Subgraph densities in K_r-free graphs

Andrzej Grzesik*
Faculty of Mathematics and Computer Science
Jagiellonian University
Kraków, Poland
Andrzej.Grzesik@uj.edu.pl

Ervin Győri[†]
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary
gyori.ervin@renyi.hu

Nika Salia[‡]
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary
Extremal Combinatorics and Probability Group
Institute for Basic Science
Daejeon, South Korea
salianika@gmail.com

Casey Tompkins[§]
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, Hungary
casey@renyi.hu

Submitted: Jun 14, 2022; Accepted: Jan 19, 2023; Published: Mar 24, 2023

Abstract

A counterexample to a recent conjecture of Lidický and Murphy on the structure of K_r-free graph maximizing the number of copies of a given graph with chromatic number at most $r-1$ is known in the case $r=3$. Here, we show that this conjecture does not hold for any r, and that the structure of extremal graphs can be richer. We also provide an alternative conjecture and, as a step towards its proof, we prove an asymptotically tight bound on the number of copies of any bipartite graph of radius at most 2 in the class of triangle-free graphs.

Mathematics Subject Classifications: 05C35

*Supported by the National Science Centre grant 2021/42/E/ST1/00193.
[†]Supported by the National Research, Development and Innovation Office NKFIH grants K132696 and SNN-135643.
[‡]Supported by the National Research, Development and Innovation Office NKFIH grants K132696 and SNN-135643, and by the Institute for Basic Science (IBS-R029-C4).
[§]Supported by the National Research, Development and Innovation Office NKFIH grant K135800.

For graphs H and F the generalized Turán number $\mathrm{ex}(n, H, F)$ is defined to be the maximum number of (not necessarily induced) copies of H in an n-vertex graph G which does not contain F as a subgraph. Estimating $\mathrm{ex}(n, H, F)$ for various pairs H and F has been a central topic of research in extremal combinatorics. The case when H and F are both cliques was settled early on by Zykov [13] and independently by Erdős [2]. The problem of maximizing 5-cycles in a triangle-free graph was a long-standing open problem. The problem was finally settled by Grzesik [5] and independently by Hatami, Hladký, Král, Norine and Razborov [9]. In the case when the forbidden graph F is a triangle and H is any bipartite graph containing a matching on all but at most one of its vertices, $\mathrm{ex}(n, H, F)$ was determined exactly by Győri, Pach and Simonovits [6] in 1991. More recently there has been extensive work on the topic following the work of Alon and Shikhelman [1], who showed various properties of the extremal function $\mathrm{ex}(n, H, F)$ for general pairs H and F.

We now introduce some further notation that we will require in the statements and proofs of our main results. For a graph G, the vertex set of G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. We also write $v(G) = |V(G)|$ and $e(G) = |E(G)|$. We denote the path, cycle, and complete graph on r vertices by P_r, C_r, and K_r, respectively. The complete multipartite graph with $r \geqslant 2$ parts of sizes $n_1, n_2, \ldots, n_r$ is denoted by $K_{n_1,n_2,\ldots,n_r}$. In the case when each n_i differs by at most one from the others the n-vertex graph is referred to as the Turán graph and is denoted by $T_r(n)$. For a graph H, the k^{th} power of H, denoted H^k, is defined to be the graph with vertex set $V(H)$ and with an edge between vertices of distance at most k in H. For graphs G and H, the number of labeled copies of H in G is denoted by $H^*(G)$, and the number of unlabelled copies of H in G is denoted by $H(G)$. In particular we we have that $H^*(G)/H(G) = |\operatorname{Aut}(H)|$ where $\operatorname{Aut}(H)$ is the set of automorphisms of H.

Recently Lidický and Murphy proposed the following natural conjecture.

