The Number of Quasi-Trees in Fans and Wheels

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Abstract

We extend the classical relation between the $2n$-th Fibonacci number and the number of spanning trees of the n-fan graph to ribbon graphs. More importantly, we establish a relation between the n-associated Mersenne number and the number of quasi-trees of the n-wheel ribbon graph. The calculations are performed by computing the determinant of a matrix associated with ribbon graphs. These theorems are also proven using contraction and deletion in ribbon graphs. The results provide neat and symmetric combinatorial interpretations of these well-known sequences. Furthermore, they are refined by giving two families of abelian groups whose orders are the Fibonacci and associated Mersenne numbers.

Mathematics Subject Classifications: 05C10, 11B39

1 Introduction

The relation between the number of spanning trees of fans and wheels with the Fibonacci and Lucas numbers seems to fascinate mathematicians in combinatorics, see [26, 31, 28, 17, 32, 25]. For example, in 1972, there were two talks at the British Combinatorial Conference about this subject, see [14, 16, 17]. As the 50-year-old gap seems meaningful, this work attempts to shed some new light on the relation between these well-known sequences and the more recent concept of quasi-trees in ribbon graphs.

The *wheel graph* with $n+1$ vertices, W_n, has n vertices in an n-cycle (the rim) plus one vertex (the hub) adjacent to the rest of the vertices. The *fan graph* with $n+1$ vertices, F_n, is obtained from W_n by deleting an edge from the rim. The basic formulae for the number of spanning trees were already known from the work of Sedláček [30, 29].

Theorem 1. *The number of spanning trees of W_n is $l_{2n}-2$, and the number of spanning trees of F_n is f_{2n}, where l_n is the n-th Lucas number and f_n is the n-th Fibonacci number.*

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The sequence of Fibonacci and Lucas numbers share the same recursive formula, $f_n = f_{n-1} + f_{n-2}$ and $l_n = l_{n-1} + l_{n-2}$. However, the initial values are different. The first Fibonacci numbers are $f_1 = 1$ and $f_2 = 1$, while the first Lucas numbers are $l_1 = 1$ and $l_2 = 3$.

The reason for this present work is twofold. First, the numbers of spanning trees of the graphs F_k are the Fibonacci numbers of even index. It is natural to ask if there is a family of graphs whose numbers of spanning trees give the entire Fibonacci sequence. A partial answer is the family $\{F_k\} \cup \{F_k^+\}$, where F_k^+ is obtained from the graph F_k by adding a parallel edge from the hub to a minimum degree vertex in the rim. This result is implicit in the work in [16, 17]. However, the arbitrary break of symmetry gives us a reason to present a family of ribbon graphs whose number of spanning quasi-trees is the Fibonacci sequence that is more natural and symmetric.

Second, the -2 in the formula for spanning trees of W_n is puzzling. It is natural to look for a family of graphs whose numbers of spanning trees give the sequence of Lucas numbers, probably with a -2 added to each term. This work presents a family of ribbon graphs whose numbers of spanning quasi-trees is the sequence of *associated Mersenne numbers* $\{a_n\}$ that naturally extends the family of wheel graphs and explains the seemingly arbitrary -2 in the well-known formula. The sequence $\{a_n\}$ was first defined in [15] as the integer sequence such that $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2} + 1 - (-1)^n$. The relation between $\{a_n\}$ and $\{l_n\}$ is folklore: $a_n = l_n - 1 - (-1)^n$.

2 Preliminaries

2.1 Graphs and ribbon graphs

An abstract graph G is just an ordered pair $G = (V, E)$ comprising a finite set V of vertices and a set E of unordered pairs of vertices called edges. For example, $P_2 \times P_n$ is the *ladder graph* with vertices $\{(1, i)|1 \leqslant i \leqslant n\} \cup \{(2, i)|1 \leqslant i \leqslant n\}$ and there is an edge joining (i, j) with (i', j') if and only if $|i - i'| + |j - j'| = 1$. For more about graph theory, see [12]

The definition of ribbon graphs is taken from [4, 10]. A ribbon graph $\mathbb{G}$ consists of two finite sets of closed disks, a set V of vertices, and a set E of edges such that their union defines a surface with boundary. The vertices and edges satisfy the following restrictions. The vertices and edges intersect in disjoint line segments; each such line segment lies on the boundary of precisely one vertex and precisely one edge; every edge contains exactly two such line segments. Note that ribbon graphs need not be connected and that edges joining a vertex to itself are allowed. If the surface is orientable, then we say that the ribbon graph is orientable.

A more intuitive construction starts with a 2-cellular embedding of a graph G in a closed compact surface Σ. Then a ribbon graph is obtained by taking a small neighborhood of the embedding of G and deleting its complement. Also, given a ribbon graph $\mathbb{G}$, if we cap each boundary component of (the surface of) $\mathbb{G}$ with a disk, we get a closed compact surface $\Sigma(\mathbb{G})$ where the abstract graph G has a natural 2-cellular embedding. Notice that we can always consider a ribbon graph $\mathbb{G}$ as an abstract graph G, by disregarding the information about the embedding.

As with graphs, we can delete edges or isolated vertices from a ribbon graph $\mathbb{G}$ and get a new ribbon graph $\mathbb{H}$, called a *ribbon subgraph* of $\mathbb{G}$. If $\mathbb{H}$ has the same vertex set as $\mathbb{G}$, it is called a *spanning ribbon subgraph*. However, care needs to be taken as $\mathbb{H}$ may not have a 2-cellular embedding in the same surface as $\mathbb{G}$. For example, the ribbon graph $\mathbb{G}$ with one vertex and two interlaced loops has a 2-cellular embedding in the torus, but the subgraph $\mathbb{H}$ obtained from $\mathbb{G}$ by deleting one loop is 2-cellular embeddable just in the sphere.

The number of spanning trees and quasi-trees

Given a connected graph G, the number of spanning subgraphs of G that are trees is a fundamental invariant associated with G, called the *complexity* of G and denoted by $\kappa(G)$.

