Monochromatic paths in 2-edge-coloured graphs and hypergraphs

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Abstract

We answer a question of Gyárfás and Sárközy from 2013 by showing that every 2-edge-coloured complete 3-uniform hypergraph can be partitioned into two monochromatic tight paths of different colours. We also give a lower bound for the number of tight paths needed to partition any 2-edge-coloured complete r-partite r-uniform hypergraph. Finally, we show that any 2-edge-coloured complete bipartite graph has a partition into a monochromatic cycle and a monochromatic path, of different colours, unless the colouring is a split colouring.

Mathematics Subject Classifications: 05C35, 05C55, 05D10

1 Introduction

Monochromatic partitions and covering problems are a part of Ramsey theory in the wider sense, and the topic has seen increased activity in the last decade. As in the classical Ramsey problem, one wishes to find certain monochromatic subgraphs in a graph G whose edges are coloured with two colours. Often this is the complete graph on n vertices, K_n. Instead of just one monochromatic copy as in Ramsey's theorem, in monochromatic partitioning problems we aim to find a collection of such copies that together cover the whole vertex set of the host graph.

At the centre of this area lies an observation by Gerencsér and Gyárfás [7], which states that in any 2-colouring of the edges of K_n there are two disjoint monochromatic paths, of different colours, that together cover the vertex set of K_n. If we allow each of $\emptyset$, a vertex, an edge to count as a cycle, then the same statement holds substituting the two

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paths with two cycles: this was conjectured by Lehel in the 1970's (see [2]), and resolved by Bessy and Thomassé in 2010 [4], with earlier asymptotic results in [1, 17].

Generalisations to hypergraphs have been considered, where several notions of cycles and paths exist, and there are some results for loose, ℓ- and Berge-paths and cycles (see e.g. [6, 9, 12, 13, 16]), but for tight paths and cycles, not much is known. A *tight path* in an r-uniform hypergraph is a ordered sequence of vertices such that each r consecutive vertices form an edge. *Tight cycles* are defined analogously on cyclic sequences of vertices. Usually, sets of size between 1 and $r-1$ are allowed as tight paths or cycles, although in this paper, this is not needed, unless $n \leqslant 5$. We will allow a tight path to be empty, i.e., to have no edges or vertices.

In 2013, Gyárfás and Sárközy [13] suggested the following problem: 'decide whether there is a partition into a red and a blue tight path in every 2-coloring of a complete 3-uniform hypergraph'. The question was reiterated in Gyárfás's survey [9], but no advances were made until 2019, when Bustamante, Hàn, and the author [5] proved that in any 2-edge-colouring of the 3-uniform complete graph $\mathcal{K}_n^{(3)}$, two tight monochromatic cycles of different colours partition almost all its vertices (all but $o(n)$ vertices, to be precise). As tight cycles contain spanning tight paths, this answers Gyárfás and Sárközy's question asymptotically.

Here, we solve the problem posed by Gyárfás and Sárközy [13] exactly.

Theorem 1. *For any $n \in \mathbb{N}$ and any 2-colouring of the edges of the 3-uniform complete hypergraph $\mathcal{K}_n^{(3)}$, there are two monochromatic vertex-disjoint tight paths, of different colours, that together cover all the vertices.*

It is not possible to improve Theorem 1 by replacing the tight paths with tight cycles: Lo and Pfenninger [15] exhibit colourings of arbitrarily large $\mathcal{K}_n^{(r)}$, for $k \geqslant 3$, which do not admit a partition into two tight cycles of different colours. However, it has been announced (see [15]) that a partition into two tight monochromatic cycles, possibly of the same colour, does exist for the case $r = 3$.

Lo and Pfenninger [15] also show an analogue of the result from [5] for 4-uniform hypergraphs (i.e., they show that for any 2-edge-colouring of the 4-uniform complete graph $\mathcal{K}_n^{(4)}$, two tight monochromatic cycles of different colours partition almost all the vertices), and indicate they believe their result should generalise to other uniformities. We believe that Theorem 1 should generalise to arbitrary uniformities as well:

Conjecture 2. For any $r, n \in \mathbb{N}$ and any 2-colouring of the edges of the r-uniform complete hypergraph $\mathcal{K}_n^{(r)}$, there are two monochromatic vertex-disjoint tight paths, of different colours, that together cover all the vertices.

