# Combinatorics on bounded free Motzkin paths and its applications

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**Abstract**

In this paper, we construct a bijection from a set of bounded free Motzkin paths to a set of bounded Motzkin prefixes that induces a bijection from a set of bounded free Dyck paths to a set of bounded Dyck prefixes. We also give bijections between a set of bounded cornerless Motzkin paths and a set of $t$-core partitions, and a set of bounded cornerless symmetric Motzkin paths and a set of self-conjugate $t$-core partitions. As an application, we get explicit formulas for the number of ordinary and self-conjugate $t$-core partitions with a fixed number of corners.

**Mathematics Subject Classifications:** 05A19, 05A17

## 1 Introduction

The main result of this paper is finding a bijection between two sets of paths in a bounded strip, which have been studied by several researchers (for example, see [1, 5, 6, 7, 10, 13]).

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A *Motzkin path* of length $n$ is a path from $(0,0)$ to $(n,0)$ which stays weakly above the $x$-axis and consists of steps $u=(1,1)$, $d=(1,-1)$, and $f=(1,0)$, called *up*, *down*, and *flat* steps, respectively. A *free Motzkin path* of length $n$ is a path which starts at $(0,0)$ or $(0,1)$, ends at $(n,0)$, and consists of $u$, $d$, and $f$. A Motzkin path with no restrictions on the end point is called a Motzkin prefix. For a given path, a *peak* is a point preceded by an up step and followed by a down step and a *valley* is a point preceded by a down step and followed by an up step. We say that a path is *cornerless* if it has no peaks or valleys.

For non-negative integers $m, r$, and $k$, let $\mathcal{F}(m,r,k)$ be the set of free Motzkin paths of length $m+r$ with $r$ flat steps that are contained in the strip $-\lfloor \frac{k}{2} \rfloor \leqslant y \leqslant \lfloor \frac{k+1}{2} \rfloor$. We denote $\mathcal{M}(m,r,k)$ the set of Motzkin prefixes of length $m+r$ with $r$ flat steps that are contained in the strip $0 \leqslant y \leqslant k$. We define $L_k$ to be one of the boundaries of each path depending on the value of $k$. More specifically, for $P \in \mathcal{F}(m,r,k)$, denote $L_k$ by

$$y = \begin{cases} \lfloor \frac{k+1}{2} \rfloor & \text{if } k \text{ is odd,} \\ -\lfloor \frac{k}{2} \rfloor & \text{if } k \text{ is even.} \end{cases}$$

Let $\overline{\mathcal{F}}(m,r,k)$ (resp. $\overline{\mathcal{M}}(m,r,k)$) be the set of paths in $\mathcal{F}(m,r,k)$ (resp. $\mathcal{M}(m,r,k)$) which touch the line $L_k$ (resp. $y=k$) so that

$$\mathcal{F}(m,r,k) = \bigcup_{i=0}^{k} \overline{\mathcal{F}}(m,r,i) \quad \text{and} \quad \mathcal{M}(m,r,k) = \bigcup_{i=0}^{k} \overline{\mathcal{M}}(m,r,i).$$

Our main theorem states the following.

*Theorem* 1. For given non-negative integers $m, r$, and $k$, there is a bijection between the sets $\overline{\mathcal{F}}(m,r,k)$ and $\overline{\mathcal{M}}(m,r,k)$.

To prove Theorem 1, we construct a map $\phi_{m,k}$ and show that it is bijective in Sections 2.1 and 2.2.

Using the adjacency matrices of path graphs, Cigler [5] showed that

$$|A_{n,k}| = |B_{n,k}| = \sum_{j \in \mathbb{Z}} (-1)^j \binom{n}{\lfloor \frac{n+(k+2)j}{2} \rfloor}$$

and expected the existence of a simple bijection between $A_{n,k}$ and $B_{n,k}$, where $A_{n,k}$ is the set of paths of length $n$ which consist of $u$ and $d$ only, start at $(0,0)$, end on height 0 or $-1$, and are contained in the strip $-\lfloor \frac{k+1}{2} \rfloor \leqslant y \leqslant \lfloor \frac{k}{2} \rfloor$ of width $k$, and $B_{n,k}$ is the set of paths of length $n$ which consist of $u$ and $d$ only, start at $(0,0)$ and are contained in the strip $0 \leqslant y \leqslant k$. Recently, Gu and Prodinger [10] and Dershowitz [7] found bijections between $A_{n,k}$ and $B_{n,k}$ independently. We note that Theorem 1 with no flat step (equivalently, $r=0$) gives a new bijection between $A_{n,k}$ and $B_{n,k}$ since $\mathcal{F}(n,0,k)$ can be obtained from $A_{n,k}$ by mirroring left and right and flipping along the $x$-axis, and $\mathcal{M}(n,0,k) = B_{n,k}$ as it is. We should mention that the bijection $\phi_{m,k}$ is inspired by the bijection due to Gu and Prodinger, but there is a property that $\phi_{m,k}$ holds whereas Gu and Prodinger's does not. This property is described in Section 2.3.

Let $\overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ be the set of cornerless free Motzkin paths in $\overline{\mathcal{F}}(m,r,k)$ that never start with a down (resp. up) step for odd (resp. even) $m$ and $\overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$ be the set of cornerless Motzkin prefixes in $\overline{\mathcal{M}}(m,r,k)$ that end with a flat step. In Section 3.1, we show that $\phi_{m,k}$ induces a bijection between $\overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ and $\overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$.

In Section 3.2, we combinatorially interpret $t$-core partitions by cornerless Motzkin paths. We describe a bijection between a set of cornerless Motzkin paths and a set of $t$-core partitions. As an application of this bijection, we count the number of $t$-core partitions with $m$ corners. In Section 3.3, we also count the number of self-conjugate $t$-core partitions with $m$ corners by constructing bijections between any pair of the following sets: a set of cornerless free Motzkin paths, a set of cornerless symmetric Motzkin paths, and a set of self-conjugate $t$-core partitions.

## 2 Bijection

In this section, we recursively define a map

$$\phi_{m,k} : \bigcup_{r\geqslant 0} \overline{\mathcal{F}}(m,r,k) \to \bigcup_{r\geqslant 0} \overline{\mathcal{M}}(m,r,k),$$

according to the values of $m$ and $k$, and then show that it is bijective. For simplicity, we define some notations first. For a path $P = p_1 p_2 \dots p_n$, where each $p_i$ denotes the $i$th step in $P$, let

$$\overline{P} := \overline{p}_1 \overline{p}_2 \dots \overline{p}_n \quad \text{and} \quad \overleftarrow{P} := \overline{p}_n \overline{p}_{n-1} \dots \overline{p}_1,$$

where $\overline{u} := d$, $\overline{d} := u$, and $\overline{f} := f$.

### 2.1 Map $\phi_{m,k}$

Now we define the map. Let $P$ be a path in the set $\overline{\mathcal{F}}(m,r,k)$ for some $r \geqslant 0$, and $\gamma \geqslant 0$ denote the maximum number such that $f^\gamma$ is a suffix of $P$.

**Case 0**. If $k = 0$ or $k = 1$, then the map is defined as

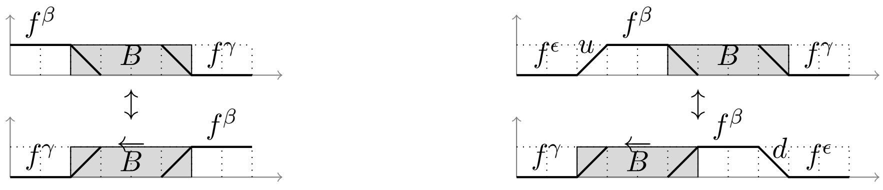
$$\phi_{m,k}(P) := \overleftarrow{P}.$$

We show the bijection $\phi_{m,1}$ in Figure 1.

Now assume $k > 1$. A *special step* of $P$ is the first step ending on the line $L_k$. We write $P$ as

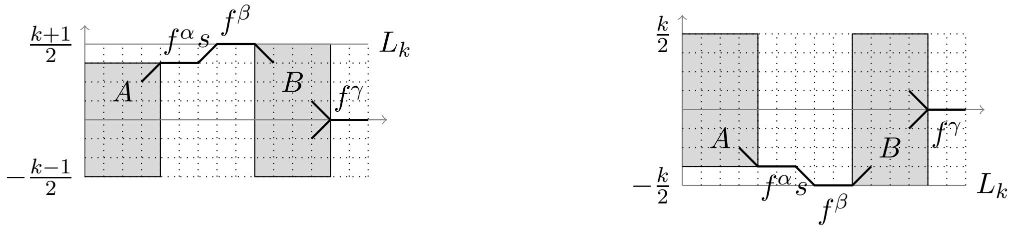
$$P = A f^\alpha s f^\beta B f^\gamma, \tag{1}$$

where $s$ is the special step, $\alpha \geqslant 0$ (resp. $\beta \geqslant 0$) is the maximum number of consecutive flat steps right before (resp. after) the step $s$, $A$ denotes the prefix of $P$ before the subpath $f^\alpha s$, and $B$ denotes the subpath between the subpaths $s f^\beta$ and $f^\gamma$. Note that $A$ and $B$ never end with a flat step (See Figure 2).