A ribbon graph with exactly one boundary component is called a *quasi-tree*. Given a connected ribbon graph $\mathbb{G}$, the number of spanning ribbon subgraphs of $\mathbb{G}$ that are quasi-trees is denoted by $\kappa(\mathbb{G})$. The first observation is that every spanning tree of the abstract graph of $\mathbb{G}$ is a quasi-tree. That quasi-trees play the same role for ribbon graphs as trees for abstract graphs is described in [10, 11].

2.2 Partial duality

For this subsection, we follow [9, 10]. Let $\mathbb{G}$ be a ribbon graph with vertices V, edges E, and f boundary components. The *geometric dual* $\mathbb{G}^*$ of $\mathbb{G}$ is constructed from $\Sigma(\mathbb{G})$ by deleting the interior of the disks in V. The new vertices are the f disks used to cap each hole in $\mathbb{G}$. Notice that the edges of $\mathbb{G}^*$ and $\mathbb{G}$ are identical. The only change is which arcs on their boundaries do and do not intersect vertices.

Let $\mathbb{G} = (V, E)$ be a ribbon graph and $A \subseteq E$ a subset of edges. The *partial dual* $\mathbb{G}^A$ of $\mathbb{G}$ is obtained in the following way. Consider the spanning ribbon subgraph $\mathbb{H} = (V, A)$. Now, take $\mathbb{G}$ and glue a disk onto each boundary component of $\mathbb{H}$; these disks are the vertices of $\mathbb{G}^A$. Removing the interior of all old vertices of $\mathbb{G}$ we get $\mathbb{G}^A$. Its edges are the same as in $\mathbb{G}$. Vertices of $\mathbb{G}$ not incident with edges in A together with edges not in A will stay the same; only the intersections of the edges from A to vertices are changed. An example is shown in Figure 1.

Some of the basic properties of partial duality enunciated in [9] are the following:

- $(\mathbb{G}^A)^{A'} = \mathbb{G}^{A \triangle A'}$, where $A \triangle A'$ is the symmetric difference of the sets;
- partial duality preserves the orientability of ribbon graphs;
- partial duality preserves the number of connected components of ribbon graphs.

Observe that the first result in the list implies that partial duality can be computed edge by edge.

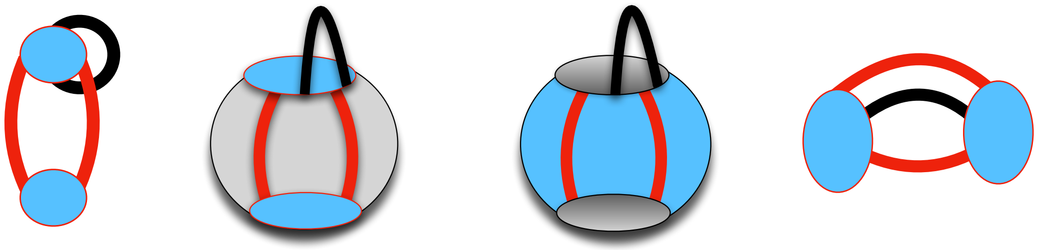


Figure 1: On the left, a ribbon graph $\mathbb{G}$ with a subset of edges A in red; the second from the left is the graph $\mathbb{G}$ with the boundary components of $\mathbb{H} = (V, A)$ capped with a disc; the second from the right is the ribbon graph obtained after deleting the original vertices of $\mathbb{G}$; on the right, the graph $\mathbb{G}^A$.

In graph theory, deletion and contraction of edges are two of the most fundamental operations. In ribbon graph theory, there are three fundamental operations: contraction, deletion and partial duality. Given a ribbon graph $\mathbb{G} = (V, E)$, and $e \in E$, the deletion of e in $\mathbb{G}$ is $\mathbb{G} \setminus e$=$(V, E \setminus e)$. The contraction of e in $\mathbb{G}$ is $\mathbb{G}^{\{e\}} \setminus e$. Table 1 shows the three operation $\mathbb{G} \setminus e$, $\mathbb{G}/e$ and $\mathbb{G}^{\{e\}}$ for a ribbon graph $\mathbb{G}$ and an edge e that can be a loop or a non-loop.

A ribbon graph with exactly one vertex is called a *bouquet* and denoted by $\mathbb{B}$. If $\mathbb{G} = (V, E)$ is a connected ribbon graph, it always contains a spanning quasi-tree $\mathbb{T} = (V, A)$ as a subgraph, for example the ribbon subgraph corresponding to a spanning tree of the abstract graph G. The partial dual $\mathbb{G}^A$ is a bouquet. One of the fundamental relations between $\mathbb{G}$ and $\mathbb{G}^A$ is that both have the same number of quasi-trees, see [10, Theorem 5.1].

A natural operation on bouquets is the *one-point join*. Here the definition is taken from [13]. For a pair of bouquets $\mathbb{B}_1$ and $\mathbb{B}_2$, the one-point join is obtained by identifying an arc on the vertex of $\mathbb{B}_1$ with an arc on the vertex of $\mathbb{B}_2$. The two arcs that are identified should not intersect any edges. Notice that the new ribbon graph is also a bouquet. It is not difficult to show that a quasi-tree in a one-point join of $\mathbb{B}_1$ and $\mathbb{B}_2$ is the union of edges of a quasi-tree of $\mathbb{B}_1$ and a quasi-tree of $\mathbb{B}_2$.

