Properly Colored Hamilton Cycles in Dirac-Type Hypergraphs

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Abstract

We consider a robust variant of Dirac-type problems in k-uniform hypergraphs. For instance, we prove that if $\mathcal{H}$ is a k-uniform hypergraph with minimum codegree at least $\left(\frac{1}{2}+\gamma\right)n$, $\gamma > 0$, and n is sufficiently large, then any edge coloring ϕ satisfying appropriate local constraints yields a properly colored tight Hamilton cycle in $\mathcal{H}$. Similar results for loose cycles are also shown.

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1 Introduction

Our study lies at the intersection of three classical areas of research – we will extend several Dirac-type theorems for hypergraphs to an edge-colored setting. The famous theorem of Dirac states that any n-vertex graph with minimum degree at least $n/2$ has a Hamilton cycle. Among the numerous research lines stemming from this theorem are so-called Dirac-type problems, where one aims to embed a spanning subgraph into a graph of given minimum degree.

Dirac-type problems in k-uniform hypergraphs, or shortly k-graphs, first considered in [3, 22], now constitute a fruitful and dynamic theory. One reason for this multitude of results is that there is a number of different notions of hypergraph degrees and cycles, and consequently several extremal constructions. Given $\ell \in [k-1]$, a (k,ℓ)-*overlapping cycle* $C_n^{(k)}(\ell)$, or shortly a (k,ℓ)-*cycle*, is an n-vertex k-graph whose vertices can be ordered

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cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges share exactly ℓ vertices. Note that $C_n^{(k)}(\ell)$ exists whenever $k-\ell$ divides n (we will further use standard notation $(k-\ell)|n$ to indicate this fact). For $k=2$ and $\ell=1$, this reduces to the graph cycle. The two extreme cases, $\ell=1$ and $\ell=k-1$, are usually referred to, respectively, as *loose* and *tight* cycles. We say that a k-graph contains a *Hamilton* (k,ℓ)*-cycle*, or is ℓ*-Hamiltonian*, if it contains an (k,ℓ)-cycle as a spanning subhypergraph.

The *degree* of a set $S \subset V(\mathcal{H})$ in a hypergraph $\mathcal{H}$, denoted by $\deg(S)$, is the number of edges of $\mathcal{H}$ containing S. For sets of order 1 or $k-1$, the terms *vertex degree* and *codegree* (respectively) are often used. *The minimum and maximum s-degree* of $\mathcal{H}$ are defined as

$$\delta_s(\mathcal{H}) = \min\{\deg(S) : S \subset V(\mathcal{H}), |S| = s\},\ \Delta_s(\mathcal{H}) = \max\{\deg(S) : S \subset V(\mathcal{H}), |S| = s\}.$$

The following three Dirac-type results are directly relevant to our topic.

Theorem 1. *Let $1 \leqslant \ell < k/2$ and $\gamma > 0$. Let $\mathcal{H}$ be a k-graph on n vertices for sufficiently large n divisible by $k-\ell$.*

(i) [29] *If $\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2}+\gamma\right)n$, then $\mathcal{H}$ contains a tight Hamilton cycle.*

(ii) [25, 15] *If $\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)}+\gamma\right)n$, then $\mathcal{H}$ contains a Hamilton (k,ℓ)-cycle.*

(iii) [5] *If $k=3$ and $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16}+\gamma\right)n^2/2$, then $\mathcal{H}$ contains a loose Hamilton cycle.*

The minimum-degree conditions in all three statements are asymptotically tight. In some cases, the corresponding exact thresholds are known [30, 17, 16]. The most recent survey covering hypergraph Dirac-type problems is [34]. The area has witnessed the development of several ubiquitous techniques such as the absorbing method and tools in hypergraph regularity, including the hypergraph blow-up lemma [20]. As we shall see in our proofs, hypergraph Turán problems also play an important role.

In our setting, the host graph will be edge-colored and, given appropriate restrictions on the coloring, we will be searching for properly colored Hamilton cycles. A hypergraph is *properly colored* if every two intersecting edges are assigned different colors. Our results can be seen as *robust* versions of the above-mentioned Dirac-type theorems.

Sufficient conditions for finding properly colored Hamilton cycles in edge-colorings of the complete graph K_n were first proposed by Daykin [10] in 1976. Let us call an edge colouring of a hypergraph $\mathcal{H}$ *locally t-bounded* if any colour appears on at most t edges incident to any given vertex. Daykin conjectured that there is a positive constant μ such that for any n, a locally μn-bounded colouring of K_n contains a properly colored Hamilton cycle. The conjecture was resolved by Chen and Daykin [7], as well as Bollobás and Erdős [2]. After a number of improvements and extensions (see, e.g., [31, 1, 4]), Lo [26] showed that Daykin's conjecture holds for any $\mu < \frac{1}{2}$ and large n, which is asymptotically optimal. Analogous 'global' sufficient conditions for rainbow embeddings (in which no two edges have the same colour) have often been proved in parallel, also motivated by famous problems which can be encoded in terms of rainbow subgraphs, such as Latin transversals. However, we will streamline most of our discussion towards the setting of proper colorings.

The question of Daykin also has natural hypergraph analogues. Dudek, Frieze and Ruciński [12] showed that for $\mu > 0$, n sufficiently large, and $\ell \in [k-1]$, every locally $\mu n^{k-\ell}$-bounded colouring of the complete k-uniform hypergraph $K_n^{(k)}$ contains a properly coloured ℓ-overlapping cycle. However, this is only known to be tight for the loose cycle $C_n^{(k)}(1)$, and the problem remains wide open for tight cycles. In [11] and [19], the boundedness condition was weakened to codegrees – in particular, if the subhypergraph induced by any color has maximum codegree at most μn, then one can find a tight Hamilton cycle.

Several authors have considered replacing the host (hyper)graph by an incomplete (hyper)graph [24, 23, 13, 6, 9, 8]. Krivelevich, Lee and Sudakov have shown that Daykin's conjecture holds even when K_n is replaced by an arbitrary graph of minimum degree at least $n/2$ (often called a *Dirac graph*), thus confirming a conjecture of Häggvist. They placed their result in the context of *robustness* of extremal and probabilistic results [32].

We will show that a similar phenomenon occurs in k-uniform hypergraphs. A *colored hypergraph* is a pair $(\mathcal{H}, \phi)$, where $\mathcal{H}$ is a k-graph and $\phi : \mathcal{H} \to \mathbf{N}$ is a coloring of the edges of $\mathcal{H}$. For each $i \in \mathbf{N}$, we denote by $\mathcal{H}_i = \{e \in \mathcal{H} : \phi(e) = i\}$ the subhypergraph of $\mathcal{H}$ consisting of the edges of color i. We now state our three new results which correspond to the three parts of Theorem 1 above. In each of them we make a suitable assumption on the coloring in terms of $\Delta_\ell(\mathcal{H}_i)$.

Theorem 2. *For every $k \geqslant 3$ and $\gamma > 0$ there exist $c > 0$ and $n_0 > 0$ such that if $(\mathcal{H}, \phi)$ is an n-vertex colored k-graph with $n \geqslant n_0$, $\delta_{k-1}(\mathcal{H}) \geqslant (1/2 + \gamma)n$ and $\Delta_{k-1}(\mathcal{H}_i) \leqslant cn$ for every $i \in \mathbf{N}$, then $(\mathcal{H}, \phi)$ contains a properly colored tight Hamilton cycle $C_n^{(k)}(k-1)$.*

Theorem 3. *For every $k \geqslant 3$, $1 \leqslant \ell < k/2$ and $\gamma > 0$ there exist $c > 0$ and $n_0 > 0$ such that if $(\mathcal{H}, \phi)$ is an n-vertex colored k-graph with $(k-\ell) \,|\, n \geqslant n_0$, $\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma\right) n$ and $\Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell}$ for every $i \in \mathbf{N}$, then $(\mathcal{H}, \phi)$ contains a properly colored Hamilton (k, ℓ)-cycle $C_n^{(k)}(\ell)$.*

Theorem 4. *For every $\gamma > 0$ there exist $c > 0$ and $n_0 > 0$ such that if $(\mathcal{H}, \phi)$ is an n-vertex colored 3-graph with $2 \,|\, n \geqslant n_0$, $\delta_1(\mathcal{H}) \geqslant (7/16 + \gamma)n^2/2$ and $\Delta_1(\mathcal{H}_i) \leqslant cn^2$ for every $i \in \mathbf{N}$, then $(\mathcal{H}, \phi)$ contains a properly colored loose Hamilton cycle $C_n^{(3)}(1)$.*

As, trivially, $\Delta_\ell(\mathcal{H}_i) \leqslant \binom{n-\ell}{k-\ell}$, the assumptions on $\Delta_\ell(\mathcal{H}_i)$ in our theorems are optimal up to a constant factor.

In our proofs, the absorbing method is adapted to the setting of proper colorings. The techniques of [5, 25, 29] mostly extend, but some new ideas are needed because the coloring can present obstacles to the standard approach. For instance, under assumptions of Theorem 4, some vertex pairs in $\mathcal{H}$ could be contained only in edges of a single color and thus could not be absorbed into a Hamilton cycle.

In fact, absorption may turn out to be a fruitful approach to similar embedding problems, seeing as its alternative do not extend straightforwardly. Indeed, a uniformly random vertex ordering analysed using the Local Lemma (used, for instance, in [4, 19]), immediately fails for host hypergraphs which are not 'almost' complete. On the other hand, Pósa rotations, a versatile constructive technique for finding Hamilton cycles in graphs utilised in [24, 26], does not have natural hypergraph analogues. Recently, the blow-up lemma for graphs has been adapted and fruitfully applied to the rainbow setting [13], but the hypergraph blow-up lemma used in [21] is significantly more intricate.

Finally, Theorems 2-4 provide new evidence in support of a meta-conjecture formulated by Coulson, Keevash, Perarnau and Yepremyan [8] in the context of rainbow problems. Namely, for a large class of Dirac-type problems for k-graphs, the rainbow counterparts for bounded colorings should have asymptotically the same *degree threshold* as the original problem. Our results show that the conjecture holds for Hamilton cycles if the rainbow colorings are replaced by less restrictive proper colorings. To our knowledge, these are the first results of this type on embedding spanning hypergraphs.

We suspect that stronger variants of Theorems 2-4 also hold, with c independent on the 'degree-excess' γ. However, our assumptions on c are still standard in dealing with edge-colored hypergraphs. For instance, in [8, 13, 33], the assumptions on the coloring also depend on the minimum degree of the hypergraph. Avoiding this dependency seems comparable to proving results under exact minimum-degree assumptions, which is difficult even in the uncolored setting as it involves dealing with specific extremal configurations.

The paper is structured as follows. Section 2 contains preliminary definitions and tools. Theorems 2, 3 and 4 are proved in Sections 3, 4 and 5 respectively.

2 Preliminaries

In this preliminary section we give some definitions and results which are relevant throughout the paper.

We begin with two simple existential results whose proofs employ the standard probabilistic method. The first of them is cited from [28] (see Lemma 3.10 therein). We will use it three times when proving respective "reservoir lemmas". However, part (c) will be needed only in Section 5.1.

Proposition 5 ([28])**.** *For every p, $0 < p < 1$, there is n_0 such that the following holds. Let $U_1, \dots, U_s$ be subsets of an n-element set V, $n \geqslant n_0$, and let $G_1, \dots, G_t$ be graphs on V, where s and t are both polynomials in n. Further, let $|U_i| \geqslant \alpha_i n$, $i = 1, \dots, s$, and $|G_j| \geqslant \beta_j \binom{n}{2}$, $j = 1, \dots, t$, for some constants $0 < \alpha_i, \beta_j < 1$. Then there exists a subset $R \subset V$ such that*

(a) $\big||R| - pn\big| \leqslant pn^{2/3}$,

(b) for all $i = 1, \dots, s$, we have $|U_i \cap R| \geqslant (\alpha_i - 2n^{-1/3})|R|$, and

(c) for all $i = 1, \dots, t$, we have $|G_j[R]| \geqslant (\beta_j - 3n^{-1/3})\binom{|R|}{2}$.

Next, we prove a simple fact which we are going to use three times when proving "absorbing lemmas".

Proposition 6. *For all $s, t \in \mathbf{N}$, $m \leqslant n^s$, and $\alpha \in (0,1)$, let $\mathcal{A}_1, \ldots, \mathcal{A}_m \subset [n]^t$ be families of t-tuples of the elements of $[n]$ with sizes $|\mathcal{A}_i| \geqslant 4\alpha t^2 n^t$, $i \in [m]$. Then, for sufficiently large n, there exists $\mathcal{F} \subset \bigcup_{i=1}^m \mathcal{A}_i$ such that $|\mathcal{F}| \leqslant \alpha n$, the t-tuples in $\mathcal{F}$ are disjoint, and $|\mathcal{F} \cap \mathcal{A}_i| \geqslant \alpha^2 t^2 n/4$ for all $i \in [m]$.*

Proof. We use the probabilistic method. Let $\mathcal{R} \subset [n]^t$ be a random family to which each t-tuple is sampled independently with probability $p = \frac{\alpha n^{-t+1}}{4}$. Using Markov's inequality, we infer that

$$\mathbf{P}(|\mathcal{R}| \geqslant \alpha n) \leqslant \frac{1}{4}$$

and, likewise, the probability that the number of intersecting pairs of t-tuples in $\mathcal{R}$, i.e. pairs sharing at least one element, exceeds $\alpha^2 t^2 n/4$ is no more than $\frac{1}{4}$. To see the latter, note that the expected number of intersecting pairs of t-tuples in $\mathcal{R}$ is at most

$$t^2 n^{2t-1} p^2 = \frac{1}{16}\alpha^2 t^2 n.$$

Now consider $i \in [m]$. The random variable $|\mathcal{R} \cap \mathcal{A}_i|$ is a binomially distributed random variable with expectation

$$|\mathcal{A}_i| p \geqslant 4\alpha t^2 n^t p = \alpha^2 t^2 n.$$

Therefore, using the well-known Chernoff bound (see, e.g., [18]), followed by the union bound over all $i \in [m]$, we infer that a.a.s.

$$|\mathcal{R} \cap \mathcal{A}_i| \geqslant \frac{1}{2}\alpha^2 t^2 n \quad \text{ for all } i \in [m].$$

Thus, the probability that $\mathcal{R}$ satisfies all three above properties is at least $\frac{1}{2} - o(1) > 0$, so the event is nonempty. Let $\mathcal{F}'$ be an instance of $\mathcal{R}$ satisfying them. Further, let $\mathcal{F}$ be obtained from $\mathcal{F}'$ by removing one t-tuple from each intersecting pair and disregarding all t-tuples of $\mathcal{F}'$ which do not belong to $\bigcup_{i=1}^m \mathcal{A}_i$. Clearly, we still have $|\mathcal{F}| \leqslant \alpha n$ and, moreover, for all $i \in [m]$,

$$|\mathcal{F} \cap \mathcal{A}_i| \geqslant \frac{1}{2}\alpha^2 t^2 n - \frac{1}{4}\alpha^2 t^2 n = \frac{1}{4}\alpha^2 t^2 n,$$

as required. □

A (k, ℓ)-*overlapping path*, or shortly (k, ℓ)-*path*, on s vertices is defined as an s-vertex k-graph whose vertices can be ordered linearly such that each of its edges consists of k consecutive vertices and every two consecutive edges share exactly ℓ vertices. The *length* of P is defined as $|V(P)| = s$. Note that many vertex orderings yield the same path. We will typically fix one of them.

