Balancing connected colourings of graphs

Freddie Illingworth* Emil Powierski
Alex Scott* Youri Tamitegama

Mathematical Institute
University of Oxford
Oxford, United Kingdom

{illingworth,powierski,scott,tamitegama}@maths.ox.ac.uk

Submitted: May 16, 2022; Accepted: Feb 12, 2023; Published: Mar 24, 2023

Abstract

We show that the edges of any graph G containing two edge-disjoint spanning trees can be blue/red coloured so that the blue and red graphs are connected and the blue and red degrees at each vertex differ by at most four. This improves a result of Hörsch. We discuss variations of the question for digraphs, infinite graphs and a computational question, and resolve two further questions of Hörsch in the negative.

Mathematics Subject Classifications: 05C02

1 Introduction

Finding edge-disjoint spanning trees in a graph has a rich history. The seminal result is the independent characterisation by Tutte [11] and Nash-Williams [7] of the presence of k edge-disjoint spanning trees in a finite graph. Much research has focussed on whether the packed spanning trees can be chosen to satisfy extra properties (for example, see [1, 2, 3]). It is folklore that the edges of any graph G can be coloured blue and red such that the blue degree and red degree of each vertex differ by at most two. The intersection of these two problems asks how well the colour-degrees can be balanced in a blue/red-edge colouring of a graph that contains a *double tree* – the union of two edge-disjoint spanning trees – subject to each colour class being connected. Kriesell [6] was the first to consider balancing colour-degrees in a blue/red-edge colouring of a double tree. Building on his work, Hörsch [5] gave the first constant bound when G is a double tree.

Theorem 1 (Hörsch [5])**.** *Let G be a finite double tree. The edges of G may be coloured blue and red such that the blue and red graphs are both spanning trees and the blue and red degrees of each vertex differ by at most five.*

*Supported by EPSRC grant EP/V007327/1

Our main result is two-fold. Firstly we reduce the above bound to four.

Theorem 2. *Let G be a finite double tree. The edges of G may be coloured blue and red such that the blue and red graphs are both spanning trees and the blue and red degrees of each vertex differ by at most four.*

Moreover, we obtain the same bound in the more general case where G is any graph containing a spanning double tree.

Theorem 3. *Let G be a finite graph containing a spanning double tree. The edges of G may be coloured blue and red such that the blue and red graphs both contain spanning trees and the blue and red degrees of each vertex differ by at most four.*

Hörsch asked whether Theorem 1 can be extended to infinite graphs. The Tutte-Nash-Williams characterisation does not hold for infinite graphs: Oxley [8] gave countable locally finite graphs satisfying the characterisation but not containing k edge-disjoint spanning trees. However, Tutte proved that the characterisation is still valid for countable graphs if one asks for the edge-disjoint subgraphs to be *semiconnected*: a subgraph $H \subset G$ is semiconnected if it contains an edge of every finite cut of G. We give an extension of this form of Theorem 3 to countably infinite graphs.

Theorem 4. *Let G be a countably infinite graph containing a spanning double tree. The edges of G may be coloured blue and red such that the blue and red graphs are both semiconnected and the blue and red degrees of each vertex differ by at most four* (*or are both infinite*). *Further, if G is a double tree, the blue and red graphs may be chosen to be acyclic.*

Hörsch asked whether digraphs can be balanced with the role of trees being played by arborescences: rooted trees where all edges are directed away from the root.

Question 5. Is there a constant C such that the following holds for every digraph D that is the union of two arc-disjoint arborescences? The edges of D can be coloured blue and red such that the blue and red graphs are both arborescences and the blue and red out-degrees of each vertex differ by at most C?

We provide an infinite family of counterexamples that need $C \geqslant |V(D)|-2$, answering Hörsch's question in the negative. We further show that the natural analogue where the role of trees is played by strongly connected digraphs is also false.

Hörsch also asked an algorithmic question.

Question 6. Does there exist a polynomial time algorithm to decide if a given Eulerian double tree has a perfectly balanced double tree decomposition?