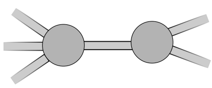
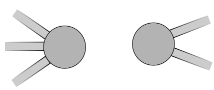
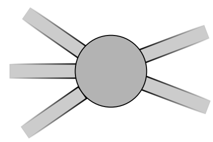
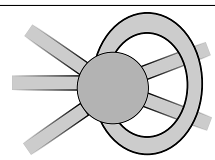
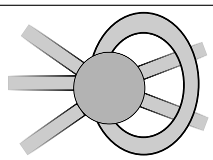
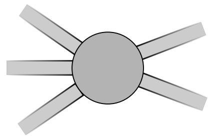
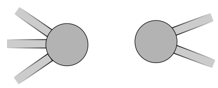
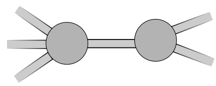
	$\mathbb{G}$	$\mathbb{G} \setminus e$	$\mathbb{G}/e$	$\mathbb{G}^{\{e\}}$
non-loop				
loop				

Table 1: The results of the operations $\mathbb{G} \setminus e$, $\mathbb{G}/e$ and $\mathbb{G}^{\{e\}}$ for a ribbon graph $\mathbb{G}$ and an edge e that can be a loop or a non-loop.

2.3 Chord diagrams and circle graphs

Because a ribbon graph is an abstract two-dimensional surface with boundary, its embedding into 3-space is not relevant for counting spanning quasi-trees. Thus, in the case of a bouquet, the cyclic order of the intersections of the only vertex and the edges determines the number of spanning quasi-trees, as explained below.

Recall that a *chord diagram* D consists of $2n$ cyclically ordered points in a circle together with n straight line segments, called *chords*, that join pairwise disjoint pairs $\{a_i, b_i\}$, $1 \leqslant i \leqslant n$. Two chords intersect if they do so as line segments, equivalently, if the four endpoints of the chord interlace. The intersection graph of the chord diagram is denoted by $G(D)$, and graphs obtained in this way are called *circle graphs*.

Thus, each bouquet $\mathbb{B} = (\{v_0\}, E)$ with n edges has an associated chord diagram $D(\mathbb{B})$ obtained by first labeling the intersections of the edges with v_0 using the labels $1, \ldots, 2n$ in a cyclic order proceeding clockwise around v_0, and then constructing a chord diagram with points $1, \ldots, 2n$ in which a and b are joined by a chord if they are the labels of the two intersections of some edge of $\mathbb{B}$ with v_0.

2.4 Counting quasi-trees in ribbon graphs

The following matrix $A(D)$ of the chord diagram $D = \{\{a_i, b_i\} : 1 \leqslant i \leqslant n\}$, was defined in [5], and it is a signing of the adjacency matrix of $G(D)$. First, choose an arbitrary ordered pair (a, b) or (b, a) for each chord $\{a, b\}$. The entry $A_{i,j}$ is zero if $i = j$ or the chords (a_i, b_i) and (a_j, b_j) do not intersect. For $i < j$, entry $A_{i,j}$ is 1 if the chords intersect and the endpoints in the corresponding ordered pairs are in cyclic order a_i, a_j, b_i, b_j, and -1 if the cyclic order is a_i, b_j, b_i, a_j. For $j < i$, $A_{i,j} = -A_{j,i}$.

Notice that although there is a unique chord diagram D associated with a bouquet $\mathbb{B}$, the matrix $A(D)$ is not uniquely determined by the chord diagram. If (a_i, b_i) is exchanged by (b_i, a_i), the diagram D does not change, but the matrix does.

The matrix $A(D)$ is sometimes called the *intersection matrix*, see [22]. Observe that $A(D)$ and any of its principal submatrices are (0,1,-1) skew-symmetric matrices. It is a classical result from the 1800's that for any skew-symmetric matrix B, $\det(B) = (pf(B))^2$. The invariant $pf(B)$ is called the Pfaffian of B. Recall that a unimodular matrix is a matrix of determinant 1 or -1. A principal unimodular matrix is such that every nonsingular principal submatrix is unimodular. It was proved by Bouchet in [5] that $A(D)$ is *principal unimodular*. Thus, any principal submatrix of $A(D)$ has determinant 0 or 1. More important for us is the following result obtained in many different contexts by different authors [5, 22, 23, 24].

Theorem 2. *Given a bouquet $\mathbb{B}$ with n edges, the number of quasi-trees of $\mathbb{B}$ equals* $\det(I_n + A(D(\mathbb{B})))$.

That $\det(I_n + A(D(\mathbb{B})))$ does not change if a and b are interchanged in the ordered pair (a, b) was implicitly proved in [27], and explicitly in [5]. To reduce notation, we write $A(\mathbb{B})$ for $A(D(\mathbb{B}))$ when the chord diagram of $\mathbb{B}$ is clear from the context.

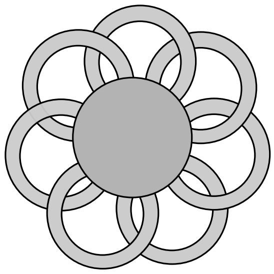


Figure 2: The bouquet $\mathbb{W}_7$.

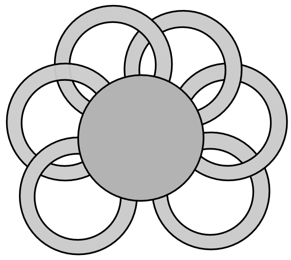


Figure 3: The bouquet $\mathbb{F}_6$.

3 Counting quasi-trees in fans and wheels

The families of ribbon graphs we are interested in are described using bouquets. In turn, each bouquet is described by a chord diagram. However, the reader may find circle graphs a better description.

3.1 Fans

The first family of ribbon graphs are the bouquets $\{\mathbb{F}_n\}$, where the (ordered pairs of the) chord diagram $D(\mathbb{F}_n)$ consists of the pairs $\{(1,3),(2,5),(4,7),\ldots,(2n-4,2n-1),(2n-2,2n)\}$. Figure 3 shows the bouquet $\mathbb{F}_6$. The corresponding circle graphs are the n-paths. The matrix $A(\mathbb{F}_n)$ is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}. \tag{1}$$

The number of quasi-trees is given by the value of the determinant of the tridiagonal matrix $A(\mathbb{F}_n)+I_n$. Starting with $\mathbb{F}_2$, the first values are $2,3,5,8,13,21,34,55,89,144,\ldots$. The following is a well-known result, see [35].

