The Davenport Constant of the Group $C_2^{r-1} \oplus C_{2k}$

Kevin Zhao
School of Mathematics and Statistics
Nanning Normal University
Nanning, China

Center for Applied Mathematics of Guangxi
Nanning Normal University
Nanning, China

zhkw-hebei@163.com

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Abstract

Let G be a finite abelian group. The Davenport constant $\mathsf{D}(G)$ is the maximal length of minimal zero-sum sequences over G. For groups of the form $C_2^{r-1} \oplus C_{2k}$ the Davenport constant is known for $r \leqslant 5$. In this paper, we get the precise value of $\mathsf{D}(C_2^5 \oplus C_{2k})$ for $k \geqslant 149$. It is also worth pointing out that our result can imply the precise value of $\mathsf{D}(C_2^4 \oplus C_{2k})$.

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1 Introduction

Let G be an additively written finite abelian group. A sequence α over G is a multi-set with elements from G, i.e., $\alpha = g_1 \cdots g_\ell$, where the repetition of elements are allowed and their order are disregarded. The number ℓ is called the length of α, also denoted by $|\alpha|$ sometimes. In particular $\ell = 0$ when α is empty. One can also write a sequence as $\alpha = \prod_{g\in G} g^{\mathsf{v}_g(\alpha)}$, where $\mathsf{v}_g(\alpha) \in \mathbb{Z}_{\geqslant 0}$ is called the multiplicity of g in α. A sequence T is called a subsequence of α if $\mathsf{v}_g(T) \leqslant \mathsf{v}_g(\alpha)$ for every $g \in G$, and T is a proper subsequence of α if $\mathsf{v}_g(T) < \mathsf{v}_g(\alpha)$ for at least one g. Althrough this paper, when we refer to sequences or subsequences, we always mean nonempty ones unless otherwise stated. A zero-sum sequence is a sequence such that the sum of all its elements is equal to the zero element of G. A minimal zero-sum sequence is a zero-sum sequence over G such that none of its proper subsequences is zero-sum. The Davenport constant of G is defined as the maximal length of all minimal zero-sum sequences over G, denoted by $\mathsf{D}(G)$.

In general it is a hard problem to determine this constant $\mathsf{D}(G)$, so far its actual value is only known for a few types of groups. For a finite abelian group G, we have $|G| = 1$ or

$G = C_{n_1} \oplus C_{n_2} \cdots \oplus C_{n_r}$ with $1 < n_1 | n_2 \cdots | n_r$. Set

$$\mathsf{D}^*(G) := 1 + \sum_{i=1}^{r} (n_i - 1).$$

It is known that $\mathsf{D}(G) \geqslant \mathsf{D}^*(G)$ for all finite abelian groups G, and the equality happens if G is a p-group or G is of rank one or two. Also the equality $\mathsf{D}(G) = \mathsf{D}^*(G)$ is conjectured to be true for groups G of rank three or $G = C_n^r$ (see, e.g.,[4] Conjecture 3.5). For more results, one can refer [1, 2, 5, 6]. In particular, van Emde Boas [1] proved the following result:

Lemma 1 ([1]). *Let p be a prime and m, n be positive integers. If $G = C_{mp^n} \oplus H$ with H being a finite abelian p-group and $p^n \geqslant D^*(H)$, then $\mathsf{D}(G) = \mathsf{D}^*(G)$.*

It is interesting to study the Davenport constant for the case $p^n < D^*(H)$ in the above lemma. Hence, the groups of the form $G = C_2^{r-1} \oplus C_{2k}$ draws much attention. For sufficiently large r, A. Plagne and W. Schmid [9] got an upper bound of $\mathsf{D}(G)$. For $r \leqslant 4$, it is known that $\mathsf{D}(G) = \mathsf{D}^*(G)$. For $r = 5$ and $k \geqslant 70$, F. Chen and S. Savchev [11] proved that $\mathsf{D}(G) = \mathsf{D}^*(G) + 1$ if k is odd, otherwise, $\mathsf{D}(G) = \mathsf{D}^*(G)$. Actually for $r \geqslant 5$ and k odd it is known that $\mathsf{D}(G) > \mathsf{D}^*(G)$, and a lower bound for the gap between these two constants is given in [8], (see also [3, 7]). In [10], W. Schmid also studied the inverse problem of $\mathsf{D}(G)$ for $r = 3$. In this paper, we determine the precise value of $\mathsf{D}(C_2^5 \oplus C_{2k})$ for $k \geqslant 149$.

Theorem 2. *For each $k \geqslant 149$, the Davenport constant of the group $C_2^5 \oplus C_{2k}$ is*

$$\mathsf{D}(C_2^5 \oplus C_{2k}) = \begin{cases} 2k + 5 = \mathsf{D}^*(C_2^5 \oplus C_{2k}), & \text{if } k \text{ is even.} \\ 2k + 6 = \mathsf{D}^*(C_2^5 \oplus C_{2k}) + 1, & \text{if } k \text{ is odd.} \end{cases}$$

In [11], the authors mainly research the structure of long minimal zero-sum sequences over $C_2^{r-1} \oplus C_{2k}$ with $k \geqslant \lceil \frac{3r-1}{r+1}(2^r - 1) \rceil - r + 2$ (the condition imposed on k occurs in section 5 of [11]). In this paper, we improve their method and have the same condition imposed on k. Besides, most of the proofs that follow require k to be relatively large as compared to r: the modest $k \geqslant 2r^2$ suffices for the purpose. Fix

$$k_0 = \max\{2r^2, \lceil \frac{3r-1}{r+1}(2^r - 1) \rceil - r + 2\}$$

and let $k \geqslant k_0$. To prove Theorem 2, we need the following result which is of general interest for the study of Davenport's constant of groups of the form $C_2^{r-1} \oplus C_{2k}$.

Theorem 3. *Let $G = C_2^{r-1} \oplus C_{2k}$ with $k \geqslant k_0$ and let α be a minimal zero-sum sequence of length $\mathsf{D}(G)$. If $\mathsf{D}(G) > \mathsf{D}^*(G)$ and there exists a unit block $U | \alpha$ with $d(U) \geqslant r - 3$, then k is odd.*

Remark: For *a unit block and* $d(U)$ in Theorem 3, one can see Definition 7 and the definition of Defect in section 2, respectivrly.

For determining the precise value of $\mathsf{D}(C_2^5 \oplus C_{2k})$, we suppose $\mathsf{D}(G) > \mathsf{D}^*(G)$ and let α be a minimal zero-sum sequence of length $\mathsf{D}(G)$ over G, where $G = \mathsf{D}(C_2^{r-1} \oplus C_{2k})$ with $r \geqslant 6$. In section 2, we improve Chen's result "$2 \leqslant d(W_{\mathscr{F}}) \leqslant r-2$" to "$3 \leqslant d(W_{\mathscr{F}}) \leqslant r-2$". In section 3, we prove that if $d(W_{\mathscr{F}}) = r-2$ or $r-3$, then k is odd, i.e., Theorem 3. Besides, we completely characterize the structure of α with $d(W_{\mathscr{F}}) = r-2$ or $r-3$. In section 4, let $r = 6$, and then we have $r-3 = 3 \leqslant d(W_{\mathscr{F}}) \leqslant r-2$, i.e., k is odd by Theorem 3. Hence, we have $\mathsf{D}(C_2^5 \oplus C_{2k}) = \mathsf{D}^*(C_2^5 \oplus C_{2k})$ for k even. By the structure of α with $d(W_{\mathscr{F}}) = r-2$ or $r-3$, we can easily prove that $\mathsf{D}(C_2^5 \oplus C_{2k}) \leqslant \mathsf{D}^*(C_2^5 \oplus C_{2k}) + 1$ for k odd. It has been known that $\mathsf{D}(C_2^{r-1} \oplus C_{2k}) \geqslant \mathsf{D}^*(C_2^{r-1} \oplus C_{2k}) + 1$ for k odd and $r \geqslant 5$. The proof is complete.

2 Preliminaries

Let

$$\alpha = g_1 \cdot \dots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(\alpha)}$$

be a sequence over G. Denote by $\mathrm{Supp}(\alpha) = \{g : \mathsf{v}_g(\alpha) \geqslant 1\}$. The sum and the sumset of a sequence α are denoted by $\sigma(\alpha)$ and $\sum(\alpha)$ respectively. For a subsequence β of α we say that α is divisible by β or β divides α, and write $\beta|\alpha$. The complementary subsequence of β is denoted by $\alpha\beta^{-1}$. For subsequences β, γ of α, if their union $\beta\gamma$ is still a subsequence of α, then we say that β, γ are disjoint subsequences of α, and call $\beta\gamma$ the product of β, γ.

Let a sequence α be the product of its disjoint subsequences $\alpha_1, \dots, \alpha_m$. We say that the α_i's form a decomposition of α with factors $\alpha_1, \dots, \alpha_m$ and write $\alpha = \prod_{i=1}^m \alpha_i$. Quite often we study the sequence with terms $\sigma(\alpha_1), \dots, \sigma(\alpha_m)$. For convenience of speech it is also said to be a decomposition of α with factors $\alpha_1, \dots, \alpha_m$; sometimes we call terms $\alpha_1, \dots, \alpha_m$ themselves.

Let H be a subgroup of G. Each sequence over G with sum in H is called an H-block. For a sequence that is an H-block, an H-decomposition of the sequence is a decomposition whose factors are H-blocks. An H-block is minimal if its projection onto the factor group G/H under the natural homomorphism is a minimal zero-sum sequence. An H-decomposition whose factors are minimal H-blocks is called an H-factorization.

Let $G = C_2^{r-1} \oplus C_{2k}$ and $a \in G$ be an element of order $2k$. We consider the subgroup $\langle a \rangle$ of G. For convenience, "$\langle a \rangle$-block", "$\langle a \rangle$-decomposition" and "$\langle a \rangle$-factorization" are usually abbreviated to "block", "decomposition", "factorization". However *decomposition* also keeps its general meaning, a partition of a sequence into arbitrary disjoint subsequences. The context excludes ambiguity. Denote by $\bar{t}$ the coset $t + \langle a \rangle$, and $u \sim v$ if $\overline{u} = \overline{v}$. For a sequence $\gamma = \prod t_i$ over G, denote by $\overline{\gamma}$ the sequence $\prod \bar{t}_i$ over $G/\langle a \rangle$, and $\langle \overline{\gamma} \rangle$ the subgroup of $G/\langle a \rangle$ generated by all terms $\overline{\gamma}$. *For any $\langle a \rangle$-block B, there exists a unique $x \in [1, 2k]$ such that $\sigma(B) = xa$. Write $x_a(B) := x$.* Let α be a minimal zero-sum sequence and $\alpha = \prod_{i=1}^n B_i$ be a $\langle a \rangle$-decomposition of α. We call $\{a\}$ a *basis* of $\prod_{i=1}^n B_i$ if $\sum_{i=1}^n x_a(B_i) = 2k$. We have the following important proposition.

Proposition 4 ([11], Proposition 4.1)**.** *Let $G = C_2^{r-1} \oplus C_{2k}$ where $r \geqslant 2$, and let α be a minimal zero-sum sequence over G with length $|\alpha| \geqslant k + \lceil \frac{3r-1}{r+1}(2^r - 1) \rceil + 1$. There exists an order-$2k$ term a of α with the following properties:*

(i) *Every $\langle a \rangle$-decomposition of α has basis $\{a\}$.*

(ii) *If $B|\alpha$ is a minimal $\langle a \rangle$-block, then $0 < x_a(B) < k$.*

(iii) *If $B|\alpha$ is a $\langle a \rangle$-block and $B = B_1 \cdots B_m$ is a $\langle a \rangle$-decomposition of B, then $x_a(B) = x_a(B_1) + \cdots + x_a(B_m)$.*

(iv) *If $\alpha = B_1 \cdots B_m$ is a $\langle a \rangle$-factorization of α, then $x_a(\alpha) = 2k = x_a(B_1) + \cdots + x_a(B_m)$ with each $x_a(B_i) \in (0, k)$.*

(v) *Every $\langle a \rangle$-block $B|\alpha$ with $x_a(B) = 1$ is minimal.*

For the rest of the paper, we let α be a minimal zero-sum sequence of maximal length in $C_2^{r-1} \oplus C_{2k}$. Obviously α has Proposition 4. It follows since by $k \geqslant k_0 = \max\{2r^2, \lceil \frac{3r-1}{r+1}(2^r - 1) \rceil - r + 2\} \geqslant \lceil \frac{3r-1}{r+1}(2^r - 1) \rceil - r + 2$

$$|\alpha| = \mathsf{D}(G) \geqslant \mathsf{D}^*(G) = 2k + r - 1 \geqslant k + \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil + 1.$$

Fix an order $2k$ term a of α as Proposition 4 predicted. Recall two unconventional notations from [11].

The DEFECT. For every $\langle a \rangle$-block $B|\alpha$, define $d(B) = |B| - x_a(B)$ and call $d(B)$ the defect of B. As indicated in Proposition 4, the defect is additive: for each $\langle a \rangle$-decomposition $B = \prod_{i=1}^m B_i$ of B one has $d(B) = \sum_{i=1}^m d(B_i)$. In particular the entire α is an $\langle a \rangle$-block with defect $d(\alpha) = |\alpha| - x_a(\alpha) = |\alpha| - 2k$ and $|\alpha| = 2k + d(\alpha)$.

The δ-QUANTITY. Let $B|\alpha$ be a $\langle a \rangle$-block and $X|B$ a proper subsequence. Then $X' = BX^{-1}$ is also proper; sometimes we say that $B = XX'$ is a proper decomposition of B. As $\sigma(X)$ and $\sigma(X')$ are in the same $\langle a \rangle$-coset, they differ by a multiple of a. Hence there is a unique integer $\delta_B(X) \in [0, k]$ such that $\sigma(X') = \sigma(X) + \delta_B(X)a$ or $\sigma(X) = \sigma(X') + \delta_B(X)a$. This $\delta_B(X)$ is called δ-quantity of $B = XX'$, and is denoted by $\delta(X)$ for short.

If, e.g., $\sigma(X') = \sigma(X) + \delta(X)a$, then $\sigma(X) + \sigma(X') = x_a(B)a$ leads to the relations $2\sigma(X') = (x_a(B) + \delta(X))a$ and $2\sigma(X) = (x_a(B) - \delta(X))a$. As $2\sigma(X) \in 2G$ and $2a$ generates $2G$, we see that $\delta(X)$ and $x_a(B)$ are of the same parity. It follows that there is an element e in the $\langle a \rangle$-coset $\overline{\sigma(X)}$ such that $2e = 0$ and

$$\{\sigma(X), \sigma(X')\} = \Big\{e + \frac{1}{2}(x_a(B) - \delta(X))a, e + \frac{1}{2}(x_a(B) + \delta(X))a\Big\}.$$

Define the lower member X^* of the decomposition $B = XX'$ (of the pair X, X'). Namely let $X^* := X$ or X' according as $\sigma(X) = e + \frac{1}{2}(x_a(B) - \delta(X))a$ or $\sigma(X') = e + \frac{1}{2}(x_a(B) - \delta(X))a$. Thus $\sigma(X^*) = e + \frac{1}{2}(x_a(B_i) - \delta(X_i))a$. Note that if $\delta(X) = 0$, then either one of X and X' can be taken as X^*.