Conjecture 1 (Lidický, Murphy [11])**.** Let H be a graph and let r be an integer such that $r > \chi(H)$. Then there exist integers $n_1, n_2, \ldots, n_{r-1}$ such that $n_1 + n_2 + \cdots + n_{r-1} = n$ and

$$\mathrm{ex}(n, H, K_r) = H(K_{n_1,n_2,\ldots,n_{r-1}}).$$

Recently Morrison, Nir, Norin, Rzążewski and Wesolek [12] showed that for any graph H and large enough r, the maximum number of copies of H in a K_r-free n-vertex graph is obtained by the Turán graph $T_{r-1}(n)$, the balanced blow-up of K_{r-1}. In other words, the above conjecture works if r is enough large comparing to $\chi(H)$.

Using the graph removal lemma one can easily show that for any graphs H and F with $\chi(F) = r$ we have $\mathrm{ex}(n, H, F) \leqslant \mathrm{ex}(n, H, K_r) + o(n^{v(H)})$ (see [4]). Therefore, the above conjecture asymptotically determines $\mathrm{ex}(n, H, F)$ in the case $\chi(F) > \chi(H)$, which shows its importance. Unfortunately, the conjecture is not true in general. Indeed a counterexample when $r = 3$ already appeared in [6]. Here we give a counterexample for arbitrary r.

Theorem 2. *For every $r \geqslant 3$ there is a counterexample to Conjecture 1.*

Proof. First, we fix some constants later used for constructing a counterexample. Let ε be a positive real number such that $\varepsilon < \frac{1}{4r}$. Take a positive integer a for which

$$2\varepsilon^{2r-2}(1-(2r-2)\varepsilon)^{2a} > \frac{1}{2^{2a}}.$$

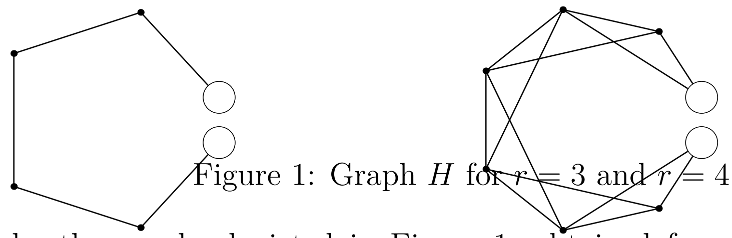

Figure 1: Graph H for $r = 3$ and $r = 4$.

Let H be the graph, depicted in Figure 1, obtained from P_{2r}^{r-2} by replacing each of the two vertices of degree $r-2$ with independent sets of size a each with the same neighborhood as the original vertex. We refer to these a vertices as copies of the terminal vertex. Note that there is a unique $(r-1)$-coloring of H, and the copies of different terminal vertices are in different color classes. For integers $n, n_1, n_2, \ldots, n_{r-1}$ such that $n = n_1 + n_2 + \cdots + n_{r-1}$, we have

$$H(K_{n_1,n_2,\ldots,n_{r-1}}) = \frac{1}{|\operatorname{Aut}(H)|} \cdot H^*(K_{n_1,n_2,\ldots,n_{r-1}}) \leqslant n^{2r-2}\left(\frac{n}{2}\right)^{2a}.$$

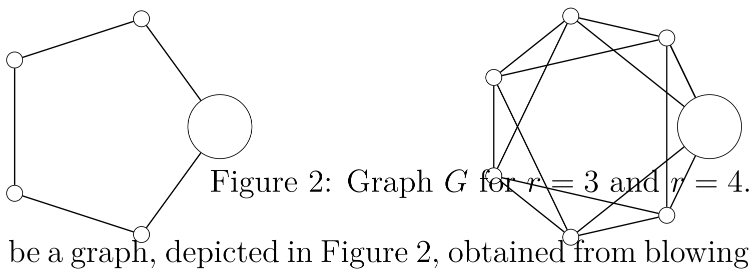

Figure 2: Graph G for $r = 3$ and $r = 4$.