(a) For odd $m$ (b) For even $m$

Figure 1: The bijections $\phi_{m,1}$ in Case 0




(a) For odd $k$ (b) For even $k$

Figure 2: The division of $P$ for $k > 1$

Let the last vertex on the line $L_k$ (resp. $y = k$) be the *turning point* of a path in $\overline{\mathcal{F}}(m, r, k)$ (resp. $\overline{\mathcal{M}}(m, r, k)$). We call the first step after the turning point starting from the $x$-axis and heading away from the line $L_k$ the *break step*, and denote it by $b$. If $P$ has the break step $b$, let $\delta \geqslant 0$ be the maximum number of consecutive flat steps right before the step $b$ and we write $B$ as $B_1 f^\delta b B_2$.

**Case 1**. Let $m$ and $k$ have the same parity with $k > 1$.

i) If there is no break step, then we write $P$ as (1) and define the map as

$$\phi_{m,k}(P) := \begin{cases} Q & \text{if } k \text{ is odd,} \\ \overline{Q} & \text{if } k \text{ is even,} \end{cases} \tag{2}$$

where

$$Q := f^\gamma \overline{B} f^\alpha s A f^\beta.$$

Note that $\phi_{m,k}(P)$ ends on the line $y = k$.

ii) If there is the break step $b$, then $P$ can be written as

$$P = A f^\alpha s f^\beta B_1 f^\delta b B_2 f^\gamma. \tag{3}$$

Note that $B_1$ is a subpath starting from the line $L_k$ and ending at the $x$-axis with a down (resp. up) step, and $B_2$ is a subpath starting from the line

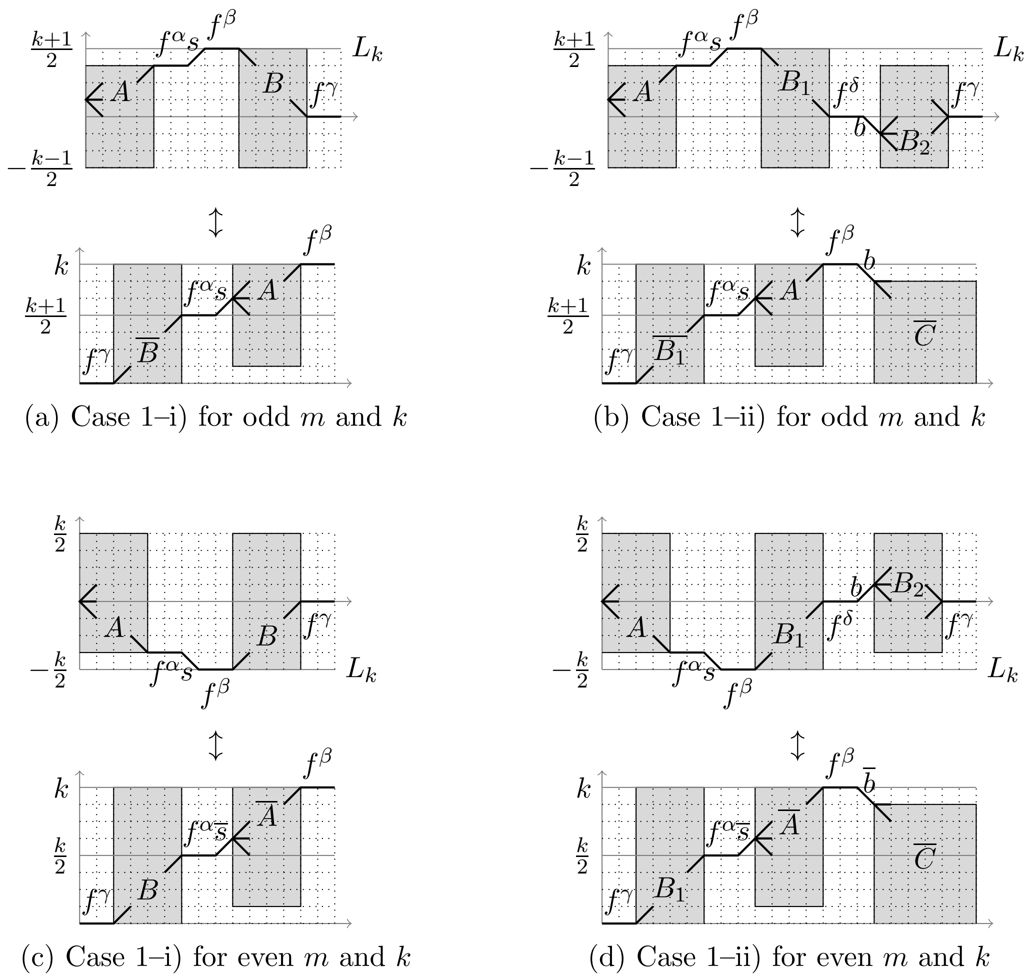
$y = (-1)^k$ and ending at the $x$-axis with a non-flat step for odd (resp. even) $k$. Define

$$\phi_{m,k}(P) := \begin{cases} Q\overline{C} & \text{if } k \text{ is odd,} \\ \overline{QC} & \text{if } k \text{ is even,} \end{cases} \tag{4}$$

where

$$Q := f^\gamma \overline{B_1} f^\alpha s A f^\beta b \quad \text{and} \quad C := \begin{cases} \phi_{m',k'}(\overline{B_2} f^\delta) & \text{if } k \text{ is odd,} \\ \phi_{m',k'}(B_2 f^\delta) & \text{if } k \text{ is even.} \end{cases}$$

Note that $m'$ is odd in this case. The bijection in Case 1 is illustrated in Figure 3.




(a) Case 1–i) for odd $m$ and $k$

(b) Case 1–ii) for odd $m$ and $k$

(c) Case 1–i) for even $m$ and $k$

(d) Case 1–ii) for even $m$ and $k$

Figure 3: The bijection $\phi_{m,k}$ in Case 1

**Case 2**. Let $m$ and $k$ have different parity with $k > 1$. In this case we write $A$ as $A_1 a A_2$, where $a$ is the first up (resp. down) step starting from the $x$-axis (resp. $y = 1$)

in $P$ for odd (resp. even) $k$. Here, $A_1$ and $A_2$ can be empty. Note that if $A_2$ is non-empty, then it never ends with a flat step. Similar to the map in Case 1–ii), we define the map as (4), where $Q$ and $C$ are given as follows.

i) If there is no break step, then $P$ can be written as

$$P = A_1 a A_2 f^{\alpha} s f^{\beta} B f^{\gamma}, \tag{5}$$

and we set

$$Q := f^{\gamma}\overline{B} f^{\alpha} s A_2 f^{\beta}\overline{a} \quad \text{and} \quad C := \begin{cases} \phi_{m',k'}(A_1) & \text{if } k \text{ is odd,} \\ \phi_{m',k'}(\overline{A_1}) & \text{if } k \text{ is even.} \end{cases} \tag{6}$$

ii) If there is the break step $b$, then $P$ can be written as

$$P = A_1 a A_2 f^{\alpha} s f^{\beta} B_1 f^{\delta} b B_2 f^{\gamma}, \tag{7}$$

and we set

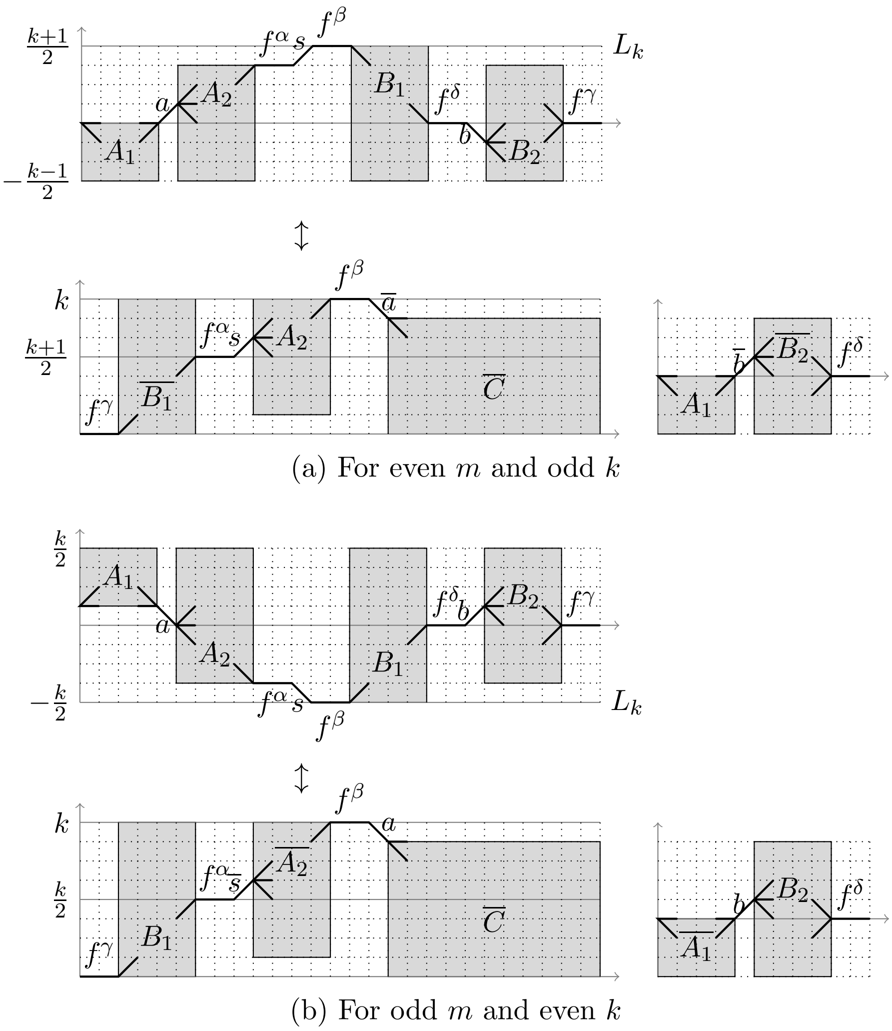
$$Q := f^{\gamma}\overline{B_1} f^{\alpha} s A_2 f^{\beta}\overline{a} \quad \text{and} \quad C := \begin{cases} \phi_{m',k'}(A_1 \overline{b B_2} f^{\delta}) & \text{if } k \text{ is odd,} \\ \phi_{m',k'}(\overline{A_1} b B_2 f^{\delta}) & \text{if } k \text{ is even.} \end{cases} \tag{8}$$

Note that $m'$ is even in this case. The bijection in Case 2–ii) is illustrated in Figure 4. By regarding $B_1$ as $B$ and $f^{\delta} b B_2$ as $\emptyset$ in this figure, we see the bijection in Case 2–i).