For the two notions, we have the following frequently-used results.

Lemma 5 ([11], Corollary 5.3)**.** *Every $\langle a \rangle$-block in α has nonnegative defect.*

Lemma 6 ([11], Lemma 4.2)**.** *Let* $B_1, \dots, B_m$ *be disjoint blocks in* α *with* $x_a(B_1) + \cdots + x_a(B_m) < k$*, and let* $B_i = X_i X_i'$ *be proper decompositions,* $i = 1, \dots, m$*, such that* $\sum_{i=1}^m \overline{\sigma(X_i)} = \overline{0}$*. Then*

(i) *The product of the lower members* $X_1^*, \dots, X_m^*$ *is a block dividing* $B_1 \cdots B_m$ *with* a*-coordinate*

$$x_a(X_1^* \cdots X_m^*) = \frac{1}{2}\Big(\sum_{i=1}^m x_a(B_i) - \sum_{i=1}^m \delta(X_i)\Big).$$

In addition $\sum_{i=1}^m \delta(X_i) \leqslant \sum_{i=1}^m x_a(B_i) - 2$*.*

(ii) *For each* $i = 1, \dots, m$ *there exists an element* $e_i \in \overline{\sigma(X_i)}$ *such that* $2e_i = 0$*,*

$$\{\sigma(X_i), \sigma(X_i')\} = \big\{e_i + \frac{1}{2}(x_a(B_i) - \delta(X_i))a, e_i + \frac{1}{2}(x_a(B_i) + \delta(X_i))a\big\},$$

and $e_1, \dots, e_m$ *satisfy* $e_1 + \cdots + e_m = 0$*.*

Definition 7. *An* (ℓ, s)*-block means a minimal* $\langle a\rangle$*-block* B *with length* ℓ *and sum* sa *with* $\ell > s$*. That is* $d(B) > 0$ *is assumed. Obviously,* $\ell \leqslant r$*. The phrase "B is an* (ℓ, s)*-block" is shortened to "B is* (ℓ, s)*" whenever convenient. We write* $(*, s)$*-block or* $(\ell, *)$*-block if* ℓ *or* s *is irrelevant. Furthermore a unit block is a product of* $(*, 1)$*-blocks.*

We have the following corollary from Lemma 6.

Corollary 8. *Let* $U \mid \alpha$ *be a unit block, and* $B_1, \dots, B_m$ *be disjoint minimal blocks in* αU^{-1} *with positive defect such that* $x_a(U) + \sum_{i=1}^m x_a(B_i) < k$*. If there exist a decomposition* $U = YY'$ *and proper decompositions* $B_i = X_i X_i' (1 \leqslant i \leqslant m)$ *such that* $Y X_1 \cdots X_m$ *is an* $\langle a\rangle$*-block, then* $\sum_{i=1}^m \delta(X_i) \leqslant \sum_{i=1}^m x_a(B_i) - 2$*.*

Proof. If $U = YY'$ is not proper, then Lemma 6 (i) completes our proof. Now suppose that $U = YY'$ is a proper decomposition. Let $U = U_1 \cdots U_n$ be a decomposition of U such that U_i is $(*, 1)$, and let $Y = Y_1 \cdots Y_n$ be a decomposition of Y such that $Y_i \mid U_i$. Let U' be the product of the U_i's such that Y_i is neither empty nor equal to U_i. Without loss of generality, suppose $U' = U_1 \cdots U_{n'}$ for some $n' \leqslant n$. Then $\delta(Y_i) \geqslant 1$ for $1 \leqslant i \leqslant n'$ since $\delta(Y_i)$ shares the same parity with $x_a(U_i)$. By Lemma 6 (i), we deduce that

$$n' + \sum_{i=1}^m \delta(X_i) \leqslant \sum_{i=1}^{n'} \delta(Y_i) + \sum_{i=1}^m \delta(X_i) \leqslant n' + \sum_{i=1}^m x_a(B_i) - 2.$$

That is $\sum_{i=1}^m \delta(X_i) \leqslant \sum_{i=1}^m x_a(B_i) - 2$. □

For circumstances it is convenient to introduce the following notation. For any sequence X, there exist an element $e \in \overline{\sigma(X)}$ of order 2 and a unique integer in $(-\frac{k-1}{2}, \frac{k+1}{2}]$, denoted by $x_a'(X)$, such that $\sigma(X) = e + x_a'(X)a$. In particular $x_a'(b)$ is defined for $b \in G$ by treating b as a sequence of length one. Note that $x_a'(B)$ may not coincide with $x_a(B)$ if B is a block. However $x_a(B) \equiv x_a'(B) \equiv \sum_{b|B} x_a'(b) \mod k$. In particular, if $B = T_1 \cdots T_\ell$ is a decomposition with each $x'_a(T_i) \in [0, \frac{k+1}{2}]$ and $\sum_{i=1}^{\ell} x'_a(T_i) \leqslant \frac{k+1}{2}$, then it is easy to see that $x_a(B) = \sum_{i=1}^{\ell} x'_a(T_i)$, which will be used repeatedly in this paper.

For minimal blocks $B_1, \ldots, B_m$ of α and proper decompositions $B_i = X_i X'_i$ satisfying the hypothesis of Lemma 6 or Corollary 8, by $\sum_{i=1}^{m} \delta(X_i) \leqslant \sum_{i=1}^{m} x_a(B_i) - 2$ and $x_a(B_i) \leqslant |B_i| \leqslant r$ we obtain

$$\begin{aligned}\frac{(1-r)(m-1)}{2} + 1 \leqslant &\frac{1}{2}\sum_{i\neq j}(\delta(X_i) - x_a(B_i)) + 1 \leqslant \frac{1}{2}(x_a(B_j) - \delta(X_j)) \\ \leqslant &\frac{1}{2}(x_a(B_j) + \delta(X_j)) \leqslant x_a(B_j) - 1 + \frac{1}{2}\sum_{i\neq j}(x_a(B_i) - \delta(X_i)) \\ \leqslant &r - 2 + \frac{(r-1)(m-1)}{2}.\end{aligned}$$

When m is small such that $r - 2 + \frac{(r-1)(m-1)}{2} \leqslant \frac{k+1}{2}$, then

$$\frac{1}{2}\sum_{i\neq j}(\delta(X_i) - x_a(B_i)) + 1 \leqslant x'_a(X_j) \leqslant x_a(B_j) - 1 + \frac{1}{2}\sum_{i\neq j}(x_a(B_i) - \delta(X_i)). \qquad (1)$$

In particular if $m = 1$, we get $1 \leqslant x'_a(X) \leqslant r - 2$. This bound will be frequently used in the next section.

The following lemma together with Proposition 4 (v) ensure that there are $(*, 1)$-blocks dividing α, hence there exist unit blocks dividing α. Actually, we may get that every term of α which is not an element of $\langle a\rangle$ is contained in a $(*, 1)$-block.

Lemma 9 ([11], Lemma 5.1). *Let G be a finite abelian group and α a minimal zero-sum sequence of maximum length over G. For each term $t|\alpha$ and each element $g \in G$ there is a subsequence of α that contains t and has sum g. In particular $\sum(\alpha) = G$.*

Note that if $\sum(\alpha) = G$, then $\langle\alpha\rangle = G$. Some results concerning unit blocks dividing α are given below.

Lemma 10 ([11], Lemma 4.8). *For each unit block $U|\alpha$, the subgroup $\langle\overline{U}\rangle$ of $G/\langle a\rangle$ has rank $d(U)$. Consequently $d(U) \leqslant r - 1$.*

Lemma 11 ([11], Lemma 4.11). (i) *Let U be a $(l, 1)$-block and B be a $(m, 2)$-block in α. If U, B are disjoint blocks such that $\bar{u} \in \langle\overline{B}\rangle$ for every term $u|U$, then the product UB is divisible by a $(*, 1)$-block V with $d(V) > d(U)$. Moreover if $m \geqslant 5$, then $d(V) > d(U)$ can be strengthened to $d(V) > d(U) + 1$.*

(ii) *Let U be a $(l, 1)$-block and B be a $(m, 3)$-block in α. If U, B are disjoint blocks such that $\bar{u} \in \langle\overline{B}\rangle$ for every term $u|U$, and UB is not divisible by a unit block V with $d(V) > d(U)$, then $l = 2$ and UB is divisible by a $(m, 2)$-block.*

Lemma 12 ([11], Corollary 4.12). *Suppose that G has rank $r \geqslant 5$. Let U_1, U_2 be both $(2, 1)$-blocks and B be a $(r, 3)$-block such that U_1, U_2, B are disjoint in α. Then the product U_1U_2B is divisible by a unit block V with $d(V) > d(U_1U_2)$.*

Fix the notation $W_{\mathscr{F}}$ for the product of all $(*,1)$-blocks in a factorization $\mathscr{F}$ of α. Let $d^*(\alpha) = \max\{d(W_{\mathscr{F}}) : \mathscr{F}$ is a factorization of $\alpha\}$.

Definition 13. A factorization $\mathscr{F}$ of α is *canonical* if $d(W_{\mathscr{F}}) = d^*(\alpha)$.

Lemma 14 ([11])**.** *Let $\mathscr{F}$ be a canonical factorization of α. Then*

(i) *The complementary block $\alpha W_{\mathscr{F}}^{-1}$ of $W_{\mathscr{F}}$ is not divisible by a unit block. More generally let $B_1, \ldots, B_m$ be blocks in $\mathscr{F}$, and let d be the combined defect of the $(*,1)$-blocks among them. Then the product $B_1 \cdots B_m$ is not divisible by a unit block V with defect $d(V) > d$.*

(ii) $2 \leqslant d(W_{\mathscr{F}}) \leqslant r-2$ *and* $d(\alpha W_{\mathscr{F}}^{-1}) \geqslant 2$.

We can strengthen Lemma 14 (ii) if $r \geqslant 4$ and $\mathsf{D}(G) > \mathsf{D}^*(G)$.

Lemma 15. *Let $r \geqslant 4$ and $\mathsf{D}(G) > \mathsf{D}^*(G)$. If α is a longest minimal zero-sum sequence over G and $\mathscr{F}$ is a canonical factorization of α, then $d(W_{\mathscr{F}}) \geqslant 3$.*

Proof. Suppose to the contrary $d(W_{\mathscr{F}}) < 3$. Then $d(W_{\mathscr{F}}) = 2$ by Lemma 14 (ii). It follows that every term of α which is not an element of $\langle a \rangle$ is a term of a $(2,1)$ or $(3,1)$-block dividing α. We show first that there is a $(3,1)$-block dividing α. Let $\mathscr{F}$ be a canonical factorization of α. Then either $W_{\mathscr{F}}$ is $(3,1)$ or $W_{\mathscr{F}} = UV$ where U, V are $(2,1)$-blocks. If $W_{\mathscr{F}}$ is $(3,1)$, then we are done. For the latter case, there exist $\langle a \rangle$-cosets $g_1 + \langle a \rangle$ and $g_2 + \langle a \rangle$ such that all terms of U and V are contained in $g_1 + \langle a \rangle$ and $g_2 + \langle a \rangle$ respectively. Then for any term g of α with $g \notin \langle g_1, g_2, a \rangle$, there is a $(*,1)$-block U' containing g. Obviously U' is not a $(2,1)$-block, or else UV and U' are disjoint and $d(UVU') = 3 > 2$ which contradicts $d(W_{\mathscr{F}}) = 2$. Hence U' is a $(3,1)$-block. This proves the existence of a $(3,1)$-block.

Now let $U = u_1 u_2 u_2'$ be a $(3,1)$-block dividing α. Since $r \geqslant 4$ and $\sum(\alpha) = G$, there is a term u_3 of α with $u_3 \notin \langle u_1, u_2, a \rangle$. Let $U_1 \mid \alpha$ be a $(*,1)$-block containing u_3. Then U_1 is $(3,1)$, or else U_1 is $(2,1)$ implying that U and U_1 are disjoint and $d(UU_1) > 2$, a contradiction. Obviously U_1 and U can not be disjoint. Then we must have $|\gcd(U, U_1)| = 1$. Without loss of generality, suppose $\gcd(U, U_1) = u_1$. Write $U_1 = u_1 u_3 u_3'$. Similarly, there exists $u_4 \notin \langle u_1, u_2, u_3, a \rangle$ such that a $(3,1)$-block U_2 contains u_4 and $|\gcd(U_2, U)| = |\gcd(U_2, U_1)| = 1$. It follows that $\gcd(U_2, U) = u_1$, since otherwise $\gcd(U_2, U_1)$ is empty. Continue this process we will find $u_1, \ldots, u_{r-1}$ such that $u_i \notin \langle u_1, \ldots, u_{i-1}, a \rangle$ for all $2 \leqslant i \leqslant r-1$ and $u_1 u_i u_i'$ are $(3,1)$-blocks. Additionally we derive that $\mathsf{v}_{u_1}(\alpha) = 1$, and for any term $u \notin \langle u_1, a \rangle$, u can not be a term of a $(2,1)$-block, instead there is a $(3,1)$-block $u_1 u u'$ dividing α.

If there are two terms g_1 and g_2 belonging to the same $\langle a \rangle$-coset other than $\langle a \rangle$ and $u_1 + \langle a \rangle$, then $g_1 g_2$ is an $\langle a \rangle$-block and $g_1 g_2$ is not $(2,1)$, hence $x_a(g_1 g_2) = 2$. In particular, if $g \notin \langle u_1, a \rangle$ and $\mathsf{v}_g(\alpha) \geqslant 2$, then $g + g = 2a$, and hence $x_a'(g) = 1$.

Consider the following decomposition of α:

$$\alpha = S_0 \cdot S_1 \cdot S_2 \cdot S_2' \cdot S_3 \tag{2}$$

where S_0 consists of terms of α that are elements of $\langle a\rangle$, S_1 consists of terms of α that are elements of $u_1+\langle a\rangle$, $S_2=\prod_{i=2}^{r-1}u_i$, $S_2'=\prod_{i=2}^{r-1}u_i'$ and $S_3=\alpha(S_0S_1S_2S_2')^{-1}$.