In a variant of the original problem in graphs, the underlying complete graph is replaced with the complete balanced bipartite graph $K_{n,n}$. Pokrovskiy [18] showed that these graphs can be partitioned into two monochromatic paths, unless the colouring is a *split colouring*, that is, a colouring where each colour induces the disjoint union of two complete bipartite graphs. (It is easy to see that if these complete bipartite graphs are sufficiently unbalanced, a partition into two monochromatic paths becomes impossible.

On the other hand, a partition into three monochromatic paths is always possible in a split colouring.)

Theorem 3 (Pokrovskiy [18]). *Any 2-colouring of the edges of $K_{n,n}$ that is not a split colouring allows for a partition of $V(K_{n,n})$ into two monochromatic paths of different colours.*

Gyárfás and Lehel [8, 10] proved earlier that a partition of all but one vertex of $K_{n,n}$ exists.

We give a shorter proof of Pokrovskiy's result, Theorem 3, and improve it to a partition into a path and a cycle, as follows.

Theorem 4. *Any 2-colouring of the edges of $K_{n,n}$ that is not a split colouring allows for a partition of $V(K_{n,n})$ into a monochromatic path and a monochromatic cycle, of different colours.*

It is not possible to obtain a partition into two cycles, or to choose the colour of the cycle in Theorem 4. This can be seen by considering a split colouring of $K_{n,n}$ with colours blue and red, and then recolouring in blue one of the red edges. If the sizes of the classes of the split colouring are chosen sufficiently different, then no partition into two monochromatic cyles exists. Moreover, given any partition into a monochromatic path and a monochromatic cycle, it is not difficult to see that the monochromatic path has to use the recoloured edge, and is therefore blue. However, our proof will show that in certain situations it is possible to choose the colour of the cycle (see Corollary 15).

Finally, we return to r-uniform hypergraphs, now for arbitrary uniformities r. Having studied the problem for the graphs K_n and $K_{n,n}$, and for the hypergraph $\mathcal{K}_n^{(r)}$, it seems natural to consider a multipartite hypergraph variant as well. We propose to replace the underlying complete r-uniform hypergraph with $\mathcal{K}_{r \times n}^{(r)}$, the complete balanced r-partite r-uniform hypergraph with n vertices in each partition class. It is not clear whether there is a function $f(r)$ such that for any 2-colouring of the edges of this hypergraph there is a partition into $f(r)$ monochromatic tight paths. But if such a function $f(r)$ exists, then $f(r) \geqslant r + 1$, as the following result shows.

Proposition 5. *For all $n, r \in \mathbb{N}$ with $n \geqslant 3^{r+2}$ and $r \geqslant 2$, one can colour the edges of $\mathcal{K}_{r \times n}^{(r)}$ with two colours so that $V(\mathcal{K}_{r \times n}^{(r)})$ cannot be partitioned into less than $r + 1$ monochromatic tight paths.*

Observe that for $r = 2$, the bound from Proposition 5 coincides with the bound coming from split colourings, which is 3. To prove the proposition, we introduce a generalisation of the split colouring to hypergraphs.

The rest of the paper is organised as follows. Section 2 contains the proof of Theorem 1, showing the slightly stronger Proposition 7. Section 3 is dedicated to r-partite r-uniform hypergraphs and split colourings, and Proposition 5 is proved. In Section 4, we give the brief proof of Theorem 4, based on Theorem 3; then offer an alternative shorter proof of Theorem 3; and finally present a quick application of Theorem 4 to 3-edge-colourings of complete and complete bipartite graphs.

2 Two tight paths for $\mathcal{K}_n^{(3)}$

In this section we show Theorem 1, thus answering the question of Gyárfás and Sárközy from [13]. Similarly as in the proof for graphs from [7], the starting point for our proof is a path, in our case a tight path, that switches colours at most once and has maximal length with this property. Let us define this object in precise terms.

Definition 6 (Tight bicoloured path). Given any 2-colouring of the edges of the 3-uniform complete hypergraph $\mathcal{K}_n^{(3)}$, we call a tight path $P = v_1 v_2 \ldots v_k$ *bicoloured* if there is an index $\ell \in \{1, \ldots, k-1\}$ such that each edge $v_{i-1} v_i v_{i+1}$ with $2 \leqslant i \leqslant \ell$ has the same colour, and all other edges of P have the other colour. Call v_ℓ the *turning point* of P.