Let G be a graph, depicted in Figure 2, obtained from blowing up C_{2r-1}^{r-2} in the following way. We replace each vertex with a disjoint independent set of size $\lfloor \varepsilon n \rfloor$ except for one vertex which we replace by an independent set A of size $n-(2r-2)\lfloor \varepsilon n \rfloor$. Note that G is an n-vertex graph and the number of labelled copies of H in G is at least

$$2(\lfloor \varepsilon n \rfloor)^{2r-2}(n-(2r-2)\lfloor \varepsilon n \rfloor)^{2a} + o(n^{2r+2a-2}).$$

Recall by the choice of a we have

$$2(\lfloor \varepsilon n \rfloor)^{2r-2}(n-(2r-2)\lfloor \varepsilon n \rfloor)^{2a} + o(n^{2r+2a-2}) > n^{2r-2}\left(\frac{n}{2}\right)^{2a}$$

for large enough n. Therefore for sufficiently large n the number of labeled copies, as well as unlabelled copies of H in G, is greater than the number in any n vertex $(r-1)$-partite graph. □

In the above counterexample when $r = 3$ a blow-up of a pentagon contains more copies of H than any complete bipartite graph. It may be natural to expect that for $r = 3$ the blow-up of a pentagon is the only obstacle, in particular, that for every bipartite graph H, $\mathrm{ex}(n, H, K_3)$ is asymptotically achieved by either a blow-up of an edge (that is, a complete bipartite graph) or a blow-up of a cycle of length five. Surprisingly this is not the case. Here we give an intuitive sketch of the argument.

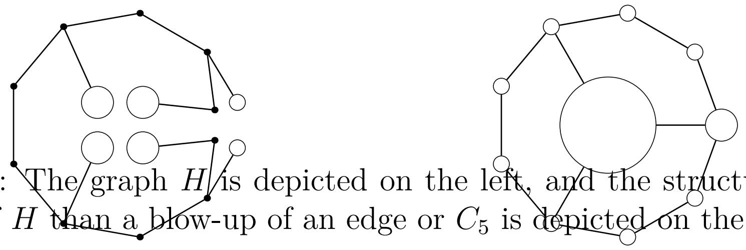

Figure 3: The graph H is depicted on the left, and the structure of a graph with more copies of H than a blow-up of an edge or C_5 is depicted on the right.

Let H be the first graph depicted in Figure 3 defined in the following way. We take a path on 10 vertices $v_1, v_2, \dots, v_{10}$, let A_2 and A_9 be big sets of y independent vertices attached to v_2 and v_9, accordingly, and let B_1, B_4, B_7 and B_{10} be huge sets of x independent vertices attached to the vertices v_1, v_4, v_7 and v_{10}, accordingly, where $x \gg y \gg 1$. If one wants to maximize the number of copies of H in a complete bipartite graph, then the huge sets B_1, B_7 will be mapped into one color class and the huge sets B_4 and B_{10} will be mapped into the other color class. Thus, the number of copies of H will be exponentially small in terms of x. If one wants to maximize the number of copies of H in a blow-up of a pentagon, then the largest number of such copies (the dominant term as a function of x) will be obtained when the vertices of big degree v_1, v_4, v_7 and v_{10} are mapped to blobs neighboring to the biggest blob. But then the two big sets A_2 and A_9 need to be mapped to different blobs and not to the largest blob. On the other hand, when one counts the number of copies of H in the graph depicted on the right in Figure 3, then the dominant term as a function of x will be obtained when the vertices of big degree v_1, v_4, v_7 and v_{10} are mapped to blobs neighboring to the largest blob, and in such a case it is still possible to map the sets A_2 and A_9 to one big part, so the dominant term as a function of y will be bigger than for the blow-up of a pentagon. Therefore, after fixing x and y to appropriate values, the maximum number of copies of H in a triangle-free graph will be achieved neither in a complete bipartite graph nor in a blow-up of a pentagon.

The main idea behind the counterexample we presented to Conjecture 1 is to have a graph with many vertices that cannot have the same color in any two-coloring but can be in the same part in a blow-up of a non-bipartite graph. One can avoid having such vertices by bounding the diameter of a graph, therefore it is natural to consider the following problem instead of Conjecture 1.

Conjecture 3. If G is a bipartite graph with diameter at most 4, then $\mathrm{ex}(n, G, K_3)$ is asymptotically achieved in a complete bipartite graph.