Let P be a (k, ℓ)-overlapping path in $\mathcal{H}$ on the vertices $v_1, \ldots, v_s$ (in this order). For $1 \leqslant p < q \leqslant s$, we say that P *connects*, or *lies between*, the segments $v_1, \ldots, v_p$ and $v_q, \ldots, v_s$. When $p = k$ and $q = s - k + 1$, these two segments span edges which we call *the end-edges* of P.

For $\ell < k/2$, we will also need the notion of *ℓ-ends* of P, defined as the ℓ-sets $\{v_1, \dots, v_\ell\}$ and $\{v_{s-\ell+1}, \dots, v_s\}$.

Our final 'meta-theorem' will be used six times (twice in the proof of each of our three main results). In all these applications either the set Q or its complement will be very small, but linear in n.

Proposition 7 (Connecting Meta-Statement)**.** *Let $(\mathcal{H}, \phi)$ be an n-vertex colored k-graph, $1 \leqslant \ell < k$, and $Q \subset V := V(\mathcal{H})$. For integers g and m, where $m = m(n)$, consider two statements.*

I. For all $Q' \subset Q$, $|Q'| \leqslant mg$, and all pairs of disjoint, properly colored (k, ℓ)-paths P_1, P_2 in $(\mathcal{H} - Q, \phi)$, each having at least $\lceil \ell/(k - \ell) \rceil$ edges, there exist vertices $v_1, \dots, v_{g'} \in Q \setminus Q'$, $g' \leqslant g$, such that $P_1 v_1 \cdots v_{g'} P_2$ is also a properly colored (k, ℓ)-path in $(\mathcal{H}, \phi)$.

II. For every collection of m disjoint, properly colored (k, ℓ)-paths $P_1, \dots, P_m$ in $(\mathcal{H} - Q, \phi)$, each having at least $\lceil \ell/(k-\ell) \rceil$ edges, there exist a properly colored (k, ℓ)-cycle C, and a properly colored (k, ℓ)-path P in $(\mathcal{H}, \phi)$ which contain all paths $P_1, \dots, P_m$ and have at most mg vertices in Q. Moreover, P connects P_1 with P_m, $P \subset C$, and $V(C) \subset Q \cup V(P_1) \cup \cdots \cup V(P_m)$.

Then, Statement I implies Statement II.

Proof. Fixing g, in order to show the existence of a path P, we perform induction on m. Trivially, it is true for $m = 2$ (as it then follows from Statement I). Assume there is a properly colored path P' which contains all paths $P_1, \dots, P_{m-1}$ and connects P_1 with P_{m-1}. Set $Q' = V(P') \cap Q$ and note that $|Q'| \leqslant (m - 1)g$. Thus, we are in position to apply Statement I to Q', P_{m-1}, P_m obtaining a desired path P which completes the induction. In order to obtain a cycle C, we apply Statement I to the pair (P_m, P_1) with $Q' = V(P) \cap Q$. $\square$

We finish this preliminary section with a couple of definitions related to the regularity method used in the proofs of 'covering lemmas'. Let $V_1, \dots, V_k \subset V$ be mutually disjoint non-empty vertex sets of a k-graph $\mathcal{H}$ on V. We define the *density* $d_{\mathcal{H}}(V_1, \dots, V_k)$ of $\mathcal{H}$ with respect to $(V_1, \dots, V_k)$ as the ratio of the number of edges in $\mathcal{H}$ with one vertex in each V_i to $|V_1||V_2| \cdots |V_k|$. We call the k-tuple $(V_1, V_2, \dots, V_k)$ *(ε, d)-regular* (for the hypergraph $\mathcal{H}$) if whenever $(A_1, \dots, A_k)$ is a k-tuple of subsets $A_i \subset V_i$ satisfying $|A_i| \geqslant \varepsilon|V_i|$ for $i \in [k]$, we have

$$|d_{\mathcal{H}}(A_1, \dots, A_k) - d| \leqslant \varepsilon.$$

For $t \in \mathbf{N}$, we call sets $V_1, \dots, V_t$ *equitable* if $||V_i| - |V_j|| \leqslant 1$ for all $i, j \in [t]$. A k-partite k-graph with equitable parts is also called *equitable*.

3 Tight Hamilton cycles with codegree conditions

This section is devoted to the proof of Theorem 2, so we will be constructing tight paths and cycles and the attribute 'tight' will sometimes be omitted. We adapt the proof in [29] to the context of colored hypergraphs.

Throughout the section, given an integer $k \geqslant 3$ and a sufficiently small $\gamma \in (0,1)$, $(\mathcal{H},\phi)$ is an n-vertex colored k-graph on vertex set V with

$$\delta_{k-1}(\mathcal{H}) \geqslant (1/2+\gamma)n \quad \text{and} \quad \Delta_{k-1}(\mathcal{H}_i) \leqslant cn \quad \text{for all} \quad i \in \mathbf{N}, \tag{1}$$

where $c > 0$ is sufficiently small with respect to γ, while n is sufficiently large. (In fact, in the actual proof of Theorem 2, Lemmas 9 and 10 will be applied to large induced sub-hypergraphs $\mathcal{H}'$ and $\mathcal{H}''$ of $\mathcal{H}$.)

We define the *end-paths* of a tight path $P = v_1, \dots, v_s$ of length $s \geqslant 2k-2$ as the sequences $(v_1, \dots, v_{2k-2})$ and $(v_{s-2k+3}, \dots, v_s)$, each spanning $k-1$ edges. This non-standard definition reflects our frequent need to preserve the property of being properly colored for the union of two properly colored paths that share one of their end-paths.

We first state three lemmas from which the proof of Theorem 2 follows in a standard way. The first of them establishes the existence of an absorbing path.

Lemma 8 (Absorbing Lemma). *Let $c = c(\gamma) > 0$ be sufficiently small. Then $(\mathcal{H},\phi)$ contains a properly colored tight path A with $|V(A)| \leqslant \frac{1}{4}(\gamma/2)^{k-2}n$ and such that for any $U \subset V$, $|U| \leqslant \frac{1}{4(4k-4)^2}(\gamma/2)^{2k}n$, there is a properly colored tight path A_U on the vertex set $V(A) \cup U$ having the same end-paths as A. (We will say that A can absorb U.)*

The next lemma sets aside a small set $R \subset V$, called *reservoir*, which retains the properties of $(\mathcal{H},\phi)$ and is, hence, useful in connecting properly colored paths into one almost spanning tight cycle, without 'interfering' with the previously built part. We remark that we could have requested the maximum degree in each color within R to 'scale' appropriately, but this is not necessary, since our proof requires $c \ll \gamma$ anyway. The role of the set R' is to prevent R from overlapping with $V(A)$. The vertices of P which do not belong to its ends are called *inner.*

Lemma 9 (Reservoir Lemma). *Let $\rho = \rho(\gamma) > 0$ be sufficiently small and $0 < c \leqslant \frac{\gamma\rho}{6(k-1)}$. Then there is a set of vertices $R \subset V$ of size $|R| \leqslant \rho n$ such that for any $R' \subset R$ with $|R'| \leqslant \frac{\gamma}{10}|R|$ and any two disjoint $(2k-2)$-tuples $\vec{v}, \vec{w} \in (V \setminus R)^{2k-2}$, both inducing properly colored tight paths in $(\mathcal{H},\phi)$, there is in $(\mathcal{H},\phi)$ a properly colored tight path of length at most $64k\gamma^{-2}$ with end-paths $\vec{v}$ and $\vec{w}$, whose all inner vertices belong to $R \setminus R'$.*

The third lemma allows us, given A and R, to cover almost all vertices of the set $V \setminus (V(A) \cup R)$ by a bounded number of disjoint, properly colored paths.

Lemma 10 (Covering Lemma). *For any $\delta > 0$, let $c = c(\gamma,\delta) > 0$ be sufficiently small and $q = q(\gamma,\delta)$ be sufficiently large. Then there is a family $\mathcal{P}$ of at most q vertex-disjoint properly colored tight paths in $(\mathcal{H},\phi)$ covering all but at most δn vertices of $\mathcal{H}$.*

The proof of Theorem 2 follows the classical outline of the absorbing method (see [29] or [27]).

Proof of Theorem 2. Set

$$\lambda = \tfrac{1}{4(4k-4)^2}(\gamma/2)^{2k}\ , \quad \rho = \min\{\rho_9(7\gamma/8), \lambda/2\}\ , \quad \text{and} \quad \delta = \lambda/2, \tag{2}$$

where $\rho_9(7\gamma/8)$ is given by Lemma 9 with $7\gamma/8$ instead of γ. Let c_i, $i = 8, 9, 10$, be a constant c yielded by, Lemma 8, Lemma 9, and Lemma 10 respectively, with appropriately altered γ – see above. Finally, let $c = \min\{c_8, c_9/2, c_{10}/2\}$ and $q = q(\gamma/2, \delta)$ be as in Lemma 10, and let n be sufficiently large. Comment: note that q and c_6 depend on δ but no changes are required here.

Let $(\mathcal{H}, \phi)$ satisfy (1) and let A be a path provided by Lemma 8, that is, a path of length

$$|V(A)| \leqslant \tfrac{1}{4}(\gamma/2)^{k-2}n \leqslant \tfrac{1}{8}\gamma n$$

which can absorb any set U of up to λn vertices.

Apply Lemma 9 to $\mathcal{H}' = \mathcal{H} - V(A)$ with $\gamma := 7\gamma/8$ and ρ and c selected above. It is feasible as, setting $n' = |V(\mathcal{H}')| = n - |V(A)| \geqslant (1 - \gamma/8)n$,

$$\delta_{k-1}(\mathcal{H}') \geqslant \delta_{k-1}(\mathcal{H}) - |V(A)|) \geqslant \left(\frac{1}{2} + \frac{7}{8}\gamma\right) n \geqslant \left(\frac{1}{2} + \frac{7}{8}\gamma\right) n'$$

and

$$\Delta_{k-1}(\mathcal{H}_i') \leqslant \Delta_{k-1}(\mathcal{H}_i) \leqslant cn \leqslant \frac{c_9}{2(1-\gamma/8)}n' \leqslant c_9 n'.$$

Let R be the resulting set of vertices of size $|R| \leqslant \rho n' \leqslant \lambda n'/2$.

Set $\mathcal{H}'' = \mathcal{H}' - R$ and note that, since $\lambda \leqslant \frac{3}{4}\gamma$ and $n'' := |V(\mathcal{H}'')| \geqslant (1 - \gamma/2)n$,

$$\delta_{k-1}(\mathcal{H}'') \geqslant \left(\frac{1}{2} + \frac{7}{8}\gamma\right) n' - |R| \geqslant \left(\frac{1}{2} + \frac{7}{8}\gamma - \frac{1}{2}\lambda\right) n' \geqslant \left(\frac{1}{2} + \frac{\gamma}{2}\right) n''$$

and

$$\Delta_{k-1}(\mathcal{H}_i'') \leqslant \Delta_{k-1}(\mathcal{H}_i) \leqslant cn \leqslant \frac{c_{10}}{2(1-\gamma/2)}n'' \leqslant c_{10} n''.$$

Apply Lemma 10 to $(\mathcal{H}'', \phi)$ with $\gamma := \gamma/2$ and $\delta = \lambda/2$ as in (2), to obtain a family $\mathcal{P}$ of at most q properly colored paths which cover all but at most $\delta n'' \leqslant \lambda n/2$ vertices of $\mathcal{H}''$. Let W be the set of all vertices of $\mathcal{H}''$ not covered by the paths in $\mathcal{P}$. Set $\mathcal{P}^A = \mathcal{P} \cup \{A\}$.

To connect the paths from $\mathcal{P}^{\mathcal{A}}$ into one cycle C, we apply Proposition 7 with $Q = R$ and $m = |\mathcal{P}^A| = q + 1$. To verify Statement I therein with $g = 64k\gamma^{-2} - (4k - 4)$, let Q', P_1 and P_2 in $(\mathcal{H} - R, \phi)$ be as in the statement. We invoke Lemma 9 with $R' = Q'$ since, for n large enough,

$$|Q'| \leqslant mg \leqslant (q + 1) \cdot 64k\gamma^{-2} \leqslant \frac{\gamma}{10}|R|.$$

Indeed, the left-hand side of the third inequality above is a constant, while the right-hand side grows linearly with n. By Lemma 9, applied to one end-path $\vec{v}$ of P_1 and one end-path $\vec{w}$ of P_2, we see that Statement I of Proposition 7 holds and thus so does Statement II.

Let U be the set of vertices of $\mathcal{H}$ not covered by C. Since $U \subset R \cup W$, we have

$$|U| \leqslant |R| + |W| \leqslant \rho n + \delta n \leqslant \lambda n.$$

Therefore, a tight Hamilton cycle can be constructed in $\mathcal{H}$ by absorbing U into the absorbing path A, which is a sub-path of the cycle C. Since the new path A_U which is replacing A in C has the same end-paths as A, the obtained Hamilton cycle is properly colored as well. $\square$

3.1 Connecting and Reservoir Lemmas

The following connecting lemma is a key tool in constructing both the absorbing path and the reservoir.

