Doubled patterns with reversal and square-free doubled patterns

Antoine Domenech
ENS de Lyon, Lyon, France
antoine.domenech@ens-lyon.fr

Pascal Ochem
LIRMM, CNRS, Université de Montpellier
Montpellier, France
ochem@lirmm.fr

Submitted: Oct 10, 2022; Accepted: Jan 12, 2023; Published: Mar 24, 2023

Abstract

In combinatorics on words, a word w over an alphabet Σ is said to avoid a pattern p over an alphabet Δ if there is no factor f of w such that $f = h(p)$ where $h : \Delta^* \to \Sigma^*$ is a non-erasing morphism. A pattern p is said to be k-avoidable if there exists an infinite word over a k-letter alphabet that avoids p. A pattern is *doubled* if every variable occurs at least twice. Doubled patterns are known to be 3-avoidable. Currie, Mol, and Rampersad have considered a generalized notion which allows variable occurrences to be reversed. That is, $h(V^R)$ is the mirror image of $h(V)$ for every $V \in \Delta$. We show that doubled patterns with reversal are 3-avoidable. We also conjecture that (classical) doubled patterns that do not contain a square are 2-avoidable. We confirm this conjecture for patterns with at most 4 variables. This implies that for every doubled pattern p, the growth rate of ternary words avoiding p is at least the growth rate of ternary square-free words. A previous version of this paper containing only the first result has been presented at WORDS 2021.

Keywords: Combinatorics on words, Pattern avoidance.

Mathematics Subject Classifications: 68R15

1 Introduction

The *mirror image* of the word $w = w_1w_2\dots w_n$ is the word $w^R = w_nw_{n-1}\dots w_1$. A pattern with reversal p is a non-empty word over an alphabet $\Delta = \{A, A^R, B, B^R, C, C^R \dots\}$ such that $\{A, B, C, \dots\}$ are the *variables* of p. An *occurrence* of p in a word w is a non-erasing morphism $h : \Delta^* \to \Sigma^*$ satisfying $h(X^R) = (h(X))^R$ for every variable X and such that $h(p)$ is a factor of w. The avoidability index $\lambda(p)$ of a pattern with reversal p is the size of the smallest alphabet Σ such that there exists an infinite word w over Σ containing no occurrence of p. A pattern p such that $\lambda(p) \leqslant k$ is said to be k-avoidable. To emphasive that a pattern is without reversal (i.e., it contains no X^R), it is said to be *classical*. A pattern is *doubled* if every variable occurs at least twice.

Our aim is to strengthen the following result.

Theorem 1. *[3, 10, 11] Every doubled pattern is* 3*-avoidable.*

First, we extend it to patterns with reversal.

Theorem 2. *Every doubled pattern with reversal is* 3*-avoidable.*

Then, we notice that all the known classical doubled patterns that are 2-unavoidable contain a square, such as $AABB$, $ABAB$, or $ABCCBADD$.

Conjecture 3. Every square-free doubled pattern is 2-avoidable.

Notice that Conjecture 3 is related to but independent of the following conjecture.

Conjecture 4. [11, 13] There exist only finitely many 2-unavoidable doubled patterns.

The proof of Conjecture 3 for patterns up to 3 variables follows from the 2-avoidability of $ABACBC$, $ABCBABC$, $ABCACB$ and $ABCBAC$ since every square-free doubled pattern with 3 variables contains one of these patterns as factor. We were able to verify it for patterns up to 4 variables.

Theorem 5. *Every square-free doubled pattern with at most 4 variables is* 2*-avoidable.*

Finally, we obtain a lower bound on the number of ternary words avoiding a doubled pattern. The factor complexity of a factorial language L over Σ is $f(n) = |L \cap \Sigma^n|$. The growth rate of L over Σ is $\lim_{n\to\infty} f(n)^{\frac{1}{n}}$. We denote by $GR_3(p)$ the growth rate of ternary words avoiding the doubled pattern p.