Note that monochromatic paths are allowed in Definition 6: namely, a bicoloured path with turning point v_ℓ is monochromatic if and only if $\ell = 1$ or $\ell = k - 1$. As another example, if we have two differently coloured edges $v_1 v_2 v_3$ and $v_2 v_3 v_4$, then they form a bicoloured path $v_1 v_2 v_3 v_4$ with turning point v_2.

Clearly, every bicoloured tight path P contains two disjoint monochromatic paths P_1, P_2 of distinct colours (and moreover, if P has at least 6 vertices, then each of P_1, P_2 is either empty or has at least an edge). So in order to prove Theorem 1, it suffices to show the following proposition.

Proposition 7. *For any $n \in \mathbb{N}$ and any colouring of the edges of the 3-uniform complete hypergraph $\mathcal{K}_n^{(3)}$, there is a spanning bicolored tight path.*

Note that Proposition 7 is slightly stronger than Theorem 1 in the sense that two monochromatic tight paths of distinct colours can not always be concatenated to a bi-coloured path.

Proof of Proposition 7. We take a longest bicoloured tight path P in $\mathcal{K}_n^{(3)}$. Say $P = v_1 v_2 \ldots v_k$, with $v_1 v_2 v_3$ red, and with turning point v_ℓ (for some $\ell \in \{1, \ldots, k-1\}$).

If $k = n$, we are done, so we assume $k < n$. Then there is a vertex w which is not on P. Because of the maximality of P, we know that w cannot be added to P via the edge $w v_1 v_2$ or via the edge $v_{k-1} v_k w$. Therefore, P is not monochromatic and thus

$$1 < \ell < k - 1.$$

Note that reversing P makes $v_{\ell+1}$ the turning point of the reversed bicoloured tight path. This means that we can invoke colour symmetry to be able to assume that

$$\text{the edge } v_\ell v_{\ell+1} w \text{ is red.} \tag{1}$$

We claim that

$$v_{\ell+1} w v_{\ell+2} \text{ is blue.} \tag{2}$$

In order to see (2) by contradiction, assume that $v_{\ell+1} w v_{\ell+2}$ is red. Consider the tight path $P' := v_1 v_2 \ldots v_\ell v_{\ell+1} w v_{\ell+2} \ldots v_{k-1} v_k$. If $\ell = k - 2$, then by (1), P' is monochromatic (and thus bicoloured). If $\ell < k - 2$, then, because of (1), no matter which colour the

edge $wv_{\ell+2}v_{\ell+3}$ has, P' is a bicoloured tight path. It is also longer than P. This is a contradiction, as P was chosen as a maximum length bicoloured tight path. Hence we proved (2).

Next, we claim that

$$v_1wv_{\ell+1} \text{ is red.} \tag{3}$$

Indeed, in order to see (3), it suffices to consider the tight path

$$P'' := v_\ell v_{\ell-1} \ldots v_2 v_1 w v_{\ell+1} v_{\ell+2} \ldots v_k.$$

If the edge $v_1wv_{\ell+1}$ was blue, then by (2), and no matter which colour the edge v_2v_1w has, P'' is a bicoloured tight path on $V(P) \cup \{w\}$. This contradicts the maximality of P. Thus we proved (3).

Similarly, by considering the tight path $v_1v_2 \ldots v_\ell v_{\ell+1}wv_kv_{k-1} \ldots v_{\ell+2}$, which cannot be bicoloured, we can use (1) to see that

$$\text{the edge } v_{\ell+1}wv_k \text{ is blue.} \tag{4}$$

Now, consider the edge v_1wv_k. If this edge is red, then because of (1) and (3), the tight path

$$v_2v_3 \ldots v_\ell v_{\ell+1}wv_1v_kv_{k-1}v_{k-2} \ldots v_{\ell+2}$$

is bicoloured (no matter which colour the edge $v_1v_kv_{k-1}$ is). Similarly, if v_1wv_k is blue, then by (4) and (2),

$$v_\ell v_{\ell-1} \ldots v_2v_1v_kwv_{\ell+1}v_{\ell+2} \ldots v_{k-2}v_{k-1}$$

is a bicoloured tight path no matter the colour of $v_2v_1v_k$. In either case, we obtain a contradiction to the maximality of the bicoloured tight path P, which concludes the proof. $\qquad\square$

3 Split colourings

3.1 Split colourings in hypergraphs

In this section, we consider 2-colourings of the edges of the r-partite r-uniform hypergraph with n vertices in each partition class, denoted by $\mathcal{K}_{r \times n}^{(r)}$. For graphs, the following definition coincides with the usual definition of split colourings.