Reducing the NP-complete problem (*cf.* Péroche [9]) of finding two edge-disjoint Hamiltonian cycles in a 4-regular graph to this decision problem, we show that Question 6 is equivalent to the problem P versus NP.

The paper is organised as follows. In Section 2 we give the proof of Theorem 2 and the main tools used to prove Theorem 3; in Section 3 we conclude the proof of Theorem 3; in Section 4 we discuss the infinite case and prove Theorem 4; in Section 5 we describe our constructions for digraphs; in Section 6 we address Question 6. Finally, in Section 7, we conclude with some natural questions of our own.

1.1 Notation

We use standard notation. Throughout we consider all graphs to be multigraphs without self-loops. For a graph $G = (V, E)$ with vertices V and edges E we write $e(G)$ for $|E|$ and $|G|$ for $|V|$. For a vertex $v \in V$, we denote the neighbourhood of v by $\Gamma(v)$. Given a partition $A \sqcup B = V$ of the vertices of G, we write $E(A, B)$ for the set of edges in E with endpoints in both A and B, and $e(A, B) = |E(A, B)|$. If $A \subset V$ we write $G[A]$ to denote the subgraph induced by A. When X is a set of vertices (respectively edges) and x is a vertex (respectively edge), we use the shorthand $X + x$ and $X - x$ to mean $X \cup \{x\}$ and $X \setminus \{x\}$. For a graph G, a vertex v and an edge e we write $G - v$ for the graph obtained by deleting v and all edges incident with it and $G - e$ for the graph obtained by deleting e. If $e \notin E(G)$, $G + e$ denotes the graph obtained by adding e to $E(G)$.

We use $\sqcup$ to denote a disjoint union. Throughout we write $G = S_1 \sqcup \cdots \sqcup S_k$ to mean that $S_1, \dots, S_k$ are spanning subgraphs of G and $E(G) = E(S_1) \sqcup \cdots \sqcup E(S_k)$. We sometimes refer to this as a *decomposition* of G. If $S_1, \dots, S_k$ are trees, we refer to it as a *k-tree decomposition*; if further $k = 2$, a *double tree decomposition*.

Definition 7. Let G be a graph, c an integer, and suppose $G = S_1 \sqcup \cdots \sqcup S_k$. A vertex $v \in V(G)$ is said to be *c-balanced* in $S_1 \sqcup \cdots \sqcup S_k$ if for all i, j,

$$|d_{S_i}(v) - d_{S_j}(v)| \leqslant c.$$

We say that the decomposition $G = S_1 \sqcup \cdots \sqcup S_k$ is *c-balanced* if every $v \in V(G)$ is c-balanced in it.

When the constant c is clear, we write *balanced* for brevity. Note, for example, that Theorem 2 can be phrased as 'every finite double tree admits a 4-balanced double tree decomposition'.

2 Balancing double trees

Fix an integer $c \geqslant 2$ and suppose there are double trees with no c-balanced double tree decomposition. Throughout this section, we take G to be such a double tree with $|G|$ minimal.

Call a vertex $v \in V(G)$ *small* if $d_G(v) \leqslant c+2$ and *big* otherwise. A simple observation that will be used throughout the section is that small vertices are balanced in any double tree decomposition. We call a vertex $v \in V(G)$ an *ℓ-vertex* if $d_G(v) = \ell$.

In Sections 2.1-2.4 we show that a minimal counterexample G satisfies a collection of structural properties. Finally, in Section 2.5 we make a discharging argument with $c = 4$ to conclude that G has too many edges for a double tree.

Our arguments to show that G cannot contain certain substructures have the following template.

1. Locally modify G to create a double tree H with $|H| < |G|$.
2. Use minimality to find a balanced decomposition for H.
3. From this decomposition recover a decomposition for G.
4. Argue that this decomposition is balanced.

Step 1 is referred to as the *reduction step*, step 3 as the *reconstruction step*. In step 4, we need only show that the vertices involved in the reduction and reconstruction steps are balanced, as all other vertices are balanced in step 2 and left untouched afterwards.