Theorem 6. *For every doubled pattern p, $GR_3(p) \geqslant GR_3(AA)$.*

Let $v(p)$ be the number of distinct variables of the pattern p. In the proof of Theorem 1, the set of doubled patterns is partitioned as follows:

1. Patterns with $v(p) \leqslant 3$: the avoidability index of every ternary pattern has been determined [10].
2. Patterns shown to be 3-avoidable with the so-called power series method:
   - Patterns with $v(p) \geqslant 6$ [3]
   - Patterns with $v(p) = 5$ and prefix ABC or length at least 11 [11]
   - Patterns with $v(p) = 4$ and prefix $ABCD$ or length at least 9 [11]
3. Ten sporadic patterns with $4 \leqslant v(p) \leqslant 5$ whose 3-avoidability cannot be deduced from the previous results: they have been shown to be 2-avoidable [11] using the method in [10].

The proofs of Theorems 2 and 6 use the same partition. Sections 3 to 5 are each is devoted to one type of doubled pattern with reversal. Theorem 5 is proved in Section 6 Theorem 6 is proved in Section 7

2 Preliminaries

A variable that appears only once in a pattern is said to be *isolated*. The *formula* f associated to a pattern p is obtained by replacing every isolated variable in p by a dot. The factors between the dots are called *fragments*. An occurrence of a formula f in a word w is a non-erasing morphism h such that the h-image of every fragment of f is a factor of w. As for patterns, the avoidability index $\lambda(f)$ of a formula f is the size of the smallest alphabet allowing the existence of an infinite word containing no occurrence of f. Recently, the avoidability of formulas with reversal has been considered by Currie, Mol, and Rampersad [6, 7] and Ochem [12].

A word w is *d-directed* if for every factor f of w of length d, the word f^R is not a factor of w.

Remark 7. If a d-directed word contains an occurrence h of $X.X^R$ for some variable X, then $|h(X)| \leqslant d-1$.

Recall that a formula is *nice* if every variable occurs at least twice in the same fragment. In particular, a doubled pattern is a nice formula with exactly one fragment.

The *avoidability exponent* $AE(f)$ of a formula f is the largest real x such that every x-free word avoids f. Every nice formula f with $v(f) \geqslant 3$ variables is such that $AE(f) \geqslant 1 + \frac{1}{2v(f)-3}$ [15].

Let $\simeq$ be the equivalence relation on words defined by $w \simeq w'$ if $w' \in \{w, w^R\}$. Avoiding a pattern up to $\simeq$ has been investigated for every binary formula [5]. Remark that for a given classical pattern or formula p, avoiding p up to $\simeq$ implies avoiding simultaneously all the variants of p with reversal.

Recall that a word is (β^+, n)-free if it contains no repetition with exponent strictly greater than β and period at least n. Finally, the repetition threshold $RT(n)$ is the smallest real number α such that there exists an infinite α^+-free word over Σ_n. We denote by w_k any infinite $RT(k)^+$-free word over Σ_k (w_k is often called a Dejean word).

A morphism is m given in the format $m(0)/m(1)/...$

We denote by b_2, b_3, b_4, and b_5, respectively, the fixed-point of the well-known morphisms:

- 01/10 [17],
- 012/02/1 [8],
- 01/21/03/23 [2]
- 01/23/4/21/0 [1]

3 Formulas with at most 3 variables

For classical doubled patterns with at most 3 variables, all the avoidability indices are known. There are many such patterns, so it would be tedious to consider all their variants with reversal.

However, we are only interested in their 3-avoidability, which follows from the 3-avoidability of nice formulas with at most 3 variables [14].

Thus, to obtain the 3-avoidability of doubled patterns with reversal with at most 3 variables, we show that every minimally nice formula with at most 3 variables is 3-avoidable up to $\simeq$.