Definition 8 (Hypergraph split colourings). Let $V_1, V_2, \ldots, V_r$ be the partition classes of $\mathcal{K}_{r \times n}^{(r)}$. For each i, partition V_i into two non-empty sets V_i^1, V_i^2. Colour each edge having an even number of vertices in $\bigcup_{1 \leqslant i \leqslant r} V_i^1$ in red, and colour all other edges blue. Any colouring obtained in this way is called a *split colouring*.

We now give the proof of Proposition 5.

Proof of Proposition 5. Consider a split colouring of $\mathcal{K}_{r \times n}^{(r)}$ with partition classes $V_1^1, V_1^2,$ $V_2^1, V_2^2, \ldots, V_r^1, V_r^2$, where for $1 \leqslant i \leqslant r$, the class V_i^1 has size 3^i, and the class V_i^2 has size $n - 3^i$. Let $\mathcal{P}$ be a set of disjoint monochromatic tight paths that cover all the vertices.

Observe that because of the structure of the split colouring, each path in $\mathcal{P}$ can only meet at most one of V_i^1, V_i^2, for each $1 \leqslant i \leqslant r$, and so,

$$\text{for each } P \in \mathcal{P} \text{ and each } i \in [r], \text{ one of } V_i^1 \cap V(P), V_i^2 \cap V(P) \text{ is empty.} \qquad (5)$$

Our aim is to show that $|\mathcal{P}| \geqslant r + 1$. For this, we let $\mathcal{P}_i$ denote the set of all paths in $\mathcal{P}$ that meet $V_1^1 \cup \ldots \cup V_i^1$ (in particular $\mathcal{P}_0 = \emptyset$), set $V_0 := \emptyset$, and use induction on i to show that for $0 \leqslant i \leqslant \ell \leqslant r$,

$$\text{at most } \sum_{j=1}^{i} |V_j^1| + i \text{ vertices in } V_\ell \text{ are covered by paths from } \mathcal{P}_i. \qquad (6)$$

This is trivially true for $i = 0$ and all $0 \leqslant \ell \leqslant r$. For $i \geqslant 1$, because of (5), the paths in $\mathcal{P}_i \setminus \mathcal{P}_{i-1}$ cannot meet V_i^2, and therefore, by induction, the number of vertices in V_ℓ covered by paths from $\mathcal{P}_i$ is at most $|V_i^1| + 1 + \sum_{j=1}^{i-1} |V_j^1| + i - 1 = \sum_{j=1}^{i} |V_j^1| + i$. This proves (6).

Now, observe that for all $1 \leqslant i \leqslant r$,

$$|V_i^1| = 3^i > \sum_{j=1}^{i-1} 3^j + i - 1 = \sum_{j=1}^{i-1} |V_j^1| + i - 1.$$

So, by (6) for $\ell = i$, we know that $\mathcal{P}_i \setminus \mathcal{P}_{i-1} \neq \emptyset$ for all $1 \leqslant i \leqslant r$. In particular, $|\mathcal{P}_r| \geqslant r$.

Moreover, by (6) for $\ell = i = r$, the paths from $\mathcal{P}_r$ cover at most

$$\sum_{j=1}^{r} |V_j^1| + r = \sum_{j=1}^{r} 3^j + r \leqslant 3^{r+2} - 3^r - 1 \leqslant n - 3^r - 1 = |V_r^2| - 1$$

vertices in V_r, where we used the fact that by assumption, $n \geqslant 3^{r+2}$. As $V_r^2 \subseteq V_r$ has to be covered by paths from $\mathcal{P}$, we deduce that $|\mathcal{P}| \geqslant |\mathcal{P}_r| + 1 \geqslant r + 1$, which is as desired. $\qquad \square$

3.2 Split colourings in graphs

For graphs, it is known that split colourings can be characterised by the existence of certain substructures. Pokrovskiy [18] showed that a 2-colouring of $K_{n,n}$ is split if and only if all vertices see both colours and there is no spanning monochromatic component. In Lemma 11 below, we give a new characterization of colourings that are not split colourings or one of their close relatives: monochromatic colourings or *V-colourings*. The latter class is defined now:

Definition 9 (V-colouring). Call a colouring of the edges of $K_{n,n}$ a *V-colouring* if it has the property that each of the colours spans a complete bipartite graph.