Lemma 11 (Connecting Lemma). *Let $(\mathcal{H}, \phi)$ satisfy (1). For any $c \leqslant \frac{\gamma}{3(k-1)}$, a subset $V' \subset V$ with $|V'| \leqslant \gamma n/10$, and any two disjoint $(2k-2)$-tuples $\vec{v}, \vec{w} \in (V \setminus V')^{2k-2}$, both inducing properly colored tight paths in $(\mathcal{H}, \phi)$, there is a properly colored tight path P in $(\mathcal{H}, \phi)$ of length at most $8(2k-1)\gamma^{-2}$ with end-paths $\vec{v}$ and $\vec{w}$, whose all vertices belong to $V \setminus V'$.*

Lemma 11 will be deduced from an adaptation of Lemma 2.4 in [29]. To state this adaptation, consider a *directed* k-graph $\overrightarrow{\mathcal{H}}$ where each edge is a *k-tuple* (a sequence of vertices), rather than a set. For a sequence of vertices $\vec{v}$ and a vertex u disjoint from $\vec{v}$, the two concatenations of u and $\vec{v}$ will be denoted by $u\vec{v}$ and $\vec{v}u$. Given a directed k-graph $\overrightarrow{\mathcal{H}}$ and a $(k-1)$-tuple $\vec{v}$ of its vertices, set

$$d^+(\vec{v}) = |\{u \in V : \vec{v}u \in \overrightarrow{\mathcal{H}}\}| \quad \text{and} \quad d^-(\vec{v}) = |\{u \in V : u\vec{v} \in \overrightarrow{\mathcal{H}}\}|.$$

Furthermore, define $d^\pm(\vec{v}) = \min(d^+(\vec{v}), d^-(\vec{v}))$ and call $\vec{v}$ *extendable* if $d^+(\vec{v}) > 0$ *or* $d^-(\vec{v}) > 0$. Finally, for two $(k-1)$-tuples $\vec{v}$ and $\vec{w}$ we say that there is a *tight path* joining them, if there is a sequence $\vec{s} = \vec{v}\vec{u}\vec{w}$, such that any k consecutive vertices in $\vec{s}$ form an edge in $\overrightarrow{\mathcal{H}}$.

A brief glance at the proof of Lemma 2.4 in [29] reveals that it goes through for any directed k-graph with high values of $d^\pm(\vec{v})$ for every extendable $(k-1)$-tuple. Consequently, the following version of that lemma is also true.

Lemma 12 ([29]). *Let $\gamma > 0$ and $\overrightarrow{\mathcal{H}}$ be a directed n-vertex k-graph. Assume that for any extendable $(k-1)$-tuple $\vec{v}$ of vertices of $\overrightarrow{\mathcal{H}}$, $d^\pm(\vec{v}) \geqslant \left(\frac{1}{2} + \gamma\right) n$. Then, for every pair of disjoint, extendable $(k-1)$-tuples $\vec{v}$ and $\vec{w}$, there is a tight path in $\overrightarrow{\mathcal{H}}$ of length at most $2k/\gamma^2$ between them.* $\square$

Proof of Lemma 11. Given $(\mathcal{H},\phi)$, define an auxiliary *directed*, $(2k-1)$-uniform hypergraph $\overrightarrow{\mathcal{H}}=\overrightarrow{\mathcal{H}}(\phi,V')$ with vertex set $V\setminus V'$ and the edge set consisting of sequences of distinct vertices $(v_1,\dots,v_{2k-1})\in(V\setminus V')^{2k-1}$ such that the sets

$$\{v_1,\dots,v_k\},\dots,\{v_k,\dots,v_{2k-1}\}$$

are edges of $\mathcal{H}$ with distinct colors. Equivalently, the edges of $\overrightarrow{\mathcal{H}}$ correspond to the properly colored tight paths of length $2k-1$ in $(\mathcal{H},\phi)$ with a fixed direction.

By this construction, one cannot exclude the possibility that a $(2k-2)$-tuple $\vec{v}$ is not a segment of an edge of $\overrightarrow{\mathcal{H}}$, and thus $\max(d^+(\vec{v}),d^-(\vec{v}))=0$ (for instance, if $\vec{v}$ does not span a path in $\mathcal{H}$). However, it turns out that $\overrightarrow{\mathcal{H}}$ satisfies the assumptions of Lemma 12 with $k:=2k-1$ and $\gamma:=\gamma/2$.

Indeed, let $\vec{v}$ be a $(2k-2)$-tuple in $\overrightarrow{\mathcal{H}}$ with $\max(d^+(\vec{v}),d^-(\vec{v}))>0$. This means, in particular, that $\vec{v}$ spans in $(\mathcal{H},\phi)$ a properly colored path. Let U be the set of vertices $u\in V\setminus V'$ such that $\{u,v_k,\dots,v_{2k-2}\}\in\mathcal{H}$ and $\phi(u,v_k,\dots,v_{2k-2})$ does not appear on the consecutive k-tuples of $\vec{v}$. Clearly, $\vec{v}u$ is an edge of $\overrightarrow{\mathcal{H}}$ for any $u\in U$, so $d^+(\vec{v})=|U|$. Using the assumed bounds $\delta_{k-1}(\mathcal{H})\geqslant(1/2+\gamma)n$ and $\Delta_{k-1}(\mathcal{H}_i)\leqslant cn$, we infer that

$$d^+(\vec{v})=|U|\geqslant\left(\frac{1}{2}+\gamma-\frac{\gamma}{10}-(k-1)c\right)n\geqslant\left(\frac{1}{2}+\frac{\gamma}{2}\right)n,$$

since $c\leqslant\frac{\gamma}{3(k-1)}$ and $|V'|\leqslant\gamma n/10$. Analogously, $d^-(\vec{v})\geqslant\left(\frac{1}{2}+\frac{\gamma}{2}\right)n$.

To deduce Lemma 11, consider vertex-disjoint $(2k-2)$-tuples $\vec{v},\vec{w}\in(V\setminus V')^{2k-2}$ spanning properly colored paths in $(\mathcal{H},\phi)$. Applying Lemma 12 to $\overrightarrow{\mathcal{H}}$ with $k:=2k-1$ and $\gamma=\gamma/2$, we get a path P in $\overrightarrow{\mathcal{H}}$ of length at most $8(2k-1)(\gamma)^{-2}$ between $\vec{v}$ and $\vec{w}$. It remains to check that P corresponds to a properly colored tight path in $\mathcal{H}$. Indeed, any two intersecting k-tuples contained in P are also contained in some edge of $\overrightarrow{\mathcal{H}}$, so they carry distinct colors by definition of $\overrightarrow{\mathcal{H}}$. $\square$

Proof of Lemma 9. Let R be a set of vertices guaranteed by Proposition 5 with $p=\frac{2}{3}\rho$, $U_S=N_{\mathcal{H}}(S)$ for $S\in\binom{V}{k-1}$ and $\alpha_S=\frac{1}{2}+\gamma$. In particular, for large n, $\frac{1}{2}\rho n\leqslant|R|\leqslant\rho n$ and, for each $S\in\binom{V}{k-1}$,

$$|N_{\mathcal{H}}(S)\cap R|\geqslant\left(\tfrac{1}{2}+\tfrac{2}{3}\gamma\right)|R|.$$

We claim that R fulfils the conclusion of Lemma 9. To see this, consider a set $R'\subset R$, $|R'|\leqslant\gamma|R|/10$, and two disjoint $(2k-2)$-tuples of vertices $\vec{v},\vec{w}\in(V\setminus R)^{2k-2}$ inducing properly colored tight paths in $(\mathcal{H},\phi)$. Let

$$\mathcal{R}=\mathcal{H}[R\cup\{\vec{v},\vec{w}\}],$$

where $\vec{v}$, $\vec{w}$ are viewed as sets. Set $r=|V(\mathcal{R})|$ and note that $\frac{1}{2}\rho n\leqslant|R|\leqslant r\leqslant|R|+4k$. Thus,

$$\delta_{k-1}(\mathcal{R})\geqslant\delta_{k-1}(\mathcal{H}[R])\geqslant\left(\frac{1}{2}+\frac{2}{3}\gamma\right)|R|\geqslant\left(\frac{1}{2}+\frac{\gamma}{2}\right)r\quad\text{and}\quad\Delta_{k-1}(\mathcal{R}_i)\leqslant cn\leqslant\frac{2c}{\rho}r.$$

Using Lemma 11 with $\mathcal{H} := \mathcal{R}$, $\phi := \phi|_{\mathcal{R}}$, $\gamma := \frac{\gamma}{2}$, $c := 2c/\rho \leqslant \gamma/(3k)$, and $V' = R'$, we get the desired path of length at most $64k\gamma^{-2}$ between $\vec{v}$ and $\vec{w}$. $\square$

3.2 Absorbing path

For a vertex $v \in V$, a *v-absorber* is a $(4k-4)$-tuple of vertices $\vec{w} = (w_1, \dots, w_{4k-4})$ spanning a tight path T in $\mathcal{H}$ such that the sequence $(w_1, \dots, w_{2k-2}, v, w_{2k-1}, \dots, w_{4k-4})$ spans another tight path T_v in $\mathcal{H}$. If both T and T_v are properly colored paths, we call the $(4k-4)$-tuple $\vec{w}$ a *properly colored v-absorber*. To construct an absorbing path, we first need to show that for every v there are many, that is, $\Theta(n^{4k-4})$, properly colored v-absorbers.

Lemma 13. *Let $c \leqslant 4(\gamma/2)^k(3k)^{-2}$. For every $v \in V(\mathcal{H})$ there are at least $4(\gamma/2)^k n^{4k-4}$ properly colored v-absorbers in $(\mathcal{H}, \phi)$.*

Proof. We begin by counting v-absorbers. A v-absorber $\vec{w} = (w_1, w_2, \dots, w_{4k-4})$ can be constructed by sequentially selecting the vertices w_i, $i = 1, \dots, 4k-4$, and each time counting the number of ways to do it. Initially, we will be allowing repetitions, $w_i = w_j$, as well as choices $w_i = v$. Those will be discarded at the end.

- For $i = 1, \dots, k-1$, there are no constraints on w_i, so the number of choices of w_i is n.
- For $i = k, \dots, 2k-3$, the k-tuple $\{w_{i-k+1}, \dots, w_i\}$ must form an edge in $\mathcal{H}$, so the number of choices of w_i is at least $\delta_{k-1}(\mathcal{H}) \geqslant (1/2+\gamma)n$.
- For $i = 2k-2, \dots, 3k-3$, vertex w_i must belong to two edges, $\{w_{i-k+1}, \dots, w_i\}$ and $\{v, w_{i-k+2}, \dots, w_i\}$, so the number of choices of w_i is at least
$$2(1/2+\gamma)n - n = 2\gamma n.$$
- For $i = 3k-2, \dots, 4k-4$, the k-tuple $\{w_{i-k+1}, \dots, w_{i-1}\}$ must form an edge, so the number of choices of w_i is, again, at least $(1/2+\gamma)n > n/2$.

Altogether, as the number of choices with vertex repetitions is $O(n^{4k-5})$, for sufficiently large n, the total number of v-absorbers in $\mathcal{H}$ is at least
$$n^{k-1}\left((1/2+\gamma)n\right)^{2k-3}(2\gamma n)^k - O(n^{4k-5}) \geqslant 8\left(\frac{\gamma}{2}\right)^k n^{4k-4}.$$

Now we have to subtract the number of v-absorbers with a color conflict. Let T and T_v be the paths as in the definition of a v-absorber $\vec{w}$. Our task is to count the v-absorbers $\vec{w}$ in which either T or T_v contains two intersecting edges with the same color.

Let us estimate from above the number of v-absorbers in which we have $\phi(e) = \phi(f)$ for a given pair (e, f) of edges in T. Let $w_j \in f \setminus e$. There are no more than n^{4k-5} choices of the vertices w_i, $i = 1, \dots, 4k-4$, $i \neq j$. However, since $\Delta_{k-1}(\mathcal{H}_{\phi(e)}) \leqslant cn$, and we want $\phi(e) = \phi(f)$, vertex w_j can be chosen in at most cn ways. Altogether, this gives us at most cn^{4k-4} v-absorbers. By the same token, the number of v-absorbers in which we have $\phi(e) = \phi(f)$ for a given pair (e, f) of edges in T_v is also at most cn^{4k-4}.

By the union bound over all possible pairwise edge intersections in T and in T_v, and using the assumption on c, the total number of properly colored v-absorbers is at least

$$8\left(\frac{\gamma}{2}\right)^k n^{4k-4} - 2 \cdot \binom{3k-2}{2} cn^{4k-4} \geqslant 4\left(\frac{\gamma}{2}\right)^k n^{4k-4}. \qquad \square$$

Proof of Lemma 8. For each $v \in V$, let $\mathcal{A}_v$ be the family of all properly colored v-absorbers in $(\mathcal{H}, \phi)$. Based on Lemma 13, we apply Proposition 6 to the family $\{\mathcal{A}_v : v \in V\}$ with $m = n$, $t = 4k - 4$, and

$$\alpha = \frac{(\gamma/2)^k}{(4k-4)^2}.$$

As an outcome, we obtain a family $\mathcal{F}$ of vertex disjoint properly colored paths of length $(4k - 4)$ with $|\mathcal{F}| \leqslant \alpha n$ and such that for each vertex v there are at least

$$\alpha^2(4k-4)^2 n/4 = \frac{1}{4(4k-4)^2}\left(\frac{\gamma}{2}\right)^{2k} n \tag{3}$$

properly colored v-absorbers in $\mathcal{F}$.

To connect the paths from $\mathcal{F}$ into one path A, we apply Proposition 7 with $Q = V \setminus \bigcup_{F\in\mathcal{F}} V(F)$ and $m = |\mathcal{F}|$. To verify Statement I therein, we invoke Lemma 11 with $V' = Q' \cup \bigcup_{F\in\mathcal{F}} V(F)$, so we set $g = 8(2k-1)\gamma^{-2} - (4k-4)$. Note that

$$|V'| \leqslant |Q'| + |\mathcal{F}|(4k-4) \leqslant \alpha n \cdot \frac{8(2k-1)}{\gamma^2} \leqslant \frac{16k}{\gamma^2}\alpha n = \frac{\gamma^{k-2}k}{2^k(k-1)^2} n < \frac{\gamma}{10} n.$$

By Lemma 11, applied to one end-path $\vec{v}$ of P_1 and one end-path of $\vec{w}$ P_2, we see that Statement I of Proposition 7 holds and thus Statement II follows. The obtained path A has length

$$|V(A)| \leqslant \alpha n \cdot \frac{8(2k-1)}{\gamma^2} \leqslant \frac{\gamma^{k-2}k}{2^k(k-1)^2} n < \frac{1}{4}(\gamma/2)^{k-2} n,$$

as required.