Our methods refine those of Hörsch [5]. There are two main novel ideas. The first is using edge swaps to control the structure around certain 3-vertices (*cf.* Lemma 13). This is used to force blue and red degrees above 1 and so aid balancedness. The second is controlling the parity of the degrees of the neighbours of 2-vertices. These ideas are crucial to most of our structural lemmas.

In figures, big vertices are black, small vertices are white. When the status of a vertex is unclear, we indicate it in grey. As a convention, when a graph has a double tree decomposition with trees labelled by 1 and 2, we use blue for tree 1, red for tree 2 and black when the colour is irrelevant.

2.1 2-vertices

Let v be a 2-vertex and x, y its (not necessarily distinct) neighbours. As v is a leaf in both trees of any double tree decomposition of G, removing it yields a double tree H, which admits a balanced decomposition by minimality of G. We refer to this as the *standard reduction for* 2*-vertices*. This reduction can be reversed in the obvious way by adding back v and the edges incident to it.

(a) Start configuration. (b) Reduced configuration.

Figure 1: Standard reduction for 2-vertices.

Lemma 8. *Let $v \in V(G)$ be a 2-vertex. Then the neighbours x, y of v are distinct, big and $d_G(x) \equiv d_G(y) \equiv c + 1 \bmod 2$.*

Proof. Let vx, vy be the edges incident to v. If $x = y$, then the standard reduction for 2-vertices immediately gives a balanced decomposition for G, a contradiction.

Suppose x is small. Apply the standard reduction for 2-vertices to obtain a double tree H with a balanced blue/red double tree decomposition. We may then add v back by giving vy a colour in which the degree of y was smallest, and vx the other colour. Since x is small, this yields a balanced decomposition for G, a contradiction.

Suppose for contradiction that $d_G(x) \equiv c \bmod 2$. Apply the standard reduction for 2-vertices to v to obtain a double tree H with balanced decomposition $H = S_1 \sqcup S_2$. In particular, $|d_{S_1}(x) - d_{S_2}(x)| \leqslant c$. By the congruence condition,

$$|d_{S_1}(x) - d_{S_2}(x)| \equiv d_{S_1}(x) + d_{S_2}(x) = d_G(x) - 1 \equiv c - 1 \bmod 2,$$

so in fact, $|d_{S_1}(x) - d_{S_2}(x)| \leqslant c - 1$. By symmetry we may assume that $d_{S_1}(y) \geqslant d_{S_2}(y)$. Put v back, adding vx to S_1 and vy to S_2. Then y is still balanced and the degree difference at x has increased by at most 1, so is at most c. This is a contradiction. □

The following observation appeared in Hörsch [5, Lemma 2] when $c = 5$; a straightforward modification of the proof gives the result for any $c \geqslant 2$.

Lemma 9. *Every big vertex $v \in V(G)$ is adjacent to at most one 2-vertex.*

2.2 Edge swaps

We remind the reader of a simple tool that will be used repeatedly in our arguments. Given a double tree decomposition $G = T_1 \sqcup T_2$ and an edge $e \in E(T_1)$ we may swap it with some $f \in E(T_2)$ such that $G = S_1 \sqcup S_2$, where $E(S_1) = E(T_1) - e + f$ and $E(S_2) = E(T_2) - f + e$, is a double tree decomposition. Indeed, T_1 splits into two components after removing e and adding e to T_2 creates a cycle C. We may thus choose f to be any edge of $C - e$ with an endpoint in each component. We will refer to this as *swapping* e. Note that if $e \in E(T_1)$ is incident to a leaf x of T_1, then f must also be incident to x. In particular, after swapping e, x is still a leaf of T_1.

Lemma 10. *Let G be a double tree with a blue/red decomposition and xy a blue edge such that x is a leaf in the blue tree. Then x remains a leaf in the blue tree after swapping xy.*

This is particularly useful for 3-vertices as in any blue/red double tree decomposition a 3-vertex must be a leaf in some colour.