Note that the set of all vertices seeing both colours coincides with one of the partition classes of $K_{n,n}$, giving the colouring the form of the letter V.

For the characterisation in Lemma 11, the following notation will be convenient.

Definition 10 (Bicoloured cycle, good cycle). In a 2-edge-coloured $K_{n,n}$, call a cycle $C = v_1 v_2 \ldots v_k v_1$ *bicoloured* if for some $\ell \in \{2, \ldots, k\}$ each edge $v_i v_{i+1}$ with $1 \leqslant i \leqslant \ell - 1$ has one colour, and all other edges have the other colour. Call v_1 and v_ℓ the *turning points of* C. If v_1 and v_ℓ lie in distinct partition classes, call C *good*.

Note that, in contrast with Definition 6, monochromatic cycles are not bicoloured.

Lemma 11. *Given a colouring c of the edges of $K_{n,n}$, the following are equivalent:*

(a) c is either a split colouring, or a V-colouring, or monochromatic;

(b) there are no good cycles for c;

(c) there is no good C_4 for c.

Proof. We only show that (c) implies (a), as the implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are straightforward. Unless c is monochromatic, there is a vertex v such that both sets $X^r := \{w \in N(v) : vw \text{ is red}\}$ and $X^b := \{w \in N(v) : vw \text{ is blue}\} = N(v) \setminus X^r$ are non-empty. Because of (c), each vertex from the partition class that contains v is monochromatic to each of X^r, X^b, but not to $X^r \cup X^b$. So c is a split colouring or a V-colouring. $\square$

4 Bipartite graphs

In Section 4.1 we prove Theorem 4, and in Section 4.2 we give a new proof of Theorem 3. We will need Definition 10 and Lemma 11 from Section 3.2. In Section 4.3, we show an application of Theorem 4 to 3-edge-colourings.

4.1 Partition into a path and a cycle

We start by observing that clearly, in any 2-edge-coloured $K_{n,n}$, every bicoloured cycle C contains two disjoint monochromatic paths of distinct colours spanning the same set of vertices as C. Also, any two disjoint monochromatic paths of distinct colours can be transformed into a bicoloured or monochromatic cycle, as follows.

Lemma 12. *Let P_1 and P_2 be disjoint monochromatic paths of distinct colours in an 2-edge-coloured $K_{n,n}$ such that $V(K_{n,n}) = V(P_1) \cup V(P_2)$. Then there is a spanning cycle that is either bicoloured or monochromatic.*

Proof. We can assume that $n \geqslant 2$ and that $P_1 = v_1 \ldots v_\ell$ has at least two vertices. If P_2 is empty, v_1 and v_ℓ are in opposite partition classes, and we can add the edge $v_\ell v_1$ to P_1 to obtain the desired cycle. If P_2 has only one vertex w, the edges $v_\ell w$ and wv_1 exist, and can be added to P_1 with the same outcome.

So assume $P_2 = w_1 \ldots w_k$ with $k \geqslant 2$. Observe that one of the endpoints of P_1, say v_1, does not lie in the same partition class as w_1 (otherwise the three vertices v_1, v_ℓ, w_1 are in the same partition class which then would be larger than the other partition class, but

both have size n). Similarly, also v_ℓ, w_k are in opposite partition classes and, therefore, we can transform the paths P_1, P_2 into a bicoloured cycle by adding the edges v_1w_1 and $v_\ell w_k$. □

From the observation above and Lemma 12, it follows that Theorem 3 is equivalent to the following.

Lemma 13. *Every 2-edge-coloured $K_{n,n}$ has a spanning bicoloured or monochromatic cycle, unless the colouring is split.*

Now Theorem 4 follows immediately from Lemma 13 and the next lemma.

Lemma 14. *If a 2-edge-coloured $K_{n,n}$ has a spanning bicoloured cycle, then it has a partition into a monochromatic cycle and a monochromatic path, of distinct colours.*

Proof. Among all spanning bicoloured cycles, we choose a cycle C having the maximum number of red edges. Say $C = v_1 v_2 \dots v_k v_1$, with $v_1 v_2$ red, and with turning points v_1, v_ℓ. If C is good, we are done, since the edge $v_1 v_\ell$ generates a monochromatic cycle with one of the two paths $v_2 v_3 \dots v_{\ell-1}$ or $v_{\ell+1} v_{\ell+2} \dots v_k$, while the other path is monochromatic in the other colour. So assume C is not good, that is, vertices v_1, v_ℓ are in the same partition class. In particular, $v_1 v_{\ell+1}$ and $v_\ell v_k$ are edges.