Recall that, by (3), for each vertex $v \in V$, the path A contains at least $\frac{1}{4(4k-4)^2}\left(\frac{\gamma}{2}\right)^{2k} n$ vertex-disjoint properly colored v-absorbers. Therefore, one can absorb into A any set of vertices U of size $|U| \leqslant \frac{1}{4(4k-4)^2}\left(\frac{\gamma}{2}\right)^{2k} n$, one by one, obtaining a new properly colored path A_U. After absorbing a vertex v into A via a v-absorber $\vec{w}$, only the edges containing $w_k, \ldots, w_{3k-3}$ are reconfigured, so the new path is properly colored, has the same end-paths, and all the other absorbers remain unaffected. In particular, the final path A_U has the same end-paths as A. $\square$

3.3 Covering by long paths

Our proof of Lemma 10 follows that in [29], Section 4. The proof in [29] relies on five technical claims and only the first two of them require, due to the coloring, certain modifications. Claim 14 below is an analogue of Claim 4.1 in [29].

Claim 14. *Let $(\mathcal{H}, \phi)$ be a colored k-partite k-graph with at most m vertices in each part and at least dm^k edges. If $\Delta_{k-1}(\mathcal{H}_i) \leqslant \frac{d}{2k^2}m$ for all $i \in \mathbf{N}$, then $\mathcal{H}$ contains a properly colored tight path on at least $\frac{d}{2}m$ vertices.*

Proof. Denote the parts of $\mathcal{H}$ by $V_1, \dots, V_k$, and call a set $S \subset V(\mathcal{H})$ *relevant* if $|S| = k-1$ and $|S \cap V_i| \leqslant 1$ for $i \in [k]$. Note that there are at most km^{k-1} relevant sets. We start by preprocessing $\mathcal{H}$ as follows. If there is a relevant set S whose degree in the *current* hypergraph is smaller than $\frac{dm}{k}$, delete all edges containing S. Repeat this step until all relevant sets have degree either 0 or at least $\frac{dm}{k}$. Denote the resulting hypergraph by $\mathcal{H}'$. Observe that the number of deleted edges is strictly smaller than $\frac{dm}{k} \cdot km^{k-1} = dm^k$ since the edges containing each relevant set were removed at most once. Hence $\mathcal{H}'$ is non-empty.

Let P be the longest properly colored tight path in $\mathcal{H}'$. Denote its vertices by $v_1, \dots, v_\ell$ in that order, and observe that since P is tight, any part V_i contains precisely every k-th vertex of P. Let $S = \{v_1, \dots, v_{k-1}\}$, and let C be the set of colors appearing on the $k-1$ edges of P intersecting S. Let U be the set of vertices v for which $e_v := S \cup \{v\}$ is an edge of $\mathcal{H}'$ with $\phi(e_v) \notin C$. By construction of $\mathcal{H}'$ and the assumption on ϕ,

$$|U| \geqslant \frac{dm}{k} - \frac{(k-1)dm}{2k^2} \geqslant \frac{dm}{2k}.$$

On the other hand, by maximality of P, $U \subset V(P)$. Notice that there is $j \in [k]$ such that $U \subset V_j$, and thus, each vertex $v \in U$ has $k-1$ predecessors on P not belonging to U. Therefore $|V(P)| \geqslant k|U| \geqslant \frac{dm}{2}$, as required. □

Our next result is a suitably modified version of Claim 4.2 in [29].

Claim 15. *For all $0 < \varepsilon < d < 1$, every ε-regular, equitable k-partite k-graph $\mathcal{H}$ on n vertices with density $d_{\mathcal{H}} \geqslant d$ and with $\Delta_{k-1}(\mathcal{H}_i) \leqslant \frac{\varepsilon d}{2k^3}n$ for all $i \in \mathbf{N}$, contains a family $\mathcal{P}$ of vertex-disjoint properly colored paths such that*

$$\text{for each } P \in \mathcal{P} \text{ we have } |V(P)| \geqslant \frac{\varepsilon(d-\varepsilon)n}{2k} \quad \text{and} \quad \sum_{P \in \mathcal{P}} |V(P)| \geqslant (1-2\varepsilon)n.$$

Proof. Let $\mathcal{P}$ be a largest family of vertex-disjoint properly colored paths with $|V(P)| \geqslant \varepsilon(d-\varepsilon)n/(2k)$ for each $P \in \mathcal{P}$. Suppose that $\sum_{P \in \mathcal{P}} |V(P)| < (1-2\varepsilon)n$. The proof goes along the lines of the one in [29] (with $\alpha := d$ and $\mathcal{Q} := \mathcal{P}$), except for the very end, where we focus on a sub-k-graph $\hat{\mathcal{H}}$ with $m \geqslant \varepsilon n/k$ vertices in each partition class which is vertex disjoint from all paths in $\mathcal{P}$. Here we have to note that

$$\Delta_{k-1}(\hat{\mathcal{H}}_i) \leqslant \Delta_{k-1}(\mathcal{H}_i) \leqslant \frac{\varepsilon d}{2k^3}n = \frac{d}{2k^2}(\varepsilon n/k) \leqslant \frac{d}{2k^2}m.$$

Moreover, by the ε-regularity of $\mathcal{H}$, we have $|\hat{\mathcal{H}}| \geqslant (d-\varepsilon)m^k$, so Claim 14 can be applied to $\hat{\mathcal{H}}$, producing a properly colored path in $\mathcal{H}$ of length at least $\varepsilon(d-\varepsilon)n/(2k)$, vertex disjoint from $\mathcal{P}$. This yields a contradiction with the maximality of $\mathcal{P}$ (for details see [29].) □

The remaining three claims from the proof in [29], combined together, imply the existence of a vertex-decomposition of our hypergraph into ε-regular k-tuples (plus some left-over vertices). This was achieved by using the Weak Hypergraph Regularity Lemma [29, Claim 4.3]. Notice that this statement concerns hypergraphs only, making no mention of the coloring. Given an ε-regular partition $(V_1, \ldots, V_t)$ of a k-graph $\mathcal{H}$ and a real $d > 0$, we denote by $\mathcal{K}(V_1, \ldots, V_t; d, \varepsilon)$ the k-graph on vertex set $[t]$ where edges correspond to ε-regular k-tuples $(V_{i_1}, \ldots, V_{i_k})$ with density at least d. Our last claim corresponds to Claims 4.3-4.5 in [29].

Claim 16. *Given $\gamma > 0$ and sufficiently small $\varepsilon = \varepsilon(\gamma) > 0$, there exist integers T_0 and n_0 such that the following holds. Any n-vertex k-graph $\mathcal{H}$, $n \geqslant n_0$, with $\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2} + \gamma\right) n$ contains an equitable partition $V = V_1 \cup \cdots \cup V_t$, $t \leqslant T_0$, such that the corresponding k-graph $\mathcal{K} = \mathcal{K}(V_1, \ldots, V_t; \frac{1}{4}, \varepsilon)$ possesses a matching covering at least $(1 - 2k\eta)\, t$ vertices, where $\eta = k(\sqrt{\varepsilon})^{1/(k-1)}$.* □

Proof of Lemma 10. For any $\gamma > 0$ and $\delta > 0$, let ε satisfy $2\varepsilon + 4k\eta \leqslant \delta$. Further, let $T_0 = T_0(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ be as in Claim 16 and set

$$c = \frac{\varepsilon}{16k^2 T_0} \quad \text{and} \quad q = \frac{16T_0}{\varepsilon}.$$

Let $\mathcal{H}$ be an n-vertex k-graph, $n \geqslant n_0$, with

$$\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2} + \gamma\right) n \quad \text{and} \quad \Delta_{k-1}(\mathcal{H}_i) \leqslant cn.$$

By Claim 16, there is an equitable partition $V = V_1 \cup \cdots \cup V_t$, $t \leqslant T_0$, such that the corresponding k-graph $\mathcal{K} = \mathcal{K}(V_1, \ldots, V_t; \frac{1}{4}, \varepsilon)$ possesses a matching $\mathcal{M}$ covering at least $(1 - 2k\eta)\, t$ vertices of $\mathcal{K}$. Note that

$$n/(2T_0) \leqslant \lfloor n/t \rfloor \leqslant |V_i| \leqslant \lceil n/t \rceil.$$

For each $e \in \mathcal{M}$, let $\mathcal{H}_e$ be the sub-k-graph of $\mathcal{H}$ induced by the partition sets constituting the edge e of $\mathcal{K}$. We have

$$\Delta_{k-1}((\mathcal{H}_e)_i) \leqslant \Delta_{k-1}(\mathcal{H}_i) \leqslant \frac{\varepsilon}{8k^2} \left(\frac{n}{2T_0}\right) \leqslant \frac{\varepsilon}{8k^3} \left(k\lfloor n/t \rfloor\right) \leqslant \frac{\varepsilon}{8k^3} |V(\mathcal{H}_e)|.$$

Thus, by Claim 15 applied to $\mathcal{H} := \mathcal{H}_e$ with $d = 1/4$, there is a family $\mathcal{P}_e$ of vertex-disjoint properly colored paths such that

$$\text{for each } P \in \mathcal{P}_e \text{ we have } |V(P)| \geqslant \frac{\varepsilon(1/4 - \varepsilon)n}{2T_0} \quad \text{and} \quad \sum_{P \in \mathcal{P}_e} |V(P)| \geqslant (1 - 2\varepsilon)|V(\mathcal{H}_e)|.$$

Applying the same argument to each $e \in \mathcal{M}$ gives a collection of paths $\mathcal{P}$ which cover all but at most $(2\varepsilon + 4k\eta)n \leqslant \delta n$ vertices of $\mathcal{H}$. Clearly,

$$|\mathcal{P}| \leqslant \frac{2T_0}{\varepsilon(1/4 - \varepsilon)} \leqslant \frac{16T_0}{\varepsilon}.$$ □

4 Hamilton (k,ℓ)-cycles with codegree conditions

This section is devoted to the proof of Theorem 3. Recall that given a k-graph $\mathcal{H}$ and a coloring ϕ of its edges, $\mathcal{H}_i = \{e \in \mathcal{H} : \phi(e) = i\}$. Throughout this section, given integers $1 \leqslant \ell < k/2$, $(\mathcal{H}, \phi)$ is an n-vertex colored k-graph on vertex set V with n sufficiently large and divisible by $k - \ell$, and such that

$$\Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell} \quad \text{for all} \quad i \in \mathbf{N}, \tag{4}$$

where $c > 0$ is sufficiently small. As all our statements below are about (k,ℓ)-paths and (k,ℓ)-cycles, we will often call them just paths and cycles. As in the previous section, we will first state three lemmas from which Theorem 3 follows.

Observe that, since $2\ell < k$, the degree of any vertex on a path or a cycle is either one or two. Recall that for $\ell < k/2$, the ℓ-ends of a (k,ℓ)-path are defined as the sets of the first and the last ℓ vertices of the path, respectively.

Lemma 17 (Absorbing Lemma). *There exist constants $a = a(k,\ell) < 1$ and $b = b(k,\ell) < 1$ such that for every $\lambda > 0$ and $c \leqslant b\lambda^5$, if $(k-\ell)|n$, $\delta_{k-1}(\mathcal{H}) \geqslant \lambda n$ and (4) holds, then there is a properly colored (k,ℓ)-path A in $\mathcal{H}$ with $|V(A)| \leqslant \lambda^5 n$ such that for every $U \subset V$ with $|U| \leqslant a(k-\ell)\lambda^{10}n$ and $(k-\ell)||U|$, there is a properly colored (k,ℓ)-path P_U in $\mathcal{H}$ with $V(P_U) = V(A) \cup U$ and with the same end-edges and ℓ-ends as A. (We will say that A can absorb U.)*

Lemma 18 (Reservoir Lemma). *For every $0 < \rho < 1$, $d > 0$, and $c \leqslant \frac{(d/2)^3}{216k!^3}(\rho/2)^{k-\ell}$, if $\delta_{k-1}(\mathcal{H}) \geqslant dn$ and (4) holds, then there is a set $R \subset V$ with $|R| \leqslant \rho n$ such that for any two disjoint ℓ-sets $X, Y \subset V \setminus R$, any two colors c_X, c_Y, and any subset $R' \subset R$, $|R'| \leqslant d|R|/20$, there exists a 3-edge properly colored (k,ℓ)-path $P = X, v_1 \cdots v_{3k-4\ell}, Y$ connecting X and Y, with $\{v_1, \ldots, v_{3k-4\ell}\} \subset R \setminus R'$ and such that $\phi(X, v_1 \cdots v_{k-\ell}) \neq c_X$ and $\phi(v_{2k-3\ell+1} \cdots v_{3k-4\ell}, Y) \neq c_Y$.*

Lemma 19 (Covering Lemma). *For every $\gamma > 0$ and $\delta > 0$, let $c = c(\gamma,\delta) > 0$ be sufficiently small and $q = q(\gamma,\delta)$ be sufficiently large. If $\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma\right) n$ and (4) holds, then there is a family $\mathcal{P}$ of at most q vertex-disjoint properly colored (k,ℓ)-paths in $(\mathcal{H}, \phi)$ covering all but at most δn vertices of $\mathcal{H}$.*

Proof of Theorem 3. Let $\gamma > 0$ be given and let a and b be the constants provided by Lemma 17. Set

$$\lambda = (\gamma/4)^{1/5}, \quad \rho = \delta = \frac{1}{2}a\lambda^{10} = \frac{a\gamma^2}{32}, \quad d \leqslant \frac{1}{2(k-\ell)}, \quad q = q_{19}(\gamma/2, \delta),$$

and

$$c = \min\left(b\lambda^5,\ \frac{1}{217k!^3} \cdot \left(\frac{d}{2}\right)^3 \left(\frac{\rho}{2}\right)^{k-\ell},\ \frac{1}{2}c_{19}(\gamma/2,\delta)\right),$$

where the subscript $_{19}$ means that the constant comes from Lemma 19. Note that the second and third ingredients of the minimum above incorporate an extra margin to accommodate the forthcoming estimates.

Let $(\mathcal{H}, \phi)$ be an n-vertex colored k-graph on vertex set V with n sufficiently large and divisible by $k - \ell$. Assume that $\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma\right) n$. Since, for sufficiently small $\gamma = \gamma(k, \ell)$ we have $\frac{1}{2(k-\ell)} + \gamma \geqslant \frac{1}{2(k-\ell)} \geqslant \lambda$, we may apply Lemma 17 to $(\mathcal{H}, \phi)$ to find an absorbing path A with $|V(A)| \leqslant \lambda^5 n = \gamma n/4$ which can absorb any set U of up to $a(k-\ell)\lambda^{10} n$ vertices and $(k-\ell)||U|$.