Theorem 3 (Strang)**.** *The determinant of the tridiagonal matrix $A(\mathbb{F}_n)+I_n$ equals the (n+1)-th Fibonacci number f_{n+1}.*

Corollary 4. *The number of quasi-trees of $\mathbb{F}_n$ equals the (n+1)-th Fibonacci number f_{n+1}.*

3.1.1 Combinatorial interpretations

Given an $n \times n$ matrix A, let us define $E_k(A)$ as the sum of its principal minors of size $k > 0$, and $E_0(A) = 1$. The following formula for the *characteristic polynomial* of A is well-known, see [18].

$$\det(tI_n - A) = E_0(A)t^n - E_1(A)t^{n-1} + \cdots + (-1)^n E_n(A). \tag{2}$$

Also, elementary linear algebra shows that a skew-symmetric matrix A of odd order is singular. It follows from subsection 2.4 that, $\det(I_n + A(\mathbb{B}))$ equals the number of principal submatrices of even size of $A(\mathbb{B})$ that are non-singular. Notice that the empty matrix corresponds to a principal submatrix of size 0 and represents the intersection matrix of the bouquet with no edges.

A proof using ribbon graphs of a well-known combinatorial interpretation of the Fibonacci numbers is given.

Theorem 5. *The (n+1)-th Fibonacci number equals the number of perfect matchings in* $P_2 \times P_n$.

Proof. A perfect matching M in $P_2 \times P_n$ is uniquely defined by the *vertically matched* vertices, that is the (possibly empty) subset of vertices $\{(1, i_j) \,|\, 1 \leqslant i_j \leqslant n\}$ that are matched with the corresponding neighboring vertices $\{(2, i_j) \,|\, 1 \leqslant i_j \leqslant n\}$. Observe that the number of interior vertices in the unique alternating path with respect to M between the vertically matched vertices $(1, i_j)$ and $(1, i_{j+1})$ is even.

Recall that the matrix $A(\mathbb{F}_k)$ is non-singular if and only if k is even. Then, a principal submatrix of $A(\mathbb{F}_n)$ is non-singular if and only if it is the intersection matrix of a ribbon subgraph $\mathbb{H} = (\{v_0\}, E)$ that is a one-point join of ribbon subgraphs of the form $\mathbb{F}_{2k}$ with $k \geqslant 0$.

Given a matching M in $P_2 \times P_n$, we delete the i_j chord of $D(\mathbb{F}_n)$ for each vertically matched vertex $(1, i_j)$ to obtain a ribbon subgraph $\mathbb{H}(M)$ of $\mathbb{F}_n$. From the discussion above, this construction defines a bijection between matchings of $P_2 \times P_n$ and non-singular principal submatrix of $A(\mathbb{F}_n)$. Now, the result follows from Theorem 3. □

The Fibonacci polynomial is defined as $f_1(x) = 1$, $f_2(x) = x$ and $f_{n+1}(x) = x f_n(x) + f_{n-1}(x)$. The first few polynomials are: $f_3(x) = x^2+1$, $f_4(x) = x^3+2x$, $f_5(x) = x^4+3x^2+1$ and $f_6(x) = x^5 + 4x^3 + 3x$. The following results appears in [19].

Theorem 6. *The characteristic polynomial of the matrix* $A(\mathbb{F}_n)$ *equals (n+1)-th Fibonacci polynomial.*

Proof. The result follows by expanding $\det(tI_n - A)$ along the first column to get the same recurrence relation as the Fibonacci polynomials. □

3.1.2 Ribbon graph theory proof

Alternative proof of Corollary 4. For $\mathbb{F}_n$, let B_o be the set of chords with an odd label. Then, it is easy to see that if $n = 2k+1$, $k \geqslant 0$, the partial dual $\mathbb{F}^{B_\mathrm{o}}$ is just the (embedding in the sphere of the) fan graph F_{k+1}. An example is shown in Figure 4. If $n = 2k + 2$, $k \geqslant 0$, the partial dual $\mathbb{F}^{B_\mathrm{o}}$ is the (embedding in the sphere of the) graph F^+_{k+1}. Now, the result follows from the classical result in [16]. □

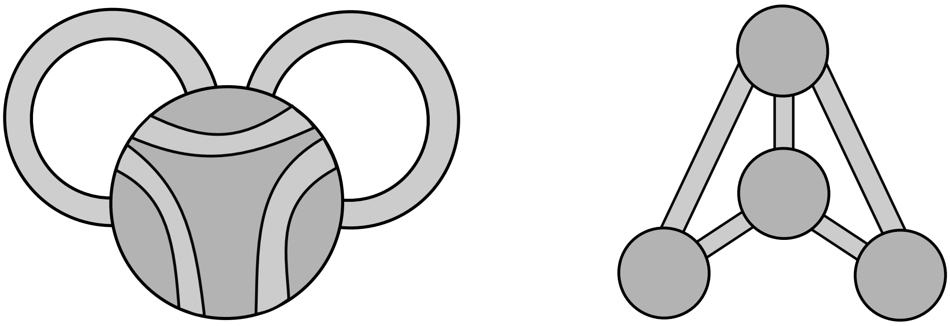


Figure 4: On the left of the figure, the bouquet $\mathbb{F}_5$ with $B_\circ$ the three edges crossing the vertex v_0; and on the right, the partial dual $\mathbb{F}_5^{B_\circ}$.

3.2 Wheels

The second family of ribbon graphs is the set $\{\mathbb{W}_n\}$ of bouquets. The (ordered pairs of the) chord diagram $D(\mathbb{W}_n)$ consists of the pairs $\{(1,4),(3,6),(5,8),\ldots,(2n-3,2n),(2n-1,2)\}$. Figure 2 shows the bouquet $\mathbb{W}_7$. The corresponding circle graphs are the n-cycles. The matrix $A(\mathbb{W}_n)$ is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

The number of quasi-trees is given by the determinant of the matrix $A(\mathbb{W}) + I_n$, that is the circulant matrix of the vector $(1,1,0,\ldots,0,-1)$. Starting with $\mathbb{W}_3$, the first values are $4,5,11,16,29,45,76,121,199,320,\ldots$. These are the first terms in the sequence of associated Mersenne numbers $\{a_n\}$.

