A Homomorphic Polynomial for Oriented Graphs

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Abstract

In this article, we define a function that counts the number of (onto) homomorphisms of an oriented graph. We show that this function is always a polynomial and establish it as an extension of the notion of chromatic polynomials. We study algebraic properties of this function. In particular we show that the coefficients of these polynomials have the alternating sign property and that the polynomials associated to the independent sets have relations with the Stirling numbers of the second kind.

Mathematics Subject Classifications: 05C15

1 Introduction

In 1994, Courcelle [4] extended the notion of vertex coloring for oriented graphs which inspired a number of works in this domain (see [20] for details). The way to view Courcelle's definition as a natural extension of vertex coloring for simple graphs is through homomorphisms.

A *homomorphism* f of a simple (oriented) graph G to a simple (oriented) *target* graph H is a function $f : V(G) \to V(H)$ such that $f(u)f(v)$ is an edge (arc) if uv is an edge (arc). A homomorphism f with an image of cardinality k is a *k-coloring (oriented k-coloring)* of G and the image of a vertex is its color[1]. The minimum k such that G admits a k-coloring

[1] This is a slightly modified, but essentially equivalent, version of the standard convention. In the standard convention, we have $k = |V(H)|$ while here we have $k = |f(V(G))|$. Importantly, this modification does not inflict any change in the standard definition of (oriented) chromatic number.

(oriented k-coloring) is the *chromatic number* $\chi(G)$ (*oriented chromatic number* $\chi_o(G)$) of G.

In 1994, Sopena [18] had defined a function, which turned out to be a polynomial, that counts the number of oriented colorings of a graph. He also noticed that this function lacks two very important properties, namely, the alternating signs of the coefficients property (present in chromatic polynomials) and the unimodal behavior of the absolute values of the coefficients property (present in chromatic polynomial, proved recently by Huh and Katz [9] through positively settling a long standing conjecture). Recently, Cox and Duffy [5] have made considerable progress on this topic by establishing several interesting properties of the chromatic polynomial. Looking back at the two decades of research on oriented coloring, it can be noted that, homomorphism, alongside coloring, is also an important aspect of oriented coloring.

Keeping that in mind, following the footsteps of Birkhoff [3] and Sopena [18], we define a function, which also turns out to be a polynomial, for counting the number of (onto) homomorphisms of an oriented graph to tournaments. This polynomial possesses the alternating signs of the coefficients property and, upon checking a number of examples, we conjecture it to have the unimodal behavior of the absolute values of the coefficients property as well.

We hope that these polynomials can become valuable tools for studying oriented chromatic number in the near future. We initiate the study by devising tools for studying these polynomials and by proving some of their interesting properties.

In this article, we motivate and introduce the notion of oriented homomorphic polynomials in Section 2 and justify its name in Section 3. In Section 4 we develop some useful tools to study the problem. In Section 5 we present interesting properties of the polynomials and prove most of them, while some of the proofs are moved to Section 6 to maintain the flow of reading. Finally in Section 7 we conclude the article.

2 Counting homomorphisms

Two functions f_1 and f_2, having the same domain D, but possibly different co-domains, are *equivalent* if $f_1(v) = f_2(v)$ for all $v \in D$, otherwise they are *distinct*.

The *chromatic polynomial* $\chi(G, x)$ of a simple graph G denotes the number of ways to obtain a coloring of G using (any subset of) the colors $\{0, 1, \ldots, x-1\}$. The function $\chi(G, x)$ is known to be a polynomial in x, hence the name.

Observe that, to distinguish between two colorings we are only interested in the images of the vertices of G, not the structure of the target graph. That is not a concern though, as any y-coloring of G, where $y \leqslant x$, can be viewed as a homomorphism of G to the complete graph K_x.

To be precise, let $\mathcal{C}_x(G)$ be the set of distinct colorings of G using any subset of the colors $\{0, 1, \ldots, x-1\}$, and $\mathcal{H}_x(G)$ be the set of distinct homomorphisms (not necessarily onto) of G to K_x.

Observation 1. *Given a simple graph G, we have $\chi(G, x) = |\mathcal{C}_x(G)| = |\mathcal{H}_x(G)|$.*

However, the scenario is completely different for oriented graphs. The major reason is that, unlike in the case of simple graphs, a maximal (with respect to number of edges) graph on x vertices is not unique for oriented graphs. To elaborate, while there is only one maximal graph K_x (the complete graph) with set of vertices $\{0, 1, \ldots, x-1\}$ in the undirected case, there are $2^{\binom{x}{2}}$ different maximal oriented graphs (the tournaments) with set of vertices $\{0, 1, \ldots, x-1\}$.

Therefore for an oriented graph G, the set $\mathcal{OC}_x(G)$ of distinct oriented colorings of G using any subset of the colors $\{0, 1, \ldots, x-1\}$ is not in a bijection with the set $\mathcal{OH}_x(G)$ of all homomorphisms of G to tournaments having x vertices $\{0, 1, \ldots, x-1\}$.

Sopena [18] defined the *oriented chromatic polynomial* of an oriented graph G as

$$P(G, x) = |\mathcal{OC}_x(G)|$$

where $\mathcal{OC}_x(G)$ denotes the set of distinct oriented colorings of G using any subset of the colors $\{0, 1, \ldots, x-1\}$.

In the following we look at an example to better understand this function and some of its limitations for motivating our "oriented version" of the chromatic polynomial for oriented graphs.

An *oclique* [10] O is an oriented graph with $\chi_o(O) = |V(O)|$. It is known that an oriented graph O is an oclique if and only if each pair of its non-adjacent vertices are connected by a directed 2-path [10]. Also, it is known [6] that the number of arcs of an oclique on n vertices may vary between $(n \log n - \frac{3n}{2})$ and $\binom{n}{2}$. However, the oriented chromatic polynomial for any oclique O on n vertices is the same [18]:

$$P(O, x) = x(x-1)\cdots(x-n+1).$$

Let P_2 and C_3 denote the directed 2-path and the directed cycle of order three having set of vertices $V(P_2) = \{a, b, c\}$ and $V(C_3) = \{x, y, z\}$, respectively (as shown in Fig. 1). There are two non-isomorphic tournaments on 3 vertices, namely, the directed tournament DT_3 and the transitive tournament TT_3.

Note that both P_2 and C_3 are ocliques on 3 vertices. Thus they both have the same oriented chromatic polynomial:

$$P(P_2, x) = P(C_3, x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x.$$

In particular, from this example we can conclude that the oriented chromatic polynomial does not determine the number of arcs of the corresponding oriented graph. Whereas, if we look at their set of homomorphisms to tournaments on 3 vertices, we observe some differences.