2.3 3-vertices

Let v be a 3-vertex with (not necessarily distinct) neighbours x, y, b where the edges vx, vy are red and the edge vb is blue as in Figure 2a. Remove v and join x and y in red to form H; we will refer to this as the *standard reduction for 3-vertices*. If v has two blue neighbours and one red neighbour, then there is an analogous reduction. Since v was a leaf in the blue tree and xvy is the only path from x to y in the red tree, the resulting blue/red decomposition is a double tree decomposition for H. Further, $|H| = |G| - 1$ so, by minimality, H has a balanced double tree decomposition. A particularly useful feature of this reduction is that it is reversible: given a double tree containing the configuration

(a) Start configuration. (b) Reduced configuration.

Figure 2: Standard reduction for 3-vertices.

shown in Figure 2b we may delete the edge xy, add a new vertex v joined to x and y in red and joined to b in blue to form another double tree.

Using this reduction, Hörsch [5, Proposition 10] showed the following.

Lemma 11. *Let $v \in V(G)$ be a 3-vertex and $G = T_1 \sqcup T_2$ a double tree decomposition. Suppose that v is a leaf in T_2 with $vb \in E(T_2)$ its unique incident edge. Then b is big.*

Suppose that v is a 3-vertex with two edges to small vertices. Let vb be the third edge incident to v. Lemma 11 shows that in any double tree decomposition of G, the edge vb is of the opposite colour to the other two edges. However, swapping the edge vb gives a contradiction. So every 3-vertex has at most one edge to a small vertex and so only the following types of 3-vertices can occur.

Definition 12 (types of **3**-vertex)**.** We say that a 3-vertex is

- *rich* if all its neighbours are big;
- *poor* if it is adjacent to three distinct vertices, two big, one small;
- *bad* if it has a small neighbour and is joined to a big vertex by a double edge.

We are ready to prove a key result that gives structure around poor 3-vertices.

Lemma 13. *Let $v \in V(G)$ be a poor 3-vertex with big neighbours x, y and small neighbour s. In any tree decomposition $G = T_1 \sqcup T_2$ where $vs \in E(T_1)$:*

- *Exactly one of vx, vy is in $E(T_1)$.*
- *If $vy \in E(T_2)$, then swapping vy gives the double tree decomposition $G = T_1' \sqcup T_2'$, where $E(T_1') = E(T_1) - vx + vy$ and $E(T_2') = E(T_2) + vx - vy$.*
- *The path from x to y in T_1 does not contain s.*

Proof. Firstly, by Lemma 11, at least one of vx, vy is in $E(T_1)$. They cannot both be otherwise v is an isolated vertex in T_2. This gives the first bullet point.

Suppose that $vy \in E(T_2)$ and so $vx \in E(T_1)$ and v is a leaf in T_2 as in Figure 3a. Consider swapping vy. Then vy becomes blue and so, by the first bullet point, vx becomes red.

We now prove the third bullet point. We may assume by symmetry that $vy \in E(T_2)$ and so $vx \in E(T_1)$. Suppose that there is a path P in T_1 from x to y that contains s and let P' be the subpath of P from s to y. Consider swapping vy: we have just shown that vx becomes red. But then $yvsP'$ forms a blue cycle, which is impossible. ☐

(a) v a leaf in T_2 with $vy \in E(T_2)$. (b) Configuration after swapping vy.

Figure 3: Edge swap in Lemma 13.

Lemma 14. *Let $v \in V(G)$ be adjacent to $\ell \geqslant 1$ bad 3-vertices via their double edges. Then,*

$$d_G(v) \geqslant 2\ell + c + 1.$$

Proof. Let v be a vertex in G with a bad neighbour u, and let $w \neq v$ be the small neighbour of u. Fix a double tree decomposition $G = T_1 \sqcup T_2$. By symmetry we may assume $uw \in E(T_1)$.