Theorem 7. *The number of quasi-trees of $\mathbb{W}_n$ equals the n-th associated Mersenne number a_n.*

Proof. Let $n \geqslant 3$ be an integer. We will prove the case for $n = 2k+1$. Let $A(\mathbb{W}_n) + I_n$ be the matrix $B = (b_{i,j})$. Let us denote by $B[i,j]$ the submatrix of B obtained by deleting column i and row j. First, we expand the determinant of B along the first column. The submatrix $B[1,1]$ is the matrix $A(\mathbb{F}_{n-1}) + I_{n-1}$ whose determinant equals f_{2k+1} by the previous subsection. Now, we expand the determinant of the submatrix $B[1,2] = B'$ along the first row, which has a 1 at the first entry and a -1 at the last entry. The submatrix $B'[1,1]$ is the matrix $A(\mathbb{F}_{n-2}) + I_{n-2}$ and the submatrix $B'[n-1,1]$ is an upper triangular matrix with all the entries at the diagonal equal to -1. We also expand the submatrix $B[1,n] = B''$ along the first row, which has a 1 at the first entry and a -1 at the last entry. The submatrix $B''[1,1]$ is a lower triangular matrix with all entries at the diagonal equal to 1. The submatrix $B''[n-1,1]$ is the matrix $A(\mathbb{F}_{n-2}) + I_{n-2}$. Thus, the determinant of B is the sum of the determinant of these matrices.

$$\begin{aligned}\det B &= f_{2k+1} + (f_{2k} + (-1)^{n-2}) + (1 + f_{2k}) \\ &= f_{2k+2} + f_{2k} + 1 + (-1)^n \\ &= l_{2k+1} \\ &= a_n\end{aligned}$$

The case for n even is similar. □

The initial values of the sequence $\{a_n\}$ suggest that a_{4k+2} is a square. This is true and it follows from the known relation among Lucas numbers that $l_{2k} = l_k^2 + (-1)^{k-1}2$.

3.2.1 Combinatorial interpretations

A combinatorial interpretation of the Mersenne numbers is given in [8]. Let N_n be the number of arrangements of black and white beads on a necklace with a total of n beads satisfying the following: there is at least one black bead; between any two black beads the number of white beads is even; rotations and flipping of the necklace are considered distinct.

Theorem 8 (Butler)**.** *The value a_n equals N_n.*

Alternative proof of Theorem 8. Notice that $\det(A(\mathbb{W}_n)) = 0$ for $n \geqslant 1$. Also, any proper principal submatrix corresponds to the intersection matrix of a ribbon subgraph $\mathbb{H} = (\{v_0\}, E)$ that is a one-point join of ribbon subgraphs of the form $\mathbb{F}_k$. Thus, the determinant of $A(\mathbb{W}_n) + I_n$ equals the number of ribbon subgraphs of $\mathbb{W}_n$ that are the one-point join of ribbon graphs of the form $\mathbb{F}_{2k}$.

The bijection between the necklaces of the statement with our quasi-trees is now obvious. The chords not present in the quasi-tree are black beads and the chords present are white beads. □

From the previous theorem it is easy to get the following combinatorial interpretation for the associated Mersenne numbers.

Theorem 9. *The n-th associated Mersenne number a_n and the number of perfect matchings in $P_2 \times C_n$, m_n are related by the equation*

$$m_n = a_n + 2 + (-1)^n 2.$$

Proof. From the first proof of Theorem 5, it follows that almost any perfect matching M in $P_2 \times C_n$ is uniquely defined by the vertically matched vertices, that is, the subset of vertices $\{(1, i_j) \,|\, 1 \leqslant i_j \leqslant n\}$ that are matched with the corresponding neighboring vertices $\{(2, i_j) \,|\, 1 \leqslant i_j \leqslant n\}$. The only perfect matchings that are missing are the perfect matchings of $P_2 \times C_n$ when n is even with no vertically matched vertices. There are 4 of these. The proof continues using the alternative proof of Theorem 8. □

The Lucas polynomials are defined by the recurrence relation $l_0(x) = 2$, $l_1(x) = x$ and $l_{n+1}(x) = xl_n(x) + l_{n-1}(x)$. The first few polynomials are: $l_2(x) = x^2 + 2$, $l_3(x) = x^3 + 3x$, $l_4(x) = x^4 + 4x^2 + 2$, $l_5(x) = x^5 + 5x^3 + 5x$ and $l_6(x) = x^6 + 6x^4 + 9x^2 + 2$.

Theorem 10. *The characteristic polynomial of the matrix $A(\mathbb{W}_n)$ equals the polynomial $l_n(x) - 1 - (-1)^n$.*

Proof. The independence polynomial of a graph G is the generating function of the numbers of independent sets of G by size, while the matching polynomial of G is the generating function of the numbers of matchings of G by size. In [7], using the result in [3], it is proved that the independence polynomial of C_n equals $l(x)$. Clearly, the independence polynomial of C_n equals the matching polynomial of C_n.

Almost any perfect matching M in $P_2 \times C_n$ corresponds to a unique k-matching of C_n. The only perfect matchings that are missing are the perfect matchings of $P_2 \times C_n$ when $n = 2p$ and have no vertically matched vertices. There are exactly 4 such matchings, corresponding to the 2 perfect matchings of C_n.