For a term g of S_0, we have $g=a$ according to Lemma 5. So $\sigma(S_0)=|S_0|a$. Write $u_i=e_i+x_a'(u_i)a$ with $2e_i=0$ for $1\leqslant i\leqslant r-1$. Then $\sigma(S_2S_2')=|S_2|(e_1+(1-x_a'(u_1))a)$. Since $u_1u_2u_2'$ and $u_1u_3u_3'$ are $(3,1)$-blocks, $V:=u_2u_2'u_3u_3'$ is an minimal $\langle a\rangle$-block with $d(V)\geqslant 0$. We have $2(1-x_a'(u_1))\equiv x_a(V)=2$ or $4 \mod 2k$. So $x_a'(u_1)=0$ or -1 since $x_a'(u_1)\in(-\frac{k-1}{2},\frac{k+1}{2}]$. We distinguish the following two cases to complete our proof:

Case 1: $x_a'(u_1)=0$. Claim that $\mathrm{Supp}(S_3)\subset\mathrm{Supp}(S_2S_2')$. Assume to the contrary there is a term u_0 of S_3 with $u_0\nmid S_2S_2'$. Then there exists a term u_0' of S_3 with $u_0'\nmid S_2S_2'$ such that $u_1u_0u_0'$ is $(3,1)$. It is easy to see either u_0 or u_0' is an element of $\langle u_2,\dots,u_{r-1},a\rangle$. Without loss of generality, suppose $u_0\in\langle u_2,\dots,u_{r-1},a\rangle$. Then there must exist a minimal block $B\mid u_0S_2$ containing u_0. For any $u_i\mid B$ $(0\leqslant i\leqslant r-1)$, $C:=Bu_i^{-1}u_1u_i'$ is also a minimal block with length $|B|+1$ satisfying that $x_a(C)\equiv x_a(B)+1-2x_a'(u_i) \mod 2k$. So by $1\leqslant x_a(C)\leqslant|B|+1\leqslant r$ we derive that $\frac{2-r}{2}\leqslant x_a'(u_i)\leqslant\frac{r-1}{2}$. Replacing $u_i\mid B$ by u_i' for all u_i with $1\leqslant x_a'(u_i)\leqslant\frac{r-1}{2}$, we get a new sequence dividing α, which by abuse of notation, is still denoted by B. Then B or Bu_1 is a minimal block. Noting that $-\frac{r-3}{2}\leqslant x_a'(u_i')\leqslant 0$ if $1\leqslant x_a'(u_i)\leqslant\frac{r-1}{2}$, we have $\frac{2-r}{2}\leqslant x_a'(b)\leqslant 0$ for each $b\mid B$. Thus

$$0\geqslant\sum_{b\mid B}x_a'(b)\geqslant|B|\cdot\frac{2-r}{2}\geqslant\frac{(2-r)(r-1)}{2}>-\frac{k-1}{2}.$$

If B is a minimal block, it follows from $x_a(B)\equiv\sum_{b\mid B}x_a'(B) \mod k$ and Proposition 4 (ii) that $x_a(B)>\frac{k}{2}>r>|B|$, which contradicts Lemma 5. By the same argument we derive $x_a(Bu_1)>\frac{k}{2}>r\geqslant|Bu_1|$ if Bu_1 is a minimal block, which also contradicts Lemma 5. Thus the claim is true. So for every $u_0\mid S_3$, $\mathsf{v}_{u_0}(\alpha)\geqslant 2$, and hence $u_0\in a+\langle e_1,\dots,e_{r-1}\rangle$. We then derive that $\sigma(S_3)\in|S_3|a+\langle e_1,\dots,e_{r-1}\rangle$.

If $|S_1|=1$, i.e., $S_1=u_1=e_1$, then $0=\sigma(\alpha)\in(|S_0|+|S_2|+|S_3|)a++\langle e_1,\dots,e_{r-1}\rangle$, which implies $|S_0|+|S_2|+|S_3|=2k$. From $|\alpha|=|S_0|+|S_1|+2|S_2|+|S_3|>\mathsf{D}^*(G)$, it follows that $|\alpha|=1+|S_2|+2k=r-1+2k>\mathsf{D}^*(G)$, a contradiction.

If $|S_1|\geqslant 2$, then for any e_1+xa contained in $S_1\cdot e_1^{-1}$, $(e_1+xa)\cdot e_1$ is a minimal $\langle a\rangle$-block, so $x=1$ or 2 from Lemma 5.

If $e_1+2a\mid S_1$, then $|S_1|=2$, since otherwise there is a minimal block $(e_1+2a)(e_1+xa)\mid S_1$ with $x=1$ or 2, which contradicts Lemma 5. Thus $\sigma(S_1)=|S_1|a$. So $2k=|S_0|+|S_1|+|S_2|+|S_3|$. From $|\alpha|=|S_0|+|S_1|+2|S_2|+|S_3|>\mathsf{D}^*(G)$, it follows that $|\alpha|=|S_2|+2k=r-2+2k>\mathsf{D}^*(G)$, a contradiction.

If $x=1$ for any e_1+xa contained in $S_1\setminus\{e_1\}$, then $2k=|S_0|+|S_1|-1+|S_2|+|S_3|$. From $|\alpha|=|S_0|+|S_1|+2|S_2|+|S_3|>\mathsf{D}^*(G)$, it follows that $|\alpha|=|S_2|+2k+1=r-1+2k>\mathsf{D}^*(G)$, a contradiction. This finishes the proof for case 1.

Case 2: $x_a'(u_1)=-1$. Let u_0 be a term of S_3. If $\mathsf{v}_{u_0}(\alpha)=1$, there exists $u_0'\mid S_3$ such that $u_1u_0u_0'$ is $(3,1)$, so $u_0+u_0'=e_1+2a$. If $\mathsf{v}_{u_0}(\alpha)\geqslant 2$, by $u_0\notin\langle u_1,a\rangle$ we have $u_0\in a+\langle e_1,\dots,e_{r-1}\rangle$. Let S_3' be products of pairs $ss'\mid S_3$ such that u_1ss' is $(3,1)$ and at least one of s and s' is of multiplicity one, and let $S_3'' = S_3 S_3'^{-1}$. Then

$$\sigma(S_3) = \sigma(S_3') + \sigma(S_3'') \in |S_3'|a + |S_3''|a + \langle e_1, \dots, e_{r-1}\rangle = |S_3|a + \langle e_1, \dots, e_{r-1}\rangle.$$

If $|S_1| = 1$, then $2k = |S_0| - |S_1| + 2|S_2| + |S_3|$. From $|\alpha| = |S_0| + |S_1| + 2|S_2| + |S_3| > \mathsf{D}^*(G)$, it follows that $|\alpha| = 2k + 2 > \mathsf{D}^*(G)$, which is impossible.

If $|S_1| \geqslant 2$, then from Lemma 5 it follows that for any $e_1 + xa$ contained in $S_1(e_1 - a)^{-1}$, $(e_1 + xa)(e_1 - a)$ is a $\langle a\rangle$-block with $x - 1 = 1$ or 2, i.e., $x = 2$ or 3. It follows that $|S_1| = 2$, since otherwise there is a minimal block of the form $(e_1 + xa)(e_1 + ya)$ contained in $S_1(e_1 - a)^{-1}$ with negative defect. So $\sigma(S_1) = a$ or $2a$. It yields that $2k = |S_0| + 2|S_2| + |S_3| + x - 1$ with $x = 2$ or 3. From $|\alpha| = 2 + |S_0| + 2|S_2| + |S_3| > \mathsf{D}^*(G)$, it follows that $|\alpha| = 2 + 2k + 1 - x > \mathsf{D}^*(G)$, which is impossible. This ends the proof of Case 2 and proves the lemma. □

Lemma 16. *Let $\mathscr{F}$ be a canonical decomposition of α and $r \geqslant 6$.*

(i) *$\mathscr{F}$ does not contains a $(r, 3)$-block.*

(ii) *If U is a $(l, 1)$-block and B is a $(r - t, 2)$-block in $\mathscr{F}$ such that U, B are disjoint and $|\langle \overline{U}\rangle \cap \langle \overline{B}\rangle| > 1$, then $\lceil \frac{r-t}{2} \rceil \leqslant t + 1$.*

Proof. (i) Suppose to the contrary that $\mathscr{F}$ contains a $(r, 3)$-block B, and let $U | W_{\mathscr{F}}$ be a $(*, 1)$-block. By Lemma 11 (ii), $W_{\mathscr{F}}$ contains only $(2, 1)$-blocks. Note that there are at least two of them by Lemma 14 (ii). Let U_1 and U_2 be such blocks. Lemma 12 states that the product $U_1 U_2 B$ is divisible by a unit block V with $d(V) > d(U_1 U_2)$, which yields a contradiction. So $\mathscr{F}$ does not contains a $(r, 3)$-block.

(ii) Suppose $\lceil \frac{r-t}{2} \rceil > t + 1$. Since $\langle \overline{B}\rangle$ is a subgroup of $G/\langle a\rangle$ with index $\frac{2^{r-1}}{2^{r-t-1}} = 2^t$ and $|\langle \overline{U}\rangle \cap \langle \overline{B}\rangle| > 1$, there exists a proper decomposition $U = X_1 \cdots X_v$ with $\sigma(\overline{X_i}) \in \langle \overline{B}\rangle$ and $|X_i| \leqslant t + 1$. By $x_a(B) = 2$, Lemma 6 implies $\delta(X_i) = 1$ and $\sigma(X_i) \in \{e_i, e_i + a\}$ for $1 \leqslant i \leqslant v$, where $e_i \in \sigma(\overline{X_i})$ is of order two. Since $x_a(U) = 1$, there is at least one X_i, say X_1, such that $\sigma(X_1) = e_1 + a$ and $\sigma(U X_1^{-1}) = e_1$, or else multiplying $\sum_{i=1}^{v} \sigma(X_i) = a$ by 2 yields the impossible $2a = 0$. Consider the proper decompositions $U = X_1(U X_1^{-1})$ and $B = Y Y'$, where $\sigma(X_1) \sim \sigma(Y)$ and $Y' = B Y^{-1}$. Lemma 6 implies that $\delta(Y) = 0$ and $\sigma(Y) = \sigma(Y') = e_1 + a$. By symmetry let $|Y| \geqslant |Y'|$. We have that $V = (U X_1^{-1}) Y$ is a block with sum $e_1 + (e_1 + a) = a$ and length $\ell' = \ell - |X_1| + |Y|$. Note that $\ell' > 1$ since $|X_1| < \ell$, so V is an $(\ell', 1)$-block dividing UB. Since $\lceil \frac{r-t}{2} \rceil > t + 1$, $|X_1| \leqslant t + 1$ and $|Y| \geqslant \lceil \frac{r-t}{2} \rceil$, we have $\ell' \geqslant \ell - (t + 1) + \lceil \frac{r-t}{2} \rceil \geqslant \ell + 1$. So $d(V) \geqslant \ell > d(U)$, a contradiction. □

3 Proof of Theorem 3

In this section we mainly prove Theorem 3. The following lemma is a key ingredient.

Lemma 17. *Let $a_1 a_1'$ be a subsequence of α such that $x_a'(a_1 a_1') = 1$. If there exists a subsequence T in $\alpha(a_1 a_1')^{-1}$ such that $x_a'(a_1 T) = x_a'(a_1' T)$, then k is odd and $x_a'(a_1) = x_a'(a_1') = \frac{k+1}{2}$.*

Furthermore,

(i) *let* $T_1 = a_1a_2$ *and* $T_2 = b_1b_2b_3$ *be two disjoint subsequences of* α *such that* $x'_a(a_1a_2) = x'_a(b_1b_2b_3) = 1$. *If* $1 \leqslant x'_a(a_ib_2b_3), x'_a(a_ib_1) \leqslant \frac{k+1}{2}$ *for* $i = 1, 2$, *then* k *is odd and* $x'_a(a_1) = x'_a(a_2) = \frac{k+1}{2}$.

(ii) *let* $T_1, \ldots, T_\ell$ *be* ℓ *disjoint subsequences of* α *of length* 2 *such that* $x'_a(T_1) = \cdots = x'_a(T_\ell) = 1$. *If* $\ell = 2$ *or* 3 *and* $1 \leqslant x'_a(t_1 \cdots t_\ell) \leqslant \frac{k+1}{2}$ *for any* $t_i \mid T_i$ $(1 \leqslant i \leqslant \ell)$, *then* k *is odd. In particular, if* $\ell = 2$, *then* $x'_a(t) = \frac{k+1}{2}$ *for any* $t|T_1T_2$.

Proof. Since $x'_a(a_1T) = x'_a(a'_1T)$, we have that $x'_a(a_1) + x'_a(T) \equiv x'_a(a'_1) + x'_a(T) \pmod{k}$, i.e., $x'_a(a_1) \equiv x'_a(a'_1) \pmod{k}$. It follows $x'_a(a_1a'_1) = 1 \equiv x'_a(a_1) + x'_a(a'_1) \equiv 2x'_a(a_1) \pmod{k}$. This implies that $x'_a(a_1) = x'_a(a'_1) = \frac{k+1}{2}$ and k is odd. We complete the proof of the first assertion.

(i) Since $x'_a(a_1a_2) = x'_a(b_1b_2b_3) = 1$ and $1 \leqslant x'_a(a_ib_2b_3), x'_a(a_ib_1) \leqslant \frac{k+1}{2}$ for $i = 1, 2$, we have $x'_a(a_1a_2b_1b_2b_3) = x'_a(a_1a_2) + x'_a(b_1b_2b_3) = 2 = x'_a(a_1b_2b_3) + x'_a(a_2b_1) = x'_a(a_2b_2b_3) + x'_a(a_1b_1)$. It follows that $x'_a(a_1b_2b_3) = x'_a(a_2b_2b_3) = 1$. The first assertion completes our proof.

(ii) Set $T_i = t_it'_i$ for $1 \leqslant i \leqslant \ell$. If $\ell = 2$, then by $x'_a(T_1) = x'_a(T_2) = 1$ and $1 \leqslant x'_a(a_1a_2) \leqslant \frac{k+1}{2}$ for any $a_1 \mid T_1, a_2 \mid T_2$, we have $x'_a(tt') = 1$ for any $tt'|T_1T_2$, since otherwise $x'_a(T_1T_2) = x'_a(T_1) + x'_a(T_2) = 2 = x'_a(tt') + x'_a(T_1T_2(tt')^{-1}) > 2$. In particular, $x'_a(t_1t_2) = x'_a(t'_1t_2) = 1$. The first assertion implies that k is odd and $x'_a(t_1) = x'_a(t'_1) = \frac{k+1}{2}$. Similarly, $x'_a(t_2) = x'_a(t'_2) = \frac{k+1}{2}$.

If $\ell = 3$, then by $x'_a(T_1) = x'_a(T_2) = x'_a(T_3) = 1$ and $1 \leqslant x'_a(a_1a_2a_3) \leqslant \frac{k+1}{2}$ for any $a_i \mid T_i$ $(i = 1, 2, 3)$, we have $x'_a(a_1a_2a_3) = 1$ or 2 for any $a_i \mid T_i$, since otherwise $x'_a(T_1T_2T_3) = x'_a(T_1) + x'_a(T_2) + x'_a(T_3) = 3 = x'_a(a_1a_2a_3) + x'_a(T_1T_2T_3(a_1a_2a_3)^{-1}) \geqslant 4$. In addition, it is easy to see that there exist $a_1 \mid T_1, a_2 \mid T_2, a_3 \mid T_3$ such that $x'_a(a_1a_2a_3) = 1$. Without loss of generality, suppose $x'_a(t_1t_2t_3) = 1$. If $x'_a(t_1t_2t_3t'_i(t_i)^{-1}) = 1$ for some $i \in [1, 3]$, the first assertion completes our proof. If $x'_a(t_1t_2t_3t'_i(t_i)^{-1}) = 2$ for all $i \in [1, 3]$, then modular k $x'_a(t_1t_2t_3) + 1 = 2 = x'_a(t_1t_2t_3t'_i(t_i)^{-1}) \equiv x'_a(t_i) + 1 + x'_a(t_1t_2t_3(t_i)^{-1}) \equiv x'_a(t'_i) + x'_a(t_1t_2t_3(t_i)^{-1})$, i.e., $x'_a(t_i) + 1 \equiv x'_a(t'_i)$. It follows that $x'_a(t_it'_i) = 1 \equiv x'_a(t_i) + x'_a(t'_i) \equiv 2x'_a(t_i) + 1 \pmod{k}$, i.e., $x'_a(t_i) = 0$ or $\frac{k}{2}$. This implies $x'_a(t_1t_2t_3) = 1 \equiv x'_a(t_1) + x'_a(t_2) + x'_a(t_3) \equiv 0$ or $\frac{k}{2} \pmod{k}$, which is impossible. This proof is complete. ☐

Lemma 18. *Let* $U \mid \alpha$ *be a unit block, and* $B \mid \alpha U^{-1}$ *be a minimal block with positive defect. Then* $\mathsf{r}(\langle \overline{UB} \rangle) \geqslant \mathsf{r}(\langle \overline{U} \rangle) + 1$.