There are exactly three distinct homomorphisms of P_2 to DT_3, namely, f_0, f_1 and f_2 given by the following:

$$f_i : a \longmapsto i, b \longmapsto i+1, c \longmapsto i+2$$

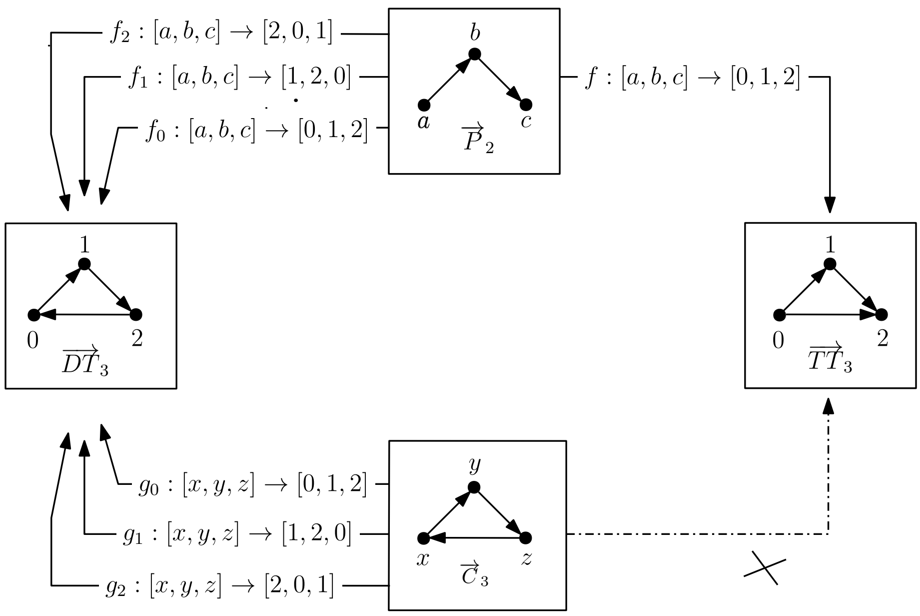

Figure 1: Counting homomorphisms of a directed 2-path and a directed 3-cycle to tournaments of order 3.

for all $i \in \{0,1,2\}$, where the + operations are taken modulo 3. Similarly, there are exactly three distinct homomorphisms of C_3 to DT_3, namely, g_0, g_1 and g_2 given by the following:

$$g_i : x \longmapsto i, y \longmapsto i+1, z \longmapsto i+2$$

for all $i \in \{0,1,2\}$, where the + operations are taken modulo 3.

On the other hand, there is exactly one homomorphism of P_2 to TT_3:

$$f : a \longmapsto 0, b \longmapsto 1, c \longmapsto 2.$$

In contrast, C_3 does not admit any homomorphism to TT_3.

Remark 2. Notice that, while counting the oriented chromatic polynomial, the oriented colorings corresponding to the homomorphisms f_0 and f are counted as the same coloring.

Note that there are two distinct directed (labeled) tournaments DT_3 and six distinct transitive (labeled) tournaments TT_3 on set of vertices $\{0,1,2\}$. Thus $|\mathcal{OH}_3(P_2)| = 12$ and $|\mathcal{OH}_3(C_3)| = 6$. Hence note that, the function defined by $|\mathcal{OH}_x(G)|$ is not the same for P_2 and C_3. However, while counting[2]

$$|\mathcal{OH}_x(P_2)| = 2^{3(x-3)+\binom{x-3}{2}} \cdot 2x(x-1)(x-2)$$

[2]For $G = P_2$ or C_3, we can have $x(x-1)(x-2)$ choices for the images of the vertices of G. Moreover, the directions of 2 or 3 arcs joining the vertices among the images of $f(V(G))$, respectively, will have fixed directions to satisfy the conditions of a homomorphism. That means, each of the remaining arcs will have two choices for direction. Notice that there are, respectively, 1 or 0 such arcs joining the vertices among the images of $f(V(G))$, $3(x-3)$ arcs with exactly one endpoint in $f(V(G))$, and $\binom{x-3}{2}$ arcs with both endpoints outside $f(V(G))$.

and

$$|\mathcal{OH}_x(C_3)| = 2^{3(x-3)+\binom{x-3}{2}} \cdot x(x-1)(x-2)$$

we are having to make a lot of redundant counts due to the vertices which are not part of the image. Furthermore, due to the redundant counts, the function is not polynomial as per our example. To avoid these redundant counts and to, hopefully, obtain a better behaved function we define a simplified version of $|\mathcal{OH}_x(G)|$ in the following.

Let $\mathcal{T}_x$ be the set of all tournaments whose set of vertices are subsets of $\{0, 1, \ldots, x-1\}$. Given an oriented graph G and a tournament T, let $\mathcal{OH}_T(G)$ be the set of all distinct onto homomorphisms of G to T. Now define

$$\mathcal{OH}^*_x(G) = \bigcup_{T \in \mathcal{T}_x} \mathcal{OH}_T(G)$$

and consider the function

$$\chi_o(G, x) = |\mathcal{OH}^*_x(G)|.$$

Reworking our example we obtain distinct polynomials[3]

$$\chi_o(P_2, x) = 2x(x-1)(x-2) = 2x^3 - 6x^2 + 4x$$

and

$$\chi_o(C_3, x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x.$$

In fact, notice that

Proposition 3. *Given two oriented graphs G and H, $|\mathcal{OH}_x(G)| = |\mathcal{OH}_x(H)|$ if and only if $|\mathcal{OH}^*_x(G)| = |\mathcal{OH}^*_x(H)|$ for all $x \in \{1, 2, \ldots\}$.*

Proof. Suppose $|\mathcal{OH}_x(G)| = |\mathcal{OH}_x(H)|$ for all $x \in \{1, 2, \ldots\}$. We want to show that $|\mathcal{OH}^*_x(G)| = |\mathcal{OH}^*_x(H)|$ for all $x \in \{1, 2, \ldots\}$. Let us prove this by induction on x.