Next, let $R \subset V \setminus V(A)$ be the set given by Lemma 18 applied to $\mathcal{H}' = \mathcal{H} - V(A)$ and to the coloring ϕ' induced in $\mathcal{H}'$ by ϕ. Such an application is feasible since, setting $n' = |V(\mathcal{H}')| = n - |V(A)| \geqslant (1 - \gamma/4)n$, we have

$$\delta_{k-1}(\mathcal{H}') \geqslant \delta_{k-1}(\mathcal{H}) - |V(A)|n \geqslant \frac{n}{2(k-\ell)} \geqslant dn'$$

and

$$\Delta_\ell(\mathcal{H}_i') \leqslant \Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell} \leqslant \frac{c}{(1-\gamma/4)^{k-\ell}}(n')^{k-\ell} \leqslant \frac{1}{216k!^3} \cdot \left(\frac{d}{2}\right)^3 \left(\frac{\rho}{2}\right)^{k-\ell} (n')^{k-\ell}.$$

Further, let $\mathcal{H}'' = \mathcal{H}' - R$ and let ϕ'' be the coloring induced in $\mathcal{H}''$ by ϕ. Set $n'' := |V(\mathcal{H}'')|$. Since $\gamma/4 + a\gamma^2/32 \leqslant \gamma/2$,

$$\delta_{k-1}(\mathcal{H}'') \geqslant \delta_{k-1}(\mathcal{H}) - (|V(A)| + |R|)n \geqslant \left(\frac{1}{2(k-\ell)} + \gamma/2\right) n''$$

and

$$\Delta_\ell(\mathcal{H}_i'') \leqslant \Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell} \leqslant \frac{c_{19}(\gamma/2, \delta)}{2(1-\gamma/2)^{k-\ell}}(n'')^{k-\ell} \leqslant c_{19}(\gamma/2, \delta)(n'')^{k-\ell}.$$

Hence, we can apply Lemma 19 to the colored hypergraph $(\mathcal{H}'', \phi'')$ with $\gamma := \gamma/2$ to obtain a family $\mathcal{P}$ of at most q properly colored paths which cover all but at most $\frac{1}{2}a\lambda^{10}n$ vertices of $\mathcal{H}''$. Let W be the set of vertices of $\mathcal{H}''$ not covered by any of the paths in $\mathcal{P}$. Include the absorbing path A into $\mathcal{P}$ to get a family of paths $\mathcal{P}^A = \mathcal{P} \cup \{A\}$.

To connect the paths from $\mathcal{P}^A$ into one cycle C, we apply Proposition 7 with $Q = R$ and $m = |\mathcal{P}^A| = q + 1$. To verify Statement I therein with $g = 3k - 4\ell$, let Q', P_1 and P_2 be as in the statement. We invoke Lemma 18 with $R' = Q'$ since, for n large enough,

$$|Q'| \leqslant mg \leqslant (q+1)(3k - 4\ell) \leqslant \frac{d}{20}|R|.$$

By Lemma 18, applied to one ℓ-end X of P_1 and one ℓ-end Y of P_2, we see that Statement I of Proposition 7 holds and so does Statement II.

Let U be the set of vertices of V not covered by C. As $|V(C)|$ is divisible by $k - \ell$, so is $|U|$. Since

$$|U| \leqslant |W| + |R| \leqslant \delta n + \rho n \leqslant a\lambda^{10} n \leqslant a(k-\ell)\lambda^{10} n,$$

U can be absorbed into A by replacing A with a properly colored path P_U. Since P_U and A have the same ℓ-ends, the absorption extends C to a Hamilton cycle in $\mathcal{H}$. Since, in addition, P_U and A have the same end-edges, the obtained Hamilton cycle remains properly colored. $\square$

4.1 Connecting and Reservoir Lemmas

In this subsection we prove first a simple connecting lemma which is used in the proofs of both the Absorbing and the Reservoir Lemma. It says, roughly, that any two disjoint ℓ-sets can be connected via a short, properly colored path which avoids a given relatively small set of vertices. In fact, although we do not need it here, we show that there are many such paths.

Lemma 20 (Connecting Lemma)**.** *For every $\kappa > 0$ and $c \leqslant \frac{1}{216}\kappa^3 k!^{-3}$, let a colored k-graph $(\mathcal{H}, \phi)$ be given with $\delta_{k-1}(\mathcal{H}) \geqslant \kappa n$ and $\Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell}$ for all $i \in \mathbb{N}$. Further, let $V' \subset V$, $|V'| \leqslant \kappa n/10$. Then, for sufficiently large n, for any pair of disjoint ℓ-sets $X, Y \subset V \setminus V'$ and any two colors c_X, c_Y, there exist at least $\kappa^3 n^{3k-4\ell}/(54k!^3)$ 3-edge properly colored paths $Xv_1 \cdots v_{3k-4\ell}Y$, where $v_1, \ldots, v_{3k-4\ell} \notin V'$, such that, in addition, $\phi(X, v_1, \ldots, v_{k-\ell}) \neq c_X$ and $\phi(v_{2k-3\ell+1}, \ldots, v_{3k-4\ell}, Y) \neq c_Y$.*

Proof. We will first estimate from below the number of paths $Xv_1 \cdots v_{3k-4\ell}Y$ with no regard to coloring. Then we will subtract an upper bound on the number of those among them which have a color conflict. Throughout we assume that n is large enough for all the estimates to hold.

Note that by the assumption $\delta_{k-1}(\mathcal{H}) \geqslant \kappa n$, every set $Z \subset V$ of size $|Z| = s < k$ is contained in at least

$$\binom{n-s}{k-1-s}\frac{\kappa n}{k-s} \geqslant \frac{\kappa n^{k-s}}{2k!} \tag{5}$$

edges of $\mathcal{H}$. Now, fix two disjoint ℓ-sets $X, Y \subset V \setminus V'$. By (5), there are at least $\frac{\kappa n^{k-\ell}}{3k!}$ edges X' of $\mathcal{H}$ such that $X \subset X'$ and $X' \cap (V' \cup Y) = \emptyset$. Similarly, there are at least $\frac{\kappa n^{k-\ell}}{3k!}$ edges Y' of $\mathcal{H}$ such that $Y \subset Y'$ and $Y' \cap (V' \cup X') = \emptyset$. For a fixed pair (X', Y'), select arbitrarily subsets $Z_X \subset X' \setminus X$ and $Z_Y \subset Y' \setminus Y$ of size $|Z_X| = |Z_Y| = \ell$. Set $Z = Z_X \cup Z_Y$ and notice that $|Z| = 2\ell < k$. Thus, there are at least $\frac{\kappa n^{k-2\ell}}{3k!}$ edges T of $\mathcal{H}$ such that $Z \subset T$ and $T \cap (X' \cup Y') = \emptyset$. Altogether, there are at least $\frac{\kappa^3 n^{3k-4\ell}}{27k!^3}$ choices of (X', T, Y') and each of them corresponds to at least one 3-edge path between X and Y.

Now we count the paths with color conflicts. In addition to $\phi(X, v_1 \ldots, v_{k-\ell}) = c_X$ and $\phi(v_{2k-3\ell+1}, \ldots v_{3k-4\ell}, Y) = c_Y$, there are two more potential conflicts, corresponding to the two pairs of edges intersecting, respectively, at Z_X and at Z_Y. Using the assumption $\Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell}$ each one of them disqualifies at most $cn^{3k-4\ell}$ paths. We conclude that the number of required paths is at least

$$\frac{\kappa^3 n^{3k-4\ell}}{27k!^3} - 4cn^{3k-4\ell} \geqslant \frac{\kappa^3 n^{3k-4\ell}}{54k!^3},$$

by our assumption on c. □

Proof of Lemma 18. Let R be a set of vertices guaranteed by Proposition 5 with $p = \frac{2}{3}\rho$, $U_S = N_\mathcal{H}(S)$ for $S \in \binom{V}{k-1}$ and $\alpha_S = d$. In particular, for large n, $\frac{1}{2}\rho n \leqslant |R| \leqslant \rho n$ and, for each $S \in \binom{V}{k-1}$,

$$|N_{\mathcal{H}}(S) \cap R| \geqslant \frac{2}{3}d|R|.$$

We claim that R fulfils the conclusion of Lemma 18. To see this, consider a set $R' \subset R$, $|R'| \leqslant d|R|/20$, two disjoint ℓ-tuples of vertices $X, Y \subset V \setminus R$ and colors c_X, c_Y. Let $\mathcal{R} = \mathcal{H}[R \cup X \cup Y]$, set $r = |V(\mathcal{R})|$ and note that $\frac{1}{2}\rho n \leqslant |R| \leqslant r \leqslant |R| + 2\ell$. Thus, for large n,

$$\delta_{k-1}(\mathcal{R}) \geqslant \frac{2}{3}d|R| \geqslant \frac{1}{2}dr \quad \text{and} \quad \Delta_{k-\ell}(\mathcal{R}_i) \leqslant cn^{k-\ell} \leqslant c\left(\frac{2}{\rho}\right)^{k-\ell} r^{k-\ell}.$$

Using Lemma 20 with $\mathcal{H} := \mathcal{R}$, $\phi := \phi|_{\mathcal{R}}$, $\kappa := \frac{d}{2}$,

$$c := c\left(\frac{2}{\rho}\right)^{k-\ell} \leqslant \left(\frac{d}{2}\right)^{3} \cdot \frac{1}{216k!^3},$$

and $V' = R'$, we get the desired 3-edge path between X and Y. □

4.2 Absorbing path

In this subsection we prove Lemma 17. As usual, the absorbing path will be built from small pieces called *absorbers*. For convenience, paths are here represented by sequences of edges (with an implicit ordering of the vertices). Given a set $S \in \binom{V}{k-\ell}$, an *S-absorber* is a 3-edge path $P = (E_1, G, E_2)$ along with its ℓ-ends F_1 and F_2 for which there exists a 4-edge path $Q = (E_1, G_1, G_2, E_2)$ with $V(Q) = V(P) \cup S$ whose ℓ-ends are F_1 and F_2.

Proposition 9 in [15] and the initial part of the proof of Lemma 5 in [15] together imply the following lower bound on the number of S-absorbers. (When citing that result, we use λ in place of ε.)

Proposition 21 ([15])**.** *For all $\lambda > 0$, if $\delta_{k-1}(\mathcal{H}) \geqslant \lambda n$, then for every $S \in \binom{V}{k-\ell}$ there are at least*

$$\frac{\lambda^5(3k-4\ell)!}{2^{6+3k}k^4(3k-2\ell)!}\binom{n}{3k-2\ell} =: \zeta\binom{n}{3k-2\ell} \geqslant \frac{\zeta}{2(3k-2\ell)!}n^{3k-2\ell} \tag{6}$$

S-absorbers in $\mathcal{H}$. □

An S-absorber is called *properly colored* if both paths, E_1, G, E_2 and E_1, G_1, G_2, E_2, are properly colored. Note that there are two intersecting pairs of edges in E_1, G, E_2 and three in E_1, G_1, G_2, E_2. Thus, arguing as in the proof of Lemma 20, there are no more than $5cn^{3k-2\ell}$ S-absorbers which are not properly colored.

Corollary 22. *For ζ as in (6) and $c \leqslant \frac{\zeta}{20(3k-2\ell)!}$, for every $S \in \binom{V}{k-\ell}$ there are at least $\frac{\zeta}{4(3k-2\ell)!}n^{3k-2\ell}$ properly colored S-absorbers.* □

Proof of Lemma 17. We are going to prove Lemma 17 with

$$a = a(k,\ell) = \frac{(3k-4\ell)!^2}{2^{22+6k}k^8(3k-2\ell)!^4(3k-2\ell)^3} \quad \text{and} \quad b = b(k,\ell) = \frac{(3k-4\ell)!}{5\cdot 2^{8+3k}k^4(3k-2\ell)!^2}.$$

Note that, with this b, the bound on c in Corollary 22 coincides with that in Lemma 17.

We apply Proposition 6, with parameters $t = 3k-2\ell$ and $\alpha = \zeta(16t!t^2)^{-1}$, where ζ is as in (6), to the families $\mathcal{A}_S$ of properly colored S-absorbers viewed as vertex sequences. Note that, by Corollary 22, we have, indeed, $|\mathcal{A}_S| \geqslant 4\alpha t^2 n^t$. As an outcome, we obtain a family $\mathcal{F}$ of disjoint absorbers, i.e. members of $\bigcup_S \mathcal{A}_S$, of size $\mathcal{F} \leqslant \alpha n$ and such that for all S we have

$$|\mathcal{F}\cap\mathcal{A}_S| \geqslant \alpha^2 t^2 n/4 = \frac{\zeta^2}{2^{10}t!t^2}n = a\lambda^{10}n.$$

To connect the paths from $\mathcal{F}$ into one path A, we apply Proposition 7 with $Q = V\setminus\bigcup_{F\in\mathcal{F}}V(F)$ and $m = |\mathcal{F}| \leqslant \alpha n$. To verify Statement I therein with $g = 3k-4\ell$, let $Q' \subset Q$ and P_1, P_2 in $(\mathcal{H}-Q,\phi)$ be as in the statement. We invoke Lemma 20 with $\kappa = \lambda$ and $V' = Q'\cup\bigcup_{F\in\mathcal{F}}V(F)$ satisfying

$$|V'| \leqslant |Q'| + |\mathcal{F}|(3k-2\ell) \leqslant mg + \alpha n(3k-2\ell) \leqslant 6(k-\ell)\alpha n \leqslant \zeta n \leqslant \lambda^5 n \leqslant \lambda n/10.$$

Moreover, the upper bound $c \leqslant b\lambda^5$ is stronger than that required for Lemma 20. Applying the lemma to one ℓ-end X of P_1 and one ℓ-end Y of P_2, we see that Statement I of Proposition 7 holds and thus Statement II follows. Note that the obtained path A has length

$$|V(A)| \leqslant |\mathcal{F}|(6k) \leqslant \zeta n \leqslant \lambda^5 n,$$

as required.

Finally, for every $U \subset V$ of size $|U| \leqslant a(k-\ell)\lambda^{10}n$ where $(k-\ell)||U$, we may partition it into sets of size $k-\ell$, say $U = S_1\cup\cdots\cup S_u$, where $u = |U|/(k-\ell) \leqslant a\lambda^{10}n$. Since for each $i = 1,\ldots,u$, there are at least $a\lambda^{10}n$ disjoint properly colored S_i-absorbers in A, we can greedily absorb each set S_i into A, obtaining a new (k,ℓ)-path P_U with the same end-edges and the same ℓ-ends as A. Note that in each step we replace a 3-edge sub-path of A by a 4-edge path with the same end-edges, so the resulting path remains properly colored. □

4.3 Covering by long paths

We emphasize that for the proof of Lemma 19 the argument from [15] goes through practically verbatim. All we have to do is to incorporate the coloring constraints into Propositions 19 and Lemma 20 in [15]. We begin with a colored version of Proposition 19.