If the edge $v_1 v_{\ell+1}$ is red, then $k > \ell + 1$ (as $v_k v_1$ is blue) and the bicoloured cycle $(C - v_k v_1 - v_\ell v_{\ell+1}) \cup v_1 v_{\ell+1} \cup v_\ell v_k$ has more red edges than C. So $v_1 v_{\ell+1}$ is blue, and $K_{n,n}$ decomposes into the red path $v_2 \dots v_\ell$ and the blue cycle on the remaining vertices (the blue cycle may be just an edge, which is allowed). □

Observe that one can easily modify the last lines of the proof of Lemma 14, arguing that we either have a red path and a blue cycle or both edges $v_1 v_{\ell+1}, v_\ell v_k$ are blue, giving a spanning bicoloured cycle that is not good and has more red edges than C. So, if at the beginning C is chosen only among the bicoloured spanning cycles that are not good (if such cycles exist), then we can choose the colour of the cycle.

Corollary 15. *Let the edges of $K_{n,n}$ be coloured with red and blue. If there is a spanning bicoloured cycle that is not good, then there is a partition of $K_{n,n}$ into a red path and a blue cycle.*

4.2 Alternative proof of Theorem 3

We give an alternative proof of Theorem 3 by proving Lemma 13. We start by excluding a certain structure outside any longest good cycle. Call a C_4 *balanced* if it has exactly two edges of each colour.

Lemma 16. *If c is a 2-edge-colouring of $E(K_{n,n})$ and $C = v_1 \dots v_k v_1$ is a longest good cycle for c, then the restriction c' of c to $K_{n,n} - V(C)$ has no balanced C_4.*

Proof. Say C has turning points $v_1 \in V_1$, $v_\ell \in V_2$, and $v_1 v_2$ is red. For contradiction, assume c' has a balanced C_4. Then, after exchanging the roles of V_1 and V_2 if needed, there are $x_i, y_i \in V_i \setminus V(C)$, for $i = 1, 2$, with $x_1 x_2$ red, $y_1 x_2$ blue, and exactly one of $x_1 y_2$, $y_1 y_2$ red. We will split our proof accordingly. But first note that by colour symmetry we can assume $v_1 x_2$ is red. As the cycle $(C - v_1 v_k) \cup v_1 x_2 x_1 v_k$ is longer than C, it is not good. So $x_1 v_k$ is red, and similarly, $x_1 v_2$ is blue.

Case 1: $y_1 y_2$ is red and $x_1 y_2$ is blue. If $v_1 y_2$ is red, then as above, also $y_1 v_k$ is red. Moreover, $\ell < k$ (replace $v_k v_1 = v_\ell v_1$ with $v_k x_1 y_2 v_1$ otherwise). Replace $v_{k-1} v_k v_1$ with $v_{k-1} y_2 y_1 v_k x_1 x_2 v_1$ to see that $y_2 v_{k-1}$ is blue. The good cycle $(C - v_k) \cup v_{k-1} y_2 x_1 x_2 v_1$ is longer than C, a contradiction. So $v_1 y_2$ is blue. Recall that also $x_1 v_2$ is blue. If $y_1 v_\ell$ is red then so is $y_2 v_{\ell+1}$ (in particular $\ell \neq k$), and the good cycle $(C - v_\ell v_{\ell+1} - v_k v_1) \cup v_\ell y_1 y_2 v_{\ell+1} \cup v_k x_1 x_2 v_1$ is longer than C, a contradiction. So $y_1 v_\ell$ is blue. By maximality of C also $x_2 v_{\ell-1}$ is blue (and thus $\ell \neq 2$), and $(C - v_1 v_2 - v_{\ell-1} v_\ell) \cup v_1 y_2 x_1 v_2 \cup v_{\ell-1} x_2 y_1 v_\ell$ is a longer good cycle, a contradiction.