The minimally nice formulas with at most 3 variables, up to symmetries, are determined in [14] and listed in the following table. Every such formula f is avoided by the image by a q-uniform morphism of either any infinite $\left(\frac{5}{4}^+\right)$-free word w_5 over Σ_5 or any infinite $\left(\frac{7}{5}^+\right)$-free word w_4 over Σ_4, depending on whether the avoidability exponent of f is smaller than $\frac{7}{5}$.

Formula f	$= f^R$	$AE(f)$	q	d	freeness
$ABA.BAB$	yes	1.5	9	9	$\left(\frac{131}{90}^+, 28\right)$
w_4; 002112201/001221122/001220112/001122012					
$ABCA.BCAB.CABC$	yes	1.333333333	6	8	$\left(\frac{4}{3}^+, 25\right)$
w_5; 021221/021121/020001/011102/010222					
$ABCBA.CBABC$	yes	1.333333333	4	9	$\left(\frac{30}{23}^+, 18\right)$
w_5; 2011/1200/1120/0222/0012					
$ABCA.BCAB.CBC$	no	1.381966011	9	4	$\left(\frac{62}{45}^+, 37\right)$
w_5; 020112122/020101112/020001222/010121222/000111222					
$ABA.BCB.CAC$[1]	yes	1.5	9	4	$\left(\frac{67}{45}^+, 37\right)$
w_4; 001220122/001220112/001120122/001120112					
$ABCA.BCAB.CBAC$	yes[2]	1.333333333	6	6	$\left(\frac{31}{24}^+, 31\right)$
w_5; 012220/012111/012012/011222/010002					
$ABCA.BAB.CAC$	yes	1.414213562	6	8	$\left(\frac{89}{63}^+, 61\right)$
w_4; 021210/011220/002111/001222					
$ABCA.BAB.CBC$	no	1.430159709	6	7	$\left(\frac{17}{12}^+, 61\right)$
w_4; 011120/002211/002121/001222					
$ABCA.BAB.CBAC$	no	1.381966011	8	7	$\left(\frac{127}{96}^+, 41\right)$
w_5; 01222112/01112022/01100022/01012220/01012120					
$ABCBA.CABC$	no	1.361103081	6	8	$\left(\frac{4}{3}^+, 25\right)$
w_5; 021121/012222/011220/011112/000102					
$ABCBA.CAC$	yes	1.396608253	6	13	$\left(\frac{4}{3}^+, 25\right)$
w_5; 022110/021111/012222/012021/011220					

[1] The formula $ABA.BCB.CAC$ seems to be also avoided up to $\simeq$ by b_3.

[2] We mistakenly said in [14] that $ABCA.BCAB.CBAC$ is different from its reverse.

In the table above, the columns indicate respectively, the considered minimally nice formula f, whether f is equivalent to its reversed formula, the avoidability exponent of f, the value q such that the corresponding morphism is q-uniform, the value such that the avoiding word is d-directed, the suitable property of (β^+, n)-freeness used in the proof that f is avoided. The second line contains the infinite ternary word avoiding f.
As an example, we show that $ABCBA.CAC$ is avoided by $g(w_5)$, where
$g = \mathtt{022110/021111/012222/012021/011220}$. First, we check that $g(w_5)$ is $\left(\frac{4}{3}^+, 25\right)$-free using the main lemma in [10], that is, we check the $\left(\frac{4}{3}^+, 25\right)$-freeness of the g-image of every $\left(\frac{5}{4}^+\right)$-free word of length at most $\frac{2\times\frac{4}{3}}{\frac{4}{3}-\frac{5}{4}} = 32$. Then we check that $g(w_5)$ is 13-directed by inspecting the factors of $g(w_5)$ of length 13. For contradiction, suppose that $g(w_5)$ contains an occurrence h of $ABCBA.CAC$ up to $\simeq$. Let us write $a = |h(A)|$, $b = |h(B)|$, $c = |h(C)|$.