By the alternative proof of Theorem 8, each k-matching of C_n corresponds to a quasi-tree of $\mathbb{W}_n$ with k edges, except for the 2 perfect matchings of C_n when n is even. The coefficient E_k in Equation 2 equals the number of quasi-trees of $\mathbb{W}_n$ with k edges. Thus, except for the constant term, the matching polynomial of C_n and the characteristic polynomial of $A(\mathbb{W}_n)$ have the same coefficients. □

3.2.2 Ribbon graph theory proof

Alternative proof of Theorem 7. For $\mathbb{W}_n$, let B_o be the set of chords with an odd label. Then, if $n = 2k$, $k \geqslant 2$, the partial dual $\mathbb{W}^{B_o}$ is just the (embedding in the sphere of the) wheel graph W_k. The reader may find the construction easy to understand by looking at Figure 4 as $\mathbb{W}_6$ and $\mathbb{F}_5$ differ by only one edge. If $n = 2k + 1$, $k \geqslant 1$, let B'_o be the set of chords with an even label. The partial dual $\mathbb{W}^{B'_o}$ does not correspond to a graph embedded in the sphere, but rather to a graph embedded in the torus. The embedded graph is obtained from embedding F_k in a disk in the torus and then adding an edge between the hub vertex and each of the minimum degree vertices in the rim (thus creating two parallel edges) in such a way that we obtain a 2-cellular embedding of the graph in the torus. Notice that if $k = 1$, we add two edges between the hub and the vertex in the rim. We will denote this ribbon graph as $\mathbb{W}_n^T$. An example of the embedding is given in Figure 5.

The proof uses contraction and deletion in ribbon graphs. For n even, the graph is W_n embedded in the sphere. As deletion and contraction preserve the embedding, the proof continues in the same manner as the proof by deletion and contraction in [16].

For n odd, let us choose an edge in $\mathbb{W}_n$ and label it e_n. The corresponding chord intersects two other chords. Let us label the associated edges e_{n-1} and e_1. Counting the quasi-trees of $\mathbb{W}_n$ requires counting those that do not contain e_n and those that do. The first set of quasi-trees corresponds to the set of quasi-trees of $\mathbb{W}_n \setminus e_n$, which are exactly those of $\mathbb{F}_{n-1}$. The second set of quasi-trees corresponds to the set of quasi-trees of $\mathbb{W}_n/e_n$. This ribbon graph is shown in the center of Figure 6. The argument continues

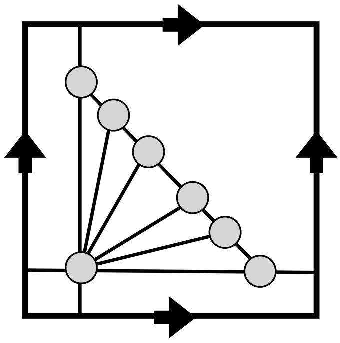


Figure 5: The ribbon graph $\mathbb{W}_{13}^T$ embedded in the torus.

by using deletion and contraction in $\mathbb{W}_n/e_n$. The number of quasi-trees of $\mathbb{W}_n/e_n$ that do not contain the edge e_{n-1} equals the number of quasi-trees of $\mathbb{F}_{n-3}$, as the edge e_1 is in every quasi-tree. The number of quasi-trees that do contain the edge e_{n-1} is the same as the number of quasi-trees of the ribbon graph $\mathbb{W}_n/e_n/e_{n-1}$ that is $\mathbb{W}_{n-2}$. Thus,

$$\begin{aligned}\kappa(\mathbb{W}_n) &= \kappa(\mathbb{F}_{n-1}) + \kappa(\mathbb{F}_{n-3}) + \kappa(\mathbb{W}_{n-2}) \\ &= f_{n-1} + f_{n-3} + a_{n-2} \\ &= l_{n-1} + a_{n-2} \\ &= a_n\end{aligned}$$

□

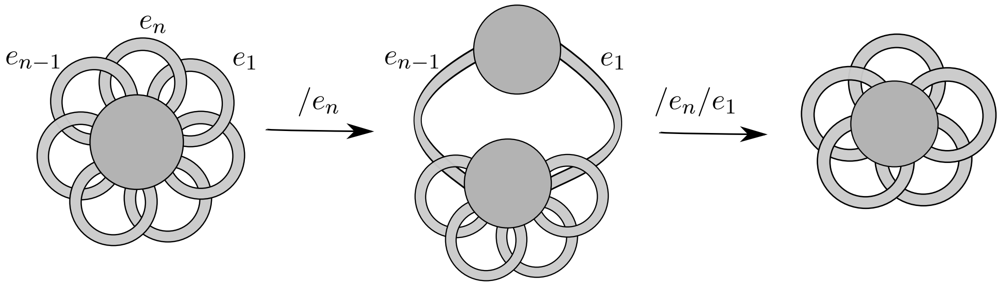


Figure 6: The graph on the left is $\mathbb{W}_7$, the graph on the center is $\mathbb{W}_7/e_n$ and the graph on the right is $\mathbb{W}_7/e_n/e_{n-1}$.

4 The critical group for fans and wheels

For an abstract graph G on $n+1$ vertices and a special vertex q, the classical critical group is the abelian group $K(G) \cong \mathbb{Z}^n/\mathbb{Z}^n L^q(G)$, where $\mathbb{Z}^n L^q(G)$ is the integer row-span of the reduced Laplacian of G. The group does not depend on the choice of q. For more on the classical critical group of a graph see [20].

Let $\mathbb{G} = (V, E)$ be a ribbon graph with n edges, $\mathbb{T} = (V, T)$ a quasi-tree of $\mathbb{G}$, and $\mathbb{B}$ the bouquet $\mathbb{G}^T$. Then, in [24], it is proved that the group $\mathbb{Z}^n/\langle I_n + A(\mathbb{B})\rangle$ does not depend on the choice of $\mathbb{T}$ or the selection of the ordered pairs for each chord in $D(\mathbb{B})$. The group obtained in this manner is called the *critical group* of the ribbon graph $\mathbb{G}$ and is denoted by $K(\mathbb{G})$. Also in [24], it is proven that any partial dual $\mathbb{G}^T$ of $\mathbb{G}$ has a critical group isomorphic to the critical group of $\mathbb{G}$.