Apply the standard reduction for 3-vertices to u and let H be the resulting graph with balanced double tree decomposition $H = S_1 \sqcup S_2$. Without loss of generality we may

Figure 4: Reduction step in Lemma 14.

assume that $vw \in E(S_1)$. Define $G = T_1' \sqcup T_2'$, where

$$E(T_1') = E(S_1) - vw + vu + uw,$$
$$E(T_2') = E(S_2) + vu,$$

reversing the reduction. All vertices, except possibly v, are balanced in $T_1' \sqcup T_2'$. Since v is adjacent to ℓ bad 3-vertices via double edges, we have $d_{T_1'}(v) \geqslant \ell$ and $d_{T_2'}(v) \geqslant \ell$. In particular, if $d_G(v) \leqslant 2\ell + c$, then v is balanced also, which is a contradiction. □

2.4 Critical vertices

A vertex v in G is said to be *critical* if $d_G(v) = c + 3$, that is, if its degree is just large enough for it to be big. A simple observation is that critical vertices are balanced in a blue/red decomposition if and only if both their blue and red degrees are at least two. We combine this observation with the final bullet point of Lemma 13 to great effect: suppose a big vertex v has a blue edge to a poor 3-vertex u and u has blue degree two in a given blue/red decomposition. Then the final bullet point of Lemma 13 guarantees that v has blue degree at least two. If further v is critical and has red degree at least two, then it is balanced.

Lemma 15. *Let $v \in V(G)$ be a critical vertex.*

(*i*) *If all neighbours of v are small, then v is not adjacent to bad 3-vertices.*
(*ii*) *At most one neighbour of v is a poor 3-vertex.*
(*iii*) *If v is adjacent to a 2-vertex, then v is not adjacent to a poor 3-vertex.*

Note that Lemmas 9, 15.(ii) and 15.(iii) yield that a critical vertex has at most one neighbour that is either a 2-vertex or a poor 3-vertex.

Proof. **(i)** Suppose not and let $G = T_1 \sqcup T_2$ be a double tree decomposition. Let $v \in V(G)$ be critical with all neighbours small and let $u \in \Gamma(v)$ be a bad 3-vertex, and $w \neq v$ be the small neighbour of u. By symmetry we may assume $uw \in E(T_1)$.

Apply the standard reduction for 3-vertices to u, let H be the resulting double tree and $H = S_1 \sqcup S_2$ a balanced double tree decomposition. Without loss of generality we may assume that $vw \in E(S_1)$.

Case 1. $d_{S_1}(v) \geqslant 2$.

Reverse the reduction to get $G = T_1' \sqcup T_2'$, where

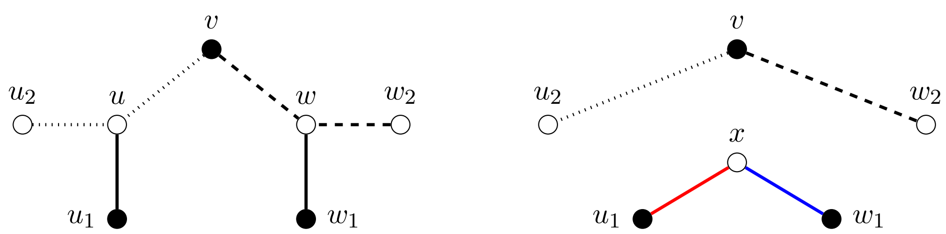
$$E(T_1') = E(S_1) - vw + vu + uw,$$
$$E(T_2') = E(S_2) + vu.$$

Then $d_{T_1'}(v) = d_{S_1}(v) \geqslant 2$ and $d_{T_2'}(v) \geqslant d_{S_2}(v) + 1 \geqslant 2$, and so, since v is critical, it is balanced and thus the decomposition $T_1' \sqcup T_2'$ is balanced, a contradiction.

Case 2. $d_{S_1}(v) = 1$.