Proof. If $\mathsf{r}(\langle \overline{UB} \rangle) < \mathsf{r}(\langle \overline{U} \rangle) + 1$, then $\mathsf{r}(\langle \overline{UB} \rangle) = \mathsf{r}(\langle \overline{U} \rangle)$, i.e., $\langle \overline{B} \rangle \subset \langle \overline{U} \rangle$. For any $b \mid B$, there is a proper subsequence $Y \mid U$ such that $Y \boldsymbol{\cdot} b$ is a block. Hence by (1) one deduces $1 \leqslant x'_a(b) \leqslant r - 2$ for all $b \mid B$. It follows that

$$k > (r-2)|B| \geqslant \sum_{b|B} x'_a(b) = x_a(B) \geqslant |B|,$$

a contradiction to $d(B) > 0$. Hence $\mathsf{r}(\langle \overline{UB} \rangle) \geqslant \mathsf{r}(\langle \overline{U} \rangle) + 1$. ☐

Lemma 19. *Let $U|\alpha$ be a unit block, B, C be two disjoint minimal blocks with positive defect in αU^{-1} such that $\langle\overline{C}\rangle \subset \langle\overline{UB}\rangle$. Let B_2 and C_2 be sequences (possibly empty) consisting of terms $b \mid B$ with $\overline{b} \in \langle\overline{U}\rangle$ and $c \mid C$ with $\overline{c} \in \langle\overline{U}\rangle$ respectively. Set $B_1 = BB_2^{-1}$ and $C_1 = CC_2^{-1}$.*

(i) *If $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{U}\rangle) + 1$, then $0 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$ for any $c_1 \mid C_1$. In addition there exists $c_1 \mid C_1$ such that $x'_a(c_1) = 0$, i.e., c_1 is of order 2.*

(ii) *If $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{U}\rangle) + 2$ and there exists some $c_1 \mid C_1$ such that $x'_a(c_1) < 0$, then $\mathsf{r}(\langle\overline{UC}\rangle) = \mathsf{r}(\langle\overline{U}\rangle) + 1$, and*

(a) $C_1 = (e + k_1a)(e' + k_2a)$, where $k_1 + k_2 = 1$ and $e, e' \in \langle\overline{UC}\rangle \setminus \langle\overline{U}\rangle$ satisfying $2e = 2e' = 0$;

(b) $C_2 = (e_1 + a)\cdots(e_{|C_2|} + a)$, where $e_i \in \langle\overline{U}\rangle$ has order 2 for $1 \leqslant i \leqslant |C_2|$;

(c) there does not exist a minimal block D with positive defect in $\alpha(UBC)^{-1}$ such that $\langle\overline{D}\rangle \subset \langle\overline{UB}\rangle$.

Proof. (i) For each term c_2 of C_2, since $\langle\overline{C_2}\rangle \subset \langle\overline{U}\rangle$, there exists a subsequence $Y \mid U$ such that Yc_2 is a block. Then (1) yields $1 \leqslant x'_a(c_2) \leqslant r - 2$. Similarly we have $1 \leqslant x'_a(b_2) \leqslant r - 2$ for $b_2 \mid B_2$.

Since $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{U}\rangle) + 1$, by $\langle\overline{C}\rangle \subset \langle\overline{UB}\rangle$ and Lemma 18 there exists e such that $\langle\overline{UB}\rangle = \langle\overline{UC}\rangle = \langle\overline{U}, e\rangle$. Obviously all terms of $\overline{B}_1$ and $\overline{C}_1$ are elements of $e + \langle\overline{U}\rangle$, and $|B_1|, |C_1| > 0$. For $c_1 \mid C_1$ and $b_1 \mid B_1$, b_1c_1 is a block or there exists proper $Y \mid U$ such that Yb_1c_1 is a block. Applying (1) we derive that

$$\frac{\delta(b_1) - x_a(B)}{2} + 1 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}.$$

Thus we get $x'_a(c_1) \leqslant \frac{3r-5}{2}$. To show $0 \leqslant x'_a(c_1)$, we suppose there exists $c_1 \mid C_1$ such that $x'_a(c_1) \leqslant -1$. Then $\frac{\delta(b_1)-x_a(B)}{2} + 1 \leqslant -1$ for each $b_1 \mid B_1$, which yields

$$2 \leqslant \frac{1}{2}(x_a(B) - \delta(b_1)) \leqslant \frac{1}{2}(x_a(B) + \delta(b_1)) \leqslant x_a(B) - 2.$$

Hence $1 \leqslant x'_a(b_1) \leqslant r - 3$. It follows that

$$2|B_1| + |B_2| \leqslant \sum_{b_1|B_1} x'_a(b_1) + \sum_{b_2|B_2} x'_a(b_2) \leqslant r(r-2) \leqslant k.$$

Consequently $x_a(B) = \sum_{b_1|B_1} x'_a(b_1) + \sum_{b_2|B_2} x'_a(b_2) > |B|$, a contradiction to $d(B) > 0$. Therefore $0 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$ for all $c_1 \mid C_1$. It is left to show there exists c_1 such that $x'_a(c_1) = 0$.

Assume to the contrary that $x'_a(c_1) \geqslant 1$ for all $c_1 \mid C_1$. Then by $1 \leqslant x'_a(c_2) \leqslant r - 2$ for $c_2 \mid C_2$ and $1 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$ for $c_1 \mid C_1$ we get $k > \sum_{c|C} x'_a(c) = x_a(C) \geqslant |C|$, which contradicts $d(C) > 0$. As a result there exists $c_1 \mid C_1$ with $x'_a(C_1) = 0$.

(ii) Since $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{U}\rangle)+2$, there are e_1, e_2 of order 2 such that $\langle\overline{UB}\rangle = \langle\overline{U}, e_1, e_2\rangle$. Let c_1 be a fixed term of C_1 with $x'_a(c_1) < 0$. Without loss of generality, one may suppose

$c_1 \in e_1 + \langle \overline{U} \rangle$. Write $B_1 = A_1A_2A_3$ with $\mathrm{Supp}(\overline{A_1}), \mathrm{Supp}(\overline{A_2})$ and $\mathrm{Supp}(\overline{A_3})$ being subsets of $e_1 + \langle \overline{U} \rangle$, $e_2 + \langle \overline{U} \rangle$ and $e_1 + e_2 + \langle \overline{U} \rangle$ respectively. By symmetry we can suppose $|A_2| \leqslant |A_3|$. Consider the decomposition $B = A_1A_2A_3'A_3''$, where A_3' is any subsequence of A_3 with $|A_3'| = |A_2|$ and $A_3'' = A_3A_3'^{-1}$. It is easy to see that $|A_3''|$ is even.

Take $X = a_1$ with $a_1 \mid A_1$ or $X = a_2a_3$ with $a_2 \mid A_2, a_3 \mid A_3'$. Then there exists a $Y \mid U$ such that XYc_1 is a block. Then (1) and $x_a'(c_1) < 0$ gives us

$$\frac{3-r}{2} \leqslant \frac{\delta(X) - x_a(B)}{2} + 1 \leqslant x_a'(c_1) \leqslant -1. \tag{3}$$

This implies $\delta(X) \leqslant x_a(B) - 4$ and hence

$$2 \leqslant \frac{1}{2}(x_a(B) - \delta(X)) \leqslant \frac{1}{2}(x_a(B) + \delta(X)) \leqslant x_a(B) - 2.$$

It follows that $2 \leqslant x_a'(X) \leqslant x_a(B) - 2$. For any $T \mid A_3''$ of length two, we have $\sigma(\overline{T}) \in \langle \overline{U} \rangle$ and hence there exists a $Y \mid U$ such that YT is a block. One deduces from (1) that $1 \leqslant x_a'(T) \leqslant r - 2$. It is worth mentioning that if there exist two disjoint subsequences of A_3'' of length two, say T_1, T_2, such that $x_a'(T_1) = x_a'(T_2) = 1$, then by Lemma 17 (ii) we have $x_a'(g) = \frac{k+1}{2}$ for any $g|T_1T_2$.

Assume $\mathsf{r}(\langle \overline{UC} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 2$. Then $\langle \overline{UC} \rangle = \langle \overline{UB} \rangle$, and hence $\langle \overline{B} \rangle \subset \langle \overline{UC} \rangle$. For each $b_1 \mid B_1$, there exist $Z \mid C$ and $Y \mid U$ such that ZYb_1 is a block, where Y is empty if Zb_1 is already a block. Then by (1)

$$1 - \frac{r-1}{2} \leqslant \frac{\delta(Z) - x_a(C)}{2} + 1 \leqslant x_a'(b_1) \leqslant \frac{3r-5}{2}. \tag{4}$$

So there is no $b_1 \mid B_1$ with $x_a'(b_1) = \frac{k+1}{2}$, and hence there exists at most one $T \mid A_3''$ satisfying $x_a'(T) = 1$. Write $A_2A_3' = Q_1 \cdots Q_s$ with each Q_i consisting of exactly one term from A_2 and one from A_3'. Let $A_3'' = T_1T_2 \cdots T_t$ be any decomposition of A_3'' with $|T_i| = 2$ for all $1 \leqslant i \leqslant t$. To sum up, we have

$$\begin{aligned} k \geqslant & \sum_{b_2|B_2} x_a'(b_2) + \sum_{a_1|A_1} x_a'(a_1) + \sum_{i=1}^{s} x_a'(Q_i) + \sum_{i=1}^{t} x_a'(T_i) \\ \geqslant & |B_2| + 2|A_1| + 2|A_2| + |A_3''| - 1 = |B| + |A_1| - 1. \end{aligned}$$

It follows that $|B| + |A_1| - 1 \leqslant x_a(B)$. To have $|B| > x_a(B)$, one must have $|A_1| = 0$, one of $T_1, \ldots, T_t$, say T_1, satisfies $x_a'(T_1) = 1$ and others satisfy $x_a'(T_i) = 2$, as well as $x_a'(Q_i) = 2$ for $1 \leqslant i \leqslant s$. By $\mathsf{r}(\langle \overline{UB} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 2$ and $|A_1| = 0$, we have $|A_2|, |A_3| > 0$. Since A_3' is arbitrarily chosen, we get $x_a'(a_2a_3) = 2$ for any $a_2 \mid A_2$ and $a_3 \mid A_3$. It follows that all $x_a'(a_3)$ are equal for $a_3 \mid A_3$. Their common value $x \in (-\frac{k-1}{2}, \frac{k+1}{2}]$ satisfies the congruence $x_a'(T_1) = 1 \equiv 2x \pmod{k}$, i.e., $x = \frac{k+1}{2}$, contradicting (4). Hence $\mathsf{r}(\langle \overline{UC} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 1$.

Recall that $\overline{c_1} \in e_1 + \langle \overline{U} \rangle$. From $\mathsf{r}(\langle \overline{UC} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 1$ we get $\mathrm{Supp}(\overline{C_1}) \subset e_1 + \langle \overline{U} \rangle$, which derives $\sigma(\overline{c_1 c_1'}) \in \langle \overline{U} \rangle$ for all $c_1' \mid C_1$. Then there exists a $Y \mid U$ such that Yc_1c_1' is a block. Consequently $1 \leqslant x_a'(c_1c_1') \leqslant r-2$ by (1). By the same argument used to derive (4), we can obtain $\frac{3-r}{2} < x_a'(c_1') < \frac{3r-5}{2}$, which together with (3) gives us $3-r < x_a'(c_1) + x_a'(c_1') \leqslant \frac{3r-7}{2}$. It implies $x_a'(c_1c_1') = x_a'(c_1) + x_a'(c_1')$ and hence

$$2 \leqslant 1 - x_a'(c_1) \leqslant x_a'(c_1') \leqslant r - 2 - x_a'(c_1) \leqslant \frac{3r-7}{2} < k.$$

It follows that $C_1 = c_1c_1'$ with $x_a'(C_1) = 1$ and $x_a'(c_2) = 1$ for any $c_2 \mid C_2$, since otherwise $k > x_a(C) = \sum_{c \mid C} x_a'(c) \geqslant |C|$, i.e., $d(C) \leqslant 0$.

Assume to the contrary that there exists a minimal block D with positive defect in $\alpha(UBC)^{-1}$ such that $\langle \overline{D} \rangle \subset \langle \overline{UB} \rangle$. Let D_2 be a sequence (possibly empty) consisting of terms $d \mid D$ with $\overline{d} \in \langle \overline{U} \rangle$. Set $D_1 = DD_2^{-1}$. Then (1) yields $1 \leqslant x_a'(d_2) \leqslant r-2$ for $d_2 \mid D_2$. For any $d_1 \mid D_1$, there exists a proper $X \mid B_1$ such that either Xd_1 is a block or XYd_1 is a block for some proper $Y \mid U$. Applying (1) we derive that

$$\frac{\delta(X) - x_a(B)}{2} + 1 \leqslant x_a'(d_1) \leqslant \frac{3r-5}{2}.$$

Obviously, there exists $d_1 \mid D_1$ such that $x_a'(d_1) \leqslant 0$, since otherwise $1 \leqslant x_a'(d_1) \leqslant \frac{3r-5}{2}$ for all $d_1 \mid D_1$, and then $x_a(D) = \sum_{d_1 \mid D_1} x_a'(d_1) + \sum_{d_2 \mid D_2} x_a'(d_2) \geqslant |D|$, a contradiction to $d(D) > 0$. If $\overline{d_1} \in e_1 + \langle \overline{U} \rangle$, by $\mathrm{Supp}(\overline{C_1}) \subset e_1 + \langle \overline{U} \rangle$ we get that c_1d_1 is a block or there exists a proper $Y \mid U$ such that Yc_1d_1 is a block, where $c_1 \mid C_1$ with $x_a'(c_1) < 0$. By $x_a'(d_1) \leqslant 0$ and $x_a'(c_1) < 0$, we have $\delta(d_1) \geqslant x_a(D)$ and $\delta(c_1) > x_a(C)$. It follows from Corollary 8 that $x_a(D) + x_a(C) + 1 \leqslant \delta(d_1) + \delta(c_1) \leqslant x_a(D) + x_a(C) - 2$, a contradiction. If $\overline{d_1} \in e_2 + \langle \overline{U} \rangle$ or $\overline{d_1} \in e_1 + e_2 + \langle \overline{U} \rangle$, then for any $b_1 \mid B_1$, one of $\{\sigma(\overline{b_1c_1}), \sigma(\overline{b_1d_1}), \sigma(\overline{b_1c_1d_1})\}$ is contained in $\langle \overline{U} \rangle$, where $c_1 \mid C_1$ with $x_a'(c_1) < 0$. Then there exists a proper $Y \mid U$ such that one of $\{Yb_1c_1, Yb_1d_1, Yb_1c_1d_1\}$ is a block. By $x_a'(d_1) \leqslant 0$, $x_a'(c_1) < 0$ and Corollary 8, we have $\delta(b_1) \leqslant x_a(B) - 2$. It implies that

$$1 \leqslant \frac{1}{2}(x_a(B) - \delta(b_1)) \leqslant \frac{1}{2}(x_a(B) + \delta(b_1)) \leqslant x_a(B) - 1.$$

Hence, $1 \leqslant x_a'(b_1) \leqslant x_a(B) - 1$ for all $b_1 \mid B_1$. Since $1 \leqslant x_a'(b_2) \leqslant r-2$ for all $b_2 \mid B_2$, we have that $x_a(B) = \sum_{b_1 \mid B_1} x_a'(b_1) + \sum_{b_2 \mid B_2} x_a'(b_2) \geqslant |B|$, a contradiction to $d(B) > 0$. The proof is completed. □

Lemma 20. *Let $U|\alpha$ be a unit block, and write $r_U := \mathsf{r}(\langle \overline{U} \rangle)$. For $1 \leqslant i \leqslant 3$, let B_i be disjoint minimal blocks with positive defect in αU^{-1} such that $\mathsf{r}(\langle \overline{UB_i} \rangle) = r_U + 1$ and*

$$\mathsf{r}(\langle \overline{UB_1B_2B_3} \rangle) = \mathsf{r}(\langle \overline{UB_iB_j} \rangle) = r_U + 2 \text{ for } 1 \leqslant i < j \leqslant 3.$$

Denote by V_i the longest subsequence of B_i with $\mathrm{Supp}(\overline{V_i}) \subset \langle \overline{U} \rangle$, and $V_i' = B_iV_i^{-1}$. Then $1 \leqslant x_a'(v) \leqslant r-2$ for all $v \mid V_i$ and $0 \leqslant x_a'(v) \leqslant 2r-3$ for all $v \mid V_i'$. In particular, there exists some $v \mid V_i'$ with $x_a'(v) = 0$.