*Lemma* 2. For given non-negative integers $m$ and $k$, the map $\phi_{m,k}$ is well-defined.

*Proof.* Let $P \in \overline{\mathcal{F}}(m,r,k)$. In Case 0, it is clear that $\phi_{m,k}(P) = \overleftarrow{P} \in \overline{\mathcal{M}}(m,r,k)$. Now consider Case 1–i). In this case, for a path $P$ as in (1), we define $\phi_{m,k}(P)$ as (2). If $k$ is odd (resp. even), then $A$ is a subpath of $P$ that starts from the line $y = 1$ (resp. $x$-axis), ends on the line $y = (-1)^{k-1}\lfloor (k-1)/2 \rfloor$, and is contained in the strip $-\lfloor (k-1)/2 \rfloor \leqslant y \leqslant \lfloor k/2 \rfloor$, while $B$ is a subpath that starts from the line $y = (-1)^{k-1}\lfloor (k+1)/2 \rfloor$, ends on the $x$-axis, and is contained in the strip $-\lfloor k/2 \rfloor \leqslant y \leqslant \lfloor (k+1)/2 \rfloor$. Hence, the prefix $f^{\gamma}\overline{B} f^{\alpha}$ (resp. $f^{\gamma} B f^{\alpha}$) of $\phi_{m,k}(P)$ is a Motzkin prefix that ends at the line $y = \lfloor (k+1)/2 \rfloor$ and is contained in the strip $0 \leqslant y \leqslant k$, and the remaining subpath $sAf^{\beta}$ (resp. $\overline{sAf^{\beta}}$) starts from the line $y = \lfloor (k+1)/2 \rfloor$, ends on the line $y = k$, and is contained in the strip $1 \leqslant y \leqslant k$ for odd (resp. even) $k$. Therefore, $\phi_{m,k}(P) \in \overline{\mathcal{M}}(m,r,k)$.

For the remaining cases, we write $A$ as $A_1 a A_2$ and $B$ as $B_1 f^{\delta} b B_2$ if necessary. Now we use the induction on $k$. For any $k' < k$, suppose that $\phi_{m',k'}(P^*) \in \overline{\mathcal{M}}(m',r',k')$ for any paths $P^* \in \overline{\mathcal{F}}(m',r',k')$. Let $P \in \overline{\mathcal{F}}(m,r,k)$, $\phi_{m,k}(P)$ is defined as $Q\overline{C}$ or $\overline{QC}$, where $Q$ and $C$ are of the forms in (4), (6), or (8). In any cases, similar to Case 1–i), $Q$ or $\overline{Q}$ is a prefix of $\phi_{m,k}(P)$ that starts from the $x$-axis, touches the line $y = k$, ends on the line $y = k-1$, and is contained in the strip $0 \leqslant y \leqslant k$. Since $C \in \overline{\mathcal{M}}(m',r',k')$ with $k' < k$, $\overline{C}$ is a suffix of $\phi_{m,k}(P)$ that starts from the line $y = k-1$ and is contained in the strip $k - k' - 1 \leqslant y \leqslant k-1$ by the induction hypothesis. Thus, we conclude that $\phi_{m,k}(P) \in \overline{\mathcal{M}}(m,r,k)$. □




(a) For even $m$ and odd $k$

(b) For odd $m$ and even $k$

Figure 4: The bijection $\phi_{m,k}$ in Case 2–ii)

*Example* 3. For given free Motzkin paths, let us apply the map $\phi_{m,k}$.

(a) For the path

$$P_1 = fduduufdfduufdff \in \overline{\mathcal{F}}(10,6,2),$$

by applying Case 1–ii) and Case 0, we get

$$\phi_{10,2}(P_1) = ffuduufdfdufuddf \in \overline{\mathcal{M}}(10,6,2)$$

since $A = \emptyset$, $\beta = \delta = 0$, and $C = \phi_{1,1}(B_2 f^\delta)$ in (3), and $\phi_{1,1}(fd) = \overleftarrow{fd} = uf$.

(b) For the path

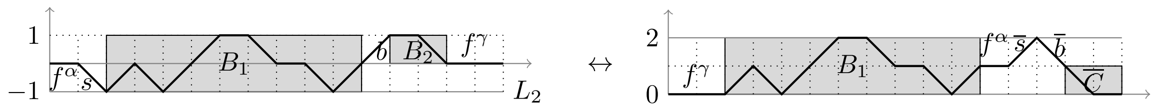
$$P_2 = fduduufudfdduufudfdffdfufuddfuf \in \overline{\mathcal{F}}(20,11,3),$$

by applying Case 2-ii), we obtain

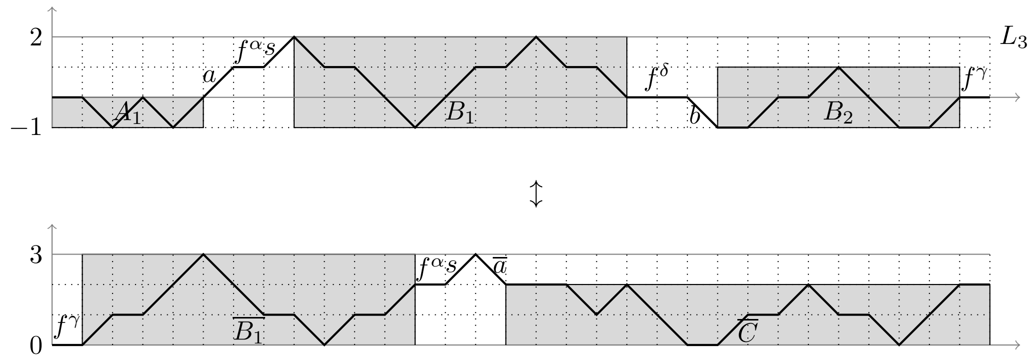
$$\phi_{20,3}(P_2) = fufuuddfdufufudffduddfufudfduuf \in \overline{\mathcal{M}}(20, 11, 3)$$

since $A_2 = \emptyset$, $\beta = 0$, and $C = \phi_{10,2}(A_1\overline{b}\overline{B_2}f^\delta) = \phi_{10,2}(P_1)$ in (7), where $P_1$ is the path given in *(a)*.

See Figure 5 for further details.




(a) A bijection $\phi_{10,2}$ in Case 1–ii)




(b) A bijection $\phi_{20,3}$ in Case 2–ii)

Figure 5: Examples of the map $\phi_{m,k}$

## 2.2 Map $\psi_{m,k}$

Now we define a map

$$\psi_{m,k} : \bigcup_{r \geqslant 0} \overline{\mathcal{M}}(m, r, k) \to \bigcup_{r \geqslant 0} \overline{\mathcal{F}}(m, r, k)$$

and show that $\psi_{m,k} = \phi_{m,k}^{-1}$. Let $S$ be a path in the set $\overline{\mathcal{M}}(m, r, k)$ for some $r \geqslant 0$.

**Case 0**. For $k = 0$ or 1, we define $\psi_{m,k}(S) = \overleftarrow{S}$.

Recall that the last vertex on the line $y = k$ is called the turning point of a path in $\overline{\mathcal{M}}(m, r, k)$. We define a *critical point* of $S$ as the rightmost point on the $x$-axis which locates before the turning point.