For the base case, observe that

$$|\mathcal{OH}_1(G)| = |\mathcal{OH}^*_1(G)| = |\mathcal{OH}^*_1(H)| = |\mathcal{OH}_1(H)|$$

is trivially true. Suppose $|\mathcal{OH}^*_y(G)| = |\mathcal{OH}^*_y(H)|$ for all $y \leqslant x-1$. We will show that it is true for $y = x$ as well. Notice that the number of homomorphisms in $\mathcal{OH}_x(G)$ (resp., $\mathcal{OH}_x(H)$) whose cardinality of the image set is equal to $y \leqslant x$ can be expressed as $\binom{x}{x-y} \cdot |\mathcal{OH}^*_y(G)|$ (resp., $\binom{x}{x-y} \cdot |\mathcal{OH}^*_y(H)|$). Therefore, using the induction hypothesis and our basic assumption

$$\begin{aligned}
|\mathcal{OH}^*_x(G)| &= |\mathcal{OH}_x(G)| - \sum_{y=1}^{x-1} \binom{x}{y} |\mathcal{OH}^*_y(G)| \\
&= |\mathcal{OH}_x(H)| - \sum_{y=1}^{x-1} \binom{x}{y} |\mathcal{OH}^*_y(H)| \\
&= |\mathcal{OH}^*_x(H)|.
\end{aligned}$$

[3]For $G = P_2$ or C_3, we can have $x(x-1)(x-2)$ choices for the images of the vertices of G. Now, among the arcs joining the vertices of $f(V(G))$, all but $3 - |A(G)|$ will have fixed direction to satisfy the conditions of a homomorphism. In this modified version, we do not care about $\{0, 1, \ldots, x-1\} \setminus f(V(G))$.

For the converse, assume that $|\mathcal{OH}_x^*(G)| = |\mathcal{OH}_x^*(H)|$ for all $x \in \{1, 2, \ldots\}$. Counting like before, we have

$$\begin{aligned} |\mathcal{OH}_x(G)| &= \sum_{y=1}^{x} \binom{x}{y} |\mathcal{OH}_y^*(G)| \\ &= \sum_{y=1}^{x} \binom{x}{y} |\mathcal{OH}_y^*(H)| \\ &= |\mathcal{OH}_x(H)|. \end{aligned}$$

That concludes the proof. □

In this sense the new function $|\mathcal{OH}_x^*(\cdot)|$ is truly a good modification of $|\mathcal{OH}_x(\cdot)|$. Upon further study, in hindsight, the function $\chi_o(\cdot, \cdot)$ turns out to be very interesting. To reiterate, for all integers $x \geqslant 0$, let $\mathcal{T}_x$ be the set of all tournaments whose vertices are a subset of $\{0, 1, \ldots, x-1\}$. We define

$$\chi_o(G, x) = |\{G \xrightarrow{onto} T : T \in \mathcal{T}_x\}|$$

as the *homomorphic polynomial* of an oriented graph G and devote this article to study this polynomial. The justification for the name is given in Theorem 7.

Remark 4. Let $\mathcal{K}_x$ be the set of all complete graphs whose set of vertices are subsets of $\{0, 1, \ldots, x-1\}$. Given a graph G and a complete graph K let $\mathcal{H}_K(G)$ be the set of all distinct onto homomorphisms of G to K. Now define $\mathcal{H}_x^*(G) = \cup_{K \in \mathcal{K}_x} \mathcal{H}_K(G)$. As any y-coloring of G, for each $y \leqslant x$, corresponds to an onto homomorphism of G to K_y, we have $\chi(G, x) = |\mathcal{H}_x^*(G)|$. Therefore, the homomorphic polynomial $\chi_o(\cdot, \cdot)$ is indeed an "oriented analogue" of the chromatic polynomial $\chi(\cdot, \cdot)$.

3 Homomorphic polynomials

We start by computing the homomorphic polynomial of a tournament.

Proposition 5. *If T is a tournament on n vertices, then*

$$\chi_o(T, x) = x(x-1)\cdots(x-n+1).$$

Proof. As a tournament T admits onto homomorphisms only to itself, it is enough to count the number of ways to label the vertices of T using $\{0, 1, \ldots, x-1\}$. □

The next result proves a formula for computing $\chi_o(G, x)$ of an oriented graph. The formula along with its derivation is similar to the deletion-contraction formula [7] for counting chromatic polynomials, even though it is not an exact analogue. For oriented chromatic polynomial, a similar formula is established by Duffy and Cox [5]. However, to prove that formula, it was necessary to extend the concept of oriented chromatic polynomials to mixed graphs.

Before presenting the formula we need to define some notation. Let G be an oriented graph and let u, v be two non-adjacent vertices of G. Then $G \cdot uv$ denotes the oriented graph obtained by identifying the vertices u, v of G and $G + uv$ denotes the oriented graph obtained by adding the arc uv to G.

Theorem 6. *Let u, v be two non-adjacent vertices of G. Then*

(i) $\chi_o(G, x) = \chi_o(G + uv, x) + \chi_o(G + vu, x)$ *where u, v are connected by a directed 2-path.*

(ii) $\chi_o(G, x) = \chi_o(G \cdot uv, x) + \chi_o(G + uv, x) + \chi_o(G + vu, x)$ *where u, v are not connected by a directed 2-path.*

Proof. Let u, v be two non-adjacent vertices of G. If f is a homomorphism of G to a tournament T, then either $f(u) = f(v)$, or $f(u)f(v)$ is an arc of T, or $f(v)f(u)$ is an arc of T.

The set of onto homomorphisms having $f(u) = f(v)$ is in a one-to-one correspondence with the set of onto homomorphisms of $G \cdot uv$.

Also the set of onto homomorphisms such that $f(u)f(v)$ is an arc is in a one-to-one correspondence with the set of onto homomorphisms of $G + uv$.

Similarly, the set of onto homomorphisms such that $f(v)f(u)$ is an arc is in a one-to-one correspondence with the set of onto homomorphisms of $G + vu$.

Moreover, it is possible to have $f(u) = f(v)$ if and only if u and v are neither adjacent, nor connected by a directed 2-path [11]. Thus the result follows. □

Now we are ready to prove that the function $\chi_o(G, x)$ is indeed a polynomial in x.

Theorem 7. *Given an oriented graph G, $\chi_o(G, x)$ is a polynomial in x.*

Proof. Let us introduce a partial order for oriented graphs to facilitate induction. Let $G \prec H$ if one of the following conditions hold:

- $|V(G)| < |V(H)|$,
- $|V(G)| = |V(H)|$ and $|E(G)| > |E(H)|$.

Now assume that G is a minimal (with respect to $\prec$) counter-example to the statement of the theorem. Thus, due to Proposition 5 G is not a tournament. Hence, there exists a pair of non-adjacent vertices u, v in G. Now apply Theorem 6 on G to obtain

$$\chi_o(G, x) = \chi_o(G + uv, x) + \chi_o(G + vu, x)$$

or

$$\chi_o(G, x) = \chi_o(G \cdot uv, x) + \chi_o(G + uv, x) + \chi_o(G + vu, x)$$

depending on whether u, v are connected by a directed 2-path or not. In any case, observe that $G + uv \prec G$, $G + vu \prec G$, and (if it exists) $G \cdot uv \prec G$. Thus, due to the minimality of G, each of $\chi_o(G + uv, x)$, $\chi_o(G + vu, x)$, and $\chi_o(G \cdot uv, x)$ (if it exists) are polynomials. Therefore, we are done using Theorem 6. □

The above result justifies the name homomorphic polynomial.