Let $\mathbb{B}$ be a bouquet with n edges. If we regard the rows of $Y = I_n + A(\mathbb{B})$, as elements of the free $\mathbb{Z}$-module $\mathbb{Z}^n$, the quotient module $\mathbb{Z}^n/\langle Y\rangle$, called the *cokernel* of Y, satisfies

$$\mathbb{Z}^n/\langle Y\rangle \cong \mathbb{Z}^{n-r} \oplus \mathbb{Z}/b_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_r\mathbb{Z}.$$

The integers b_i are called the *invariant factors* and satisfy that b_i divides b_{i+1} for $1 \leqslant i \leqslant r-1$. The invariant factors can be computed by the formula $b_i = d_i/d_{i-1}$, where $d_i = \gcd\{\det(B) : B \text{ is a } i \times i \text{ minor of } Y\}$ and $d_0 = 1$. The subgroup $K = \mathbb{Z}/b_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_r\mathbb{Z}$ is called the *torsion* subgroup. Notice that when Y is nonsingular, $n - r = 0$ and $\mathbb{Z}^n/\langle Y\rangle$ is the finite abelian group K. Also, if K is not trivial and the first t invariant factors are equal to 1, then $K \cong \mathbb{Z}/b_{t+1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_r\mathbb{Z}$. All these results can be consulted in the beautiful exposition rendered in [34].

Computing the critical groups of fans is easy. We have the following:

Theorem 11. *The critical group of the fan ribbon graph $\mathbb{F}_n$ is cyclic and $K(\mathbb{F}_n) \cong \mathbb{Z}/f_{n+1}\mathbb{Z}$.*

Proof. The $(n-1)\times(n-1)$ submatrix obtained by deleting the first column and last row of $A(\mathbb{F}_n) + I_n$ has determinant 1. Then, $d_{n-1} = 1$ and $b_{n-1} = 1$. Thus, as $b_1|b_2|\cdots|b_{n-1}$, the group is cyclic and the result follows directly from Theorem 3. □

This theorem is not surprising. As mentioned above, $\mathbb{F}_{2k+1}$ has a partial dual $\mathbb{F}^{B_o}$ which is the (embedding in the sphere of the) fan graph F_{k+1}. It was proved in [24] that the critical group of a plane ribbon graph is isomorphic to the classical critical group of the corresponding abstract graph. The critical group of the fan graph F_n is known to be $\mathbb{Z}/f_{2n}\mathbb{Z}$, see [21].

Computing the critical group of $\mathbb{W}_n$ is not so straightforward. We can proceed as above and notice that the $(n-2)\times(n-2)$ submatrix obtained by deleting the first and last columns and the last two rows of $I_n + A(\mathbb{W}_n)$ has determinant 1. Thus, the critical group of $\mathbb{W}_n$ has at most two generators. However, the explicit structure of $K(\mathbb{W}_n)$ is not so easy. The first groups are $K(\mathbb{W}_3) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $K(\mathbb{W}_4) = \mathbb{Z}/5\mathbb{Z}$, $K(\mathbb{W}_5) = \mathbb{Z}/11\mathbb{Z}$, $K(\mathbb{W}_6) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $K(\mathbb{W}_7) = \mathbb{Z}/29\mathbb{Z}$, $K(\mathbb{W}_8) = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$, $K(\mathbb{W}_9) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/38\mathbb{Z}$, $K(\mathbb{W}_{10}) = \mathbb{Z}/11\mathbb{Z} \oplus \mathbb{Z}/11\mathbb{Z}$. The heavy lifting was already made in [2, Section 4] for, what turns out to be, an isomorphic group studied in a different setting.

Theorem 12. *The critical group of the wheel ribbon graph $\mathbb{W}_n$ is given by the following expression.*

$$K(\mathbb{W}_n) \cong \mathbb{Z}/\alpha_n\mathbb{Z} \oplus \mathbb{Z}/(a_n/\alpha_n)\mathbb{Z},$$

where $\alpha_n = \gcd\{f_n, f_{n-1} - 1\}$.

Proof. By moving the last row to the first row, we obtain the circulant matrix of the vector $(-1,1,1,0,\dots,0)$. Therefore, the critical group of $\mathbb{W}_n$ has presentation $\langle x_1,\dots,x_n \,|\, x_i = x_{i+1}+x_{i+2}\rangle$, where the subscripts are interpreted modulo n. The result now follows from the work in [2]. $\square$

4.1 The Eulerian digraph group of fans and wheels

Given a chord diagram $D(\mathbb{B})$ with n chords, construct the following 2-in, 2-out Eulerian digraph $\vec{G}(\mathbb{B})$. First, consider the oriented cycle with vertices $\{1,2,\dots,2n\}$ and arcs $\{(i,i+1) : 1 \leqslant i \leqslant 2n-1\} \cup \{(2n,1)\}$. Now, take the Eulerian digraph obtained by identifying the vertices a_i and b_i, $1 \leqslant i \leqslant n$, corresponding to the chords in $D(\mathbb{B})$. It is implicitly proven in [6] that the number of Eulerian circuits in $\vec{G}(\mathbb{B})$ is equal to the number of quasi-trees of $\mathbb{B}$. Thus, we have the following theorem, also mention in [22, 23].

Theorem 13. *The number of Eulerian circuits in $\vec{G}(\mathbb{B})$ is equal to* $\det(A(\mathbb{B})+I_n)$.