**Case I**. For $k > 1$, assume that $S$ is a path which ends on the line $y = k$. Note that $m$ and $k$ have the same parity and we write

$$S = f^\gamma B^* f^\alpha u^* A^* f^\beta,$$

where $u^*$ is the first up step starting from the line $y = \lfloor (k+1)/2 \rfloor$ after the critical point of $S$, $\gamma \geqslant 0$ (resp. $\beta \geqslant 0$) is the maximum number of consecutive initial (resp. final) flat steps of $P$, and $\alpha \geqslant 0$ is the maximum number of consecutive flat steps before the step $u^*$. Hence, $B^*$ (resp. $A^*$) is the subpath of $S$ such that it starts from the $x$-axis (resp. $y = \lfloor (k+1)/2 \rfloor + 1$), ends on the line $y = \lfloor (k+1)/2 \rfloor$ (resp. $y = k$), and is contained in the strip $0 \leqslant y \leqslant k$ (resp. $1 \leqslant y \leqslant k$). We define

$$\psi_{m,k}(S) := \begin{cases} A^* f^\alpha u^* f^\beta \overline{B^*} f^\gamma & \text{if } k \text{ is odd,} \\ \overline{A^*} f^\alpha \overline{u^*} f^\beta B^* f^\gamma & \text{if } k \text{ is even.} \end{cases} \tag{9}$$

**Case II**. Suppose that $S$ is a path which does not end on the line $y = k$ for $k > 1$. In this case, we write

$$S = f^\gamma B^* f^\alpha u^* A^* f^\beta d^* \overline{C^*},$$

where $u^*, A^*, B^*, \alpha, \beta, \gamma$ is defined as in Case I, $d^*$ is the last down step starting from the line $y = k$, and $\overline{C^*}$ is a suffix of $S$ after the step $d^*$. Note that $C^* \in \overline{\mathcal{M}}(m', r', k')$ for some $k' < k$ since $C^*$ is contained in the strip $0 \leqslant y \leqslant k-1$.

i) Let $m$ and $k$ have the same parity, which follows that $m'$ is odd. We write $\psi_{m',k'}(C^*) = B^\bullet f^\delta$, where $\delta \geqslant 0$ is the maximum number of consecutive flat steps at the suffix of $\psi_{m',k'}(C^*)$. We set

$$\psi_{m,k}(S) := \begin{cases} A^* f^\alpha u^* f^\beta \overline{B^*} f^\delta d^* \overline{B^\bullet} f^\gamma & \text{if } k \text{ is odd,} \\ \overline{A^*} f^\alpha \overline{u^*} f^\beta B^* f^\delta \overline{d^*} B^\bullet f^\gamma & \text{if } k \text{ is even.} \end{cases} \tag{10}$$

ii) Let $m$ and $k$ have different parity. In this case, $m'$ is even. We divide two cases whether $\psi_{m',k'}(C^*)$ goes above the $x$-axis or not.

If $\psi_{m',k'}(C^*)$ does not go above the $x$-axis, then we write $\psi_{m',k'}(C^*) = A^\bullet$ and define

$$\psi_{m,k}(S) := \begin{cases} A^\bullet \overline{d^*} A^* f^\alpha u^* f^\beta \overline{B^*} f^\gamma & \text{if } k \text{ is odd,} \\ \overline{A^\bullet} d^* \overline{A^*} f^\alpha \overline{u^*} f^\beta B^* f^\gamma & \text{if } k \text{ is even.} \end{cases} \tag{11}$$

If $\psi_{m',k'}(C^*)$ goes above the $x$-axis, then we write $\psi_{m',k'}(C^*) = A^\bullet u^\bullet B^\bullet f^\delta$, where $u^\bullet$ is the first up step starting from the $x$-axis and $\delta \geqslant 0$ is the maximum number of consecutive flat steps at the suffix of $\psi_{m',k'}(C^*)$. We define

$$\psi_{m,k}(S) := \begin{cases} A^\bullet \overline{d^*} A^* f^\alpha u^* f^\beta \overline{B^*} f^\delta \overline{u^\bullet} \overline{B^\bullet} f^\gamma & \text{if } k \text{ is odd,} \\ \overline{A^\bullet} d^* \overline{A^*} f^\alpha \overline{u^*} f^\beta B^* f^\delta u^\bullet B^\bullet f^\gamma & \text{if } k \text{ is even.} \end{cases} \tag{12}$$

*Lemma* 4. The map $\psi_{m,k}$ is the inverse map of $\phi_{m,k}$.

*Proof.* For $k = 0$ or 1, it is clear that $\psi_{m,k}(\phi_{m,k}(P)) = P$ for any path $P \in \overline{\mathcal{F}}(m, r, k)$ by the construction.

From now on, we set $k > 1$. Let $P \in \overline{\mathcal{F}}(m, r, k)$ when $m$ and $k$ have the same parity and there is no break step in $P$ so that $P$ is represented as $P = Af^{\alpha}sf^{\beta}Bf^{\gamma}$. When $k$ is odd (resp. even), it follows from (2) and (9) that $\psi_{m,k}(\phi_{m,k}(P)) = P$ since $A = A^*$ (resp. $A = \overline{A^*}$), $s = u^*$ (resp. $s = \overline{u^*}$), and $B = \overline{B^*}$ (resp. $B = B^*$), where $S = \phi_{m,k}(P) \in \overline{\mathcal{M}}(m, r, k)$.

Now, we assume that $\psi_{m',k'}(S^*) \in \overline{\mathcal{F}}(m', r', k')$ for any path $S^* \in \overline{\mathcal{M}}(m', r', k')$ with $k' < k$. Let $P \in \overline{\mathcal{F}}(m, r, k)$ when $m$ and $k$ have the same parity and there is a break step $b$ in $P$ so that $P$ is represented as $P = Af^{\alpha}sf^{\beta}B_1f^{\delta}bB_2f^{\gamma}$. For odd (resp. even) $k$, according to (4) and (10), $\psi_{m,k}(\phi_{m,k}(P)) = P$ since $A = A^*$ (resp. $A = \overline{A^*}$), $s = u^*$ (resp. $s = \overline{u^*}$), $B_1 = \overline{B^*}$ (resp. $B_1 = B^*$), $b = d^*$ (resp. $b = \overline{d^*}$), and $B_2 = \overline{B^\bullet}$ (resp. $B_2 = B^\bullet$).

Similarly, by (6), (8), (11), and (12), we can see that $\psi_{m,k}(\phi_{m,k}(P)) = P$, where $P \in \overline{\mathcal{F}}(m, r, k)$ when $m$ and $k$ have different parity with $k > 1$. □

*Example* 5. For given Motzkin prefixes,

$$S_1 = uufufddduf uuf \in \overline{\mathcal{M}}(9, 4, 3),$$
$$S_2 = uufuufdddfdfuuudfddf \in \overline{\mathcal{M}}(14, 6, 4),$$
$$S_3 = fuuufuduufuffdddfddfuuufufddduf uuf \in \overline{\mathcal{M}}(23, 11, 6),$$

we have

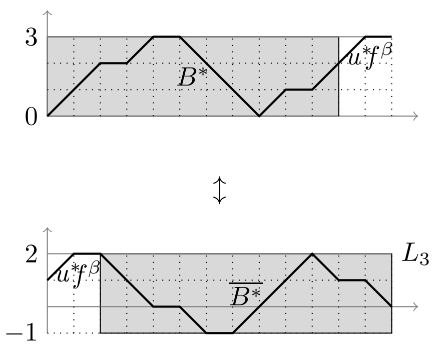
$$\psi_{9,3}(S_1) = ufddfdfuuudfd \in \overline{\mathcal{F}}(9, 4, 3),$$
$$\psi_{14,4}(S_2) = dfdfuuuufddfdfuuudfd \in \overline{\mathcal{F}}(14, 6, 4),$$
$$\psi_{23,6}(S_3) = ufufddduddfdfdffuuuuufddfdfuuudfdf \in \overline{\mathcal{F}}(23, 11, 6).$$

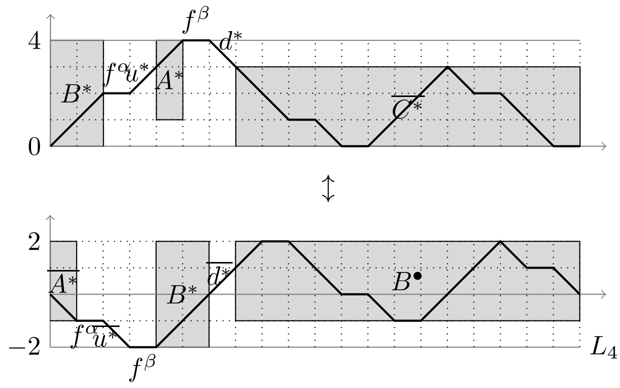
See Figure 6 for further details.

### 2.3 A property of $\phi_{m,k}$

For a free Motzkin path $P$, a maximal subpath in $P$ with no down (resp. up) step is called an *upward* (resp. *downward*) *run* if it contains at least one up (resp. down) step. Let $\mathrm{run}(P)$ denote the total number of runs in $P$. If $P$ has no flat step, then the total number of peaks and valleys of $P$ is counted by $\mathrm{run}(P) - 1$. For example, the path $P = uufufddduf uuf$ has two upward runs, $uufuf$ and $ufuuf$, and one downward run $fddd$ so that $\mathrm{run}(P) = 3$. Note that $\mathrm{run}(P) = 0$ if and only if $P$ is empty or a path consisting of flat steps only, and $\mathrm{run}(P) = \mathrm{run}(\overline{P})$. For a path $P \in \overline{\mathcal{F}}(m, r, k)$, the following proposition shows that $\mathrm{run}(P)$ and $\mathrm{run}(\phi_{m,k}(P))$ are differ by at most 1.