Proof. For $v \mid V_i$, there exists a proper subsequence $W \mid U$ such that Wv is a block. Applying (1) to the decompositions $V_i = v \boldsymbol{\cdot} V_i v^{-1}$ and $U = W \boldsymbol{\cdot} (UW^{-1})$ one deduces that $1 \leqslant x'_a(v) \leqslant r-2$.

For $1 \leqslant i \leqslant 3$, let v_i be any term of V'_i. Then $\sigma(\overline{v_1 v_2 v_3}) \in \langle \overline{U} \rangle$. So there exists a subsequence $W \mid U$ such that $Wv_1v_2v_3$ is a block, where W is empty if $v_1v_2v_3$ is a block. Then by (1) we derive

$$3 - r \leqslant \frac{\delta(v_h) - x_a(B_h)}{2} + \frac{\delta(v_j) - x_a(B_j)}{2} + 1 \leqslant x'_a(v_i) \leqslant 2r - 3 < \frac{k+1}{2}, \qquad (5)$$

where $1 \leqslant h, i, j \leqslant 3$ are different integers. Hence $3 - r \leqslant x'_a(v_i) \leqslant 2r - 3$.

Assume that there is a $v_1 \mid V'_1$ with $x'_a(v_1) \leqslant -1$. Then by (5) we have

$$\frac{\delta(v_2) - x_a(B_2)}{2} + \frac{\delta(v_3) - x_a(B_3)}{2} + 1 \leqslant -1$$

for all $v_2 \mid V'_2$ and $v_3 \mid V'_3$.

If $x'_a(v_2) \geqslant 1$ for all $v_2 \mid V'_2$, then $|B_2| \geqslant x_a(B_2) \geqslant |V_2| + |V'_2| = |B_2|$, contradiction to $d(B_2) > 0$. If $x'_a(v_2) \leqslant 0$ for some $v_2 \mid V'_2$, then $\delta(v_2) \geqslant x_a(B_2)$. It follows that $\delta(v_3) \leqslant x_a(B_3) - 4$, and hence

$$2 \leqslant \frac{1}{2}(x_a(B_3) - \delta(v_3)) \leqslant \frac{1}{2}(x_a(B_3) + \delta(v_3)) \leqslant x_a(B_3) - 2.$$

Thus $2 \leqslant x'_a(v_3) \leqslant x_a(B_3) - 2$ for all $v_3 \mid V'_3$. This together with $1 \leqslant x'_a(v_3) \leqslant r - 2$ for $v_3 \mid V_3$ implies that $x_a(B_3) \geqslant |V_3| + 2|V'_3| > |B_3|$, a contradiction. Hence we conclude that $x'_a(v_1) \geqslant 0$ for all $v_1 \mid V'_1$. Similarly we can prove $x'_a(v) \geqslant 0$ for v dividing V'_2 or V'_3.

Finally, if there exists no $v \mid V'_i$ with $x'_a(v) = 0$, then $1 \leqslant x'_a(v) \leqslant 2r - 3$ for all $v \mid V'_i$. Consequently $x_a(B_i) = \sum_{v|V_i} x'_a(v) + \sum_{v|V'_i} x'_a(v) \geqslant |B_i|$, a contradiction. This proves the existence of $v \mid V'_i$ with $x'_a(v) = 0$. ☐

Lemma 21. *Let $U|\alpha$ be a unit block. If there is a minimal block $B|\alpha U^{-1}$ with $d(B) \geqslant 2$ and $\mathsf{r}(\langle \overline{UB} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 1$. Then k is odd.*

Proof. Since $\mathsf{r}(\langle \overline{UB} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 1$, there is an $e \mid B$ such that $\langle \overline{UB} \rangle = \langle \overline{U}, \overline{e} \rangle$. Write $B = B_1 B_2$ with $\mathrm{Supp}(\overline{B_1}) \subset \overline{e} + \langle \overline{U} \rangle$ and $\mathrm{Supp}(\overline{B_2}) \subset \langle \overline{U} \rangle$. Then $|B_1| \geqslant 2$ is even and each pair of terms of B_1 has sum in $\langle \overline{U} \rangle$. Consider any decomposition $B_1 = T_1 \cdots T_m$ with $|T_i| = 2$. For each $T_i \mid B_1$, since $\sigma(\overline{T_i}) \in \langle \overline{U} \rangle$, there exists a subsequence W of U such that $T_i W$ is a block. Then from (1) it follows that $1 \leqslant x'_a(T_i) \leqslant r - 2$. On the other hand, we can similarly get $1 \leqslant x'_a(b_2) \leqslant r - 2$ for any $b_2 \mid B_2$. If there exists at most one T_i, say T_1, such that $x'_a(T_1) = 1$, then $2 \leqslant x'_a(T_i) \leqslant r - 2$ for $2 \leqslant i \leqslant m$. It follows that

$$x_a(B) = \sum_{i=1}^{m} x'_a(T_i) + \sum_{b|B_2} x'_a(b) \geqslant 1 + 2(m-1) + |B_2| = |B| - 1,$$

contradicting $d(B) \geqslant 2$. So there exist T_i and T_j such that $x'_a(T_i) = x'_a(T_j) = 1$. Then Lemma 17 (ii) tells that k is odd. ☐

Lemma 22. *Let $U|\alpha$ be a unit block with $d(U)=r-2$. Then there exists exactly one minimal block with positive defect in αU^{-1}.*

Proof. Since $d(\alpha)=|\alpha|-2k\geqslant r$ and $d(U)=r-2$, by the additivity of defect we have $d(\alpha U^{-1})=d(\alpha)-d(U)\geqslant 2$, i.e., there exists at least one minimal block with positive defect in αU^{-1}. Assume to the contrary that there exist two disjoint minimal blocks B and C with positive defect in αU^{-1}. Combining Lemma 10 with Lemma 18 yields that $\mathsf{r}(\langle\overline{UB}\rangle)\geqslant\mathsf{r}(\langle\overline{U}\rangle)+1=d(U)+1=r-1$ and $\mathsf{r}(\langle\overline{UB}\rangle)\leqslant\mathsf{r}(\overline{G})=r-1$. Then $\langle\overline{UB}\rangle=\overline{G}$. Similarly, $\langle\overline{UC}\rangle=\overline{G}$. By Lemma 19 (i) there exist b_1 and c_1 of order 2 of $\overline{G}\setminus\langle\overline{U}\rangle$ contained in B and C respectively. Then $\delta(b_1)=x_a(B)$ and $\delta(c_1)=x_a(C)$. Since $\langle\overline{U}\rangle$ is an index-2 subgroup of $\overline{G}$, there exists a $Y\mid U$ such that Yb_1c_1 is a $\langle a\rangle$-block. It follows from Corollary 8 that $\delta(b_1)+\delta(c_1)=x_a(B)+x_a(C)\leqslant x_a(B)+x_a(C)-2$, a contradiction. This proves the lemma. □

Lemma 23. *Let $U|\alpha$ be a unit block with $d(U)=r-3$. Then there exists at most two disjoint minimal blocks in αU^{-1} with positive defect.*

Furthermore if there exist two minimal blocks B,C in αU^{-1} with positive defect, then $\langle\overline{UB}\rangle\neq\langle\overline{UC}\rangle$ and one of the following two holds:

(i) *if $\langle\overline{UB}\rangle$ and $\langle\overline{UC}\rangle$ do not contain each other, then $\mathsf{r}(\langle\overline{UB}\rangle)=\mathsf{r}(\langle\overline{UC}\rangle)=r-2$.*

(ii) *if $\langle\overline{UC}\rangle\subset\langle\overline{UB}\rangle$, then $\mathsf{r}(\langle\overline{UB}\rangle)=\mathsf{r}(\langle\overline{UC}\rangle)+1=r-1$, $d(B)\geqslant 2, d(C)=1$ and*

$$C=(e_1'+k_1a)\cdot(e_2'+k_2a)\cdot(e_3'+a)\cdot\cdots\cdot(e_{|C|}'+a),$$

where $k_1+k_2=1$, $k_1\leqslant 0$, $(e_1'+k_1a)\mid C_1$ and $e_i'\in G$ has order two.

Proof. Suppose that there exist two disjoint minimal blocks B,C in αU^{-1} with positive defect. Let B_2 and C_2 be sequences (possibly empty) consisting of terms $b\mid B$ with $\overline{b}\in\langle\overline{U}\rangle$ and $c\mid C$ with $\overline{c}\in\langle\overline{U}\rangle$ respectively. Set $B_1=BB_2^{-1}$ and $C_1=CC_2^{-1}$.

Claim: Suppose $\langle\overline{UC}\rangle\subset\langle\overline{UB}\rangle$.

(a) If there exists some $c_1\mid C_1$ such that $x_a'(c_1)<0$, then $\mathsf{r}(\langle\overline{UB}\rangle)=\mathsf{r}(\langle\overline{U}\rangle)+2=r-1$. In particular, $\langle\overline{UB}\rangle=\overline{G}$.

(b) If $x_a'(c_1)\geqslant 0$ for any $c_1\mid C_1$, then $0\leqslant x_a'(c_1)\leqslant\frac{3r-5}{2}$ for any $c_1\mid C_1$. In addition, there exists $c_1\mid C_1$ such that $x_a'(c_1)=0$, i.e., c_1 is of order 2

(a) Suppose to the contrary $\mathsf{r}(\langle\overline{UB}\rangle)<r-1$. By Lemma 18 we have $\mathsf{r}(\langle\overline{UB}\rangle)\geqslant\mathsf{r}(\langle\overline{U}\rangle)+1=d(U)+1=r-2$. Then $\mathsf{r}(\langle\overline{UB}\rangle)=r-2=\mathsf{r}(\langle\overline{U}\rangle)+1$. By Lemma 19 (i) we get $0\leqslant x_a'(c_1)\leqslant\frac{3r-5}{2}$ for any $c_1\mid C_1$, a contradiction.

(b) Since $\langle\overline{UC}\rangle\subset\langle\overline{UB}\rangle$, by (1) we get $1\leqslant x_a'(b_2),x_a'(c_2)\leqslant r-2$ for all $b_2\mid B_2$ and $c_2\mid C_2$. In addition, for any $c_1\mid C_1$ there exist proper $X\mid B$ and $Y\mid U$ (Y may be empty) such that XYc_1 is a block. Applying (1) we derive that

$$\frac{\delta(X)-x_a(B)}{2}+1\leqslant x_a'(c_1)\leqslant\frac{3r-5}{2}.$$

Hence, $0 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$. If $1 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$ for all $c_1 \mid C_1$, then by $1 \leqslant x'_a(c_2) \leqslant r-2$ for $c_2 \mid C_2$ we get $k > \sum_{c|C} x'_a(c) = x_a(C) \geqslant |C|$, which contradicts $d(C) > 0$. We complete the proof of the claim.

Step 1 : $\langle \overline{UB} \rangle \neq \langle \overline{UC} \rangle$.

Assume to the contrary that $\langle \overline{UB} \rangle = \langle \overline{UC} \rangle$, i.e., $\langle \overline{B} \rangle \subset \langle \overline{UC} \rangle$ and $\langle \overline{C} \rangle \subset \langle \overline{UB} \rangle$. If there exists some $c_1 \mid C_1$ such that $x'_a(c_1) < 0$, then by Claim (a) we have $\mathsf{r}(\langle \overline{UB} \rangle) = \mathsf{r}(\langle \overline{U} \rangle) + 2$. It follows from Lemma 19 (ii) that $\mathsf{r}(\langle \overline{UC} \rangle) = \mathsf{r}(\langle \overline{U} \rangle)+1$, which implies that $\langle \overline{UB} \rangle \neq \langle \overline{UC} \rangle$.

If $x'_a(c_1) \geqslant 0$ for any $c_1 \mid C_1$, then Claim (b) yields $0 \leqslant x'_a(b_1), x'_a(c_1) \leqslant \frac{3r-5}{2}$ for all $b_1 \mid B_1$, $c_1 \mid C_1$ and there exists $e \mid B_1$, $e' \mid C_1$ such that e, e' are of order 2. If $\mathsf{r}(\langle \overline{UB} \rangle) = r-2$, by Lemma 18 we have $\mathsf{r}(\langle \overline{UB} \rangle) = r-2 \geqslant \mathsf{r}(\langle \overline{U} \rangle)+1 = d(U)+1 = r-2$, i.e., $\mathsf{r}(\langle \overline{UB} \rangle) = \mathsf{r}(\langle \overline{U} \rangle)+1 = r-2$. Then there exists e_1 such that $\langle \overline{UB} \rangle = \langle \overline{UC} \rangle = \langle \overline{U}, e_1 \rangle$. It follows that $e, e' \in e_1 + \langle \overline{U} \rangle$. Then ee' is a block. Since $\delta(e) = x_a(B)$ and $\delta(e') = x_a(C)$, applying Corollary 8 we derive that $x_a(B) + x_a(C) = \delta(e) + \delta(e') \leqslant x_a(B) + x_a(C) - 2$, a contradiction.