Let $H = S_1' \sqcup S_2'$ be the decomposition obtained after swapping vw. Since v is a leaf in S_1, it remains a leaf in S_1'. Moreover, every vertex in the neighbourhood of v is small, so every big vertex is balanced in $S_1' \sqcup S_2'$. Now $vw \in E(S_2')$ and $d_{S_2'}(v) \geqslant 2$, so by Case 1 we can find a balanced decomposition $G = T_1' \sqcup T_2'$, a contradiction.

(ii) Suppose that $v \in V(G)$ is critical and $u, w \in \Gamma(v)$ are distinct poor 3-vertices. Let the other neighbours of u and w be u_1, u_2 and w_1, w_2 (not necessarily distinct) respectively, as in Figure 5a, where u_2, w_2 are small. By Lemma 13 we may perform edge swaps to ensure $\{vw, ww_2\} \subset E(T_i)$ and $\{uv, uu_2\} \subset E(T_j)$ for some $i, j \in \{1, 2\}$. Apply the



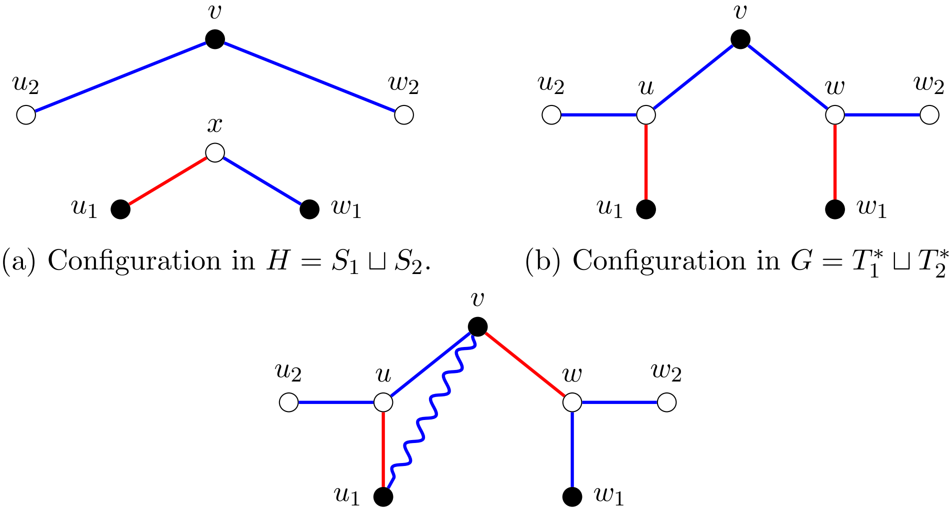
(a) Configuration in G. (b) Reduction to H.

Figure 5: Reduction step in Lemma 15.(ii). Dashed edges are in the same tree, dotted edges are in the same tree.

standard reduction for 3-vertices to u and w and add a 2-vertex x adjacent to both u_1 and w_1, yielding a double tree H which by induction has a balanced double tree decomposition

$H = S_1 \sqcup S_2$. Without loss of generality we may assume that $vw_2 \in E(S_1)$. We consider multiple cases.

Case 1. $vu_2 \in E(S_1)$.



(a) Configuration in $H = S_1 \sqcup S_2$. (b) Configuration in $G = T_1^* \sqcup T_2^*$.

(c) Configuration in $G = T_1' \sqcup T_2'$.

Figure 6: Reconstruction in Case 1.

By symmetry we may assume that $xw_1 \in E(S_1)$. Reverse the reductions and delete x to give $G = T_1^* \sqcup T_2^*$, where

$$E(T_1^*) = E(S_1) - xw_1 - vu_2 - vw_2 + vu + uu_2 + vw + ww_2,$$
$$E(T_2^*) = E(S_2) - xu_1 + uu_1 + ww_1,$$

as in Figure 6b. Consider swapping ww_1. Lemma 13 implies that we get the decomposition $G = T_1' \sqcup T_2'$ shown in Figure 6c, where