The BEST theorem [1] implies that in a 2-regular connected digraph the number of Eulerian circuits equals the number of arborescences, that is the number of rooted trees at a fixed vertex q. This number is independent of the vertex q. The matrix-tree theorem for digraphs and the previous comment imply that the number of arborescences equals the common value of all the cofactors of the Laplacian matrix $L(\mathbb{G}) = 2I_n - A(\vec{G}(\mathbb{B}))$, where $A(\vec{G}(\mathbb{B}))$ is the adjacency matrix of the digraph $\vec{G}(\mathbb{B})$, see [33]

For the ribbon graph $\mathbb{F}_n$, the matrix $L(\mathbb{F}_n)$ is

$$\begin{pmatrix} 2 & -1 & -1 & \cdots & 0 & 0 \\ -1 & 2 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -1 & 2 & 0 & -1 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{3}$$

For the ribbon graph $\mathbb{W}_n$, the matrix $L(\mathbb{W}_n) = 2I_n - A(\vec{G}(\mathbb{W}_n))$ is the circulant matrix of the vector $(2,0,-1,0,\dots,0,-1)$.

Proposition 14. *The value of any cofactor of the matrix $L(\mathbb{F}_n)$ equals the (n+1)-th Fibonacci number. The value of any cofactor of $L(\mathbb{W}_n)$ equals the n-th associated Mersenne number.*

Let $\mathbb{G} = (V,E)$ be a ribbon graph with n edges, $\mathbb{T} = (V,T)$ a quasi-tree of $\mathbb{G}$, and $\mathbb{B}$ the bouquet $\mathbb{G}^T$. The digraph $\vec{G}(\mathbb{B})$, and $L(\mathbb{B})$, does not depend on $\mathbb{T}$ as it is shown in a more general context in [13]. In [24], the torsion subgroup of the group $\bar{K}(\mathbb{G}) = \mathbb{Z}^n/\langle L(\mathbb{B})\rangle$ is proven to be isomorphic to $K(\mathbb{G})$. We will call the group $\bar{K}(\mathbb{G})$ the Eulerian digraph group of $\mathbb{G}$ and prove directly the corresponding results for fans and wheels.

Theorem 15. *The Eulerian digraph group of the fan ribbon graph $\mathbb{F}_n$ is $\bar{K}(\mathbb{F}_n) \cong \mathbb{Z} \oplus K(\mathbb{F}_n)$.*

Proof. We know that the group $K(\mathbb{F}_n)$ is the cyclic group $\mathbb{Z}/f_{n+1}\mathbb{Z}$. The determinant of the matrix $L(\mathbb{F}_n)$ is 0. The determinant of any $(n-1)\times(n-1)$ submatrix of $L(\mathbb{F}_n)$ equals f_{n+1} by Theorem 3 and Theorem 13. The determinant of the $(n-2)\times(n-2)$ submatrix obtained by deleting the first two columns and the last two rows of $L(\mathbb{F}_n)$ has determinant 1. Thus, $d_n = 0$, $d_{n-1} = f_{n+1}$ and $d_{n-2} = 1$. We conclude that the (torsion-free) rank is 1 and the torsion subgroup is cyclic of order f_{n+1}. □

Theorem 16. *The Eulerian digraph group of the fan ribbon graph* $\mathbb{W}_n$ *is* $\bar{K}(\mathbb{W}_n) \cong \mathbb{Z} \oplus K(\mathbb{W}_n)$.

Proof. The determinant of the matrix $L(\mathbb{W}_n)$ is 0. The determinant of any $(n-1)\times(n-1)$ submatrix of $L(\mathbb{W}_n)$ equals a_n by Theorem 7 and Theorem 13. The determinant of the $(n-3)\times(n-3)$ submatrix obtained by deleting the first three columns and first row and last two rows of $L(\mathbb{W}_n)$ is $(-1)^{n-3}$. Thus, $d_n = 0$, $d_{n-1} = a_n$ and $d_{n-3} = 1$. We conclude that the (torsion-free) rank is 1 and the torsion subgroup has order a_n and has at most two generators.

Now, we show that the group $K(\mathbb{W}_n)$ is a subgroup of $\bar{K}(\mathbb{W}_n)$, thus, proving the assertion of the theorem as the order of $K(\mathbb{W}_n)$ is a_n. Recall that the matrix $A(\mathbb{W}_n) + I_n$ gives us a presentation $\langle x_1, \ldots, x_n \mid x_i = x_{i+1} + x_{i+2}\rangle$, where the subscripts are interpreted modulo n. From this set of relations we get a new set of relations: $x_i = x_{i+1} + x_{i+2} = x_{i+2} + x_{i+3} + x_{i+2} = 2x_{i+2} + x_{i+3}$. Thus, we get $2x_i = x_{i-2} - x_{i+1}$. We get the same set of relations from the matrix $L(\mathbb{W}_n)$ by first multiplying each odd row and odd column by -1. The effect of these is to change the sign of the entries in the off-diagonal below the main diagonal of the matrix $L(\mathbb{W}_n)$. Second, make the change of variables $y_i = x_{n+1-i}$. Thus, the image of $A(\mathbb{W}_n) + I_n$ has a subspace isomorphic to the image of $L(\mathbb{W}_n)$. We conclude that the cokernel $\mathbb{Z}^n/\langle L(\mathbb{W}_n)\rangle$ contains as a subgroup the cokernel $\mathbb{Z}^n/\langle A(\mathbb{W}_n) + I_n\rangle = K(\mathbb{W}_n)$. □

5 Conclusion

The paper relates the Fibonacci and associated Mersenne numbers with the number of quasi-trees of two families of ribbon graphs. This relation gives combinatorial interpretations for these well-known sequences of numbers. Two families of abelian groups whose order are the Fibonacci and associated Mersenne numbers are also obtained.

Some of these results are well-known, but our approach is novel in using ribbon graphs. Also, this approach provides a neat interpretation of Fibonacci and associated Mersenne numbers as enumerations of substructures in symmetric structures.

Finding the Lucas numbers as the number of quasi-trees of a family of ribbon graphs is a question that was not pursued in this paper. However, it seems like a natural question. It is worth noticing that ribbon graphs give rise to a class of delta-matroids. Matroids are a particular type of delta matroid. Thus, an even more general answer for all three sequences might lie in the world of delta-matroids.

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