Since $\mathsf{r}(\langle \overline{UB} \rangle) \geqslant \mathsf{r}(\langle \overline{U} \rangle)+1 = r-2$ and $\mathsf{r}(\langle \overline{UB} \rangle) \leqslant \mathsf{r}(\overline{G}) = r-1$, we have $\mathsf{r}(\langle \overline{UB} \rangle) = r-2$ or $r-1$. Then it suffices to prove our result if $\mathsf{r}(\langle \overline{UB} \rangle) = r-1$. By $\langle \overline{UC} \rangle = \langle \overline{UB} \rangle$ there exist e_1, e_2 such that $\langle \overline{UB} \rangle = \langle \overline{UC} \rangle = \langle \overline{U}, e_1, e_2 \rangle = \overline{G}$. It follows that there exists exactly one element of order 2 in B_1, C_1 respectively. Assume to the contrary that there exist two elements c_1, c'_1 of order 2 in C_1. Let b_1 be an element of order 2 in B_1. Obviously, one of $\{c_1c'_1, b_1c_1, b_1c'_1, b_1c_1c'_1\}$ is contained in $\langle \overline{U} \rangle$. Since $\delta(b_1) = x_a(B)$ and $\delta(c_1) = \delta(c'_1) = \delta(c_1c'_1) = x_a(C)$, by Corollary 8 we get that either $x_a(C) = \delta(c_1c'_1) \leqslant x_a(C) - 2$ or $x_a(B)+x_a(C) = \delta(b_1)+\delta(X) \leqslant x_a(B)+x_a(C)-2$ for $Xb_1 \in \{b_1c_1, b_1c'_1, b_1c_1c'_1\}$ contained in $\langle \overline{U} \rangle$. This is a contradiction. Let e and e' are elements of order 2 in B_1, C_1 respectively. Then we have $x'_a(e) = x'_a(e') = 0$ and $x'_a(b) \geqslant 1$, $x'_a(c) \geqslant 1$ for all $b \mid B_1e^{-1}$ and $c \mid C_1e'^{-1}$. Hence, by $0 \leqslant x'_a(b_1), x'_a(c_1) \leqslant \frac{3r-5}{2}$ for all $b_1 \mid B_1$, $c_1 \mid C_1$ and $1 \leqslant x'_a(b_2), x'_a(c_2) \leqslant r-2$ for all $b_2 \mid B_2$, $c_2 \mid C_2$, we get

$$\frac{k+1}{2} > \frac{3r-5}{2}(r-1) \geqslant \sum_{b|Be^{-1}} x'_a(b) = x_a(B) \geqslant |B|-1 \geqslant x_a(B) \ \text{and}$$

$$\frac{k+1}{2} > \frac{3r-5}{2}(r-1) \geqslant \sum_{c|Ce'^{-1}} x'_a(c) = x_a(C) \geqslant |C|-1 \geqslant x_a(C).$$

It follows that $|B| = x_a(B)+1$, $|C| = x_a(C)+1$ and $x'_a(b) = x'_a(c) = 1$ for all $b \mid Be^{-1}$ and $c \mid Ce'^{-1}$, which implies that $d(B) = d(C) = 1$. In addition, by the proof of $\mathsf{r}(\langle \overline{UB} \rangle) = r-2$, it is easy to see that e and e' can not be contained in the same $\langle \overline{U} \rangle$-coset, i.e., $\overline{e} \neq \overline{e'}$. Since $d(\alpha U^{-1}) \geqslant 3$, there exists a minimal block D in $\alpha(UBC)^{-1}$ with positive defect. Since $\langle \overline{UB} \rangle = \overline{G}$, we have $\langle \overline{D} \rangle \subset \langle \overline{UB} \rangle$. Repeat the reasoning of C and we have that $\langle \overline{UB} \rangle = \langle \overline{UD} \rangle = \langle \overline{U}, e_1, e_2 \rangle = \overline{G}$, $d(D) = 1$ and there exists exactly one element of order 2 in D_1. Set e'' is the order-2 element of D_1 and we have that $\overline{e}, \overline{e'}, \overline{e''}$ are pairwise distinct contained in $\langle e_1, e_2 \rangle$. Hence, $\sigma(\overline{ee'e''}) = \overline{0}$. Since $\delta(e) = x_a(B)$, $\delta(e') = x_a(C)$ and $\delta(e'') = x_a(D)$, by Corollary 8 we get that $x_a(B)+x_a(C)+x_a(D) = \delta(e)+\delta(e')+\delta(e'') \leqslant x_a(B)+x_a(C)+x_a(D)-2$, a contradiction.

By step 1 it is easy to see that any two disjoint minimal blocks B, C in αU^{-1} with positive defect satisfy $\langle\overline{UB}\rangle \neq \langle\overline{UC}\rangle$.

Step 2: If $\langle\overline{UC}\rangle \subset \langle\overline{UB}\rangle$, then (ii) holds and there exist exactly two disjoint minimal blocks in αU^{-1} with positive defect.

If there exists some $c_1 \mid C_1$ such that $x'_a(c_1) < 0$, then by $\langle\overline{UC}\rangle \subset \langle\overline{UB}\rangle$ and Claim (a) we have $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{U}\rangle) + 2$ and $\langle\overline{UB}\rangle = \overline{G}$. It follows from Lemma 19 (ii) that (1) $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{UC}\rangle) + 1 = \mathsf{r}(\langle\overline{U}\rangle) + 2 = r - 1$; (2) $C = (e'_1 + k_1 a) \cdot (e'_2 + k_2 a) \cdot (e'_3 + a) \cdots (e'_{|C|} + a)$, where $k_1 + k_2 = 1$, $k_1 < 0$ and $e'_i \in G$ has order two, and this implies $d(C) = 1$; (3) there does not exist a minimal block D with positive defect in $\alpha(UBC)^{-1}$ ($\langle\overline{D}\rangle \subset \langle\overline{UB}\rangle = \overline{G}$), which implies that $d(B) = d(\alpha) - d(U) - d(C) \geqslant 2$ by the additivity of defect. Hence, our result is true.

Now suppose $x'_a(c_1) \geqslant 0$ for any $c_1 \mid C_1$. Since $\langle\overline{UC}\rangle \subset \langle\overline{UB}\rangle$ and $\langle\overline{UC}\rangle \neq \langle\overline{UB}\rangle$, by Lemma 18 we have $\mathsf{r}(\langle\overline{UB}\rangle) \geqslant \mathsf{r}(\langle\overline{UC}\rangle) + 1 \geqslant \mathsf{r}(\langle\overline{U}\rangle) + 2 = d(U) + 2 = r - 1$. By $\mathsf{r}(\langle\overline{UB}\rangle) \leqslant \mathsf{r}(\overline{G}) = r - 1$, we derive that $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{UC}\rangle) + 1 = \mathsf{r}(\langle\overline{U}\rangle) + 2 = r - 1$, $\langle\overline{UB}\rangle = \overline{G}$ and there exists e_1, e_2 such that $\langle\overline{UB}\rangle = \langle\overline{U}, e_1, e_2\rangle$ and $\langle\overline{UC}\rangle = \langle\overline{U}, e_1\rangle$. By (1) and Claim (b) we get $1 \leqslant x'_a(c_2) \leqslant r - 2$, $0 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$ for all $c_2 \mid C_2$, $c_1 \mid C_1$, and there exists $c_1 \mid C_1$ such that $x'_a(c_1) = 0$. It follows that there exists exactly one element of order 2 in C_1. Assume to the contrary that there exist two elements c_1, c'_1 of order 2 in C_1. It follows from $\langle\overline{UC}\rangle = \langle\overline{U}, e\rangle$ that $\overline{c_1}, \overline{c'_1} \in e_1 + \langle\overline{U}\rangle$. Then $c_1 c'_1$ is a block. Since $\delta(c_1 c'_1) = x_a(C)$, applying Corollary 8 we derive that $x_a(C) = \delta(c_1 c'_1) \leqslant x_a(C) - 2$, a contradiction. Let e'_1 be the element of order 2 in C_1. Then we have $x_a(e'_1) = 0$ and $x_a(c) \geqslant 1$ for all $c \mid C_1 {e'_1}^{-1}$. Hence, by $0 \leqslant x'_a(c_1) \leqslant \frac{3r-5}{2}$ for all $c_1 \mid C_1$ and $1 \leqslant x'_a(c_2) \leqslant r - 2$ for all $c_2 \mid C_2$, we get

$$\frac{k+1}{2} > \frac{3r-5}{2}(r-1) \geqslant \sum_{c \mid C{e'_1}^{-1}} x'_a(c) = x_a(C) \geqslant |C| - 1 \geqslant x_a(C).$$

It follows that $|C| = x_a(C) + 1$ and $x'_a(c) = 1$ for all $c \mid C{e'_1}^{-1}$, which implies that $d(C) = 1$ and $C = e'_1 \cdot (e'_2 + a) \cdot (e'_3 + a) \cdot \ldots \cdot (e'_{|C|} + a)$, where $e'_i \in G$ has order two.

If there does not exist minimal blocks in $\alpha(UBC)^{-1}$ with positive defect, then by the additivity of defect, $d(B) = d(\alpha) - d(U) - d(C) \geqslant 2$. Hence, it suffices to prove that there does not exist minimal blocks in $\alpha(UBC)^{-1}$ with positive defect. Assume to the contrary that there exists a minimal block D in $\alpha(UBC)^{-1}$ with positive defect. Let D_2 be the sequence (possibly empty) consisting of terms $d \mid D$ with $\overline{d} \in \langle\overline{U}\rangle$. Set $D_1 = DD_2^{-1}$. By step 1 we can see that $\langle\overline{UD}\rangle \neq \langle\overline{UB}\rangle$ and $\langle\overline{UD}\rangle \neq \langle\overline{UC}\rangle$. Since $\langle\overline{UB}\rangle = \overline{G}$, we have $\langle\overline{UD}\rangle \subset \langle\overline{UB}\rangle$. By the proof of the structure of C, we can derive that

$$D = (e''_1 + k'_1 a) \cdot (e''_2 + k'_2 a) \cdot (e''_3 + a) \cdot \ldots \cdot (e''_{|D|} + a),$$

where $k'_1 + k'_2 = 1$, $k'_1 \leqslant 0$, $(e''_1 + k'_1 a) \mid D_1$ and $e''_i \in G$ has order two. Since $\mathsf{r}(\langle\overline{UB}\rangle) = r - 1$, $\langle\overline{UB}\rangle = \overline{G}$, we have $r - 1 > \mathsf{r}(\langle\overline{UD}\rangle) \geqslant \mathsf{r}(\langle\overline{U}\rangle) + 1 = r - 2$, i.e., $\mathsf{r}(\langle\overline{UD}\rangle) = r - 2$. Since $\langle\overline{UB}\rangle = \langle\overline{U}, e_1, e_2\rangle$, $\langle\overline{UC}\rangle = \langle\overline{U}, e_1\rangle$ and $\langle\overline{UD}\rangle \neq \langle\overline{UC}\rangle$, we must have either $\langle\overline{UD}\rangle = \langle\overline{U}, e_2\rangle$ or $\langle\overline{UD}\rangle = \langle\overline{U}, e_1 + e_2\rangle$. By $\overline{e'_1} \in e_1 + \langle\overline{U}\rangle$ and $\overline{e''_1 + k'_1 a} \in e_2 + \langle\overline{U}\rangle$ or

$e_1+e_2+\langle\overline{U}\rangle$, we have that for any $b_1 \mid B_1$ there exists a proper $Y \mid U$ such that one of $\{Yb_1e_1', Yb_1(e_1''+k_1'a), Yb_1e_1'(e_1''+k_1'a)\}$ is a block. By $\delta(e_1')=x_a(C)$, $\delta(e_1''+k_1'a)\geqslant x_a(D)$ and Corollary 8, we get that

$$\delta(b_1)+x_a(C)=\delta(b_1)+\delta(e_1')\leqslant x_a(B)+x_a(C)-2 \text{ or}$$

$$\delta(b_1)+x_a(D)\leqslant \delta(b_1)+\delta(e_1''+k_1'a)\leqslant x_a(B)+x_a(D)-2 \text{ or}$$

$$\delta(b_1)+x_a(C)+x_a(D)\leqslant \delta(b_1)+\delta(e_1')+\delta(e_1''+k_1'a)\leqslant x_a(B)+x_a(C)+x_a(D)-2.$$

This implies $\delta(b_1)\leqslant x_a(B)-2$ and hence

$$1\leqslant \frac{1}{2}(x_a(B)-\delta(b_1))\leqslant \frac{1}{2}(x_a(B)+\delta(b_1))\leqslant x_a(B)-1.$$

It follows that $1\leqslant x_a'(b_1)\leqslant x_a(B)-1$ for any $b_1\mid B_1$. By (1) we have $1\leqslant x_a'(b_2)\leqslant r-2$ for any $b_2\mid B_2$. Hence, $k>\sum_{b_1\mid B_1}x_a'(b_1)+\sum_{b_2\mid B_2}x_a'(b_2)=x_a(B)\geqslant |B|$, a contradiction to $d(B)>0$. We complete the proof of step 2.

By step 2 we can suppose that any two disjoint minimal blocks B, C in αU^{-1} with positive defect satisfy that $\langle\overline{UB}\rangle$ and $\langle\overline{UC}\rangle$ do not contain each other.

Step 3: If $\langle\overline{UB}\rangle$ and $\langle\overline{UC}\rangle$ do not contain each other, then (i) holds and there exist exactly two disjoint minimal blocks in αU^{-1} with positive defect.

Since $\langle\overline{UB}\rangle\nsubseteq\langle\overline{UC}\rangle$ and $\langle\overline{UC}\rangle\nsubseteq\langle\overline{UB}\rangle$, we have $\mathsf{r}(\langle\overline{UB}\rangle), \mathsf{r}(\langle\overline{UC}\rangle)<\mathsf{r}(\langle\overline{UBC}\rangle)\leqslant \mathsf{r}(\overline{G})=r-1$. By $\mathsf{r}(\langle\overline{UB}\rangle), \mathsf{r}(\langle\overline{UC}\rangle)\geqslant \mathsf{r}(\langle\overline{U}\rangle)+1=r-2$, we derive that $\mathsf{r}(\langle\overline{UB}\rangle)=\mathsf{r}(\langle\overline{UC}\rangle)=r-2$ and $\mathsf{r}(\langle\overline{UBC}\rangle)=r-1$.