$$E(T_1') = E(T_1^*) - vw + ww_1,$$
$$E(T_2') = E(T_2^*) - ww_1 + vw.$$

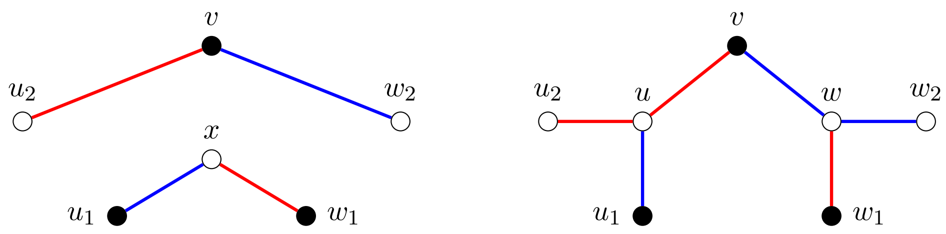
We claim that $G = T_1' \sqcup T_2'$ is balanced. All degree differences are the same as in $S_1 \sqcup S_2$ at all big vertices except v where a blue edge has become red. By Lemma 13, there is a path in T_1' from v to u_1 that does not use u_2. Since also $uv \in E(T_1')$, we get $d_{T_1'}(v) \geqslant 2$. As v is critical this means it is balanced, and therefore $G = T_1' \sqcup T_2'$ is balanced, as required.

Case 2.i. $vu_2 \in S_2$ and $xu_1 \in S_1$.

Reverse the reductions to get $G = T_1' \sqcup T_2'$, where

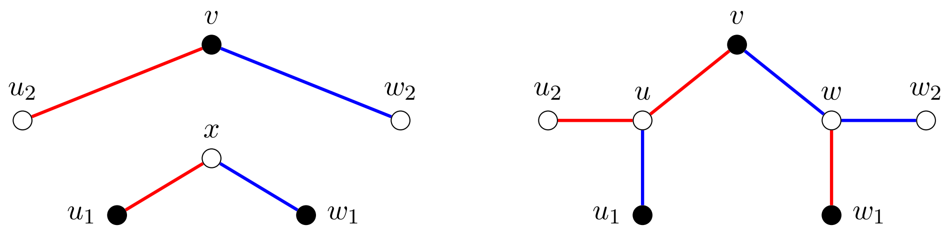
$$E(T_1') = E(S_1) - xu_1 - vw_2 + uu_1 + vw + ww_2,$$
$$E(T_2') = E(S_2) - xw_1 - vu_2 + ww_1 + uv + uu_2,$$

as in Figure 7b. Further, since $S_1 \sqcup S_2$ is balanced and degree differences of big vertices remained unchanged, $T_1' \sqcup T_2'$ is balanced, a contradiction.

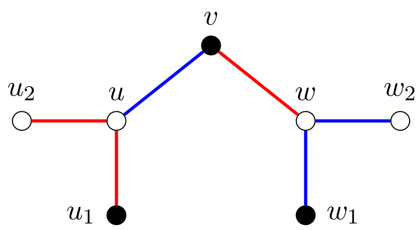


(a) Configuration in $H = S_1 \sqcup S_2$. (b) Reconstruction in $G = T_1' \sqcup T_2'$.

Figure 7: Reconstruction in Case 2.i.



(a) Configuration in $H = S_1 \sqcup S_2$. (b) Configuration in $G = T_1^* \sqcup T_2^*$.



(c) Configuration in $G = T_1' \sqcup T_2'$.

Figure 8: Reconstruction in Case 2.ii.

Case 2.ii. $vu_2 \in E(S_2)$ and $xu_1 \in E(S_2)$.
Reverse the reductions to get $G = T_1^* \sqcup T_2^*$, where