Suppose that $a \geqslant 25$. Since $g(w_5)$ is 13-directed, all occurrences of $h(A)$ are identical. Then $h(ABCBA)$ is a repetition with period $|h(ABCB)| \geqslant 25$. So the $\left(\frac{4}{3}^+, 25\right)$-freeness implies the bound $\frac{2a+2b+c}{a+2b+c} \leqslant \frac{4}{3}$, that is, $a \leqslant b + \frac{1}{2}c$.

In every case, we have
$$a \leqslant \max\left\{b + \tfrac{1}{2}c, 24\right\}.$$
Similarly, the factors $h(BCB)$ and $h(CAC)$ imply
$$b \leqslant \max\left\{\tfrac{1}{2}c, 24\right\}$$
and
$$c \leqslant \max\left\{\tfrac{1}{2}a, 24\right\}.$$
Solving these inequalities gives $a \leqslant 36$, $b \leqslant 24$, and $c \leqslant 24$. Now we can check exhaustively that $g(w_5)$ contains no occurrence up to $\simeq$ satisfying these bounds.

Except for $ABCBA.CBABC$, the avoidability index of the nice formulas in the above table is 3. So the results in this section extend their 3-avoidability up to $\simeq$.

4 The power series method

The so-called power series method has been used [3, 11] to prove the 3-avoidability of many classical doubled patterns with at least 4 variables and every doubled pattern with at least 6 variables, as mentioned in the introduction.

Let p be such a classical doubled pattern and let p' be a doubled pattern with reversal obtained by adding some $-^R$ to p. Without loss of generality, the leftmost appearance of every variable X of p remains free of $-^R$ in p'. Then we will see that p' is also 3-avoidable. The power series method is a counting argument that relies on the following observation. If the h-image of the leftmost appearance of the variable X of p is fixed, say $h(X) = w_X$, then there is exactly one possibility for the h-image of the other appearances of X, namely $h(X) = w_X$. This observation can be extended to p', since there is also exactly one possibility for $h(X^R)$, namely $h(X^R) = w_X^R$.

Notice that this straightforward generalization of the power series method from classical doubled patterns to doubled patterns with reversal cannot be extended to avoiding a doubled pattern up to $\simeq$. Indeed, if $h(X) = w_X$ for the leftmost appearance of the variable X and w_X is not a palindrome, then there exist two possibilities for the other appearances of X, namely w_X and w_X^R.

5 Sporadic patterns

Up to symmetries, there are ten doubled patterns whose 3-avoidability cannot be deduced by the previous results. They have been identified in [11] and are listed in Table 1.

Doubled pattern	Avoidability exponent
$ABACBDCD$	1.381966011
$ABACDBDC$	1.333333333
$ABACDCBD$	1.340090632
$ABCADBDC$	1.292893219
$ABCADCBD$	1.295597743
$ABCADCDB$	1.327621756
$ABCBDADC$	1.302775638
$ABACBDCEDE$	1.366025404
$ABACDBCEDE$	1.302775638
$ABACDBDECE$	1.320416579

Table 1: The seven sporadic patterns on 4 variables and the three sporadic patterns on 5 variables

Let h be the 9-uniform morphism

$$\texttt{020022221/011111221/010202110/010022112/000022121}.$$

Using the same method as in Section 3, we show that $h(w_5)$ avoids up to $\simeq$ these ten sporadic patterns simultaneously. The suitable properties are that $h(w_5)$ is 7-directed and $\left(\frac{139}{108}^+, 46\right)$-free.

6 Square-free doubled patterns with at most 4 variables

Here we show Theorem 5, that is, every square-free doubled pattern with at most 4 variables is 2-avoidable. We list them as follows:

- Among patterns that are equal up to letter permutation, we only list the lexicographically least.

- If a pattern is distinct from its mirror image, we only list the lexicographically least among the pattern and its mirror image.
- We do not include the seven sporadic patterns on 4 variables from Table 1, which are 2-avoidable [11].
- We do not list patterns that contain an occurrence of a (strictly smaller) square-free doubled pattern.