4 Associated $(2,3)$-rooted tree and factorial form

Notice that Theorem 6 enables us to compute the homomorphic polynomial of any oriented graph. Using that idea we will describe a rooted $(2,3)$-tree, not necessarily unique, associated to each oriented graph G which will enable us to compute its homomorphic polynomial.

Given an oriented graph G, we describe the design of a rooted $(2,3)$-tree $\mathcal{T}_G$ as follows: the vertices of $\mathcal{T}_G$ are oriented graphs with its root being G. Now take two non-adjacent vertices u, v of G. If u, v are connected by a directed 2-path, then there are two children $G+uv$ and $G+vu$ of G and if u, v are not connected by a directed 2-path, then there are three children $G+uv$, $G+vu$ and $G \cdot uv$ of G. We recursively continue this process for each node of the tree unless that node is a tournament. Thus, finally, we obtain a rooted $(2,3)$-tree $\mathcal{T}_G$ whose leaves are tournaments. Observe that such a tree is not unique.

Note that if we stop the recursive process of obtaining a $\mathcal{T}_G$ for an oriented graph G at any point and the leaves at that point of time are $G_1, G_2, \ldots, G_t$, then

$$\chi_o(G,x) = \chi_o(G_1,x) + \chi_o(G_2,x) + \cdots + \chi_o(G_t,x).$$

As the collection of leaves of the tree is a multiset of tournaments after finishing the whole process, the homomorphic polynomial $\chi_o(G,x)$ of G can be expressed as a sum of homomorphic polynomials of some tournaments. In other words, let such a multiset $L(G)$ have a_i tournaments of order i for each $i \in \{1, 2, \ldots, n\}$. Then the homomorphic polynomial of G is

$$\chi_o(G,x) = a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x^{(1)}$$

where $x^{(i)} = x(x-1)\cdots(x-i+1)$ is the i^{th} falling factorial of x. The above representation of $\chi_o(G,x)$ is called its *factorial form*.

From now on, for convenience, we will denote such a multiset $L(G)$ by

$$\{a_n \cdot T_n, a_{n-1} \cdot T_{n-1}, \ldots, a_1 \cdot T_1\}.$$

Observe that as the homomorphic polynomial of an oriented graph G is well defined, the multiset $L(G)$ will not depend on the choice of the $(2,3)$-rooted tree associated to G.

5 Properties

Throughout this section, the set of vertices and arcs of an oriented graph G is denoted by $V(G)$ and $A(G)$, respectively. Also we would like to suggest the master's thesis of Hubai Tamás [8] as a detailed survey on properties of chromatic polynomials for simple graphs.

As hinted in the introduction, the homomorphic polynomial carries information about the *order* (number of vertices) and *size* (number of arcs) of an oriented graph.

Theorem 8. *Let G be an oriented graph with homomorphic polynomial*

$$\chi_o(G,x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$. Then $|V(G)| = n$ and $|A(G)| = \binom{n}{2} - \log_2 a_n$.

Proof. Recall the partial order $\prec$ defined in the proof of Theorem 7 and assume that G is a minimal (with respect to $\prec$) counter-example to this theorem.

Notice that G cannot be a tournament due to Proposition 5. Hence, there exists a pair of non-adjacent vertices u, v in G. Now apply Theorem 6 on G to obtain

$$\chi_o(G,x) = \chi_o(G+uv,x) + \chi_o(G+vu,x)$$

or

$$\chi_o(G,x) = \chi_o(G \cdot uv,x) + \chi_o(G+uv,x) + \chi_o(G+vu,x)$$

depending on whether u, v are connected by a directed 2-path or not. In any case, observe that $G+uv \prec G$, $G+vu \prec G$, and (if it exists) $G \cdot uv \prec G$.

Now, notice that the number of vertices in G is the same as the number of vertices in $G+uv$ or $G+vu$ and one more than the number of vertices in $G \cdot uv$. Due to the minimality of G, the polynomials $\chi_o(G+uv,x)$, $\chi_o(G+vu,x)$, and $\chi_o(G \cdot uv,x)$ (if it exists) must have degree n, n, and $n-1$, respectively. Therefore, the polynomial $\chi_o(G,x)$ also has degree $n = |V(G)|$.

Now let us concentrate on the other part of the statement. Notice that, as $\chi_o(G \cdot uv, x)$ has degree $n-1$, its coefficients do not contribute in the value of a_n. To be precise, the leading coefficient a_n is a sum of the leading coefficients a_{uv} and a_{vu} (say) of $\chi_o(G+uv,x)$ and $\chi_o(G+vu,x)$, respectively. Also, as both $G+uv$ and $G+vu$ have one arc more than the number of arcs in G, due to the minimality of G, we must have $a_{uv} = a_{vu}$. Hence,

$$\binom{n}{2} - \log_2 a_n = \binom{n}{2} - \log_2 2a_{uv} = \binom{n}{2} - \log_2 a_{uv} - 1 = |A(G+uv)| - 1 = |A(G)|.$$

This completes the proof. □

Clearly, $\chi_o(G,x)$ is not a monic polynomial like the chromatic polynomial [7] and the oriented chromatic polynomial [18]. Regardless, it is possible to characterize all the monic homomorphic polynomials as a simple corollary of Theorem 8.

Corollary 9. *The homomorphic polynomial $\chi_o(G,x)$ is monic if and only if G is a tournament.*

Thus, this seems like the right time for computing homomorphic polynomial of an oclique.

Proposition 10. *If O is an oclique on n vertices and m arcs, then*

$$\chi_o(O,x) = 2^t \cdot x(x-1)\cdots(x-n+1)$$

where $t = \binom{n}{2} - m$.

Proof. Recall the partial order $\prec$ defined in the proof of Theorem 7 and assume that the oclique G is a minimal (with respect to $\prec$) counter-example to this proposition.

Notice that G cannot be a tournament due to Proposition 5. Hence, there exists a pair of non-adjacent vertices u, v in G. Now apply Theorem 6 on G to obtain

$$\chi_o(G, x) = \chi_o(G + uv, x) + \chi_o(G + vu, x)$$

as u, v are connected by a directed 2-path due to G being an oclique. In any case, observe that $G + uv \prec G$ and $G + vu \prec G$.