Case 2: $x_1 y_2$ is red and $y_1 y_2$ is blue. If $y_1 v_\ell$ is blue, then by maximality of C, so is $y_2 v_{\ell-1}$, and $(C - v_k v_1 - v_{\ell-1} v_\ell) \cup v_k x_1 x_2 v_1 \cup v_\ell y_1 y_2 v_{\ell-1}$ is a longer good cycle, so $y_1 v_\ell$ is red. Note that $\ell \neq k$, as otherwise $(C - v_\ell v_1 \cup v_\ell y_1 x_2 v_1)$ is longer than C. Also, $v_{\ell+1} y_2$ is red, as otherwise $(C - v_k v_1 - v_\ell v_{\ell+1}) \cup v_k x_1 x_2 v_1 \cup v_\ell y_1 y_2 v_{\ell+1}$ is a longer good cycle. Now, $v_k y_1$ is blue, as otherwise $(C - v_k v_1 - v_\ell v_{\ell+1}) \cup v_k y_1 v_\ell \cup v_1 x_2 x_1 y_2 v_{\ell+1}$ is a longer good cycle. Consider $(C - v_k v_1 - v_\ell) \cup v_k x_1 x_2 v_1 \cup v_{\ell-1} y_2 v_{\ell+1}$ to see that $v_{\ell-1} y_2$ is blue. If $x_1 v_\ell$ is blue, add $v_2 x_1 v_\ell$ and $v_{\ell-1} y_2 y_1 v_k$ to $C - v_1 - v_{\ell-1} v_\ell$ for a longer good cycle, and if $x_1 v_\ell$ is red, replace $v_k v_1$ and $v_\ell v_{\ell+1}$ with $v_k y_1 x_2 v_1$ and $v_\ell x_1 y_2 v_{\ell+1}$ for a longer good cycle, a contradiction. $\square$

Now we determine which colourings avoid balanced C_4's. Call an edge-coloring c of $K_{n,n}$ *near-monochromatic* if there is a colour that appears on at most one edge.

Lemma 17. *Any edge-colouring c of $K_{n,n}$ having no balanced C_4 is near-monochromatic.*

Proof. Assume c is not monochromatic. Then some vertex v sees both colours. Let V_1, V_2 be the partition classes of $K_{n,n}$, and use symmetry to assume that $v \in V_1$, and that $|R_v| \geqslant |B_v| \geqslant 1$, where $R_v := \{u \in V_2 : uv \text{ is red}\}$ and $B_v := V_2 \setminus R_v$.

If $|R_v| = 1$, then $|B_v| = 1$, and since c has no balanced C_4, it is near-monochromatic. So assume $|R_v| \geqslant 2$. Then, repeatedly using the fact that there is no balanced C_4, we see that each $x \in V_1 \setminus \{v\}$ sends at least one red edge to R_v, and thus only red edges to B_v, and therefore, only red edges to R_v. In particular $|B_v| = 1$ (otherwise v, x and any two vertices from B_v span a balanced C_4). Thus c has exactly one blue edge (from v to B_v), and is therefore near-monochromatic. $\square$

We are ready to prove Lemma 13.

Proof of Lemma 13. We can assume the colouring c is neither a V-colouring nor monochromatic, as otherwise it is easy to find a spanning bicoloured or monochromatic cycle. By Lemma 11, c has a good cycle, let C_1 be a largest such cycle.

For any good cycle C in c, let $c(C)$ denote the restriction of c to $K_{n,n} - V(C)$. By Lemmas 16 and 17, $c(C_1)$ is near-monochromatic, and thus has a monochromatic spanning path, say in red. Among all good cycles C with the property that $c(C)$ has a red spanning path, choose C_2, maximising the number of blue edges. Say $C_2 = v_1 \ldots v_k v_1$ with $v_1 v_2$ red and turning points v_1 and v_ℓ, and let $P = x_1 \ldots x_h$ be a red spanning path in $c(C_2)$. As v_1 and v_ℓ are in distinct partition classes, one of them, say v_1, is in the same partition class as x_1. If both $x_1 v_\ell$, $v_1 x_h$ are blue, the cycle $v_1 x_h \cup P \cup x_1 v_\ell v_{\ell+1} \ldots v_k v_1$ contradicts our choice of C_2. So at least one of the two edges, say $x_1 v_\ell$, is red. Then $(C_2 - v_\ell v_{\ell+1}) \cup v_\ell x_1 P x_h v_{\ell+1}$ is a spanning bicolored cycle. $\qquad\square$

4.3 Three colours

We now turn to colourings of complete and complete bipartite graphs with three colours. Then, it is known that the edges of K_n cannot always be partitioned into three paths of *distinct* colours, so one drops this condition. In any 3-edge-colouring of K_n there is a partition into three monochromatic paths [18] and a partition of almost all vertices into three monochromatic cycles [11]. Pokrovskiy [18] found 3-colourings that do not admit a partition of all vertices into three monochromatic cycles, however, he conjectures [19] that three cycles and one vertex are always sufficient.