Suppose that there exists a minimal block D in $\alpha(UBC)^{-1}$ with positive defect. Let D_2 be the sequence (possibly empty) consisting of terms $d\mid D$ with $\overline{d}\in\langle\overline{U}\rangle$. Set $D_1=DD_2^{-1}$. By step 2 we can see that any two of $\{\langle\overline{UB}\rangle, \langle\overline{UC}\rangle, \langle\overline{UD}\rangle\}$ do not contain each other. Hence, $\mathsf{r}(\langle\overline{UD}\rangle)=r-2$ and $\mathsf{r}(\langle\overline{UBC}\rangle)=\mathsf{r}(\langle\overline{UBD}\rangle)=\mathsf{r}(\langle\overline{UCD}\rangle)=r-1$. By $\mathsf{r}(\langle\overline{U}\rangle)=r-3$ and Lemma 20, we get that $1\leqslant x_a'(v_2)\leqslant r-2$, $0\leqslant x_a'(v_1)\leqslant 2r-3$ for all $v_2\mid B_2C_2D_2$, $v_1\mid B_1C_1D_1$, and there exist some $b_1\mid B_1, c_1\mid C_1, d_1\mid D_1$ with $x_a'(b_1)=x_a'(c_1)=x_a'(d_1)=0$. In addition, there exist e_1, e_2 such that $\overline{G}=\langle\overline{U}, e_1, e_2\rangle$. Without loss of generality, we can suppose $\langle\overline{UB}\rangle=\langle\overline{U}, e_1\rangle$, $\langle\overline{UC}\rangle=\langle\overline{U}, e_2\rangle$ and $\langle\overline{UD}\rangle=\langle\overline{U}, e_1+e_2\rangle$. It follows that $\overline{b_1}=e_1$, $\overline{c_1}=e_2$ and $\overline{d_1}=e_1+e_2$. Hence, $b_1c_1d_1$ is a block and $\delta(b_1)=x_a(B)$, $\delta(c_1)=x_a(C)$, $\delta(d_1)=x_a(D)$. By Corollary 8 we have $x_a(B)+x_a(C)+x_a(D)=\delta(b_1)+\delta(c_1)+\delta(d_1)\leqslant x_a(B)+x_a(C)+x_a(D)-2$, a contradiction. □

Lemma 24. *If there is a unit block U of α with $d(U)=r-3$, then k is odd.*

Proof. Since $d(U)=r-3$, from Lemma 23 it follows that there exist at most two disjoint minimal blocks in αU^{-1} with positive defect. Since $\mathsf{r}(\langle\overline{U}\rangle)=r-3$, there exist e_1, e_2 such that $\overline{G}=\langle\overline{U}, e_1, e_2\rangle$. We consider the following two cases to complete our proof:

Case 1 : There exist two disjoint minimal blocks B, C in αU^{-1} with positive defect.

Let B_2 and C_2 be sequences (possibly empty) consisting of terms $b\mid B$ with $\overline{b}\in\langle\overline{U}\rangle$ and $c\mid C$ with $\overline{c}\in\langle\overline{U}\rangle$ respectively. Set $B_1=BB_2^{-1}$ and $C_1=CC_2^{-1}$. By the additivity of defect $d(B)+d(C)=d(\alpha)-d(U)\geqslant 3$. By $d(B)>0$ and $d(C)>0$, we have $d(B)\geqslant 2$

or $d(C) \geqslant 2$. If $\langle\overline{UB}\rangle$ and $\langle\overline{UC}\rangle$ do not contain each other, then Lemma 23 (i) tells us that $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{UC}\rangle) = \mathsf{r}(\langle\overline{U}\rangle) + 1 = r - 2$. Lemma 21 yields that k is odd.

If $\langle\overline{UC}\rangle \subset \langle\overline{UB}\rangle$, then by Lemma 23 (ii) we have that $\mathsf{r}(\langle\overline{UB}\rangle) = \mathsf{r}(\langle\overline{UC}\rangle) + 1 = r - 1$, $d(B) \geqslant 2, d(C) = 1$ and

$$C = (e_1' + k_1 a) \cdot (e_2' + k_2 a) \cdot (e_3' + a) \cdot \dots \cdot (e_{|C|}' + a),$$

where $k_1 + k_2 = 1$, $k_1 \leqslant 0$, $(e_1' + k_1 a) \mid C_1$ and $e_i' \in G$ has order two. It follows that $\langle\overline{UB}\rangle = \overline{G} = \langle\overline{U}, e_1, e_2\rangle$. Without loss of generality, we can suppose $\langle\overline{UC}\rangle = \langle\overline{U}, e_1\rangle$ with $\mathrm{Supp}(\overline{C_1}) \subset e_1 + \langle\overline{U}\rangle$. Write $B_1 = A_1 A_2 A_3$ with $\mathrm{Supp}(\overline{A_1}), \mathrm{Supp}(\overline{A_2})$ and $\mathrm{Supp}(\overline{A_3})$ being subsets of $e_1 + \langle\overline{U}\rangle$, $e_2 + \langle\overline{U}\rangle$ and $e_1 + e_2 + \langle\overline{U}\rangle$ respectively. By symmetry we can suppose $|A_2| \leqslant |A_3|$. Consider the decomposition $B = A_1 A_2 A_3' A_3''$, where A_3' is any subsequence of A_3 with $|A_3'| = |A_2|$ and $A_3'' = A_3 A_3'^{-1}$. It is easy to see that $|A_3''|$ is even and $|A_2| + |A_3| \geqslant 2$ is also even.

Take $X = a_1$ with $a_1 \mid A_1$ or $X = a_2 a_3$ with $a_2 \mid A_2, a_3 \mid A_3$. Then there exists a $Y \mid U$ such that $XY(e_1' + k_1 a)$ is a block. Then (1) and $x_a'(e_1' + k_1 a) \leqslant 0$ give us

$$\frac{3-r}{2} \leqslant \frac{\delta(X) - x_a(B)}{2} + 1 \leqslant x_a'(e_1' + k_1 a) \leqslant 0.$$

This implies $\delta(X) \leqslant x_a(B) - 2$ and hence

$$1 \leqslant \frac{1}{2}(x_a(B) - \delta(X)) \leqslant \frac{1}{2}(x_a(B) + \delta(X)) \leqslant x_a(B) - 1.$$

It follows that $1 \leqslant x_a'(X) \leqslant x_a(B) - 1$. It is worth mentioning that if there exist two disjoint subsequences T_1, T_2 of $A_2 A_3$ of length two such that $x_a'(T_1) = x_a'(T_2) = 1$, then by Lemma 17 (ii) and $1 \leqslant x_a'(a_2 a_3) \leqslant x_a(B) - 1$ for all $a_2 \mid A_2, a_3 \mid A_3$ we have that k is odd. In addition, it is easy to see that the above conditional assumption must hold. Assume to the contrary and then for a decomposition $A_2 A_3 = T_1 \cdots T_\ell$ with each $|T_i| = 2$, there exists at most one T_i, say T_1, such that $x_a'(T_1) = 1$ and $2 \leqslant x_a'(T_i) \leqslant x_a(B) - 1$ for $2 \leqslant i \leqslant \ell$. For any $T = b_2 \mid B_2$ or $T \mid A_i$ of length two ($i = 2, 3$), we have $\sigma(\overline{T}) \in \langle\overline{U}\rangle$ and hence there exists a $Y \mid U$ such that YT is a block. One deduces from (1) that $1 \leqslant x_a'(T) \leqslant r - 2$. It follows from $d(B) \geqslant 2$ that

$$\begin{aligned} |B| - 2 \geqslant x_a(B) &= \sum_{a_1 \mid A_1} x_a'(a_1) + \sum_{i=1}^{\ell} x_a'(T_i) + \sum_{b_2 \mid B_2} x_a'(b_2) \\ &\geqslant |A_1| + |B_2| + |A_2| + |A_3| - 1 = |B| - 1. \end{aligned}$$

This is a contradiction.

Case 2 : There exists exactly one minimal block B in αU^{-1}.

Then $d(B) = d(\alpha) - d(U) \geqslant 3$. Since $r(\langle\overline{UB}\rangle) \geqslant r(\langle\overline{U}\rangle) + 1 = r - 2$ and $r(\langle\overline{UB}\rangle) \leqslant r(\overline{G}) = r - 1$, by Lemma 21 we can suppose $r(\langle\overline{UB}\rangle) = r - 1$. It follows that $\langle\overline{UB}\rangle = \overline{G} = \langle\overline{U}, e_1, e_2\rangle$. Let B_2 be a sequence (possibly empty) consisting of terms $b \mid B$ with $\overline{b} \in \langle\overline{U}\rangle$. Set $B_1 = B B_2^{-1}$. Write $B_1 = A_1 A_2 A_3$ with $\mathrm{Supp}(\overline{A_1}), \mathrm{Supp}(\overline{A_2})$ and $\mathrm{Supp}(\overline{A_3})$

being subsets of $e_1+\langle\overline{U}\rangle$, $e_2+\langle\overline{U}\rangle$ and $e_1+e_2+\langle\overline{U}\rangle$ respectively. Take $T=b_2\mid B_2$ or $T=a_1a_2a_3$ for $a_i\mid A_i$ $(i=1,2,3)$ or $T\mid A_i$ of length two for $1\leqslant i\leqslant 3$, we have $\sigma(\overline{T})\in\langle\overline{U}\rangle$ and hence there exists a $Y\mid U$ such that YT is a block. One deduces from (1) that $1\leqslant x'_a(T)\leqslant r-2$. It is easy to see that at least two A_i are nonempty for $1\leqslant i\leqslant 3$, and either all $|A_i|$ are even or all $|A_i|$ are odd.

If all $|A_i|$ are even, then let $A_i=T_{i1}\cdots T_{it_i}$ be a product of some subsequences of length two. We can find three subsequences of length two, say T_1,T_2,T_3, such that $x'_a(T_1)=x'_a(T_2)=x'_a(T_3)=1$. Assume to the contrary there exist at most two subsequences of length two, say T_1,T_2, such that $x'_a(T_1)=x'_a(T_2)=1$. Since $1\leqslant x'_a(T_{ij})\leqslant r-2$ for all T_{ij}, we have $2\leqslant x'_a(T_{ij})\leqslant r-2$ except for T_1,T_2. By $d(B)\geqslant 3$ we have that

$$\begin{aligned}|B|-3\geqslant x_a(B)&=\sum_{T_{ij}\neq T_1,T_2}x'_a(T_{ij})+x'_a(T_1)+x'_a(T_2)+\sum_{b_2|B_2}x'_a(b_2)\\&\geqslant |A_1|+|A_2|+|A_3|-2+|B_2|=|B|-2.\end{aligned}$$

This is a contradiction. If there exist two T_i, say T_1,T_2, in $\{T_1,T_2,T_3\}$ contained in the same A_j for $1\leqslant j\leqslant 3$, then for any $t_1\mid T_1,t_2\mid T_2$ there exists a $Y\mid U$ such that Yt_1t_2 is a block. It follows from (1) that $1\leqslant x'_a(t_1t_2)\leqslant r-2$. From Lemma 17 (ii) one deduces that k is odd. If T_1,T_2,T_3 are contained in distinct A_j respectively, then by $1\leqslant x'_a(a_1a_2a_3)\leqslant r-2$ for any $a_i\mid A_i$ $(i=1,2,3)$, Lemma 17 (ii) yields that k is odd.

If all $|A_i|$ are odd, then let $A_ia_i^{-1}=T_{i1}\cdots T_{it_i}$ be a product of some subsequences of length two for $a_i\mid A_i$. By Lemma 17 (i) we can suppose that either $2\leqslant x'_a(T_{ij})\leqslant r-2$ for all T_{ij} or $2\leqslant x'_a(a_1a_2a_3)\leqslant r-2$ for any $a_i\mid A_i$ $(i=1,2,3)$. If the former holds, then by $d(B)\geqslant 3$ we have

$$\begin{aligned}|B|-3\geqslant x_a(B)=x'_a(a_1a_2a_3)+\sum_{i,j}x'_a(T_{ij})+\sum_{b_2|B_2}x'_a(b_2)\\ \geqslant 1+2(\frac{|B_1|-3}{2})+|B_2|=|B|-2.\end{aligned}$$

This is a contradiction. If the latter holds, then by Lemma 17 (ii) we can suppose that there exists at most one T_{ij} in each A_i, say T_{i1}, such that $x'_a(T_{i1})=1$. By $1\leqslant x'_a(a_1a_2a_3)\leqslant r-2$ for any $a_i\mid A_i$ $(i=1,2,3)$ and Lemma 17 (ii) we can again suppose that at most two of $\{x'_a(T_{11}),x'_a(T_{21}),x'_a(T_{31})\}$ equal 1. It follows from $d(B)\geqslant 3$ that

$$\begin{aligned}|B|-3\geqslant x_a(B)=x'_a(a_1a_2a_3)+\sum_{i,j}x'_a(T_{ij})+\sum_{b_2|B_2}x'_a(b_2)\\ \geqslant 2+2+2(\frac{|B_1|-3}{2}-2)+|B_2|=|B|-3.\end{aligned}$$

Then we must have that $x'_a(a_1a_2a_3)=2$ for any $a_i\mid A_i$ $(i=1,2,3)$ and there exist exactly two of $\{x'_a(T_{11}),x'_a(T_{21}),x'_a(T_{31})\}$, say T_{11},T_{21}, such that $x'_a(T_{11})=x'_a(T_{21})=1$. Set $T_{11}=t_1t'_1\mid A_1$ and we have $x'_a(t_1a_2a_3)=x'_a(t'_1a_2a_3)=2$ for $a_2\mid A_2$, $a_3\mid A_3$. Lemma 17 implies that k is odd. We complete the proof. □

Proof of Theorem 3: Immediately from Lemma 21 and Lemma 24.

4 Proof of Theorem 2

Proof of Theorem 2. Set $C_2^5 \oplus C_{2k} = \langle e\rangle \oplus G_1$, where $2e = 0$ and $G_1 \cong C_2^4 \oplus C_{2k}$. We have known that $\mathsf{D}(C_2^4 \oplus C_{2k}) = 2k+5$, if k is odd with $k \geqslant 70$. Thus there exists a zero-sum free sequence T of length $2k+4$ over G_1, if k is odd with $k \geqslant 70$. It follows that $S = eT$ is a zero-sum free sequence of length $2k+5$ over $C_2^5 \oplus C_{2k}$, i.e, $\mathsf{D}(C_2^4 \oplus C_{2k}) \geqslant 2k+6$, if k is odd with $k \geqslant 70$.

Suppose that a group $C_2^5 \oplus C_{2k}$ with $k \geqslant 149$ satisfies the excessive inequality $\mathsf{D}(C_2^5 \oplus C_{2k}) > \mathsf{D}^*(C_2^5 \oplus C_{2k}) = 2k+5$. Let $r = 6$ be the rank of $C_2^5 \oplus C_{2k}$, let α be an arbitrary minimal zero-sum sequence of maximum length over this group, and let $a \mid \alpha$ be a distinguished term, i.e., a is a generator of C_{2k}. Then $d(\alpha) \geqslant 6$. By Lemma 15 and Lemma 14 (ii), we have $r-3 = 3 \leqslant d(W_{\mathscr{F}}) \leqslant r-2 = 4$. It follows from Theorem 3 that k is odd. Hence, if $k \geqslant 149$ is even, then $\mathsf{D}(C_2^5 \oplus C_{2k}) = \mathsf{D}^*(C_2^5 \oplus C_{2k}) = 2k+5$. Let $W_{\mathscr{F}} = T_1T_2\dots T_m$ be a product of $(*,1)$-blocks T_i. By Proposition 4 we have that $x_a(W_{\mathscr{F}}) = x_a(T_1) + x_a(T_2) + \cdots + x_a(T_m) = m$. It follows from Lemma 18 that for any (ℓ, s)-block B with positive defect in $\alpha W_{\mathscr{F}}^{-1}$, we have $2 \leqslant s < \ell \leqslant r-1 = 5$, since otherwise $\ell = 6$ and then $r(\overline{C_2^5 \oplus C_{2k}}) = 5 = r(\langle \overline{B}\rangle) \leqslant r(\langle \overline{W_{\mathscr{F}}B}\rangle) \leqslant r(\overline{C_2^5 \oplus C_{2k}})$, i.e., $r(\langle \overline{B}\rangle) = r(\langle \overline{W_{\mathscr{F}}B}\rangle)$.