In the initial version of this paper, we proposed a more general conjecture for all graphs with the chromatic number r. In particular, our conjecture stated that for every graph G with the diameter at most $2r-2$ and $\chi(G) < r$ the maximum number of G in a K_r-free graph is asymptotically achieved by a blow-up of K_{r-1}. This conjecture was subsequently disproved by Keat and Mergoni in [10].

A first step towards Conjecture 3 for $r = 3$ is to prove it for all bipartite graphs of radius 2. Each such graph can be viewed as a star with additional adjacent vertices. Here we prove a slightly more general result, i.e., for bipartite graphs consisting of some complete bipartite graph and additional adjacent vertices.

Theorem 4. *Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$. Assume the distance of each vertex $v \in V(H)$ to $V(K)$ is at most one. Then the maximum number of copies of H in a triangle-free n-vertex graph is obtained asymptotically by a complete bipartite graph.*

Proof. We start proof with a simple observation. Let us assume that the maximum number of copies of a connected graph H' in a triangle-free n-vertex graph is obtained by a blow-up of an edge. Then for every bipartite graph H such that $H' \subseteq H$ we have that the maximum number of copies of H in a triangle-free n-vertex graph is obtained by a blow-up of an edge. Therefore we may assume that H consists of a complete bipartite graph $K_{s,t}$ with color classes S and T and some pendant edges. The number of pendant edges attached to the vertices of S are denoted $a_1, a_2, \dots, a_s$ and the number of pendant edges attached to vertices of T are denoted $b_1, b_2, \dots, b_t$.

For a graph H, we estimate the number of labeled copies of H in a graph G. First we fix a set of size s in G say $\{x_1, x_2, \dots, x_s\}$, onto which we will map the color class S of H. Let us denote the common neighborhood of $\{x_1, x_2, \dots, x_s\}$ in G by $X = \bigcap_{i=1}^{s} N_G(x_i)$. In the estimates below the vertices $x_1, x_2, \dots, x_s$ are variables, and therefore the common neighborhood is another variable. After the set $\{x_1, x_2, \dots, x_s\}$ is chosen we choose a permutation $\sigma \in S_s$ to map vertices of $\{x_1, x_2, \dots, x_s\}$ to the vertices of S. Next we choose vertices $y_1, y_2, \dots, y_t \in X$ as representatives of T. Finally, we choose the endpoints of the pendant edges. Note that during this process it is possible that we have chosen a vertex of G as a representative of more than one vertex of H, which does not qualify as a copy of H in G. Hence we overestimate here by $o(n^{v(H)})$. We have

$$H^*(G) = \sum_{\{x_1,\dots,x_s\} \subset V(G)} \left(\sum_{\sigma \in S_s} \prod_{i=1}^{s} d(x_{\sigma(i)})^{a_i} \right) \left(\sum_{y_1,\dots,y_t \in X} \prod_{j=1}^{t} d(y_j)^{b_j} \right) + o(n^{v(H)}). \quad (1)$$

We use Muirhead's inequality [8, Theorem 45] to estimate both terms of the product above. For the degrees of $x_1, x_2, \dots, x_s$ since the sequence $(a_1, a_2, \dots, a_s)$ is majorized by the sequence $(\sum_{i=1}^{s} a_i, 0, \ldots, 0)$ we have

$$\sum_{\sigma \in S_s} \prod_{i=1}^{s} d(x_{\sigma(i)})^{a_i} \leqslant (s-1)! \sum_{x \in \{x_1, \ldots, x_s\}} d(x)^{\sum_{i=1}^{s} a_i}. \tag{2}$$

Moreover for the degrees of all vertices of X the sequence $(b_1, b_2, \ldots, b_t, 0, 0 \ldots, 0)$ is majorized by the sequence $(\sum_{i=j}^{t} b_j, 0, \ldots, 0)$ we have

$$\sum_{y_1, \ldots, y_t \in X} \prod_{j=1}^{t} d(y_j)^{b_j} \leqslant \frac{(|X|-1)!}{(|X|-t)!} \sum_{y \in X} d(y)^{\sum_{j=1}^{t} b_j}. \tag{3}$$

