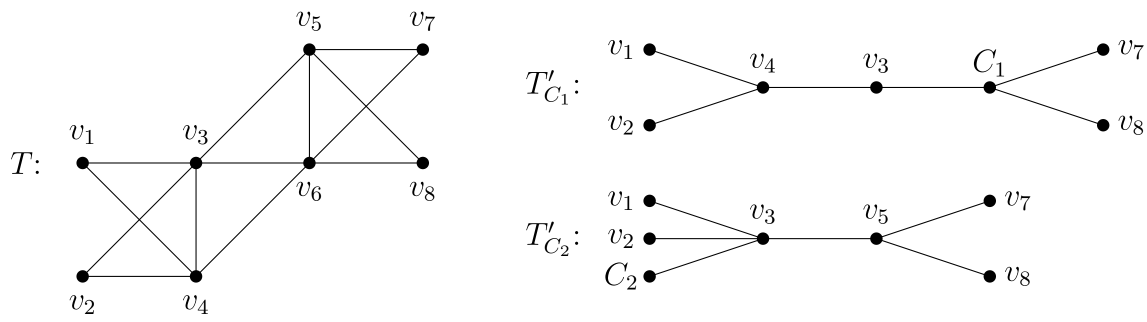

Figure 2: A 2-tree T and its two characteristic 1-trees T'_{C_1}, T'_{C_2}, where $C_1 = \{v_5, v_6\}$ and $C_2 = \{v_4, v_6\}$.

Lemma 31. *Let T be a k-tree, R a sub-k-tree of T, and C any k-clique of R. Then R'_C is a subtree of T'_C.*

Proof. If R is a k-clique of T, then the statement clearly holds. Now suppose that R is a non-trivial sub-k-tree of T. Let v be a k-leaf of R that is not in C. By Lemma 28, there is a corresponding unique sequence $A_R(C, v)$ and a path-type k-tree $P_R(C, v)$. Note that v also belongs to T. It follows from Lemma 29 that v belongs to $P_T(C, u)$ for some k-leaf u of T. Then it follows from Lemma 28 that if $A_T(C, u) = (C, w_1, w_2, \ldots, w_s)$, where $w_s = u$, then we must have $v = w_i$ for some $1 \leqslant i \leqslant s$, i.e., $A_R(C, v) = A_T(C, v) = (C, w_1, w_2, \ldots, w_i)$. Hence $P'_R(C, v)$ is a subpath of $P'_T(C, u)$, which implies that R'_C is a subtree of T'_C from the fact that each k-tree has a path-type representation. $\square$

For any k-tree T with a sub-k-tree R, recall that $\mathcal{S}(T;R)$ is the set of all sub-k-trees of T containing the sub-k-tree R, and $|\mathcal{S}(T;R)| = N(T;R)$. The next result is a generalization of the result in [20, Theorem 9].

Lemma 32. *Let T be a k-tree, R a sub-k-tree of T, and C any k-clique of R. Then there is a one-to-one correspondence between $\mathcal{S}(T;R)$ and $\mathcal{S}(T'_C;R'_C)$.*

Proof. Define a function $f : \mathcal{S}(T;R) \mapsto \mathcal{S}(T'_C;R'_C)$ by the rule $f(X) = X'_C$ for all $X \in \mathcal{S}(T;R)$. We first show that f is well-defined. Let $X \in \mathcal{S}(T;R)$. Then $R \subseteq X \subseteq T$. By Lemma 31, we have $R'_C \subseteq X'_C \subseteq T'_C$. Therefore $X'_C \in \mathcal{S}(T'_C;R'_C)$, and thus f is well-defined.

Then we show that f is a bijection. Suppose that $f(X) = f(Y)$ for some sub-k-trees $X, Y \in \mathcal{S}(T;R)$. Then $X'_C = Y'_C$. It follows that $L(X'_C) = L(Y'_C)$, implying $L(X) = L(Y)$. By Lemma 28, we have $P'_X(C,v) = P'_Y(C,v)$ for any leaf v of $f(X) = f(Y)$, which implies $P_X(C,v) = P_Y(C,v)$ for any k-leaf v of X or Y. Thus we have $(\cup V(P_X(C,v)), \cup E(P_X(C,v))) = (\cup V(P_Y(C,v)), \cup E(P_Y(C,v)))$, where the unions on the left (resp., right) are over all leaves v of X (resp., Y). It follows from Lemma 29 that $X = Y$. Thus f is injective.

Now let Y be a subtree of $\mathcal{S}(T'_C;R'_C)$. For each leaf v of Y, Lemma 28 shows that there is a corresponding unique sequence $A_Y(C,v)$ and a path $P_Y(C,v)$. Note that there also exist $P_T(C,v)$ and $A_T(C,v)$. And it follows from Lemma 28 that $A_T(C,v) = A_Y(C,v)$. Let $X = (\cup V(P_T(C,v)), \cup E(P_T(C,v)))$, where the unions are over the leaves v of Y. We shall show that $f(X) = Y$. First we prove that $X \in \mathcal{S}(T;R)$. Let u be a leaf of R'_C. In terms of Lemma 29, u belongs to $P_Y(C,v)$ for some leaf v of Y. It follows from Lemma 28 that if $A_Y(C,v) = (C, w_1, w_2, \ldots, w_s)$, where $w_s = v$, then we must have $u = w_i$ for some $1 \leqslant i \leqslant s$, i.e., $A_{R'_C}(C,u) = A_Y(C,u) = (C, w_1, w_2, \ldots, w_i)$. This implies that $P_T(C,u)$ is a sub-k-tree of $P_T(C,v)$. Further, Lemma 29 gives the path representation $R = (\cup V(P_T(C,u)), \cup E(P_T(C,u)))$, where the unions are over the leaves u of R. Note that $L(X) = L(Y)$. Hence R is a sub-k-tree of X and thus $X \in \mathcal{S}(T;R)$. In addition, we can see that $P'_X(C,v) = P_Y(C,v)$ for all $v \in L(Y)$. Therefore by Lemma 29, we have