Observe that both $G+uv$ and $G+vu$ are ocliques as they are each obtained by adding an arc to an oclique. Thus, due to the minimality of G, the fact that $|A(G + uv, x)| = |A(G + vu, x)| = m + 1$, we must have

$$\chi_o(G + uv, x) = \chi_o(G + vu, x) = 2^{t'} \cdot x(x - 1) \cdots (x - n + 1)$$

where $t' = \binom{n}{2} - (m + 1)$. Now using the above equation along with the relation

$$\chi_o(G, x) = \chi_o(G + uv, x) + \chi_o(G + vu, x),$$

we have

$$\chi_o(G, x) = 2^t \cdot x(x - 1) \cdots (x - n + 1)$$

where $t = (t' + 1) = \binom{n}{2} - m$. □

Let us look at some other properties of the homomorphic polynomial. It is easy to show that the constant term of a homomorphic polynomial is 0. This is also a property of chromatic polynomials [7] and oriented chromatic polynomials [18].

Proposition 11. *The constant term of a homomorphic polynomial $\chi_o(G, x)$ is 0.*

Proof. As any oriented graph G admits 0 oriented 0-colorings, $\chi_o(G, 0) = 0$. On the other hand, $\chi_o(G, 0)$ is the constant term of the polynomial. □

Among the other standard graphs, we want to compute the homomorphic polynomial for the independent set I_n of cardinality n. Surprisingly, it turns out to be a challenging problem. For convenience, we will figure out the factorial form of the polynomial instead of its standard form.

Theorem 12. *For an independent set I_n on n vertices,*

$$\chi_o(I_n, x) = a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x^{(1)}$$

where $a_n = 2^{\binom{n}{2}}$, $a_i = 2^{\binom{i}{2}} \sum_{1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-i} \leqslant i} (x_1 \cdot x_2 \cdot x_3 \cdots x_{n-i})$ *for all* $i \in \{2, 3, \ldots, n-1\}$ *and* $a_1 = 1$.

The proof of Theorem 12 is provided in Section 6.

Let $\left\{ {n \atop k} \right\}$ denote the number of ways to partition a set of n labeled objects into k non-empty unlabeled subsets. These numbers are called the Stirling numbers of the second kind.

Using the polynomial $\chi_o(I_n, x)$, it is possible to prove the following result involving Stirling numbers of the second kind.

Theorem 13. *For any $k \geqslant 1$,*

$$\sum_{i=1}^{k} 2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\} \cdot k^{(i)} = \chi_o(I_n, k).$$

Proof. To count the number of onto homomorphisms of I_n to a fixed labeled tournament on i vertices, we partition the n vertices in i subsets in $\left\{ {n \atop i} \right\}$ ways and can map a particular partition to separate vertices in $i!$ ways. Furthermore, there are in total $2^{\binom{i}{2}}$ labeled tournaments on i vertices. Therefore, the coefficient of $x^{(i)}$ in the homomorphic polynomial of I_n is $2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\}$. Thus the result follows. □

Furthermore, using Theorem 13 one can compute the Stirling numbers of the second kind using the polynomial $\chi_o(I_n, x)$. The *forward difference operator* [17] of a real function $f(x)$ is

$$(\Delta f)(x) = f(x+1) - f(x).$$

It is known [17] that

$$(\Delta x^{(i)})(x) = i \cdot x^{(i-1)} \tag{1}$$

and that the operator commutes with the operation $+$ as well as scalar multiplication.

Corollary 14. *For $1 \leqslant k \leqslant n$,*

$$\left\{ {n \atop k} \right\} = \frac{\chi_o(I_n, k) - \chi_o(I_n, k-1) - (\Delta \chi_o(I_n, \cdot))(k-1)}{2^{\binom{k}{2}} \cdot (k!)^2}.$$

Proof. Note that, by Theorem 13 we have

$$\begin{aligned} \chi_o(I_n, k) - \chi_o(I_n, k-1) &= \sum_{i=1}^{k} 2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\} \cdot k^{(i)} - \sum_{i=1}^{k-1} 2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\} \cdot (k-1)^{(i)} \\ &= 2^{\binom{k}{2}} \cdot k! \cdot \left\{ {n \atop k} \right\} \cdot k^{(k)} + \sum_{i=1}^{k-1} 2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\} \cdot [k^{(i)} - (k-1)^{(i)}]. \end{aligned}$$

Observe the following before continuing with the calculation steps.

$$\begin{aligned} &[k^{(i)} - (k-1)^{(i)}] \\ &= k(k-1)\cdots(k-i+2)(k-i+1) - (k-1)(k-2)\cdots(k-1-i+2)(k-1-i+1) \\ &= k(k-1)\cdots(k-i+1) - (k-1)(k-2)\cdots(k-i+1)(k-i) \\ &= [k-(k-i)][(k-1)(k-2)\cdots(k-i+1)] \\ &= i \cdot (k-1)^{(i-1)}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\chi_o(I_n, k) - \chi_o(I_n, k-1) &= 2^{\binom{k}{2}} \cdot k! \cdot \left\{ {n \atop k} \right\} \cdot k^{(k)} + \sum_{i=1}^{k-1} 2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\} \cdot [k^{(i)} - (k-1)^{(i)}] \\
&= 2^{\binom{k}{2}} \cdot (k!)^2 \cdot \left\{ {n \atop k} \right\} + \sum_{i=1}^{k-1} 2^{\binom{i}{2}} \cdot i! \cdot \left\{ {n \atop i} \right\} \cdot i \cdot (k-1)^{(i-1)} \\
&= 2^{\binom{k}{2}} \cdot (k!)^2 \cdot \left\{ {n \atop k} \right\} + (\Delta \chi_o(I_n, \cdot))(k-1).
\end{aligned}
$$

The final equality is justified by applying the formula given in (1) to the equality from Theorem 13. Thus we are done. □

Computation of the chromatic polynomial of a disconnected undirected simple graph has a nice relation with the chromatic polynomials of its components.

Proposition 15. [7] *Let G be an undirected simple graph with c connected components $G_1, G_2, \dots, G_c$. Then*

$$\chi(G, x) = \chi(G_1, x) \cdot \chi(G_2, x) \cdots \chi(G_c, x).$$

However, the above condition does not hold at all in the oriented case. Thus, if we mandate an oriented analogue of the Tutte polynomial to retain the important property of factoring through components, then we can conclude that the homomorphic polynomial cannot be a valid candidate for it. However, it is worth studying in detail possible natural analogues of the Tutte polynomial in the context of oriented coloring. On the other hand, in pursuit of a similar result for homomorphic polynomials, we found a particular type of construction where such a result holds.