Table 2 contains every pattern p in this list with an infinite binary word avoiding p. Let us detail how to read Table 2:

- If the avoiding word is a pure morphic word $m^\omega(0)$, then m is given.
- If the avoiding word is a morphic word $f(m^\omega(0))$, then we write m; f.
- If the avoiding word is of the form $f(w_k)$, then we write w_k; f.

The proofs that a (pure) morphic word avoids a pattern use Cassaigne's algorithm [4] and the proofs that a morphic image word a Dejean word avoids a pattern use the technique described in Section 3.

7 Growth rate of ternary words avoiding a doubled pattern

Theorem 6 obviously holds for $p = AA$. Without loss of generality, we do not need to consider a doubled pattern p that contains an occurrence of another doubled pattern. In particular, p is square-free. So we need to show that $GR_3(p)$ is at least $GR_3(AA)$, which is close to 1.30176 [16].

If p is 2-avoidable, then p is avoided by sufficiently many ternary words. By Lemma 4.1 in [10], $\lambda(p) = 2$ implies that $GR_3(p) \geqslant 2^{\frac{1}{2}} > GR_3(AA)$. Thus, Conjecture 3 implies Theorem 6. By Theorem 5, we can assume that $v(p) \geqslant 5$. We can also rule out the three sporadic patterns on 5 variables from Table 1, which are 2-avoidable [11].

According to the partition of the set of doubled patterns mentioned in the introduction, there remains to consider the doubled patterns p whose 3-avoidability has been obtained via the power series method. In that case, we even get $GR_3(p) > 2 > GR_3(AA)$.

8 Conclusion

Unlike classical formulas, we know that there exist avoidable formulas with reversal of arbitrarily high avoidability index [12]. Maybe doubled patterns and nice formulas are easier to avoid. We propose the following open problems.

- Are there infinitely many doubled patterns up to $\simeq$ that are not 2-avoidable?
- Is there a nice formula up to $\simeq$ that is not 3-avoidable?