For disjoint subsets $V_1, V_2, \ldots, V_k \subset V$ of the vertex set of a k-graph $\mathcal{J}$, a (k,ℓ)-path $P = v_1v_2\ldots v_s$ is called *canonical* if for each $j = i,\ldots,s$,

$$v_i \in (V_1\cup\cdots\cup V_\ell)\cup(V_{k-\ell+1}\cup\cdots\cup V_k) \quad \text{if and only if} \quad deg_P(v_i) = 2.$$

The following result is an analog of Claim 14. Note that, trivially, $\Delta_{k-1}(\mathcal{J}) \leqslant dm$ implies $\Delta_\ell(\mathcal{J}) \leqslant dm^{k-\ell}$.

Claim 23. *Let $(\mathcal{J}, \phi)$ be a colored k-partite k-graph with partition classes $W_1, \dots, W_k$, $|W_i| \leqslant m$ for all $i \in [k]$, and at least dm^k edges. If $\Delta_\ell(\mathcal{J}_i) \leqslant dm^{k-\ell}/4$ for all i, then $\mathcal{J}$ contains a properly colored canonical path on at least $dm/4$ vertices.*

Proof. As in the proof of Proposition 19 in [15], by deleting iteratively some edges of $\mathcal{J}$, we construct a (non-empty) sub-hypergraph $\mathcal{J}'$ of $\mathcal{J}$ in which all ℓ-element sets L with exactly one vertex in each W_i, $i = 1, \dots, \ell$, as well as, all ℓ-element sets L with exactly one vertex in each W_i, $i = k - \ell + 1, \dots, k$, satisfy: $deg_{\mathcal{J}'}(L) = 0$ or $deg_{\mathcal{J}'}(L) \geqslant dm^{k-\ell}/2$.

Let $P = v_1 v_2 \cdots v_s$ be the longest properly colored canonical path in $\mathcal{J}'$. The set $L = \{v_{s-\ell+1}, \dots, v_s\}$ has, clearly, a nonzero degree in $\mathcal{J}'$, and so it is contained in at least $dm^{k-\ell}/2$ edges of $\mathcal{J}'$. Moreover, by our assumption, at most $dm^{k-\ell}/4$ of them have the same color as the last edge of P. Hence, there are at least $dm^{k-\ell}/2 - dm^{k-\ell}/4 = dm^{k-\ell}/4$ edges in $\mathcal{J}'$ which contain L and have a different color than the last edge of P. On the other hand, by the maximality of P, each one of these edges must intersect $V(P) \setminus L$ and so there cannot be more than $sm^{k-\ell+1}$ such edges. Consequently, $dm^{k-\ell}/4 \leqslant sm^{k-\ell+1}$ which implies that $s \geqslant dm/4$, as required. $\square$

The outcome of the proof of Lemma 7 in [15], excluding the last two sentences, is summarized in the following lemma.

Lemma 24. *For sufficiently small $\varepsilon := \varepsilon(\gamma) > 0$, there exists an integer T_0 such that the following holds. Any n-vertex k-graph $\mathcal{H}$ with $\delta_{k-1}(\mathcal{H}) \geqslant (1/2(k-\ell) + \gamma)n$ contains a collection $\mathcal{C}$ of at most T_0 vertex-disjoint $(\varepsilon, \gamma/6)$-regular k-tuples $(U_1^j, \dots, U_{k-1}^j, U_k^j)$, $1 \leqslant j \leqslant |\mathcal{C}| \leqslant T_0$, of sets of sizes $|U_1^j| = \cdots = |U_{k-1}^j| = (2k-2\ell-1)m$ and $|U_k^j| = (k-1)m$, for some m, which cover all but εn vertices of $\mathcal{H}$.* $\square$

In fact, in [15] the number of such collections is at most $\frac{2k-2\ell-1}{2k-2\ell}t$, where $t \leqslant T_{14}$ and T_{14} is a constant delivered by the Weak Regularity Lemma (Lemma 14 in [15]).

As a final ingredient of the proof of Lemma 19, we now prove a colored modification of Lemma 20 in [15].

Lemma 25. *For all $\beta > 0$ and $d > 0$ there exist $c', \varepsilon > 0$ and $q \in \mathbb{N}$ such that the following holds for sufficiently large m. Let $(\mathcal{J}, \phi)$ be a colored k-partite k-graph with the partition classes forming an (ε, d)-regular k-tuple $(U_1, \dots, U_{k-1}, U_k)$ with $|U_1| = \cdots = |U_{k-1}| = (2k - 2\ell - 1)m$ and $|U_k| = (k-1)m$. If $\Delta_\ell(\mathcal{J}_i) \leqslant c' m^{k-\ell}$, then there is a family of at most q vertex-disjoint, properly colored paths which cover all but at most βm vertices of $\mathcal{J}$.*

Proof. A brief analysis of the proof of Lemma 20 in [15] reveals that the only alteration is an application of our Claim 23 instead of their Proposition 19. To do so, we need to set $c' \leqslant \varepsilon(2k\varepsilon)^{k-\ell}/4$. Indeed, then for the sets $W_i \subset U_i$ defined in the proof of Lemma 20 in [15], setting $m' = 2k\varepsilon m$ and assuming $\varepsilon \leqslant d/2$, we have $|W_i| = m'$, $i = 1, \dots, k$, and

$$e(W_1, \dots, W_k) \geqslant (d - \varepsilon)(m')^k \geqslant \varepsilon(m')^k,$$

while

$$\Delta_\ell(\mathcal{J}[W_1,\dots,W_k]) \leqslant c'm^{k-\ell} = \frac{c'}{(2k\varepsilon)^{k-\ell}}(m')^{k-\ell} \leqslant \frac{\varepsilon}{4}(m')^{k-\ell}.$$

Thus, by Claim 23 with $d := \varepsilon$, $\mathcal{J}[W_1,\dots,W_k]$ contains a properly colored canonical path on at least $\varepsilon m'/4$ vertices, which leads to a contradiction in the proof of Lemma 20 in [15] (note that in [15] the length of a path is measured by the number of edges rather than vertices). The rest of that proof carries on unchanged. □

Proof of Lemma 19. For any $\gamma > 0$ and $\delta > 0$, let ε_{24} and T_0 be as in Lemma 24 and let $\beta \leqslant \delta k^2/(2T_0)$. Further, set $d = \gamma/6$ and let $\varepsilon_{25}, c', q_{25}$ be as in Lemma 25. Finally, set $\varepsilon = \min\{\varepsilon_{24}, \varepsilon_{25}, \delta/2, \gamma/12\}$. We are going to prove Lemma 19 with $c = c'/(4T_0k^2)^{k-\ell}$ and $q = T_0 q_{25}$.

By Lemma 24, $\mathcal{H}$ contains a collection of vertex-disjoint $(\varepsilon, \gamma/6)$-regular k-tuples $\mathcal{C} = \{(U_1^j, \dots, U_{k-1}^j, U_k^j) : 1 \leqslant j \leqslant |\mathcal{C}|\}$, such that $|\mathcal{C}| \leqslant T_0$, the sets U_i^j cover all but εn vertices of $\mathcal{H}$ and satisfy $|U_1^j| = \dots = |U_{k-1}^j| = (2k-2\ell-1)m$ and $|U_k^j| = (k-1)m$ for an integer m. Note that

$$(1-\varepsilon)n \leqslant |\mathcal{C}|\left[(k-1)(2k-2\ell-1) + (k-1)\right]m \leqslant 2T_0k^2m.$$

Since $\varepsilon \leqslant 1/2$, the above estimate implies that $n \leqslant (4T_0k^2)m$. On the other hand, we also have

$$|\mathcal{C}|\left[(k-1)(2k-2\ell-1) + (k-1)\right]m \leqslant n,$$

and so $m \leqslant n/k^2$.

We now verify the hypothesis of Lemma 25. Setting $\mathcal{J}^j = \mathcal{H}[U_1^j, \dots, U_k^j]$, we have

$$\Delta_\ell(\mathcal{J}_i^j) \leqslant \Delta_\ell(\mathcal{H}_i) \leqslant cn^{k-\ell} \leqslant c(4T_0k^2)^{k-\ell}m^{k-\ell} = c'm^{k-\ell}.$$

Thus, by Lemma 25, for each j there is a family $\mathcal{P}^j$ of at most q_{25} vertex-disjoint, properly colored paths in $\mathcal{J}^j$ which cover all but at most βm vertices of $(U_1^j, \dots, U_{k-1}^j, U_k{}^j)$.

Consider the family $\bigcup_{j=1}^{|\mathcal{C}|} \mathcal{P}^j$. It consists of at most $|\mathcal{C}|q_{25} \leqslant T_0 q_{25} = q$ vertex-disjoint, properly colored paths. Moreover, the number of vertices of V not covered by these paths, by our estimates on β, m, and ε, is at most

$$T_0(\beta m) + \varepsilon n \leqslant \frac{\delta}{2}k^2\frac{n}{k^2} + \frac{\delta}{2}n = \delta n.$$

This completes the proof of Lemma 19. □

5 Loose Hamilton cycles with degree conditions

This section is devoted to the proof of Theorem 4. As all our statements below are about loose paths and cycles, the attribute 'loose' will sometimes be dropped. Recall that given a 3-graph $\mathcal{H}$ and a coloring ϕ of its edges, $\mathcal{H}_i = \{e \in \mathcal{H} : \phi(e) = i\}$. Throughout this section, given a sufficiently small $\gamma \in (0,1)$, $(\mathcal{H}, \phi)$ is an n-vertex colored 3-graph on vertex set V with n sufficiently large,

$$\delta_1(\mathcal{H}) \geqslant (7/16+\gamma)n^2/2 \quad \text{and} \quad \Delta_1(\mathcal{H}_i) \leqslant cn^2 \quad \text{for all} \quad i \in \mathbf{N}, \tag{7}$$

where $c > 0$ is sufficiently small with respect to γ and some other constants introduced later.

Our proof follows, again, the standard absorbing method. However, there is a serious obstacle related to the act of absorption which affects two of the three crucial lemmas.

Due to the structure of loose paths, one cannot absorb vertices into P one by one, but rather in pairs. On the other hand, the proper-coloring constraint results in not every pair of vertices being absorbable. Thus, we introduce an auxiliary graph G on V whose edges represent the pairs of vertices which can be absorbed into P. Recall that the 1-ends of a loose path $P = v_1 \cdots v_s$ are just v_1 and v_s.

Lemma 26 (Absorbing Lemma)**.** *For every $0 < \lambda \leqslant 10^{-14}$ and $c \leqslant 30^{-9}$, if (7) holds (even with $\gamma = 0$), then there is a properly colored loose path A in $\mathcal{H}$ of order at most $12\lambda n$ and a graph $G = G(\mathcal{H}, \phi)$ on V with $\delta(G) \geqslant \frac{3}{4}n$ such that the following holds.*

For every $U \subset V$ with $|U| \leqslant 2\lambda^2 n$, if $G[U]$ has a perfect matching, then there is a properly colored loose path P_U in $\mathcal{H}$ with $V(P_U) = V(A) \cup U$ and with the same end-edges and 1-ends as A. (We will say that A can absorb U.)

Part (b) of the next lemma ensures that the vertices left outside the long cycle at the end of the proof induce a perfect matching in the graph G guaranteed by Lemma 26.

Lemma 27 (Reservoir Lemma)**.** *Let G be a graph on V with $\delta(G) \geqslant 0.71n$. For every $\rho > 0$ and $c \leqslant \frac{\rho^2}{4 \cdot 10^5}$, if (7) holds (even with $\gamma = 0$), then there is a set $R \subset V$ with $\rho n/2 \leqslant |R| \leqslant \rho n$, which has the following properties:*

(i) *For any two vertices $x, y \notin R$, any two colors c_x, c_y, and any subset $R' \subset R$, $|R'| \leqslant |R|/100$, there exist $v_1, v_2, v_3, v_4, v_5 \in R \setminus R'$ such that $xv_1v_2v_3v_4v_5y$ is a properly colored loose path in $\mathcal{H}$ with $\phi(xv_1v_2) \neq c_x$ and $\phi(v_4v_5y) \neq c_y$.*

(ii) *If $U \subset V$, $|U|$ even, $|R \setminus U| \leqslant |R|/100$, and $|U \setminus R| \leqslant |R|/100$, then $G[U]$ has a perfect matching.*

Only the covering lemma is not affected by the above-mentioned problem with absorption.

Lemma 28 (Covering Lemma)**.** *For every $\gamma > 0$ and $\delta > 0$, let $c = c(\gamma, \delta) > 0$ be sufficiently small and $q = q(\gamma, \delta)$ be sufficiently large. If (7) holds, then there is a family $\mathcal{P}$ of at most q vertex-disjoint properly colored loose paths in $(\mathcal{H}, \phi)$ covering all but at most δn vertices of $\mathcal{H}$.*

Proof of Theorem 4. Set

$$\lambda = \min\left(10^{-14}, \gamma/25\right), \quad \rho = \lambda^2, \quad \delta = \lambda^2/200, \quad q = q_{28}(\gamma/2, \delta),$$

and

$$c = \min\left(30^{-9}, \rho^2/(5\cdot 10^5), \frac{1}{2}c_{28}(\gamma/2,\delta)\right),$$

where the subscript $_{28}$ means that the constant comes from Lemma 28.

Apply Lemma 26 to find a graph G with $\delta(G) \geqslant \frac{3}{4}n$ and an absorbing path A with $|V(A)| \leqslant 12\lambda n$ which can absorb any set U of up to $2\lambda^2 n$ vertices such that $G[U]$ contains a perfect matching.