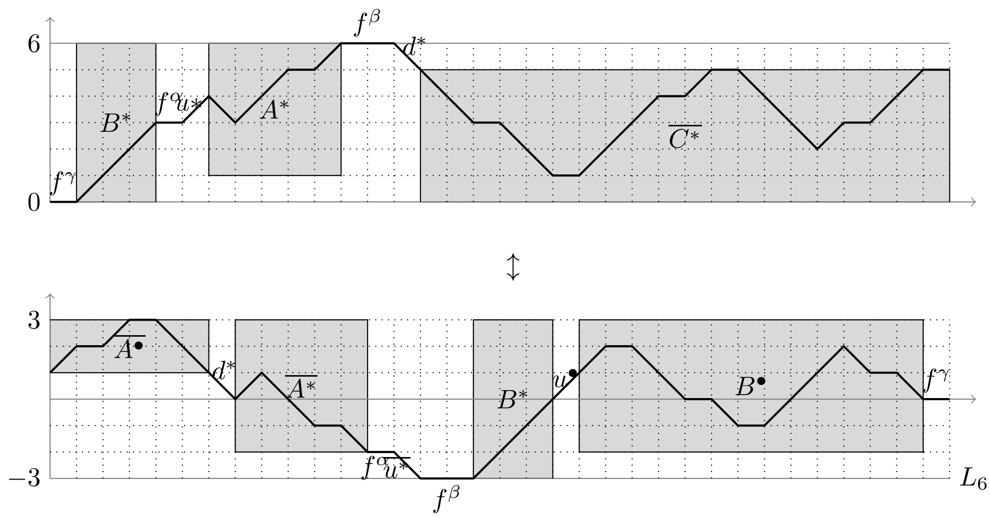
*Proposition* 6. For positive integers $m$ and $k$, let $P \in \overline{\mathcal{F}}(m, r, k)$ be given.




(a) A bijection $\psi_{9,3}$ in Case I

(b) A bijection $\psi_{14,4}$ in Case II–i)




(c) A bijection $\psi_{23,6}$ in Case II–ii)

Figure 6: Examples of the map $\psi_{m,k}$

(a) If $P$ starts with an upward run, then

$$\text{run}(\phi_{m,k}(P)) = \text{run}(P) - \{1 - (-1)^m\}/2.$$

(b) If $P$ starts with a downward run, then

$$\text{run}(\phi_{m,k}(P)) = \text{run}(P) - \{1 + (-1)^m\}/2.$$

*Proof.* As erasing any number of flat steps do not change the number of runs, it suffices to show that this proposition holds when $r = 0$. We prove it by using induction on $k$.

For the initial step with $k = 1$, we consider Case 0. Recall that $\phi_{m,1}(P) = \overleftarrow{P}$. If $P$ starts with an up step, $m$ must be even and $\text{run}(\overleftarrow{P}) = \text{run}(P)$. When $P$ starts with a down step, $m$ is odd and $\text{run}(\overleftarrow{P}) = \text{run}(P)$.

Now we assume $k > 1$ and suppose that this proposition holds for any $P^* \in \overline{\mathcal{F}}(m', 0, k')$ with $k' < k$. Here we give a detailed proof for (a) and the proof for (b) comes out similarly.

Suppose that $P \in \overline{\mathcal{F}}(m, 0, k)$ starts with an up step and let $S = \phi_{m,k}(P)$. We need to show that $\mathrm{run}(S)$ is given by $\mathrm{run}(P) - 1$ (resp. $\mathrm{run}(P)$) if $m$ is odd (resp. even).

In Case 1–i), we write $P = AuB$ (resp. $P = AdB$) and $S = \overline{B}uA$ (resp. $S = Bu\overline{A}$), where $\overline{B}$ (resp. $B$) ends with an up step if $m$ is odd (resp. even). If $A$ is empty, then $m$ must be odd so that $\mathrm{run}(S) = \mathrm{run}(B) = \mathrm{run}(P) - 1$ as we desire. Now assume that $A$ starts with an up step. In this case, $\mathrm{run}(P) = \mathrm{run}(A) + \mathrm{run}(B)$ and

$$\mathrm{run}(S) = \begin{cases} \mathrm{run}(A) + \mathrm{run}(B) - 1 & \text{if } m \text{ is odd,} \\ \mathrm{run}(A) + \mathrm{run}(B) & \text{if } m \text{ is even,} \end{cases}$$

so we are done.

In Case 1–ii), we write $P = AuB_1dB_2$ (resp. $P = AdB_1uB_2$) and $\overline{B_2} \in \overline{\mathcal{F}}(m', 0, k')$ (resp. $B_2 \in \overline{\mathcal{F}}(m', 0, k')$) for some $k' < k$ and odd $m'$, where $m$ is odd (resp. even). Let $r := \mathrm{run}(A) + \mathrm{run}(B_1)$. Note that if $m$ is odd (resp. even), then

$$\mathrm{run}(P) = \begin{cases} r + \mathrm{run}(B_2) & \text{if } B_2 \text{ starts with an up (resp. down) step,} \\ r + \mathrm{run}(B_2) - 1 & \text{if } B_2 \text{ starts with a down (resp. up) step.} \end{cases}$$

In this case, if $m$ is odd (resp. even), then $S = \overline{B_1}uAd\overline{C}$ (resp. $S = B_1u\overline{A}d\overline{C}$), where $C = \phi_{m',k'}(\overline{B_2})$ (resp. $C = \phi_{m',k'}(B_2)$). By the induction hypothesis, if $m$ is odd (resp. even), then

$$\mathrm{run}(C) = \begin{cases} \mathrm{run}(B_2) & \text{if } B_2 \text{ starts with an up (resp. down) step,} \\ \mathrm{run}(B_2) - 1 & \text{if } B_2 \text{ starts with a down (resp. up) step.} \end{cases}$$

Hence, if $m$ is odd, then $\mathrm{run}(S) = r + \mathrm{run}(C) - 1$ so that

$$\mathrm{run}(S) = \begin{cases} r + \mathrm{run}(B_2) - 1 & \text{if } B_2 \text{ starts with an up step,} \\ r + \mathrm{run}(B_2) - 2 & \text{if } B_2 \text{ starts with a down step,} \end{cases}$$

which means that $\mathrm{run}(S) = \mathrm{run}(P) - 1$. Similarly, we show that $\mathrm{run}(S) = \mathrm{run}(P)$ for even $m$.

The proofs of Case 2–i) and Case 2–ii) are similar, so we only prove Case 2–ii). We divide this case into two cases depending on the parity of $m$.

When $m$ is odd, we write $P = A_1dA_2dB_1uB_2$ and $S = B_1u\overline{A_2}d\overline{C}$, where $A_1$ starts with an up step and $C = \phi_{m',k'}(\overline{A_1}uB_2)$ for some $k' < k$ and even $m'$. Note that $\mathrm{run}(P) = \mathrm{run}(A_1) + \mathrm{run}(dA_2dB_1) + \mathrm{run}(uB_2) - 2$, $\mathrm{run}(S) = \mathrm{run}(B_1u\overline{A_2}d) + \mathrm{run}(C) - 1$, and $\mathrm{run}(C) = \mathrm{run}(\overline{A_1}uB_2) - 1$ by the induction hypothesis. We have

$$\mathrm{run}(P) = \begin{cases} \mathrm{run}(A_1) + \mathrm{run}(dA_2dB_1) + \mathrm{run}(B_2) - 1 & \text{if } B_2 \text{ starts with a down step,} \\ \mathrm{run}(A_1) + \mathrm{run}(dA_2dB_1) + \mathrm{run}(B_2) - 2 & \text{if } B_2 \text{ starts with an up step,} \end{cases}$$

and

$$\operatorname{run}(S) = \begin{cases} \operatorname{run}(B_1 u\overline{A_2} d) + \operatorname{run}(A_1) + \operatorname{run}(B_2) - 2 & \text{if } B_2 \text{ starts with a down step,} \\ \operatorname{run}(B_1 u\overline{A_2} d) + \operatorname{run}(A_1) + \operatorname{run}(B_2) - 3 & \text{if } B_2 \text{ starts with an up step.} \end{cases}$$

Since $\operatorname{run}(dA_2dB_1) = \operatorname{run}(B_1 u\overline{A_2} d)$ whenever $A_2$ starts with $u$ or $d$, we get $\operatorname{run}(S) = \operatorname{run}(P) - 1$.

For even $m$, we write $P = uA_2uB_1dB_2$ and $S = \overline{B_1}uA_2d\overline{C}$, where $C = \phi_{m',k'}(u\overline{B_2})$ for some $k' < k$ and even $m'$. We have $A_1 = \emptyset$ because $P$ starts with $u$. We also get $\operatorname{run}(S) = \operatorname{run}(P)$ in a similar manner. □

*Remark* 7. Proposition 6 confirms that $|\operatorname{run}(P) - \operatorname{run}(\phi_{m,k}(P))| \leqslant 1$, which shows that the map $\phi_{m,k}$ is structurally distinguishable from the map from Gu and Prodinger [10]. For example, Gu and Prodinger's map sends

$$P = dduuuuudddddduuu \mapsto S = uuuuuudddduuuddud,$$

which shows that their map has paths satisfying $|\operatorname{run}(P) - \operatorname{run}(S)| = 2$ (one can obtain this example by putting $A = dddddduuu$, $B = uu$, $C = du$, and $D = \emptyset$ in Figure 2.5 in [10]).

Let $\overline{\mathcal{F}}(m, r, k; i)$ denote the set of paths in $\overline{\mathcal{F}}(m, r, k)$ with $\lceil i/2 \rceil$ downward (resp. upward) runs for odd (resp. even) $m$, and let $\overline{\mathcal{M}}(m, r, k; i)$ denote the set of paths in $\overline{\mathcal{M}}(m, r, k)$ with $i$ runs. By Proposition 6, it is straightforward to get the following corollary.

*Corollary* 8. For non-negative integers $i$ and $m$ of the same (resp. different) parity, the map $\phi_{m,k}$ induces a bijection between the set $\overline{\mathcal{M}}(m, r, k; i)$ and the set of paths in $\overline{\mathcal{F}}(m, r, k; i)$ that end with a downward (resp. upward) run.