We will describe the construction before stating the result. Let $G_1, G_2, \dots, G_c$ be c oriented graphs. Let T be a tournament on c vertices $\{1, 2, \dots, c\}$. Now construct the oriented graph G as follows. Take the disjoint union of $G_1, G_2, \dots, G_c$. For each arc ij of T, put an arc from each vertex of G_i to each vertex of G_j. The so obtained oriented graph is G. It is worth mentioning that this construction appears frequently while studying oriented coloring. For instance, such constructions are used to prove lower bounds for oriented chromatic number of several graph families, such as outerplanar graphs [19], partial 2-trees [14], planar graphs [15], graphs on surfaces [2], etc.

Theorem 16. *Let G be an oriented graph as described above. Then*

$$\chi_o(G, x) = \chi_o(G_1, x) \cdot \chi_o(G_2, x) \cdots \chi_o(G_c, x).$$

Proof. Observe that for any $i \neq j$, a vertex of G_i is adjacent to a vertex of G_j. Thus, the images of G_i and G_j are disjoint under any homomorphism. Hence, the formula described in Theorem 6 can be independently applied to G_i and G_j for any distinct $i, j \in \{1, 2, \dots, c\}$. □

We are going to present a few more results in the same direction. But before presenting them, let us describe a graph construction.

Let G_1 and G_2 be two oriented graphs. First take the disjoint union of G_1, G_2 and then add a new vertex v to it. After that for each vertex u of G_1, add an arc uv and for each vertex w of G_2, add an arc vw. Let us denote the so-obtained oriented graph by $G_1 \ltimes_v G_2$.

Theorem 17. *For any non-negative integer k, the homomorphic polynomial*

$$\chi_o(G_1 \ltimes_v G_2, k+1)$$

is equal to

$$\sum_{\{k_1,k_2 | 0 \leqslant k_1+k_2 \leqslant k\}} \frac{(k+1)!}{k_1! k_2! (k-k_1-k_2)!} \cdot 2^{k_1 k_2} \cdot n(G_1, k_1) \cdot n(G_2, k_2)$$

where $n(G, j)$ satisfies the recurrence relation

$$n(G, j) = \chi_o(G, j) - \sum_{i=1}^{j-1} \binom{j}{i} n(G, j)$$

with the initial condition $n(G, 1) = \chi_o(G, 1)$.

Proof. We want to count the number of ways to choose an onto homomorphism f of $G_1 \ltimes_v G_2$ to distinct tournaments on vertices labeled by subset of $\{0, 1, \ldots, k\}$.

Note that the vertex v will always have a distinct image to any other vertices of the graph with respect to any homomorphism. Thus we can choose the image $f(v)$ of v from the set $\{0, 1, \ldots, k\}$ in $k+1$ different ways.

Also any vertex of G_1 will have a distinct image to that of any vertex of G_2 with respect to any homomorphism as they are end points of a directed 2-path whose middle vertex is v.

Let $n(G_i, k_i)$ denote the number of onto homomorphisms of G_i to all possible tournaments on k_i vertices. Notice that, $n(G_i, 1) = \chi_o(G_i, 1)$ and

$$n(G_i, j) = \chi_o(G_i, j) - \sum_{i=1}^{k-1} \binom{k}{j} n(G_i, j).$$

Therefore, given a subset $S_i \subseteq \{0, 1, \ldots, k\}$ having cardinality k_i, there are exactly $n(G_i, k_i)$ onto homomorphisms of G_i to tournaments of order k_i with vertices labeled by elements of S for each $i \in \{1, 2\}$. Furthermore, we can choose a subset S_1 of cardinality k_1 from the set $\{0, 1, \ldots, k\} \setminus \{f(v)\}$ in $\binom{k}{k_1}$ ways and we can choose a subset S_2 of cardinality k_2 from the set $\{0, 1, \ldots, k\} \setminus (S_1 \cup \{f(v)\})$ in $\binom{k-k_1}{k_2}$ ways.

There can be k_1k_2 different arcs between the vertices of S_1 and S_2. Those arcs can be chosen in $2^{k_1k_2}$ ways. Thus

$$\begin{aligned}\chi_o(G_1 \ltimes_v G_2, k+1) &= \sum_{\{k_1,k_2 \mid 0 \leqslant k_1+k_2 \leqslant k\}} (k+1)\binom{k}{k_1}\binom{k-k_1}{k_2} 2^{k_1k_2}\, n(G_1,k_1)\, n(G_2,k_2) \\ &= \sum_{\{k_1,k_2 \mid 0 \leqslant k_1+k_2 \leqslant k\}} \frac{(k+1)!}{k_1!k_2!(k-k_1-k_2)!} 2^{k_1k_2}\, n(G_1,k_1)\, n(G_2,k_2).\end{aligned}$$

Hence we are done. □

In the construction of $G_1 \ltimes_v G_2$, if we choose G_2 (or G_1) to be the null graph, then the resultant graph we obtain is denoted by G_1^+ (or G_2^-). A weaker version of the above theorem is the following.

Corollary 18. *Let G be an oriented graph. Then*

$$\chi_o(G^+, k) = \chi_o(G^-, k) = k \cdot \chi_o(G, k-1).$$

Above we tried to investigate the nature of the homomorphic polynomials of graphs with a dominating vertex. Now let us probe the opposite aspect.

Theorem 19. *Let G_v be the disjoint union of an oriented graph G and an isolated vertex v. If the factorial form of the homomorphic polynomial of G is*

$$\chi_o(G, x) = a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x^{(1)},$$

then

$$\chi_o(G_v, x) = x \cdot [(\Delta\chi_o(G, \cdot))(x-1) + \sum_{i=1}^{n} a_i \cdot 2^i \cdot (x-1)^{(i)}].$$

The proof of Theorem 19 is provided in Section 6.

The chromatic polynomial for a simple graph has alternative signs of its coefficients [7]. Sopena [18] showed that this property does not hold for oriented chromatic polynomials. However, the above property holds for homomorphic polynomials.

Theorem 20. *Let G be an oriented graph with homomorphic polynomial*

$$\chi_o(G, x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x$$

where $a_n \neq 0$. Then $a_i \geqslant 0$ for all $i \equiv n \pmod 2$ and $a_i \leqslant 0$, otherwise.

The proof of Theorem 20 is provided in Section 6.

An immediate corollary of the above result is the following:

Corollary 21. *A homomorphic polynomial does not have any negative real root.*

Proof. Let G be an oriented graph with homomorphic polynomial $\chi_o(G,x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$ where $a_n \neq 0$ and let y be a negative real number.