Next, let $R \subset V \setminus V(A)$ be the set given by Lemma 27 applied to $\mathcal{H}' = \mathcal{H} - V(A)$, $G' = G - V(A)$ and to the coloring ϕ' induced in $\mathcal{H}'$ by ϕ. Such an application is feasible, because, setting $n' = |V(\mathcal{H}')| = n - |V(A)|$, we have

$$\delta_1(\mathcal{H}') \geqslant \delta_1(\mathcal{H}) - |V(A)|n \geqslant \left(\frac{7}{16}+\gamma\right)\frac{n^2}{2} - 12\lambda n^2 \geqslant \frac{7}{32}(n')^2,$$

$$\Delta_1(\mathcal{H}_i') \leqslant \Delta_1(\mathcal{H}_i) \leqslant cn^2 \leqslant \frac{\rho^2}{5\cdot 10^5(1-12\lambda)^2}(n')^2 \leqslant \frac{\rho^2}{4\cdot 10^5}(n')^2,$$

and

$$\delta_1(G') \geqslant \delta(G) - |V(A)| \geqslant \frac{3}{4}n - 12\lambda n \geqslant 0.71n \geqslant 0.71n'.$$

Further, let $\mathcal{H}'' = \mathcal{H}' - R$ and let ϕ'' be the coloring induced in $\mathcal{H}''$ by ϕ. Set $n'' := |V(\mathcal{H}'')|$. Since, for $\lambda \leqslant \frac{\gamma}{25}$, we have $\lambda^2 + 12\lambda \leqslant \gamma/2$,

$$\delta_1(\mathcal{H}'') \geqslant \delta_1(\mathcal{H}) - (|V(A)| + |R|)n \geqslant \left(\frac{7}{16}+\frac{\gamma}{2}\right)(n'')^2/2$$

and

$$\Delta_1(\mathcal{H}_i'') \leqslant \Delta_1(\mathcal{H}_i) \leqslant cn^2 \leqslant \frac{c_{28}}{2(1-\gamma/2)^2}(n'')^2 \leqslant c_{28}(n'')^2.$$

Hence, we can apply Lemma 28 to the colored hypergraph $(\mathcal{H}'', \phi'')$ with $\gamma := \gamma/2$ to obtain a family $\mathcal{P}$ of at most q properly colored paths which cover all but at most $\lambda^2 n''/200$ vertices of $\mathcal{H}''$. Let W be the set of vertices of $\mathcal{H}''$ not covered by any of the paths in $\mathcal{P}$. Include the absorbing path A into $\mathcal{P}$ to get a family of paths $\mathcal{P}^A = \mathcal{P} \cup \{A\}$.

To connect the paths from $\mathcal{P}^A$ into one cycle C, we apply Proposition 7 with $Q = R$ and $m = |\mathcal{P}^A| = q+1$. To verify Statement I therein, we invoke Lemma 27 with $R' = Q'$, so we set $g = 5$. For n large enough, we have

$$|Q'| \leqslant mg \leqslant 5(q+1) \leqslant \frac{1}{100}|R|.$$

By Lemma 27(i), applied to one 1-end x of P_1 and one 1-end y of P_2, we see that Statement I of Proposition 7 holds and thus Statement II follows.

Let U be the set of vertices of V not covered by C. Since $|V(C)|$ is even, so is $|U|$. We have

$$|U \setminus R| \leqslant |W| \leqslant \frac{\lambda^2 n''}{200} \leqslant \frac{|R|}{100}, \quad \text{and} \quad |R \setminus U| \leqslant |R'| \leqslant \frac{|R|}{100}.$$

Thus, by Lemma 27(ii), the graph $G[U]$ has a perfect matching. Moreover, since

$$|U| \leqslant |W| + |R| \leqslant \delta n + \rho n < 2\lambda^2 n,$$

U can be absorbed into A (by replacing A with a properly colored path P_U), and thus into C, forming a Hamilton cycle in $\mathcal{H}$. Since P_U and A have the same end-edges, the obtained Hamilton cycle remains properly colored. $\square$

5.1 Connecting and Reservoir Lemmas

The following lemma states that if x and y are vertices already present in edges of given colors c_x and c_y, respectively, then one can connect them via a short properly colored path which avoids a given relatively small set of vertices without creating a color conflict. In fact, although we do not need it here, we show that there are many such paths.

Lemma 29 (Connecting Lemma)**.** *Let a colored 3-graph $(\mathcal{H}, \phi)$ be given with $\delta_1(\mathcal{H}) \geqslant n^2/5$ and $\Delta_1(\mathcal{H}_i) \leqslant 10^{-5}n^2$. Further, let $Z \subset V(\mathcal{H})$, $|Z| \leqslant 0.01n$. Then, for any pair of vertices $x, y \in V(\mathcal{H}) \setminus Z$, and any two colors c_x, c_y, there exist at least $4(n/10)^5$ properly colored paths $xv_1v_2v_3v_4v_5y$, where $v_1, v_2, v_3, v_4, v_5 \notin Z$, such that, in addition, $\phi(xv_1v_2) \neq c_x$ and $\phi(v_4v_5y) \neq c_y$.*

Proof. We will first estimate from below the number of paths $xv_1v_2v_3v_4v_5y$ with no regard to coloring. Then we will subtract an upper bound on the number of those among them which have a color conflict. Given a vertex $u \in V(\mathcal{H})$, we define its *neighborhood* in $\mathcal{H}$ as $N_{\mathcal{H}}(u) = \{S \subset V(H) : S \cup \{v\} \in \mathcal{H}\}$ and denote $L_u = N_{\mathcal{H}}(u)$. Note that under our assumptions, L_u is a graph with $|L_u| \geqslant n^2/5$.

Choose $\{v_4, v_5\} \in L_y$, $v_4, v_5 \neq x$, arbitrarily. There are at least $n^2/5 - n$ ways to do so. Turning, for a moment, to the other end of the to-be-path, let $X = \{v \in V : \deg_{L_x}(v) \leqslant n/20\}$ and set $|X| = \tau n$. Then

$$\frac{1}{5}n^2 - n \leqslant |L_x| \leqslant |X|n/20 + \binom{|V \setminus X|}{2} \leqslant \left(\frac{1}{10}\tau + (1-\tau)^2\right) n^2/2,$$

which implies (since n is arbitrarily large) that

$$f(\tau) := \frac{1}{10}\tau + (1-\tau)^2 \geqslant \frac{2}{5}.$$

Note that f has the unique minimum at $\frac{19}{20}$, while $f(1) = 1/10 < 2/5$ and $f(2/5) = 2/5$. It follows that $\tau \leqslant 2/5$. As the same is true for L_{v_4}, we infer that there are at least $n/5$ vertices $v \in V$ with both $\deg_{L_x}(v) \geqslant n/20$ and $\deg_{L_{v_4}}(v) \geqslant n/20$. Thus, avoiding $Z \cup \{x, y, v_4, v_5\}$, we have at least

$$(n/5 - |Z| - 4)(n/20 - |Z| - 4)(n/20 - |Z| - 5) \geqslant \frac{3}{10^4}n^3$$

triples (v_1, v_2, v_3) with $v_1, v_2, v_3 \notin Z$, $\{v_1, v_2\} \in L_x$ and $\{v_2, v_3\} \in L_{v_4}$. Consequently, factoring in the number of ordered pairs (v_4, v_5), there are at least

$$2\left(\frac{n^2}{5} - n\right)\frac{3}{10^4}n^3 \geqslant \frac{8}{10^5}n^5$$

paths $xv_1v_2v_3v_4v_5y$ with $v_1, v_2, v_3, v_4, v_5 \notin Z$.

Now, we count the paths with color conflicts. On top of the forbidden conflicts $\phi(xv_1v_2) = c_x$ and $\phi(v_4v_5y) = c_y$, there are two more corresponding to the two pairs of edges intersecting, respectively, at v_2 and at v_4. Using the assumption $\Delta_1(\mathcal{H}_i) \leqslant 10^{-5}n^2$, each one of them disqualifies at most $(n/10)^5$. Altogether, the number of required paths is at least

$$8\left(\frac{n}{10}\right)^5 - 4\left(\frac{n}{10}\right)^5 \geqslant 4\left(\frac{n}{10}\right)^5. \qquad \square$$

Proof of Lemma 27. Let $p = \frac{2}{3}\rho$. Applying Proposition 5 with $U_v = N_G(v)$, $G_v = N_{\mathcal{H}}(v)$, $\alpha_v = 0.71$, $\beta_v = \frac{7}{16}$, for all $v \in V$, we obtain a subset $R \subset V$ such that $\frac{1}{2}\rho n \leqslant |R| \leqslant \rho n$, and, for all $v \in V$,

(1) $|N_G(v) \cap R| \geqslant (0.71 - 2n^{-1/3})|R| \geqslant \frac{2}{3}|R|$, and

(2) $|N_{\mathcal{H}}(v)[R]| \geqslant (\frac{7}{16} - 3n^{-1/3})\binom{|R|}{2} \geqslant \frac{13}{32}\binom{|R|}{2}$.

We claim that for any set R satisfying properties (1) and (2) above, conditions (i) and (ii) of Lemma 27 hold.

To prove (i) consider a pair of vertices $x, y \notin R$, two colors c_x, c_y, and a subset $R' \subset R$ with $|R'| \leqslant |R|/100$. Let $\mathcal{R} = \mathcal{H}[R \cup \{x, y\}]$ and set $r = |V(\mathcal{R})| = |R| + 2$. Since $r \geqslant \rho n/2$ and n is sufficiently large, by (2),

$$\delta_1(\mathcal{R}) \geqslant \frac{13}{32}\binom{|R|}{2} \geqslant \frac{r^2}{5},$$

and, by our assumption on c, for each i,

$$\Delta_1(\mathcal{R}_i) \leqslant cn^2 \leqslant \frac{4c}{\rho^2}|R|^2 \leqslant 10^{-5}r^2.$$

Thus, we are in position to apply Lemma 29 with $\mathcal{H} := \mathcal{R}$, $\phi := \phi|_{\mathcal{R}}$, $Z := R'$, and $c := 4c/\rho^2$, obtaining the desired $v_1, v_2, v_3, v_4, v_5 \in R \setminus R'$.

To prove (ii), it is enough to show that $\delta(G[U]) \geqslant \frac{1}{2}|U|$, as this, recalling that $|U|$ is even, implies the existence of a perfect matching in $G[U]$. To this end, note that, since $|U \setminus R| \leqslant |R|/100$, we have $|U| \leqslant |R| + |R|/100$, or equivalently, $|R| \geqslant \frac{100}{101}|U|$. Thus, for any $u \in U$, using (1) and the inequality $|R \setminus U| \leqslant |R|/100$,

$$|N_G(u) \cap U| \geqslant |N_G(u) \cap R| - |R \setminus U| \geqslant \frac{2}{3}|R| - \frac{1}{100}|R| = \frac{197}{300}|R| \geqslant \frac{197}{303}|U| > \frac{1}{2}|U|,$$

which completes the proof of the lemma. $\square$

5.2 Absorbing path

In this subsection we prove Lemma 26. As usual, the absorbing path will be built from small pieces called *absorbers*. For a pair of distinct vertices $x, y \in V$, an (x, y)-*absorber* is a 7-tuple $\vec{v} = (v_1, v_2, \ldots, v_7)$ of vertices of $\mathcal{H}$ such that:

(i) $v_1v_2v_3, v_3v_4v_5, v_5v_6v_7 \in \mathcal{H}$,

(ii) $v_2xv_4, v_4yv_6 \in \mathcal{H}$.

In other words, both $\vec{v}$ and $\vec{v}_{x,y} := (v_1, v_3, v_2, x, v_4, y, v_6, v_5, v_7)$ induce loose paths in $\mathcal{H}$ in that ordering. If both $\vec{v}$ and $\vec{v}_{x,y}$ are properly colored in $\mathcal{H}$, then $\vec{v}$ is called a *properly colored* (x, y)*-absorber*. Note that $\vec{v}$ and $\vec{v}_{x,y}$ have the same end-edges and 1-ends, though the order of vertices in the end-edges does change.

In ([5], Proposition 8) it was shown that under some milder degree assumptions, for any pair of vertices $x, y \in V(\mathcal{H})$, the number of (x, y)-absorbers is $\Theta(n^7)$.

Proposition 30 ([5])**.** *For every $\xi \in (0, 3/8)$ there exists n_0 such that the following holds. Suppose that $\mathcal{H}$ is a 3-graph on $n > n_0$ vertices with $\delta_1(\mathcal{H}) \geqslant (\frac{5}{8} + \xi)^2\binom{n}{2}$. Then for every pair of vertices $x, y \in V(\mathcal{H})$ the number of (x, y)-absorbers is at least $(\xi n)^7/8$.*

Under our stronger assumption on $\delta_1(\mathcal{H})$, we may take $\xi = 1/30$ and deduce the following.

Corollary 31. *There exists n_0 such that if $\mathcal{H}$ be a 3-graph on $n > n_0$ vertices with $\delta_1(\mathcal{H}) \geqslant \frac{7}{16}\binom{n}{2}$, then for every pair of vertices $x, y \in V(\mathcal{H})$ the number of (x, y)-absorbers is at least $\frac{1}{8}(n/30)^7$.*

Consider a pair of vertices $x, y \in V(\mathcal{H})$. Although there are many (x, y)-absorbers, there is no guarantee that any of them are properly colored. Indeed, it may be the case that for every choice of $v_2, v_4, v_6 \in V(\mathcal{H})$, we have $\phi(v_2xv_4) = \phi(v_4yv_6)$. The following proposition states that for every vertex x there are many vertices y such that the number of properly colored (x, y)-absorbers is of the order of magnitude as large as possible. A pair of distinct vertices $x, y \in V(\mathcal{H})$ is called *pc-absorbable* if there exists at least $\frac{1}{16}(n/30)^7$ properly colored (x, y)-absorbers.

Proposition 32. *Let $c \leqslant 30^{-9}$. For every $x \in V$, there are at least $\frac{3}{4}n$ vertices $y \in V$ such that the pair (x, y) is pc-absorbable.*

Proof. Given $x \in V$ it is enough to show that for each $Y \subset V$, $|Y| = n/4$, there is a vertex $y \in Y$ such that the pair (x, y) is pc-absorbable.

Set $\beta = \frac{1}{16}(1/30)^7$ for convenience. By Corollary 31, the number of pairs $(y, \vec{v})$, where $y \in Y$ and $\vec{v} = (v_1, \ldots, v_7)$ is an (x, y)-absorber is at least $\frac{1}{2}\beta n^8$. We wish to count for how many of these pairs $\vec{v}$ is not a properly colored (x, y)-absorber, i.e. at least two intersecting edges in $\vec{v}$ or in $\vec{v}_{x,y}$ have the same color. As there are in total five intersecting pairs among the edges of the two paths (three in $\vec{v}$ and two in $\vec{v}_{x,y}$), this number can be crudely bounded from above by $5cn^8$.

Indeed, fixing two intersecting edges, say, $v_1v_2v_3$ and $v_3v_4v_5$, the number of pairs $(y, \vec{v})$ in question can be bounded from above by first bounding crudely the number of choices of $y, v_1, v_2, v_3, v_6, v_7$ by n^6 and then, setting $i := \phi(v_1v_2v_3)$, bounding the number of choices of v_4, v_5 by $\Delta(\mathcal{H}_i) \leqslant cn^2$, as $v_3v_4v_5$ must be of the same color as $v_1v_2v_3$. If one of the intersecting edges involves x, as in the pair v_2xv_4, v_4yv_5, then the bound cn^2 is applied to the other edge in the pair (in the given example, we would use this bound to the number of choices of y, v_5).