*Remark* 9. It is clear that the set of paths in $\overline{\mathcal{M}}(m, 0, k; i)$ is in bijection with the set of symmetric Dyck paths of length $2m$ with $i$ peaks which touch the line $y = k$ and is contained in the strip $0 \leqslant y \leqslant k$. By Corollary 8, the set of symmetric Dyck paths of length $2m$ with $i$ peaks corresponds to the set of paths in $\cup_k \overline{\mathcal{F}}(m, 0, k; i)$ that end with a downward (resp. upward) run whenever $i$ and $m$ have the same (resp. different) parity. This one-to-one correspondence gives a combinatorial proof of the well-known fact that the number of symmetric Dyck paths of length $2m$ with $i$ peaks is given by

$$\binom{\lfloor \frac{m-1}{2} \rfloor}{\lfloor \frac{i-1}{2} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{i}{2} \rfloor}. \tag{13}$$

## 3 Cornerless free Motzkin paths

In this section, we combinatorially interpret cornerless Motzkin paths as a $t$-core partitions. First let us consider the restriction of the map $\phi_{m,k}$.

### 3.1 Restriction to cornerless free Motkzin paths

Recall that $\overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ is the set of cornerless free Motzkin paths in $\overline{\mathcal{F}}(m,r,k)$ that never start with a down (resp. up) step for odd (resp. even) $m$ and $\overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$ is the set of cornerless Motzkin prefixes in $\overline{\mathcal{M}}(m,r,k)$ that end with a flat step. Now we show that the map $\phi_{m,k}$, defined in Section 2.1, gives a one-to-one correspondence between these sets.

*Proposition* 10. For given non-negative integers $m, r,$ and $k$, $\phi_{m,k}$ induces a bijection between the sets $\overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ and $\overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$.

*Proof.* Let $P \in \overline{\mathcal{F}}(m,r,k)$ and $\phi_{m,k}(P) \in \overline{\mathcal{M}}(m,r,k)$. For $k = 0$ or $1$, $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ when it is cornerless and starts with a flat step, and $\overleftarrow{P} \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$ when it is cornerless and ends with a flat step. Hence, $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ if and only if $\phi_{m,k}(P) = \overleftarrow{P} \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$.

Let $m$ and $k$ have the same parity and there is no break step in $P$ with $k > 1$. It follows from (1) and (2) that $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ and $\phi_{m,k}(P) \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$ have the same restriction such that $A$ and $B$ are cornerless, $A$ does not start with a down (resp. up) step for odd (resp. even) $m$, and $\beta > 0$. Hence, $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ if and only if $\phi_{m,k}(P) \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$.

For the remaining cases, we assume that $\phi_{m',k'}$ induces a bijection between $\overline{\mathcal{F}}_{\mathrm{c}}(m',r',k')$ and $\overline{\mathcal{M}}_{\mathrm{c}}(m',r',k')$ for $k' < k$. We consider the case when $m$ and $k$ have the same parity and there is a break step $b$ in $P$. By (3) and (4), $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ and $\phi_{m,k}(P) \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$ have the same condition such that $A$ and $B_1$ are cornerless, $A$ does not start with a down (resp. up) step, $\overline{B_2}f^{\delta}$ (resp. $B_2 f^{\delta}$) $\in \overline{\mathcal{F}}_{\mathrm{c}}(m',r',k')$ for some $k' < k$ when $m$ is odd (resp. even), and $\beta > 0$. Hence, $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ if and only if $\phi_{m,k}(P) \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$.

Similarly, we can show that $P \in \overline{\mathcal{F}}_{\mathrm{c}}(m,r,k)$ if and only if $\phi_{m,k}(P) \in \overline{\mathcal{M}}_{\mathrm{c}}(m,r,k)$ when $m$ and $k$ have different parity by considering (5), (6), (7), and (8). □

### 3.2 Cornerless Motkzin paths and $t$-cores

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a non-increasing positive integer sequence. The *Young diagram* of $\lambda$ is an array of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$th row. An *inner corner* of a Young diagram is a box that can be removed from the Young diagram and the rest of the Young diagram is still the Young diagram of a partition. We say that $\lambda$ has $m$ corners if its Young diagram has $m$ inner corners. For a given Young diagram, the *hook length* of a box at the position $(i,j)$, denoted by $h(i,j)$, is the number of boxes on the right, in the below, and itself. For a partition $\lambda$, the *beta-set* of $\lambda$, denoted by $\beta(\lambda)$, is the set of hook lengths of boxes in the first column of the Young diagram of $\lambda$. A partition is called a $t$*-core* if its Young diagram has no box of hook length $t$. We mainly consider $t$-core partitions with $m$ corners and use the abacus diagram introduced by James and Kerber [12] to count them. The $t$*-abacus diagram* is a diagram to be the bottom and left-justified diagram with infinitely many rows labeled by $i \in \mathbb{N} \cup \{0\}$ and $t$ columns labeled by $j = 0, 1, \ldots, t-1$ whose position $(i,j)$ is labeled by $ti + j$. The $t$*-abacus* of a partition $\lambda$ is obtained from the $t$-abacus diagram by placing a bead on each position labeled by $h$, where $h \in \beta(\lambda)$. A position without bead is called a *spacer*. The following lemma is useful to determine whether a given partition is a $t$-core or not.

*Lemma* 11. [12, Lemma 2.7.13] A partition $\lambda$ is a $t$-core if and only if $h \in \beta(\lambda)$ implies $h - t \in \beta(\lambda)$ whenever $h > t$. Equivalently, $\lambda$ is a $t$-core if and only if the $t$-abacus of $\lambda$ has no spacer below a bead in any column.

From the above lemma, we easily obtain a simple bijection between the set of $t$-core partitions and the set of non-negative integer sequences $(n_0, n_1, \ldots, n_{t-1})$, where $n_0 = 0$ and $n_j$ is the number of beads in column $j$ for $j = 1, \ldots, t-1$. Using the bijection between the bar graphs and cornerless Motzkin paths, introduced by Deutsch and Elizalde [8], we give a path interpretation of the $t$-core partitions restricted by the number of corners and the first hook length $h(1,1)$.

*Theorem* 12. For non-negative integers $t$, $m$, and $k$, there is a bijection between any pair of the following sets.

(a) The set of $t$-core partitions with $m$ corners such that $h(1,1) < kt$.

(b) The set of non-negative integer sequences $(n_0, n_1, \ldots, n_{t-1})$ satisfying that $n_0 = 0$, $n_i \leqslant k$ for all $i$, and

$$\sum_{i=1}^{t} |n_i - n_{i-1}| = 2m,$$

where we set $n_t := 0$.

(c) The set of cornerless Motzkin paths of length $2m + t - 1$ with $t - 1$ flat steps that are contained in the strip $0 \leqslant y \leqslant k$.

*Proof.* Let $A, B$, and $C$ be the set described in (a), (b), and (c), respectively. Set the maps $\phi_1 : A \to B$ and $\phi_2 : B \to C$. For a partition $\lambda \in A$, let $n_i$ be the number of beads in the $i$th column of the $t$-abacus of $\lambda$. Given $\lambda \in A$, define $\phi_1(\lambda) = (n_0, n_1, \ldots, n_{t-1})$. Then, by the definition of the $t$-abacus and the fact that $h(1,1) < kt$, it is given that $n_0 = 0$ and $n_i \leqslant k$ for each $i$. Moreover, we get one inner corner for each maximal sequence of consecutive numbers in the beta-set $\beta(\lambda)$. Note that $\sum_{i=1}^{t} \max(n_i - n_{i-1}, 0)$ counts the number of hook lengths which is the smallest among each maximal sequence of consecutive numbers in the beta-set, so we get $\sum_{i=1}^{t} |n_i - n_{i-1}| = 2m$. Let $\psi_1 : B \to A$ and $\mathbf{n} = (n_0, n_1, \ldots, n_{t-1}) \in B$. Define $\psi_1(\mathbf{n}) = \lambda$, where $\lambda$ is the partition obtained from the $t$-abacus diagram with $n_i$ beads in the $i$th column. We place the beads on the elements of $\beta(\lambda)$ in the $t$-abacus diagram. Then, since column 0 has no bead and each $n_i \leqslant k$ for all $i$, the largest element in $\beta(\lambda)$ is less than $kt$, meaning that $\lambda$ is a $t$-core partition with $h(1,1) < kt$. Also, the fact that the sum of $|n_i - n_{i-1}|$ is $2m$ implies that there are $m$ piles of beads which are placed on $m$ maximal consecutive numbers, so $\lambda$ has $m$ corners.

For $\mathbf{n} = (n_0, n_1, \ldots, n_{t-1}) \in B$, let $\phi_2(\mathbf{n}) = P_{\mathbf{n}}$, where $P_{\mathbf{n}}$ be the cornerless Motzkin path which starts at $(0,0)$, ends at $(2m + t - 1, 0)$, and has $t - 1$ flat steps at height

$n_1, n_2, \cdots, n_{t-1}$ with proper up and/or down steps connecting those flat steps. Due to the fact that $n_i \leqslant k$, it is given that $P_{\mathbf{n}}$ is contained in the strip $0 \leqslant y \leqslant k$.