In 3-edge-colourings of the complete bipartite graph $K_{n,n}$, almost all vertices can be partitioned into five monochromatic cycles [14], while folklore 3-colour split colourings[1] show that a partition into fewer than five monochromatic cycles or paths is not always possible. It is conjectured that five is indeed the optimal number for partitions into monochromatic paths [18]. For more details and variants with more colours, see [9].

In [18, 19], Pokrovskiy shows how the following two lemmas for 2-edge-colourings can be used to obtain partitions of 3-edge-coloured graphs.

Lemma 18. [3, 18] *For any red-blue colouring of the edges of K_n there is a partition of the vertices into a red path and a blue balanced complete bipartite graph.*

Lemma 19. [19] *For any red-blue colouring of the edges of $K_{n,n}$ there is a partition of the vertices into a red path and two blue balanced complete bipartite graphs.*

We can use Pokrovskiy's approach from [19] together with our Theorem 4 to obtain partitions of a 3-edge-coloured K_n or $K_{n,n}$ into a mix of monochromatic cycles and paths. For K_n, we start with any 3-colouring of its edges, say in colours red, blue and green. We apply Lemma 18 with blue and green merged, which gives a red path and a balanced complete bipartite graph that only uses blue and green. To this graph we apply Theorem 4 to find a partition into a monochromatic path and a monochromatic cycle, unless the graph is split-coloured, in which case we easily find a partition into three monochromatic cycles. In total, we obtain a partition of K_n into either two monochromatic paths and one monochromatic cycle, or one monochromatic path and three monochromatic cycles.

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[1] A 3-colour split colouring is obtained from a proper edge colouring of $K_{3,3}$ by replacing each vertex with a set of vertices and each edge with a complete bipartite graph in the colour of the edge.

Apply Lemma 19 and Theorem 4 in the same way to obtain a partition of $K_{n,n}$ into either three monochromatic paths and two monochromatic cycles, or into two monochromatic paths and four monochromatic cycles, or into one monochromatic path and six monochromatic cycles. We can reduce the number of elements of the partition by analysing the edges between the two balanced blue-green complete bipartite graphs that are given by Lemma 19:

Lemma 20. *For any 3-edge-colouring of $K_{n,n}$ there is a partition of the vertices into either three monochromatic paths and two monochromatic cycles or into two monochromatic paths and four monochromatic cycles.*

Proof. Say colours are red, blue, green. Lemma 19 gives a red path and four sets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$ (where V_1, V_2 are the partition classes of $K_{n,n}$), with $|A_1| = |A_2| \leqslant |B_1| = |B_2|$, and no red edge between A_1 and A_2 or between B_1 and B_2. We can assume the colourings in green and blue on A_1, A_2 and B_1, B_2 are both split, as by Theorem 4, any non-split colouring allows for a partition into a monochromatic cycle and a monochromatic path, while a split colouring allows for a partition into three monochromatic cycles, giving a total outcome which is as desired for the lemma.

First assume there is a non-red edge e between A_j and B_{3-j} for $j = 1$ or $j = 2$. Say e is blue. Then we can find a blue path through e such that the remainder of each of $A_1 \cup A_2$ and $B_1 \cup B_2$ is a balanced complete bipartite graph with a V-colouring and can thus be partitioned into two monochromatic cycles. This gives a total of two paths and four cycles, which is as desired.

On the other hand, if all edges between A_1, B_2, and between B_1, A_2 are red, then there are two disjoint red cycles covering all of $A_1 \cup A_2$ and part of $B_1 \cup B_2$. Now either the remainder B of $B_1 \cup B_2$ is split, or we can apply Theorem 4. In both cases B partitions into at most two monochromatic cycles and a monochromatic path. Together with the red path and the two red cycles, we obtain the desired partition. $\square$

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