If $d(W_{\mathscr{F}}) = 4$, then $d(\alpha W_{\mathscr{F}}^{-1}) \geqslant 2$ and $|W_{\mathscr{F}}| = x_a(W_{\mathscr{F}}) + d(W_{\mathscr{F}}) = m+4$. By Lemma 22 there exists exactly a (ℓ, s)-block B with $2 \leqslant s < \ell \leqslant 5$ in $\alpha W_{\mathscr{F}}^{-1}$. Thus $\alpha = W_{\mathscr{F}}B\alpha'$ with $d(\alpha W_{\mathscr{F}}^{-1}) = d(B) = \ell - s \geqslant 2$, where α' is a product of some minimal block D with $d(D) = 0$. It follows that $5 \geqslant \ell \geqslant s+2 \geqslant 4$ and $x_a(\alpha') = |\alpha'|$. This implies that B is $(5,3)$, $(5,2)$, or $(4,2)$. Combining with Lemma 16 yields that B is not $(5,2)$, i.e., B is $(s+2, s)$ with $2 \leqslant s \leqslant 3$. Since $x_a(\alpha') = |\alpha'|$ and $x_a(\alpha) = 2k$, by Proposition 4 we have

$$x_a(\alpha) = 2k = x_a(W_{\mathscr{F}}) + x_a(B) + x_a(\alpha') = m + s + |\alpha'|.$$

Hence,

$$|\alpha| = |W_{\mathscr{F}}| + |B| + |\alpha'| = (m+4) + (s+2) + (2k-s-m) = 2k+6.$$

If $d(W_{\mathscr{F}}) = 3$, then $d(\alpha W_{\mathscr{F}}^{-1}) \geqslant 3$ and $|W_{\mathscr{F}}| = x_a(W_{\mathscr{F}}) + d(W_{\mathscr{F}}) = m+3$. By Lemma 23 there exist at most two disjoint minimal blocks with positive defect in $\alpha W_{\mathscr{F}}^{-1}$. If there exists exactly a (ℓ, s)-block B with positive defect in $\alpha W_{\mathscr{F}}^{-1}$, then $2 \leqslant s < \ell \leqslant 5$ and $d(B) = d(\alpha W_{\mathscr{F}}^{-1}) \geqslant 3$. It follows that $\ell = s+3 = 5$, i.e., B is $(5,2)$. This is a contradiction to Lemma 16.

If there exist a (ℓ, s)-block B and a (ℓ_1, s_1)-block C with positive defects in $\alpha W_{\mathscr{F}}^{-1}$ such that B, C are disjoint, then $2 \leqslant s < \ell \leqslant 5$ and $2 \leqslant s_1 < \ell_1 \leqslant 5$. Set $\alpha = W_{\mathscr{F}}BC\alpha'$, where α' is a product of some minimal block D with $d(D) = 0$. It follows from Lemma 23 that either $d(B) \geqslant 2$, $d(C) = 1$ or

$$r(\langle \overline{W_{\mathscr{F}}B}\rangle) = r(\langle \overline{W_{\mathscr{F}}C}\rangle) = 4,\ \langle \overline{C}\rangle \nsubseteq \langle \overline{W_{\mathscr{F}}B}\rangle \text{ and } \langle \overline{B}\rangle \nsubseteq \langle \overline{W_{\mathscr{F}}C}\rangle.$$

If the former holds, then $\ell_1 - s_1 = 1$ and $4 \leqslant s+2 \leqslant \ell \leqslant 5$, i.e., B is $(5,3)$, $(5,2)$ or $(4,2)$. Combining with Lemma 16 yields that B is not $(5,2)$, i.e., B is $(s+2, s)$ with $2 \leqslant s \leqslant 3$.

Since $x_a(\alpha) = 2k$ and $x_a(\alpha') = |\alpha'|$, by Proposition 4 we have

$$x_a(\alpha) = 2k = x_a(W_{\mathscr{F}}) + x_a(B) + x_a(C) + x_a(\alpha') = m + s + s_1 + |\alpha'|.$$

Hence,

$$|\alpha| = |W_{\mathscr{F}}| + |B| + |C| + |\alpha'| = (m+3) + (s+2) + \ell_1 + (2k - m - s - s_1) = 2k + 6.$$

If the latter holds, then $2 \leqslant s \leqslant \ell \leqslant 5$ and $2 \leqslant s_1 \leqslant \ell_1 \leqslant 5$, i.e., B and C are contained in $\{(4,2),(4,3),(5,2),(5,3),(5,4)\}$. By Lemma 16 B is not $(5,2)$. If B is $(5,3)$, then $r(\langle\overline{B}\rangle) = 4 = r(\langle\overline{W_{\mathscr{F}}B}\rangle)$, a contradiction to Lemma 18. Hence, B is $(4,2)$, $(4,3)$ or $(5,4)$. Similarly, C is $(4,2)$, $(4,3)$ or $(5,4)$. From $d(\alpha W_{\mathscr{F}}^{-1}) = d(B) + d(C) \geqslant 3$ it is easy to see that one of B, C must be $(4,2)$. Without loss of generality, suppose B is $(4,2)$.

If C is $(4,3)$ or $(5,4)$, then by $x_a(\alpha) = 2k$, $x_a(\alpha') = |\alpha'|$ and Proposition 4 we have

$$x_a(\alpha) = 2k = x_a(W_{\mathscr{F}}) + x_a(B) + x_a(C) + x_a(\alpha') = m + 2 + s_1 + |\alpha'|.$$

Hence,

$$|\alpha| = |W_{\mathscr{F}}| + |B| + |C| + |\alpha'| = (m+3) + 4 + \ell_1 + (2k - s_1 - m - 2) = 2k + 6.$$

If C is $(4,2)$, then by $d(W_{\mathscr{F}}) = r(\langle\overline{W_{\mathscr{F}}}\rangle) = 3$ and $r(\overline{G}) = 5$, there exist e_0, e_0', e_1, e_2, e_3 such that $\overline{G} = \langle e_1, e_2, e_3, e_0, e_0'\rangle$, where $\langle\overline{W_{\mathscr{F}}}\rangle = \langle e_1, e_2, e_3\rangle$. Since $r(\langle\overline{W_{\mathscr{F}}B}\rangle) = r(\langle\overline{W_{\mathscr{F}}C}\rangle) = 4$, $\langle\overline{C}\rangle \nsubseteq \langle\overline{W_{\mathscr{F}}B}\rangle$ and $\langle\overline{B}\rangle \nsubseteq \langle\overline{W_{\mathscr{F}}C}\rangle$, without loss of generality we can suppose $\langle\overline{W_{\mathscr{F}}B}\rangle = \langle\overline{W_{\mathscr{F}}}, e_0\rangle$ and $\langle\overline{W_{\mathscr{F}}C}\rangle = \langle\overline{W_{\mathscr{F}}}, e_0'\rangle$. Let B_2 and C_2 be sequences (possibly empty) consisting of terms $b \mid B$ with $\overline{b} \in \langle\overline{W_{\mathscr{F}}}\rangle$ and $c \mid C$ with $\overline{c} \in \langle\overline{W_{\mathscr{F}}}\rangle$ respectively. Set $B_1 = BB_2^{-1}$ and $C_1 = CC_2^{-1}$. It is easy to see that $\mathrm{Supp}(\overline{B_1}) \subset e_0 + \langle\overline{W_{\mathscr{F}}}\rangle$, $\mathrm{Supp}(\overline{C_1}) \subset e_0' + \langle\overline{W_{\mathscr{F}}}\rangle$ and $|B_1|, |C_1| \in \{2,4\}$. Let $X = b_1b_1' \mid B_1$ or $X = c_1c_1' \mid C_1$ or $X = b_2 \mid B_2$ or $X = c_2 \mid C_2$ and we have $\sigma(\overline{X}) \in \langle\overline{W_{\mathscr{F}}}\rangle$. Then there is a proper subsequence $Y \mid W_{\mathscr{F}}$ such that YX is a block. By (1) one deduces $1 \leqslant x_a'(X) \leqslant r - 2 = 4$. It follows that $|B_2| = 0$, since otherwise $|B_2| = |B_1| = 2$ and then $x_a(B) = 2 = x_a'(b_1b_1') + x_a'(b_2) + x_a'(b_2') > 2$, where $B_1 = b_1b_1'$ and $B_2 = b_2b_2'$. Set $B = b_1b_2b_3b_4$ with all $\overline{b_i} \in e_0 + \langle\overline{W_{\mathscr{F}}}\rangle$ and we have $1 \leqslant x_a'(b_ib_j) \leqslant 4$ for $1 \leqslant i < j \leqslant 4$. Then $x_a(B) = 2 = x_a'(b_ib_j) + x_a'(B(b_ib_j)^{-1}) \geqslant 2$ i.e., $x_a'(b_ib_j) = 1$. It follows from Lemma 17 (ii) that $x_a'(b_i) = \frac{k+1}{2}$ for $1 \leqslant i \leqslant 4$. Without loss of generality, we can set

$$B = e_0 + (e_1' + \frac{k+1}{2}a)(e_2' + \frac{k+1}{2}a)(e_3' + \frac{k+1}{2}a)(e_1' + e_2' + e_3' + \frac{k+1}{2}a),$$

where each e_i' is of order two with $\langle\overline{e_1'}, \overline{e_2'}, \overline{e_3'}\rangle = \langle\overline{W_{\mathscr{F}}}\rangle = \langle e_1, e_2, e_3\rangle$. Similarly, we can set

$$C = e_0' + (e_1'' + \frac{k+1}{2}a)(e_2'' + \frac{k+1}{2}a)(e_3'' + \frac{k+1}{2}a)(e_1'' + e_2'' + e_3'' + \frac{k+1}{2}a),$$

where each e_i'' is of order two with $\langle\overline{e_1''}, \overline{e_2''}, \overline{e_3''}\rangle = \langle\overline{W_{\mathscr{F}}}\rangle = \langle e_1, e_2, e_3\rangle$.

We claim that at least one of $\{\overline{e_1'' + e_2''}, \overline{e_1'' + e_3''}, \overline{e_2'' + e_3''}\}$ equal to $\overline{e_i' + e_j'}$ for some $1 \leqslant i < j \leqslant 3$. If not, then we have $\{\overline{e_1'' + e_2''}, \overline{e_1'' + e_3''}, \overline{e_2'' + e_3''}\} \subset \{\overline{e_1'}, \overline{e_2'}, \overline{e_3'}, \overline{e_1' + e_2' + e_3'}\}$.

Since $\overline{e_1''}, \overline{e_2''}, \overline{e_3''}$ are distinct, we have that $\overline{e_1''+e_2''}, \overline{e_1''+e_3''}, \overline{e_2''+e_3''}$ are distinct. Thus two of $\{\overline{e_1''+e_2''}, \overline{e_1''+e_3''}, \overline{e_2''+e_3''}\}$ are contained in $\{\overline{e_1'}, \overline{e_2'}, \overline{e_3'}\}$, say $\overline{e_1''+e_2''} = \overline{e_1'}$ and $\overline{e_1''+e_3''} = \overline{e_2'}$, which implies that $\overline{(e_1''+e_2'')} + \overline{(e_1''+e_3'')} = \overline{e_2''+e_3''} = \overline{e_1'+e_2'}$. This is a contradiction. Without loss of generality, let $\overline{e_1''+e_2''} = \overline{e_1'+e_3'}$. Furthermore, we have $e_1''+e_2'' = e_1'+e_3'$. Assume to contrary that $e_1''+e_2'' = e_1'+e_3'+ka$. Take $X = (e_0+e_1'+\frac{k+1}{2}a)(e_0+e_3'+\frac{k+1}{2}a) \mid B$ and $Z = (e_0'+e_1''+\frac{k+1}{2}a)(e_0'+e_2''+\frac{k+1}{2}a) \mid C$. Then $\sigma(XZ) = (k+2)a$, i.e., XZ is $(4, k+2)$. Lemma 5 implies that $4 \geqslant k+2$, which is impossible. Hence,

$$\begin{aligned}\sigma((e_0 + e_1'+\frac{k+1}{2}a)(e_0 + e_2' + \frac{k+1}{2}a)\\ (e_0' + e_1'' + \frac{k+1}{2}a)(e_0' + e_2'' + \frac{k+1}{2}a)) = e_2' + e_3' + 2a.\end{aligned} \tag{6}$$

Take $X = (e_0 + e_2' + \frac{k+1}{2}a)(e_0 + e_3' + \frac{k+1}{2}a) \mid B$. Since $\sigma(\overline{X}) \in \langle \overline{W_{\mathscr{F}}} \rangle$, there exists a proper $Y \mid W_{\mathscr{F}}$ such that YX is a block. From $\sigma(X) = (e_2' + e_3' + ka) + a$ it is easy to see that $x_a'(X) = 1$ and $\delta(X) = 0$. Set $\sigma(X) = e+a$ with $e = e_2'+e_3'+ka$. Let $Y = Y_1^* \cdots Y_m^*$, where $Y_i^* \mid T_i$. Since each T_i is a $(*, 1)$-block, we have $\delta(Y_i^*) \geqslant 1$ is odd. Since $\delta(X) = 0$, $x_a(B) = 2$ and $x_a(W_{\mathscr{F}}) = m$, by Lemma 6 (i) we have that

$$m \leqslant \sum_{i=1}^{m} \delta(Y_i^*) = \delta(Y) + \delta(X) \leqslant x_a(W_{\mathscr{F}}) + x_a(B) - 2 = m,$$

i.e., $\delta(Y) = m$. It follows from Lemma 6 (ii) that

$$\{\sigma(Y), \sigma(W_{\mathscr{F}}Y^{-1})\} = \{e + \frac{1}{2}(x_a(W_{\mathscr{F}}) - \delta(Y))a, e + \frac{1}{2}(x_a(W_{\mathscr{F}}) + \delta(Y))a\} = \{e, e + ma\}.$$

Then $\sigma(Y) = e$. Combining (6) yields that

$$\begin{aligned}&\sigma(Y(e_0 + e_1' + \frac{k+1}{2}a)(e_0 + e_2' + \frac{k+1}{2}a)\\ &(e_0' + e_1'' + \frac{k+1}{2}a)(e_0' + e_2'' + \frac{k+1}{2}a)) = e + e_2' + e_3' + 2a = (k+2)a,\end{aligned}$$

i.e, $Y(e_0 + e_1' + \frac{k+1}{2}a)(e_0 + e_2' + \frac{k+1}{2}a)(e_0' + e_1'' + \frac{k+1}{2}a)(e_0' + e_2'' + \frac{k+1}{2}a) = (|Y| + 4, k+2)$. Since $d(W_{\mathscr{F}}) = 3$ and $d(T_i) \geqslant 1$, by the additivity of defect, we have $d(W_{\mathscr{F}}) = 3 = |W_{\mathscr{F}}| - x_a(W_{\mathscr{F}}) = |W_{\mathscr{F}}| - m = \sum_{i=1}^{m} d(T_i) \geqslant m$. This implies that $m \leqslant 3$ and $|W_{\mathscr{F}}| \leqslant 6$. By Lemma 5 we have that $|Y|+4 \geqslant k+2$, i.e., $k-2 \leqslant |Y| \leqslant |W_{\mathscr{F}}| \leqslant 6$. This is impossible and the proof is completed. □

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