Let $\psi_2 : C \to B$ and $P \in C$. Define $\psi_2(P) = (n_0, n_1, \ldots, n_{t-1})$, where $n_0 = 0$ and, for $1 \leqslant i \leqslant t-1$, each $n_i$ represents the height of the $i$th flat step in $P$. We know that $P$ is contained in the strip $0 \leqslant y \leqslant k$, which implies $n_i \leqslant k$. On the path $P$, there are $2m$ many up and down steps. The number $|n_i - n_{i-1}|$ represents the difference of the height of the $(i-1)$st flat step and the $i$th flat step, so it counts the number of up or down steps in between those two flat steps. Since $\sum_{i=1}^{t} \max(n_i - n_{i-1}, 0) = \sum_{i=1}^{t} |\min(n_i - n_{i-1}, 0)|$, we get $\sum_{i=1}^{t} |n_i - n_{i-1}| = 2m$. □

For example, there are sixteen 4-core partitions with 2 corners. By letting $t = 4$ and $m = 2$ in Theorem 12, we get the correspondence between these partitions, abaci, non-negative integer sequences, and cornerless Motzkin paths as described in Figure 7.

We denote that a partition $\lambda$ is a $(t_1, t_2, \ldots, t_p)$-core if $\lambda$ is a $t_i$-core for all $i = 1, \ldots, p$. It is known that the number of $t$-core partitions is infinite, and the number of $(t_1, t_2, \ldots, t_p)$-cores is finite for relatively prime $t_1, \ldots, t_p$. Huang and Wang [11] enumerated the number of $(t, t+1)$-cores, $(t, t+1, t+2)$-cores with the fixed number of corners, where these results are generalized to $(t, t+1, \cdots, t+p)$-cores in [4]. As far as we know, it seems new to get the formula for the number of $t$-core partitions with the fixed number of corners, which we enumerate this by using the path interpretation.

*Proposition* 13. The number of $t$-core partitions with $m$ corners is given by

$$\operatorname{cc}(t, m) := \sum_{i=1}^{\min(m, \lfloor t/2 \rfloor)} N(m, i) \binom{t + 2m - 2i}{2m},$$

where $N(m, i) = \frac{1}{m}\binom{m}{i}\binom{m}{i-1}$ denotes the Narayana number.

*Proof.* By Theorem 12, $\operatorname{cc}(t, m)$ is equal to the number of cornerless Motzkin paths of length $2m + t - 1$ with $t - 1$ flat steps. Let a Dyck path consisting of $m$ up steps and $m$ down steps with $i$ peaks be given. The number of ways of inserting $t - 1$ flat steps such that the resultant path becomes a cornerless Motzkin path is $\binom{t+2m-2i}{2m}$ since we have to insert at least one flat steps at the positions of $i$ peaks and $i - 1$ valleys. As the number of Dyck paths consisting of $m$ up steps and $m$ down steps with $i$ peaks is counted by the Narayana number $N(m, i)$, the proof is followed. □

The numbers of $t$-core partitions with $m$ corners for $2 \leqslant t \leqslant 6$ and $1 \leqslant m \leqslant 8$ are given in Table 1. Clearly, $\operatorname{cc}(2, m) = 1$, $\operatorname{cc}(3, m) = 2m + 1$, and $\operatorname{cc}(4, m) = (5m^2 + 5m + 2)/2$. See sequences A063490 and A160747 in [14] for more the values of $\operatorname{cc}(t, m)$ for $t = 5$ and $t = 6$, respectively.

### 3.3 Cornerless symmetric Motzkin paths and self-conjugate $t$-cores

For a partition $\lambda$, its *conjugate* is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where each $\lambda'_j$ is the number of boxes in the $j$th column of the Young diagram of $\lambda$. A partition $\lambda$ is called

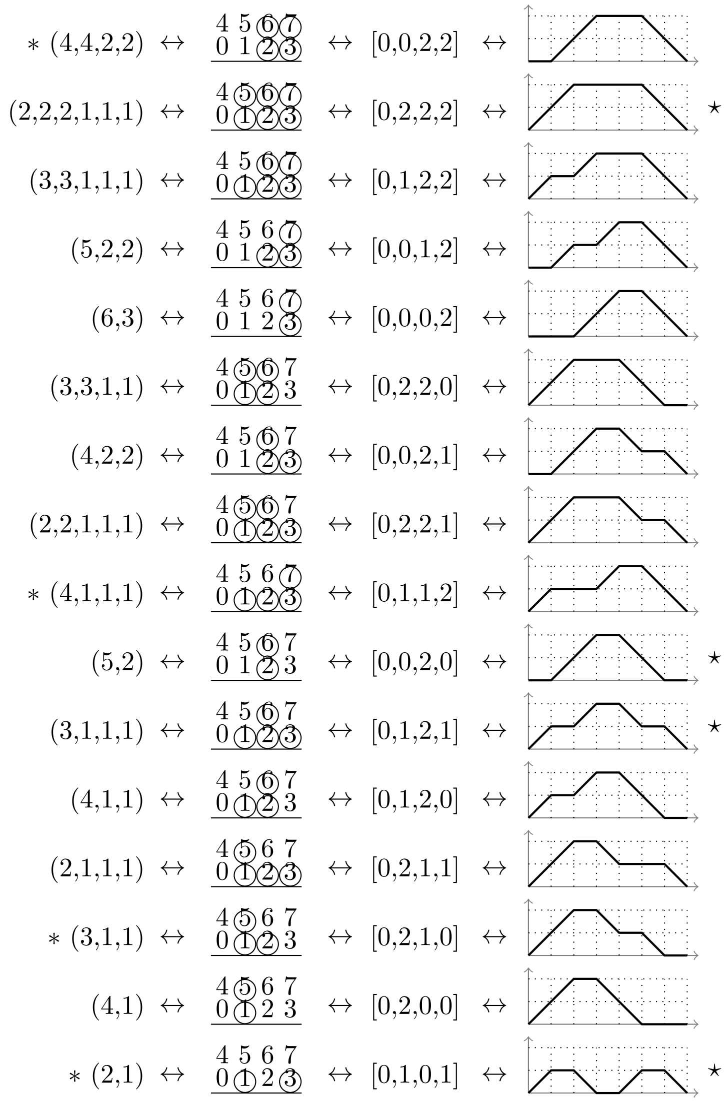


Figure 7: 4-cores with 2 corners and the corresponding objects

| $t\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| 4 | 6 | 16 | 31 | 51 | 76 | 106 | 141 | 181 |
| 5 | 10 | 40 | 105 | 219 | 396 | 650 | 995 | 1445 |
| 6 | 15 | 85 | 295 | 771 | 1681 | 3235 | 5685 | 9325 |

Table 1: The numbers $\mathrm{cc}(t,m)$ of $t$-cores with $m$ corners

*self-conjugate* if $\lambda = \lambda'$. Let $\mathrm{MD}(\lambda)$ denote the set of the main diagonal hook lengths of $\lambda$. Note that if $\lambda$ is a self-conjugate partition, then the elements in $\mathrm{MD}(\lambda)$ are all distinct and odd. Similar to Lemma 11, Ford, Mai, and Sze [9] gave a useful result to determine whether a given partition is a self-conjugate $t$-core or not.

*Proposition* 14. [9, Proposition 3] Let $\lambda$ be a self-conjugate partition. Then $\lambda$ is a $t$-core if and only if both of the following hold:

(a) For $h > t$, if $h \in \mathrm{MD}(\lambda)$, then $h - 2t \in \mathrm{MD}(\lambda)$.

(b) If $h_1, h_2 \in \mathrm{MD}(\lambda)$, then $h_1 + h_2 \not\equiv 0 \pmod{2t}$.

We slightly modify the $t$-abacus to get the *$t$-doubled abacus*, which is useful when we deal with a self-conjugate $t$-core partition. Let the *$t$-doubled abacus diagram* is a left-justified diagram with infinitely many rows labeled by $i \in \mathbb{Z}$ and $\lfloor t/2 \rfloor$ columns labeled by $j = 0, 1, \ldots, \lfloor t/2 \rfloor - 1$ whose position $(i, j)$ is labeled by $|2(ti + j) + 1|$. The *$t$-doubled abacus* of a self-conjugate partition $\lambda$ is obtained from the $t$-doubled abacus diagram by placing a bead on each position labeled by $h$, where $h \in \mathrm{MD}(\lambda)$. From Proposition 14, we have the following lemma.

*Lemma* 15. A self-conjugate partition $\lambda$ is a $t$-core if and only if the $t$-doubled abacus diagram of $\lambda$ satisfies both of the following.

(a) If a bead is placed on position $(i, j)$ with $i > 0$ (resp. $i < 0$), then a bead is also placed on position $(0, j)$ (resp. $(-1, j)$) and there is no spacer between them in any column $j$.

(b) A bead can be placed on at most one of the two positions $(-1, j)$ and $(0, j)$ in any column $j$.

From the above lemma, we easily obtain a simple bijection between the set of self-conjugate $t$-core partitions and the set of integer sequences $(n_0, \ldots, n_{\lfloor t/2 \rfloor - 1})$, where the number of beads in column $j$ is denoted by either $n_j$ or $-n_j$ for $j = 0, 1, \ldots, \lfloor t/2 \rfloor - 1$ if a bead is placed in position $(0, j)$ or not, respectively. Now we give a path interpretation of the self-conjugate $t$-core partitions restricted by the number of corners and the first hook length $h(1, 1)$. We define

$$\mathcal{F}_c(m, r, k) := \bigcup_{i=0}^{k} \overline{\mathcal{F}}_c(m, r, i) \quad \text{and} \quad \mathcal{M}_c(m, r, k) := \bigcup_{i=0}^{k} \overline{\mathcal{M}}_c(m, r, i).$$

*Theorem* 16. For non-negative integers $t$, $m$, and $k$, there is a bijection between any pair of the following sets.