Thus $\chi_o(G,y) > 0$ if n is even and $\chi_o(G,y) < 0$ if n is odd due to Theorem 20. □

It is worth noting that oriented chromatic polynomials do not posses this property [5]. One other important property of chromatic polynomials that was not satisfied by oriented chromatic polynomials is the unimodal property [21]. That the coefficients of a chromatic polynomial have unimodal property was observed by Read [16], later conjectured by Nijenhuis and Wilf [13], and finally proved by Huh and Katz [9].

Even though we have not proved the unimodal property for homomorphic polynomials, this property was observable throughout the many examples we encountered, which makes us propose the following conjecture.

Conjecture 22. Let G be an oriented graph with homomorphic polynomial

$$\chi_o(G,x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x$$

where $c_n \neq 0$. Then there exists $i < n$ such that

$$|c_n| \leqslant |c_{n-1}| \leqslant \cdots \leqslant |c_{n-i}| \geqslant |c_{n-i-1}| \geqslant \cdots |c_1|.$$

6 The proofs

We open our proof section with an important lemma.

Lemma 23. *Let G be the union of a tournament T of order k and an isolated vertex v. Then $L(G) = \{2^k \cdot T_{k+1}, k \cdot T_k\}$.*

Proof. Assume that the vertices of T are $v_1, v_2, \ldots, v_k$. Now let us construct a $(2,3)$ rooted tree $\mathcal{T}_T$ as follows. We will recursively use the formula given by Theorem 6 on the pair of vertices v and v_i, where $i \in \{1, 2, \ldots, k\}$ starts with the initial value 1 and gets incremented in each step until it reaches k.

Let T^i denote the mixed graph obtained by adding the edges vv_i for all $i \in \{1, 2, \ldots, i\}$ in G. An orientation of T^i refers to an oriented graph obtained by replacing its edges with arcs (in any possible direction). That means, there can be exactly 2^i distinct orientations of T^i. We claim that the level $i+1$ of the rooted tree consists of 2^i distinct orientations of T^i and the tournament T for all $i \in \{1, 2, \ldots, k+1\}$. We will use induction to prove the correctness of this claim.

For the base case, observe that the root of $\mathcal{T}_T$ is G. It will have three children $G + vv_1$, $G + v_1 v$ and $G \cdot vv_1$, which will be the level 2 of the tree. Note that $G \cdot vv_1$ is exactly the tournament T and $G + vv_1$, $G + v_1 v$ are the two distinct orientations of T^1. Hence our claim is correct for $i = 1$.

For the induction step, let the claim be true for all $j \leqslant i$. We want to prove it for $j = i + 1$. For that, notice that among all distinct orientations of T^{i-1}, there exists exactly one in which v and v_i are not connected by a directed 2-path. Thus for that orientation of T^{i-1} we will apply the part (ii) of the formula given in Theorem 6, while we will apply part (i) of the formula for the rest. The tournament T will be a child of the particular orientation of T^{i-1} where v and v_i are not connected by a directed 2-path. The other children of it, and the children of the rest of the orientations of T^{i-1} will be all the orientations of T^i. Finally, the tournament T in the i^{th} level will not have any child. This proves our claim.

The construction of the tree will be complete after k steps, that is, at level $k+1$. In the level $k+1$, we have 2^k distinct orientations of T^k, which are tournaments on $k+1$ vertices. Moreover, every level, except for the level 0, contains a leaf which is the tournament T. This completes the proof. □

Now we are ready to prove Theorem 19.

Proof of Theorem 19. Note that to obtain a $(2,3)$-rooted tree $\mathcal{T}_{G_v}$ associated to G_v, we can simply include an isolated vertex to each node of $\mathcal{T}_G$ and continue the branching process as described in Section 4.

Thus using Lemma 23 we obtain

$$\begin{aligned}\chi_o(G_v,x) &= \sum_{i=1}^{n} i\cdot a_i\cdot x^{(i)} + \sum_{i=1}^{n} a_i\cdot 2^i\cdot x^{(i+1)}\\ &= x\cdot\sum_{i=1}^{n} i\cdot a_i\cdot (x-1)^{(i-1)} + x\cdot\sum_{i=1}^{n} a_i\cdot 2^i\cdot (x-1)^{(i)}\\ &= x\cdot[(\Delta\chi_o(G,\cdot))(x-1) + \sum_{i=1}^{n} a_i\cdot 2^i\cdot (x-1)^{(i)}].\end{aligned}$$

This concludes the proof. □

For proving Theorem 12 we are going to introduce some notation. Let $\mathbb{H}_k$ denote the set of all (labeled) tournaments of order k. Furthermore, let $a\cdot\mathbb{H}_k$ denote the multiset having a copies of each elements belonging to $\mathbb{H}_k$. Note that $|\mathbb{H}_k| = 2^{\binom{k}{2}}$ and $|a\cdot\mathbb{H}_k| = a\cdot 2^{\binom{k}{2}}$.

Now we are ready to prove Theorem 12.

Proof of Theorem 12. To prove the theorem, it is sufficient to prove that the multiset $L(I_n)$ of leaves of a $(2,3)$-rooted tree $\mathcal{T}_{I_n}$ associated to I_n is $L(I_n) = \{a_n\cdot T_n, a_{n-1}\cdot T_{n-1}, \ldots, a_1\cdot T_1\}$ where the a_i's are as in the statement of Theorem 12.

We will prove this by induction. For $n \leqslant 2$, the result is trivially true.

Assume that the statement is true for all $n \leqslant t$ for some fixed t. Then we want to show that the statement is true for $n = t+1$ as well.

Note that to obtain a $(2,3)$-rooted tree $\mathcal{T}_{I_{t+1}}$ associated to I_{t+1}, we can simply include an isolated vertex to each node of $\mathcal{T}_{I_t}$ and continue the branching process as described in Section 4.

For convenience, assume that $L(I_t) = \{a_t \cdot T_t, a_{t-1} \cdot T_{t-1}, \ldots, a_1 \cdot T_1\}$ where a_i's are as in the statement of Theorem 12 and $L(I_{t+1}) = \{b_{t+1} \cdot T_{t+1}, b_t \cdot T_t, \ldots, b_1 \cdot T_1\}$.

Due to Lemma 23 $b_{t+1} = 2^t \cdot a_t = 2^t \cdot 2^{\binom{t}{2}} = 2^{\binom{t+1}{2}}$. Furthermore, $b_1 = 1$ as there is only one tournament on 1 vertex and there is exactly one homomorphism of I_{t+1} to that tournament.