Doubled pattern p	Binary word avoiding p
ABCABDCBD	w_5; 0010101110/0010011000/0001111110/0001110101/0000011001
ABCACDCBD	w_5; 000101010111/000100110111/000011001111/000001011111/000000111111
ABCBABDBCBD	b_4; 01/00/10/11
ABCBADCBCD	b_4; 0000/0011/1111/1010
ABCBDACBCD	b_4; 01/00/10/11
ABCBDBCACBCD	b_2
ABCBDCBACBCD	b_4; 1000/0111/0110/0010
ABCDACBD	w_5; 0010011011111000/0010011011011000/ 00011110110101010/0000111111011010/0000101010101111
ABCDBACBD	w_6; 010101111100/010010100000/001001110111/000111111101/000101010111/000100011011
ABCDBADC	w_5; 10001000101111101010110/00000110110101000111111/ 00000101011100100111111/00000011101010010011111/00000011011000101011111
ABCDBCBACBD	001/011
ABCDCACBD	b_5; 0011110110000/0011010100110/0001111100111/0001110001000/0001101101111
ABCDCBABCD	Every occurrence h of p is a repetition with period $\|h(ABCDCB)\| \geqslant 6$ and exponent $\frac{\|h(ABCDCBABCD)\|}{\|h(ABCDCB)\|} = \frac{2a+3b+3c+2d}{a+2b+2c+d} = \frac{3}{2} + \frac{a+d}{2(a+2b+2c+d)} > \frac{3}{2}$. Thus, every $\left(\frac{4}{3}^+, 6\right)$-free binary word avoids p [9].
ABCDCBCACBD	b_5; 00/01/10/110/111
ABACDCBCD	w_5; 10011011000/01011111000/00111010100/00100100111/00001111111
ABCABDBCD	w_5; 0010111111/0010011110/0010011100/0000010101/0000001101
ABCADBCBD	w_5; 001011010000/001001111000/000110011001/000011101010/000010111111
ABCADCBCD	w_5; 001101111000/001101101000/001001111111/000101110101/000001100101
ABCBADBDC	w_5; 0011111110110/0001010111100/0000101101110/0000011010111/0000001011111
ABCBDABCD	w_4; 1111/1101/0010/0000
ABCBDABDC	w_5; 101110000001/101100100001/011111110100/010001111110/010001101110
ABCBDACBD	b_5; 00/01/10/110/111
ABCBDADBC	w_5; 00110111010010/00110000000010/00011111111011/00011110101000/00010101100011
ABCBDADBDC	b_5; 111/101/000/011/001
ABCBDBABDBC	b_5; 00/01/10/110/111
ABCBDBABDC	b_5; 000/011/001/111/101
ABCBDBACBCD	b_4; 01/00/10/11
ABCBDBACD	w_5; 0001111101010/0001110111000/0001011111111/0000111001111/0000011011001
ABCBDBADBDC	011/100
ABCBDBADC	w_5; 00111101110000/00111011000010/00111010100000/00011001001111/00010101111111
ABCBDBCABCD	b_5; 00/01/10/110/111
ABCBDBCACBD	b_5; 00/01/10/110/111
ABCBDBCAD	w_5; 00011110110011/00011101101001/00011011010100/00010111111110/00000011111010
ABCBDBCBABCD	b_4; 000/111/10/01
ABCBDBCBACBCD	b_2
ABCBDBCBACBD	b_4; 00/01/10/11
ABCBDBCBACD	001/110
ABCBDCABCD	b_5; 00/10/111/01/011
ABCBDCABD	w_5; 10000000011/01111010010/01101100010/01011111110/00001010101
ABCBDCACBD	b_5; 111/101/000/100/110
ABCBDCBABCD	b_5; 00/01/10/110/111
ABCBDCBACBD	b_5; 00/01/10/110/111
ABCBDCBACD	b_5; 00/01/10/1100/111
ABCBDCBAD	w_5; 001101101100/001011111111/001001111100/000110010100/000001110100
ABCBDCBCABD	b_4; 000/111/10/01
ABCBDCBCAD	w_5; 1111100/1100110/0110101/0010010/0000101
ABCDADCB	w_5; 0000010001111110101000100111110111/0000010001111100100001100101101111/ 0000001001111111010000110101111011/0000001001111110110100010101111011/ 000000010111001000011111010010111
ABCDBABDC	w_5; 0011111110101/0010110111010/0010101110000/0000111111001/0000110110001
ABCDBADBC	w_5; 01011111111/01001000111/00101000011/00011110101/00000001011
ABCDBCACBD	b_5; 101/000/110/111/100
ABCDBCBACD	w_5; 0110101/0100000/0011110/0001111/0000111
ABCDBCBAD	w_5; 00010111001010/00001111010101/00001110001010/00001100111111/00001100010110
ABCDBDAC	w_5; 00000011011011001110001111011010110000101111010100100101110111/ 00000011011011000010011110110101000010101111010100100101110111/ 00000010110011110101010011000111000010101111010100100101110111/ 00000010101101101000100011111101000010101111010100100101110111/ 00000010101011001110001111010011000010101111010100100101110111
ABCDBDADBC	w_5; 01111101/00111100/00111001/00110110/00000101
ABCDCACDB	w_5; 00110001000110/00101011111110/00011111010011/00010101011111/00000001010011