(a) The set of self-conjugate $t$-cores with $m$ corners such that $h(1, 1) < kt$.

(b) The set of integer sequences $(n_0, n_1, \ldots, n_{\lfloor t/2 \rfloor - 1})$ satisfying that for odd (resp. even) $m$, $n_0$ is positive (resp. non-positive); for all $i$, $-\lfloor k/2 \rfloor \leqslant n_i \leqslant \lfloor (k+1)/2 \rfloor$; and

$$\sum_{i=0}^{\lfloor t/2 \rfloor} |n_i - n_{i-1}| = \begin{cases} m+1 & \text{for odd } m, \\ m & \text{for even } m, \end{cases}$$

where we set $n_{-1} := 0$ and $n_{\lfloor t/2 \rfloor} := 0$.

(c) The set of cornerless free Motzkin paths in $\mathcal{F}_{\rm c}(m, \lfloor t/2 \rfloor, k)$.

(d) The set of cornerless Motzkin prefixes in $\mathcal{M}_{\rm c}(m, \lfloor t/2 \rfloor, k)$.

(e) The set of cornerless symmetric Motzkin paths of length $2m+t-1$ with $t-1$ flat steps that are contained in the strip $0 \leqslant y \leqslant k$.

*Proof.* Let $A, B, C, D$, and $E$ be the set described in (a), (b), (c), (d), and (e), respectively. By similar argument to the proof of Proposition 10, we know that there is a bijection between $C$ and $D$. Now we set $\phi_1 : A \to B, \phi_2 : B \to C$, and $\phi_3 : D \to E$ and show that $\phi_1, \phi_2, \phi_3$ are bijections.

Given $\lambda \in A$, let $\phi_1(A) = (n_0, n_1, \dots, n_{\lfloor t/2 \rfloor -1})$, where each $n_i$ is the highest or lowest row that the bead is placed in the $i$th column depending on the sign of $n_i$. We get that $1 \in MD(\lambda)$ when the number of corners $m$ is odd and $1 \notin MD(\lambda)$ otherwise. Thus, $n_0$ is positive when $m$ is odd and non-positive otherwise. This map gives a bijection between $A$ and $B$.

Let $\mathbf{n} = (n_0, n_1, \dots, n_{\lfloor t/2 \rfloor -1})$. For odd (resp. even) $m$, let $\phi_2(\mathbf{n})$ be the cornerless free Motzkin path that starts at $(0,1)$ (resp. $(0,0)$), ends at $(m + \lfloor t/2 \rfloor, 0)$, has $i$th flat step at height $n_{i-1}$ with proper up and down steps between them. Then, the map $\phi_2$ describes a bijection between $B$ and $C$.

Denote a path by $P = p_1 p_2 \cdots p_{m+\lfloor t/2 \rfloor} \in D$. We set

$$\phi_3(P) = \begin{cases} p_1 p_2 \cdots p_{m+\lfloor t/2 \rfloor} \overline{p_{m+\lfloor t/2 \rfloor}} \cdots \overline{p_2 p_1} & \text{if } t \text{ is odd,} \\ p_1 p_2 \cdots p_{m+\lfloor t/2 \rfloor -1} p_{m+\lfloor t/2 \rfloor} \overline{p_{m+\lfloor t/2 \rfloor -1}} \cdots \overline{p_2 p_1} & \text{if } t \text{ is even.} \end{cases}$$

Then, the map $\phi_3$ is a bijective. □

Note that Figure 7 shows that there are four self-conjugate 4-core partitions with 2 corners and four cornerless symmetric Motzkin paths of length 7 with 3 flat steps, which are marked by $*$ and $\star$, respectively. The correspondences between the sets described in Theorem 16 for $t = 4, m = 2$ and $t = 5, m = 3$ are given in Figure 8.

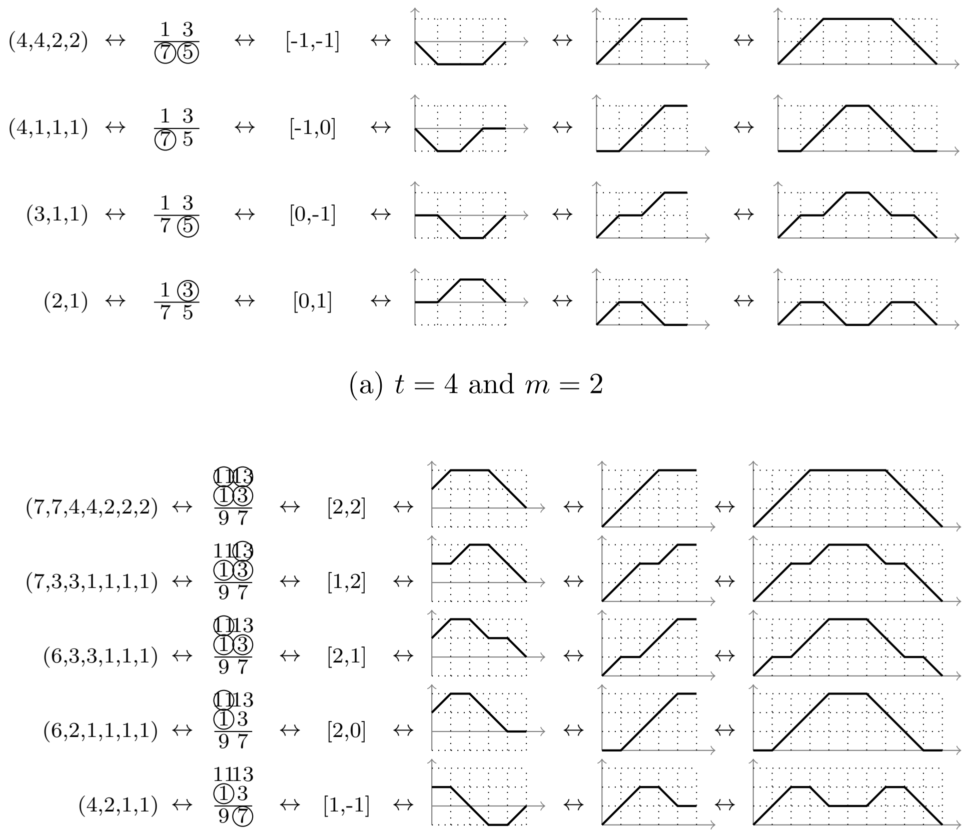
Although the number of self-conjugate $(t, t+1, \cdots, t+p)$-cores with the fixed number of corners is unknown in general, it is enumerated in [2, 3] when $p = 1, 2$, and 3. The number of self-conjugate $t$-core partitions with $m$ corners can be counted by using these path interpretations.

*Proposition* 17. The number of self-conjugate $t$-core partitions with $m$ corners is given by

$$\mathrm{scc}(t, m) := \sum_{i=1}^{\min(m, \lfloor t/2 \rfloor)} \binom{\lfloor \frac{m-1}{2} \rfloor}{\lfloor \frac{i-1}{2} \rfloor} \binom{\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{i}{2} \rfloor} \binom{\lfloor \frac{t}{2} \rfloor + m - i}{m}$$

for $m > 0$ and $\mathrm{scc}(t, 0) = 1$. In addition, $\mathrm{scc}(t, m) = \mathrm{scc}(t+1, m)$ for even $t$.

*Proof.* By Theorem 16, $\mathrm{scc}(t, m)$ also counts the number of cornerless symmetric Motzkin paths of length $2m + t - 1$ with $t - 1$ flat steps. Let a symmetric Dyck path consisting of $m$ up steps and $m$ down steps with $i$ peaks with $2i \leqslant t$ be given. The number of ways




(a) $t = 4$ and $m = 2$

(b) $t = 5$ and $m = 3$

Figure 8: Examples of self-conjugate $t$-cores with $m$ corners and the corresponding objects

inserting $t-1$ flat steps such that the resultant path becomes a cornerless symmetric Motzkin path is $\binom{\lfloor t/2\rfloor+m-i}{m}$. The proof is followed since the number of symmetric Dyck paths consisting of $m$ up steps and $m$ down steps with $i$ peaks is given by (13). □

The numbers of self-conjugate $t$-core partitions with $m$ corners for $2 \leqslant t \leqslant 11$ and $1 \leqslant m \leqslant 8$ are given in Table 2. Clearly, $\mathrm{scc}(2,m) = \mathrm{scc}(3,m) = 1$, $\mathrm{scc}(4,m) = \mathrm{scc}(5,m) = \lfloor 3m/2\rfloor + 1$, and $\mathrm{scc}(6,m) = \mathrm{scc}(7,m) = (10m(m+1) + (-1)^m(2m+1) + 7)/8$.

| $t\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 2,3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4,5 | 2 | 4 | 5 | 7 | 8 | 10 | 11 | 13 |
| 6,7 | 3 | 9 | 15 | 27 | 37 | 55 | 69 | 93 |
| 8,9 | 4 | 16 | 34 | 76 | 124 | 216 | 309 | 471 |
| 10,11 | 5 | 25 | 65 | 175 | 335 | 675 | 1095 | 1875 |

Table 2: The numbers $\mathrm{scc}(t,m)$ of self-conjugate $t$-cores with $m$ corners

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