$$\begin{aligned} f(X) = X'_C &= \left(\bigcup_{v \in L(X)} V(P'_X(C,v)), \bigcup_{v \in L(X)} E(P'_X(C,v)) \right) \\ &= \left(\bigcup_{v \in L(Y)} V(P_Y(C,v)), \bigcup_{v \in L(Y)} E(P_Y(C,v)) \right) = Y. \end{aligned}$$

Thus f is a surjective function. Consequently, we conclude that f is a bijection and the desired result follows. $\square$

Theorem 33. *If $R \subset S$ are sub-k-trees of a k-tree T, then*

$$\mu(T;R) < \mu(T;S) \leqslant \mu(T;R) + \frac{|S| - |R|}{2}.$$

Proof. Let C be a k-clique both belonging to R and S. It follows from Lemma 32 that there is a one-to-one correspondence between $\mathcal{S}(T;S)$ and $\mathcal{S}(T'_C;S'_C)$, which implies $N(T;S) = N(T'_C;S'_C)$. Observe that for each sub-k-tree X in $\mathcal{S}(T;S)$ with its corresponding subtree Y in $\mathcal{S}(T'_C;S'_C)$, we have $|X| = |Y| + k - 1$. Hence

$$\mu(T;S) = \mu(T'_C;S'_C) + k - 1.$$

Similarly, we have

$$\mu(T;R) = \mu(T'_C;R'_C) + k - 1.$$

It follows that

$$\mu(T;S) - \mu(T;R) = \mu(T'_C;S'_C) - \mu(T'_C;R'_C).$$

Since R is a proper sub-k-tree of S, by Lemma 31, R'_C is a proper subtree of S'_C. Note that $|S| - |R| = |S'_C| - |R'_C|$. Combining with Theorem 25, we obtain

$$0 < \mu(T;S) - \mu(T;R) = \mu(T'_C;S'_C) - \mu(T'_C;R'_C) \leqslant \frac{|S'_C| - |R'_C|}{2} = \frac{|S| - |R|}{2},$$

which completes the proof. □

Theorem 34. *For any sub-k-tree R of a proper sub-k-tree S of a k-tree T, we have*

$$\mu(S;R) < \mu(T;R).$$

Proof. It suffices to establish the result in the case that S is obtained by deleting a k-leaf v from T. The general result will then follow by induction.

Choose an arbitrary k-clique C of R. By Lemma 28, there exists a unique sequence $A_T(C,v)$ and a path-type k-tree $P_T(C,v)$. Now we show that each sub-k-tree of T containing both v and C must contain $P_T(C,v)$. Set $B = P_T(C,v)$. Note that B'_C is a path with $V(B'_C) = V(P'_T(C,v))$. By Lemma 32, there is a one-to-one correspondence between $\mathcal{S}(T;B)$ and $\mathcal{S}(T'_C;B'_C)$. Since each subtree in $\mathcal{S}(T'_C)$ containing both v and C (a vertex of T'_C) must contain the path B'_C, it follows that each sub-k-tree in $\mathcal{S}(T)$ containing both v and C (a k-clique of T) must contain the path-type k-tree B. Let Q be the graph induced by $V(B) \cup V(R)$. Then Q is the smallest sub-k-tree containing both v and R.

Note that each sub-k-tree of T containing R either contains v or not. Then

$$\mu(T;R) = \lambda_1 \mu(S;R) + \lambda_2 \mu(T;Q),$$

where $\lambda_1 = \frac{N(S;R)}{N(T;R)}$ and $\lambda_2 = \frac{N(T;Q)}{N(T;R)}$ satisfy $\lambda_1 + \lambda_2 = 1$. It follows that $\mu(T;R)$ is a convex combination of $\mu(S;R)$ and $\mu(T;Q)$. By Theorem 33, $\mu(T;Q) > \mu(T;R)$. Hence we have $\mu(S;R) < \mu(T;R)$. □

5 Final remarks

Theorem 20 shows that the path-type-k-trees have the smallest global mean sub-k-tree orders among all k-trees of a given order. This result generalizes Jamison's result [11] that the path P_n has the smallest mean subtree order among all trees of a fixed order n. However, the problem of determining the structure of those k-trees of a given order with maximum global mean order remains open even for $k = 1$. It was conjectured by Jamison [11] that the maximum mean subtree order is attained by a caterpillar (i.e., a tree that becomes a path when all leaves are removed) for every tree of given order. This is known as Jamisons Caterpillar Conjecture. We define a *caterpillar-type k-tree* as a k-tree that becomes a path-type k-tree when all k-leaves are removed. For the general k-trees, we have the following natural question:

Question 35. Among all k-trees of a given order, is the k-tree with the largest global mean sub-k-tree order necessarily a caterpillar-type k-tree?

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