Table 2: Binary words avoiding doubled patterns

A first step would be to improve Theorem 2 by generalizing the 3-avoidability of doubled patterns with reversal to doubled patterns up to $\simeq$. Notice that the results in Sections 3 and 5 already consider avoidability up to $\simeq$. However, the power series method gives weaker results. Classical doubled patterns with at least 6 variables are 3-avoidable because

$$1 - 3x + \left(\frac{3x^2}{1-3x^2}\right)^v$$

has a positive real root for $v \geqslant 6$. The (basic) power series for doubled patterns up to $\simeq$ with v variables would be

$$1 - 3x + \left(\frac{6x^2}{1-3x^2} - \frac{3x^2+3x^4}{1-3x^4}\right)^v.$$

The term $\frac{6x^2}{1-3x^2}$ counts for twice the term $\frac{3x^2}{1-3x^2}$ in the classical setting, for $h(V)$ and $h(V)^R$. The term $\frac{3x^2+3x^4}{1-3x^4}$ corrects for the case of palindromic $h(V)$, which should not be counted twice. This power series has a positive real root only for $v \geqslant 10$. This leaves many doubled patterns up to $\simeq$ whose 3-avoidability must be proved with morphisms.

Looking at the proof of Theorem 2, we may wonder if a doubled pattern with reversal is always easier to avoid than the corresponding classical pattern. This is not the case: backtracking shows that $\lambda(ABCA^RC^RB) = 3$, whereas $\lambda(ABCACB) = 2$ [10].

To get a more precise version of both conjectures 3 and 4, we plan to obtain the (conjectured) list of all 2-unavoidable doubled patterns, which should be a finite list containing no square-free pattern.

References

[1] G. Badkobeh and P. Ochem. Characterization of some binary words with few squares. *Theoret. Comput. Sci.* **588** (2015), 73–80.

[2] K.A. Baker, G.F. McNulty, and W. Taylor. Growth problems for avoidable words, *Theoret. Comput. Sci.* **69** (1989), 319–345.

[3] J. Bell, T. L. Goh. Exponential lower bounds for the number of words of uniform length avoiding a pattern. *Inform. and Comput.* **205** (2007), 1295-1306.

[4] J. Cassaigne. *An Algorithm to Test if a Given Circular HD0L-Language Avoids a Pattern. IFIP Congress*, pages 459–464, 1994.

[5] J. Currie and L. Mol. The undirected repetition threshold and undirected pattern avoidance. *Theor. Comput. Sci.* **866** (2021), 56–69.

[6] J. Currie, L. Mol, and N. Rampersad. A family of formulas with reversal of high avoidability index. *International Journal of Algebra and Computation* **27(5)** (2017), 477–493.

[7] J. Currie, L. Mol, and N. Rampersad. Avoidance bases for formulas with reversal. *Theor. Comput. Sci.* **738** (2018), 25–41.

[8] M. Hall, Jr., "Generators and relations in groups–The Burnside problem". In *Lectures on Modern Mathematics, Vol. 2*, Wiley, New York, pp. 42–92.

[9] L. Ilie, P. Ochem, and J.O. Shallit. A generalization of repetition threshold. *Theoret. Comput. Sci.* **92(2)** (2004), 71–76.

[10] P. Ochem. A generator of morphisms for infinite words. *RAIRO: Theoret. Informatics Appl.* **40** (2006), 427–441.

[11] P. Ochem. Doubled patterns are 3-avoidable. *Electron. J. Combinatorics.* **23(1)** (2016), #P1.19.

[12] P. Ochem. A family of formulas with reversal of arbitrarily high avoidability index. *Theoret. Comput. Sci.* **896** (2021), 168–170.

[13] P. Ochem and A. Pinlou. Application of entropy compression in pattern avoidance. *Electron. J. Combinatorics.* **21(2)**:#RP2.7, (2014).

[14] P. Ochem and M. Rosenfeld. On some interesting ternary formulas. *Electron. J. Combininatorics.* **26(1)**:#P1.12 (2019).

[15] P. Ochem and M. Rosenfeld. Avoidability of palindrome patterns. *Electron. J. Combininatorics.* **28(1)**:#P1.4, (2021).

[16] A. Shur. Growth rates of complexity of power-free languages. *Theoret. Comput. Sci.* **411(34-36)** (2010), 3209–3223.

[17] A. Thue. Über unendliche Zeichenreihen. *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiania*, 7:1–22, 1906.