For an $i \in \{2, 3, \ldots, t\}$, we have

$$\begin{aligned} b_i &= 2^{i-1} \cdot a_{i-1} + i \cdot a_i \\ &= 2^{i-1} \cdot 2^{\binom{i-1}{2}} \sum_{1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{t-(i-1)} \leqslant i-1} (x_1 \cdot x_2 \cdot x_3 \cdots x_{t-(i-1)}) \\ &\quad + 2^{\binom{i}{2}} i \sum_{1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{t-i} \leqslant i} (x_1 \cdot x_2 \cdot x_3 \cdots x_{t-(i-1)}) \\ &= 2^{\binom{i}{2}} \sum_{1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{t+1-i} \leqslant i} (x_1 \cdot x_2 \cdot x_3 \cdots x_{t+1-i}). \end{aligned}$$

This concludes the proof. □

For proving Theorem 20 we need another lemma. Note that we can write

$$x^{(n)} = \sum_{k=0}^{n} (-1)^k C_{n,k} x^{n-k},$$

where the coefficients $C_{n,k}$ are elementary symmetric polynomials in the first positive $n-1$ integers, given by

$$C_{n,k} = \sum_{1 \leqslant j_1 < \ldots < j_k \leqslant n-1} j_1 \cdots j_k.$$

For $k < 0$ or $k \geqslant n$ we will set $C_{n,k} = 0$. The following lemma proves a recurrence relation on the coefficients $C_{n,k}$.

Lemma 24. *For all $n, k \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$C_{n,k} = (n-1)C_{n-1,k-1} + C_{n-1,k}.$$

Proof. The term $C_{n,k}$ can be written as a sum of two expressions $e_1 + e_2$, such that every additive term in e_1 has $n-1$ as a factor and every additive term in e_2 does not have $n-1$ as a factor. In e_1, $n-1$ can be factored out to give $(n-1) \cdot C_{n-1,k-1}$ and e_2 is $C_{n-1,k}$ by definition. □

Now we are ready to prove Theorem 20.

Proof of Theorem 20. We will prove this by the method of strong induction on $|V(G)|$. For the base case, observe that the statement is correct for all $|V(G)| \leqslant 2$. This is verifiable using Proposition 5 and Theorem 12. As induction hypothesis, assume that the statement holds for all $|V(G)| \leqslant n-1$. Thus, if we can show that the statement is true for $|V(G)| = n$, we will be done.

Let $r = \binom{|V(G)|}{2} - |E(G)|$ be the number of missing arcs in G and let p be the number of pairs of vertices in G that are neither adjacent nor connected by a directed 2-path. Note that $p \leqslant r$. We refer to those p pairs as *identifiable pairs*.

Let $\{v_1, v_2, \ldots, v_d\}$ be the d distinct vertices among the identifiable pairs. Moreover, let d_i be the number of vertices identifiable with v_i for each $i \in \{1, 2, \ldots, d\}$. We know that $\chi_o(G, x) = \sum_{i=1}^{n} a_i x^{(i)}$ where a_i is the number of tournaments on i vertices in $L(G)$ and $|V(G)| = n$.

Observe that while constructing an associated $(2,3)$-rooted tree $\mathcal{T}_G$ of G we can stop in between in such a way that a leaf is either a tournament of order n or a tournament of order $n-1$ or an oriented graph of order $n-2$. Therefore we can express the homomorphic polynomial of G as follows:

$$\chi_o(G, x) = a_n x^{(n)} + a_{n-1} x^{(n-1)} + W$$

where W is a sum of some homomorphic polynomials of oriented graphs of order $n-2$. Notice that the polynomial W has alternating signs due to the the induction hypothesis. Hence if we can show that the polynomial $a_n x^{(n)} + a_{n-1} x^{(n-1)}$ has alternating signs, then we will be done.

For this to be true, the absolute value of the i^{th} term in $a_n x^{(n)}$ should be greater than that of the $(i-1)^{\text{th}}$ term in $a_{n-1} x^{(n-1)}$. In other words we need to show that

$$a_n C_{n,i} \geqslant a_{n-1} C_{n-1,i-1}.$$

Furthermore, Lemma 24 implies that it is enough to show

$$a_n((n-1)C_{n-1,i-1} + C_{n-1,i}) \geqslant a_{n-1} C_{n-1,i-1},$$

that is,

$$C_{n-1,i} \geqslant \left(\tfrac{a_{n-1}}{a_n} - (n-1)\right) C_{n-1,i-1}. \tag{2}$$

While constructing an associated $(2,3)$-rooted tree $\mathcal{T}_G$ of G, if v_i is identified with another vertex to obtain the oriented graph G', then the number of missing arcs in G' is at most $r - d_i$. Let $\mathcal{T}_{G'}$ be the subtree of $\mathcal{T}_G$ rooted at G' and let $L(G')$ be the multiset of the leaves in $\mathcal{T}_{G'}$. Thus, tournaments of order $n-1$ in $L(G')$ is at most 2^{r-d_i}.

There are d_i such vertices with which v_i can identify. Hence the number of tournaments having $n-1$ vertices that have v_i identified with another vertex is at most $d_i \cdot 2^{r-d_i}$. Moreover,

$$d_i \cdot 2^{r-d_i} \leqslant 2^{r-1}.$$

As there are d distinct vertices among the identifiable pairs in G, the number of tournaments having $n-1$ vertices is at most $\frac{d}{2} \cdot 2^{r-1}$.

On the other hand, $a_n = 2^r$ by Theorem 8. So the ratio $\frac{a_{n-1}}{a_n}$ is at most $\frac{d}{4}$. With this bound, (2) is satisfied, proving that $a_n x^{(n)} + a_{n-1} x^{(n-1)}$ has alternating signs. □

7 Conclusions

In this article we introduced and studied homomorphic polynomials for oriented graphs. To state a few highlights of this work, we would like to mention that we found an interesting relation between homomorphic polynomials for independent sets and Stirling numbers of the second kind. We also showed that the polynomials possess the alternating signs of the coefficients property. Moreover, we conjecture homomorphic polynomials to be unimodal.

While working through several examples, one question has peeped into our mind: is our homomorphic polynomial a refinement of the oriented chromatic polynomial? A precise restatement of the question is the following.

Question 25. Given two oriented graphs G and H, does $\chi_o(G, x) = \chi_o(H, x)$ imply $P(G, x) = P(H, x)$?

We have already noted that the converse is false.

Recently, Beaton, Cox, Duffy, and Zolkavich [1] have introduced and studied chromatic polynomials for 2-edge-colored graphs. One may also consider the analogue of homomorphic polynomials in that set up. It can be an interesting task to generalize these notions for (m, n)-colored mixed graphs [12].

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