Note that we have $\frac{(|X|-1)!}{(|X|-t)!} \leqslant |X|^{t-1} \leqslant d(x)^{t-1}$ for all x in $\{x_1, x_2, \ldots, x_s\}$. Putting together the bounds (1), (2) and (3) we obtain

$$H^*(G) \leqslant \sum_{x \in \{x_1, \ldots, x_s\} \subset V(G)} (s-1)! d(x)^{t-1+\sum_{i=1}^{s} a_i} \sum_{y \in X} d(y)^{\sum_{j=1}^{t} b_j} + o(n^{v(H)}) = F^*(G) + o(n^{v(H)}) \tag{4}$$

where F is a double-star with central vertices v and u joined by an edge, $\sum_{i=1}^{s} a_i + t - 1$ pendant edges attached to v and $\sum_{i=1}^{t} b_i + s - 1$ pendant edges attached to u. Here we explain the last equality. Let us fix a set S' in $V(F)$ containing the vertex v and $s-1$ leaves adjacent with u. In order to find a copy of F in G first we choose vertices $x_1, x_2, \ldots, x_s$ of G, then we map vertices from S' to it and choose representatives of all vertices adjacent to v in F except u. Then we fix a vertex y representing u, and finally, we choose the remaining leaves adjacent to it.

For a given n and F, Győri, Wang and Woolfson [7] proved that there exists n' such that for all triangle-free graphs G on n vertices we have $F(G) \leqslant F(K_{n',n-n'}) + o(n^{v(F)})$. Therefore we have $F^*(G) \leqslant F^*(K_{n',n-n'}) + o(n^{v(F)})$. Hence the maximum number of labeled copies of H in G is also asymptotically attained when $G = K_{n',n-n'}$, so

$$H(G) \leqslant H(K_{n',n-n'}) + o(n^{v(H)}). \qquad \square$$

In a follow-up work [3], Gerbner sharpened the above mentioned result of Győri, Wang and Woolfson, and as a consequence proved that Theorem 4 holds as an exact result.

Acknowledgments

We would like to thank Daniel Gerbner for some useful remarks on the manuscript.

References

[1] Noga Alon and Clara Shikhelman. Many T copies in H-free graphs. *Journal of Combinatorial Theory, Series B*, 121:146–172, 2016.

[2] Paul Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7(3):459–464, 1962.

[3] Dániel Gerbner. On weakly Turán-good graphs. arXiv:2207.11993, 2022.

[4] Dániel Gerbner and Cory Palmer. Counting copies of a fixed subgraph in F-free graphs. *European Journal of Combinatorics*, 82:103001, 2019.

[5] Andrzej Grzesik. On the maximum number of five-cycles in a triangle-free graph. *Journal of Combinatorial Theory, Series B*, 102(5):1061–1066, 2012.

[6] Ervin Győri, János Pach, and Miklós Simonovits. On the maximal number of certain subgraphs in K_r-free graphs. *Graphs and Combinatorics*, 7(1):31–37, 1991.

[7] Ervin Győri, Runze Wang, and Spencer Woolfson. Extremal problems of double stars. arXiv:2109.01536, 2021.

[8] Godfrey Harold Hardy, John Edensor Littlewood, and George Pólya. *Inequalities*. Cambridge University Press, 1952.

[9] Hamed Hatami, Jan Hladký, Daniel Král, Serguei Norine, and Alexander Razborov. On the number of pentagons in triangle-free graphs. *Journal of Combinatorial Theory, Series A*, 120(3):722–732, 2013.

[10] Eng Keat Hng and Domenico Mergoni Cecchelli. Density of small diameter subgraphs in K_r-free graphs. arXiv:2207.14297, 2022.

[11] Bernard Lidický and Kyle Murphy. Maximizing five-cycles in K_r-free graphs. *European Journal of Combinatorics*, 97:103367, 2021.

[12] Natasha Morrison, JD Nir, Sergey Norin, Paweł Rzążewski, and Alexandra Wesolek. Every graph is eventually Turán-good. arXiv:2208.08499, 2022.

[13] Alexander Aleksandrovich Zykov. On some properties of linear complexes. *Matematicheskii sbornik*, 66(2):163–188, 1949.