Consequently, the number of pairs $(y, \vec{v})$, where $y \in Y$ and $\vec{v}$ is a properly colored (x, y)-absorber is at least

$$\left(\frac{1}{2}\beta - 5c\right) n^8.$$

By averaging, there exists $y \in Y$ for which the number of properly colored (x, y)-absorbers is at least

$$(2\beta - 20c)n^7 \geqslant \beta n^7 = \frac{1}{16}(n/30)^7,$$

where we also used the inequality $20c < \beta$. Thus, the pair (x, y) is pc-absorbable. $\square$

Proof of Lemma 26. By definition, for any pc-absorbable pair (x, y), the family $\mathcal{A}_{(x,y)}$ of properly colored (x, y)-absorbers has size

$$|\mathcal{A}_{(x,y)}| \geqslant 2^{-4}(n/30)^7 > 4 \cdot 7^2\lambda n,$$

where the last inequality follows by our assumption on λ. The elements of $\bigcup_{(x,y)} \mathcal{A}(x, y)$ will be called *absorbers*. Applying Proposition 6 to the families $\mathcal{A}_{(x,y)}$ with $t = 7$ and $\alpha = \lambda$, one can find a family $\mathcal{F}$ of absorbers satisfying $|\mathcal{F}| \leqslant \lambda n$ and $|\mathcal{F} \cap \mathcal{A}_{(x,y)}| \geqslant \frac{49}{4}\lambda^2 n$ for all pc-absorbable pairs (x, y).

To connect the paths from $\mathcal{F}$ into one path A, we apply Proposition 7 with $Q = V \setminus \bigcup_{F\in\mathcal{F}} V(F)$ and $m = |\mathcal{F}| \leqslant \lambda n$. To verify Statement I therein, let Q', P_1 and P_2 in $(\mathcal{H} - R, \phi)$ be as in the statement. We invoke Lemma 29 with $Z = Q' \cup \bigcup_{F\in\mathcal{F}} V(F)$, so we set $g = 5$. Note that, for n large enough,

$$|Z| \leqslant |Q'| + 7|\mathcal{F}| \leqslant mg + 7\lambda n \leqslant 12\lambda n \leqslant 0.01n.$$

By Lemma 29, applied to one 1-end x of P_1 and one 1-end y of P_2, we see that Statement I of Proposition 7 holds and thus Statement II follows. Note that the obtained path A has length

$$|V(A)| \leqslant 12|\mathcal{F}| \leqslant 12\lambda n,$$

as required.

Let G be an auxiliary graph on vertex set V whose edges correspond to pc-absorbable pairs. Observe that there is no ambiguity in this definition, since, by reversing the 7-tuples, (x, y) is a pc-absorbable pair if and only if (y, x) is. Proposition 32 implies that $\delta(G) \geqslant \frac{3}{4}n$. Consider a set $U \subset V$ of size $|U| \leqslant 2\lambda^2 n$ which induces a matching M in G. For each edge xy of M, the path A contains

$$|\mathcal{A}_{(x,y)}| \geqslant \frac{49}{4}\lambda^2 n \geqslant \lambda^2 n$$

properly colored (x, y)-absorbers. Hence, one can absorb the pairs in M greedily to obtain a properly colored path P_U on the vertex set $V(A) \cup U$. Note that, by construction of absorbers, in each step the 1-ends and end-edges of the resulting path stay intact. Thus, the final path P_U has the same 1-ends and end-edges as the path A. □

5.3 Covering by long paths

The proof of Lemma 28 follows closely the proof of Lemma 10 in [5] which, in turn, is based partly on Lemma 20 from [15]. Since we already have a colored version of that lemma, namely Lemma 25 in Section 4, all we need is a decomposition result analogous to Lemma 24. Such a result can be deduced from the results in [5]. It is based on the Weak Hypergraph Regularity Lemma (Proposition 15 in [5]) and an extremal result developed in [5] (Lemma 11 therein); this deduction is shown in the proof of Lemma 10 in [5]. (In our version, we apply Lemma 11 with $\alpha := \varepsilon$.) Notice that this statement makes no reference to edge coloring.

Lemma 33 ([5])**.** *For a sufficiently small $\varepsilon = \varepsilon(\gamma) > 0$, there exists an integer T_0 such that the following holds. Any n-vertex 3-graph $\mathcal{H}$ with $\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right)\binom{n}{2}$ contains a collection $\mathcal{C}$ of at most T_0 vertex-disjoint $(\varepsilon, \gamma/3)$-regular triples (U_1^j, U_2^j, U_3^j), $j = 1, \dots, |\mathcal{C}| \leqslant T_0$, which cover all but εn vertices of $\mathcal{H}$. Moreover, for some m, half of the triples satisfy the identity $|U_1^j| = |U_2^j| = \frac{3}{2}|U_3^j| = 3m$, while for the other half $|U_1^j| = |U_2^j| = \frac{3}{2}|U_3^j| = 6m$.* □

Proof of Lemma 28. In this proof, which is very similar to that of Lemma 19, we utilize Lemma 33 combined with Lemma 25 for $k = 3$ and $\ell = 1$. For any $\gamma > 0$ and $\delta > 0$, let ε_{33} and T_0 be as in Lemma 33 and let $\beta \leqslant 6\delta$. Further, set $d = \gamma/3$ and let $\varepsilon_{25}, c', q_{25}$ be as in Lemma 25 with $k = 3$ and $\ell = 1$. Finally, set $\varepsilon = \min\{\varepsilon_{33}, \varepsilon_{25}, \delta/2, \gamma/6\}$. We are going to prove Lemma 28 with $c = c'/(24T_0)^2$ and $q = T_0 q_{25}$.

By Lemma 33, $\mathcal{H}$ contains a collection $\mathcal{C}$ of size $|\mathcal{C}| \leqslant T_0$ of vertex-disjoint $(\varepsilon, \gamma/3)$-regular triples of prescribed sizes which cover all but εn vertices of $\mathcal{H}$. Note that

$$(1 - \varepsilon)n \leqslant 12|\mathcal{C}|m \leqslant 12T_0 m.$$

Since $\varepsilon \leqslant 1/2$, the above estimate implies that $n \leqslant 24T_0 m$. On the other hand, we also have $12|\mathcal{C}|m \leqslant n$.

In order to apply Lemma 25, we need to verify its assumption (with $k = 3$ and $\ell = 1$). Setting, $\mathcal{J}^j = \mathcal{H}[U_1^j, U_2^j, U_3^j]$, we have

$$\Delta_1(\mathcal{J}_i^j) \leqslant \Delta_1(\mathcal{H}_i) \leqslant cn^2 \leqslant c(24T_0)^2 m^2 = c' m^2.$$

Thus, by Lemma 25, for each j there is a family $\mathcal{P}^j$ of at most q_{25} vertex-disjoint, properly colored paths in $\mathcal{J}^j$ which cover all but at most βm vertices of (U_1^j, U_2^j, U_3^j).

Consider the family $\bigcup_{j=1}^{|\mathcal{C}|} \mathcal{P}^j$. It consists of at most $|\mathcal{C}|q \leqslant T_0 q_{25} = q$ vertex-disjoint, properly colored paths. Moreover, the number of vertices of V not covered by these paths, by our estimates on β, m, and ε, is at most

$$|\mathcal{C}|(\beta m) + \varepsilon n \leqslant \beta \frac{n}{12} + \frac{\delta}{2} n \leqslant \delta n.$$

This completes the proof of Lemma 28.

□

6 Concluding remarks

We point out an extension and some directions for further research.

Compatibility systems: Theorems 2-4 actually hold in the setting of 'compatibility systems' which generalise edge colorings. Compatible cycles were considered for instance in [24, 14].

Dirac hypergraphs: Do our theorems hold for Dirac hypergraphs, that is, with the constant $\gamma = 0$? Analogous results are presently only known for graphs [24, 13].

The rainbow setting: It is likely that the rainbow analogues of Theorems 2-4 hold, where an additional assumption on the number of occurrences of each colour is needed. This was already asked by [8], but let us spell out the conjecture for tight cycles. For loose cycles, the conjecture may even hold without the local-boundedness assumption, as in [12, 19].

Conjecture 34. For every $k \geqslant 3$ and $\gamma > 0$ there exist $c > 0$ and $n_0 > 0$ such that if $(\mathcal{H}, \phi)$ is an n-vertex colored k-graph with $n \geqslant n_0$, $\delta_{k-1}(\mathcal{H}) \geqslant (1/2+\gamma)n$, $\Delta_0(\mathcal{H}_i) \leqslant cn^{k-1}$ and $\Delta_{k-1}(\mathcal{H}_i) \leqslant cn$ for every $i \in \mathbf{N}$, then $(\mathcal{H}, \phi)$ contains a rainbow colored tight Hamilton cycle $C_n^{(k)}(k-1)$.

Existing rainbow results for incomplete hypergraphs [8, 33] are achieved using the method of switchings, which is suitable for embedding H-factors because it exploits their symmetry. In other words, there are many ways to locally modify an H-factor and obtain another H-factor. It is likely that new ideas are needed for connected spanning hypergraphs.

References

[1] N. Alon and G. Gutin. Properly colored Hamilton cycles in edge-colored complete graphs. *Random Structures Algorithms*, 11(2):179–186, 1997.

[2] B. Bollobás and P. Erdős. Alternating Hamiltonian cycles. *Israel J. Math.*, 23(2):126–131, 1976.

[3] J.C. Bermond, A. Germa, M.C. Heydemann, and D. Sotteau, Hypergraphes Hamiltoniens. In *Problemes Combinatoires et Theorie des Graphes*, Colloq. Internat. CNRS No. 260, Orsay, 1976.

[4] J. Böttcher, Y. Kohayakawa, and A. Procacci. Properly coloured copies and rainbow copies of large graphs with small maximum degree. *Random Structures Algorithms*, 40(4):425–436, 2012.

[5] E. Buß, H. Hàn, and M. Schacht. Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 103(6):658–678, 2013.

[6] P. Cano, G. Perarnau, and O. Serra. Rainbow spanning subgraphs in bounded edge–colourings of graphs with large minimum degree. *Electronic Notes in Discrete Mathematics*, 61:199–205, 2017.

[7] C. C. Chen and D. E. Daykin. Graphs with Hamiltonian cycles having adjacent lines different colors. *J. Combin. Theory Ser. B*, 21(2):135–139, 1976.

[8] M. Coulson, P. Keevash, G. Perarnau, and L. Yepremyan. Rainbow factors in hypergraphs. *J. Combin. Theory Ser. A*, 172: 105184, 14, 2020.

[9] M. Coulson and G. Perarnau. A Rainbow Dirac's Theorem. *SIAM J. Disc. Math.*, 34(3): 1670–1692, 2020.

[10] D. E. Daykin. Graphs with cycles having adjacent lines of different colors. *J. Combin. Theory Ser. B*, 20(2):149–152, 1976.

[11] A. Dudek and M. Ferrara. Extensions of results on rainbow Hamilton cycles in uniform hypergraphs. *Graphs Combin.*, 31(3):577–583, 2015.

[12] A. Dudek, A. Frieze, and A. Ruciński. Rainbow Hamilton cycles in uniform hypergraphs. *Electron. J. Combin.*, 19(1):#P46, 11, 2012.

[13] S. Glock and F. Joos. A rainbow blow-up lemma. *Random Structures Algorithms*, 56(4): 1031–1069, 2020.

[14] Z. Guo, X. Li, C. Xu, and S. Zhang. Compatible Eulerian circuits in Eulerian (di) graphs with generalized transition systems. *Discrete Math.*, 341(7):2104–2112, 2018.

[15] H. Hàn and M. Schacht. Dirac-type results for loose Hamilton cycles in uniform hypergraphs. *J. Combin. Theory Ser. B*, 100(3):332–346, 2010.

[16] J. Han and Y. Zhao. Minimum codegree threshold for Hamilton ℓ-cycles in k-uniform hypergraphs. *J. Combin. Theory Ser. A*, 132:194–223, 2015.

[17] J. Han and Y. Zhao. Minimum vertex degree threshold for loose Hamilton cycles in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 114:70–96, 2015.

[18] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[19] N. Kamčev, B. Sudakov, and J. Volec. Bounded colorings of multipartite graphs and hypergraphs. *European J. Combin.*, 66:235–249, 2017.

[20] P. Keevash. A hypergraph blow-up lemma. *Random Structures Algorithms*, 39(3):275–376, 2011.

[21] P. Keevash, D. Kühn, R. Mycroft, and D. Osthus. Loose Hamilton cycles in hypergraphs. *Discrete Math.*, 311(7):544–559, 2011.

[22] G. Y. Katona, H. A. Kierstead. Hamiltonian chains in hypergraphs. *J. Graph Theory*, 30(3):205–212, 1999.

[23] M. Krivelevich, C. Lee, and B. Sudakov. Compatible Hamilton cycles in random graphs. *Random Structures Algorithms*, 49(3):533–557, 2016.

[24] M. Krivelevich, C. Lee, and B. Sudakov. Compatible Hamilton cycles in Dirac graphs. *Combinatorica*, 37(4):697–732, 2017.

[25] D. Kühn, R. Mycroft, and D. Osthus. Hamilton ℓ-cycles in uniform hypergraphs. *J. Combin. Theory Ser. A*, 117(7):910–927, 2010.

[26] A. Lo. Properly coloured Hamiltonian cycles in edge-coloured complete graphs. *Combinatorica*, 36(4):471–492, 2016.

[27] V. Rödl and A. Ruciński. Dirac-type questions for hypergraphs—a survey (or more problems for Endre to solve). In *An irregular mind*, volume 21 of *Bolyai Soc. Math. Stud.*, pages 561–590. János Bolyai Math. Soc., Budapest, 2010.

[28] V. Rödl, A. Ruciński, M. Schacht, and E. Szemerédi. On the Hamiltonicity of triple systems with high minimum degree. *Ann. Comb.*, 21(1):95–117, 2017.

[29] V. Rödl, A. Ruciński, and E. Szemerédi. An approximate Dirac-type theorem for k-uniform hypergraphs. *Combinatorica*, 28(2):229–260, 2008.

[30] V. Rödl, A. Ruciński, and E. Szemerédi. Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs. *Adv. Math.*, 227(3):1225–1299, 2011.

[31] J. Shearer. A property of the colored complete graph. *Discrete Math.*, 25(2):175–178, 1979.

[32] B. Sudakov. Robustness of graph properties. *Surveys in Combinatorics*, 440 (2017): 372, 2017.

[33] R. Yuster. Rainbow H-factors, *Electron. J. Combin.* 13:#R13, 2006.

[34] Y. Zhao. Recent advances on Dirac-type problems for hypergraphs. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 145–165. Springer, 2016.