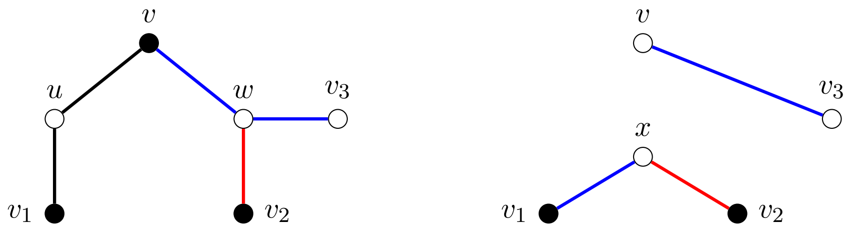
$$E(T_1^*) = E(S_1) - xw_1 - vw_2 + uu_1 + vw + ww_2,$$
$$E(T_2^*) = E(S_2) - xu_1 - vu_2 + vu + uu_2 + ww_1,$$

as in Figure 8b. By edge swapping uu_1 and ww_1 and noting that u, w are leaves in T_1^*, T_2^* respectively, Lemma 11 guarantees that we obtain the double tree decomposition $G = T_1' \sqcup T_2'$, where

$$E(T_1') = E(T_1^*) - uu_1 - vw + uv + ww_1,$$
$$E(T_2') = E(T_2^*) - uv - ww_1 + uu_1 + vw.$$

Vertices v, u_1, w_1 are balanced in $S_1 \sqcup S_2$, hence with respect to $T_1' \sqcup T_2'$ as well, as degree differences remained unchanged. All other degree differences at big vertices were preserved, so $G = T_1' \sqcup T_2'$ is balanced, a contradiction.

(iii) Let $v \in V(G)$ be critical, $G = T_1 \sqcup T_2$ be a double tree decomposition and suppose that $u, w \in \Gamma(v)$ are a 2-vertex and a poor 3-vertex, respectively. Let v_3 be the small neighbour of w, and $v_1, v_2 \neq v$ the other big neighbours of u, w respectively, where Lemma 8 ensures $v_1 \neq v$. By symmetry we may assume that w is a leaf in T_2. By Lemma 13, we may swap edges to ensure that $wv_2 \in E(T_2)$. Apply the standard

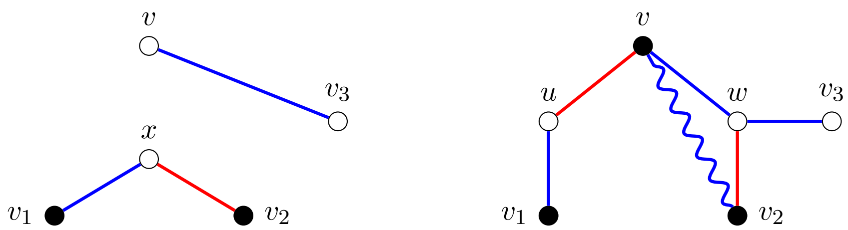


(a) Configuration in G. (b) Reduction to H.

Figure 9: Reduction step in Lemma 15.(iii).

reduction for 2-vertices and 3-vertices to u and w respectively, and add a new 2-vertex x joined to v_1 and v_2, yielding a double tree H as shown in Figure 9b. By minimality of G, H has a balanced double tree decomposition $H = S_1 \sqcup S_2$. By symmetry we may assume that $vv_3 \in E(S_1)$. We treat two cases separately.

Case 1. $xv_2 \in E(S_2)$.



(a) Configuration in $H = S_1 \sqcup S_2$. (b) Reconstruction in $G = T_1' \sqcup T_2'$.

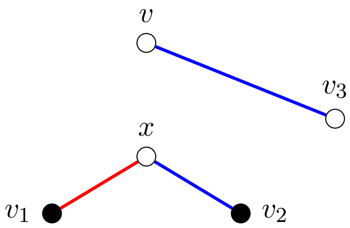
Figure 10: Reconstruction step in Case 1.

Reverse the reductions to get $G = T_1' \sqcup T_2'$, where

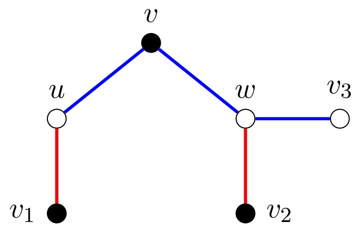
$$E(T_1') = E(S_1) - xv_1 - vv_3 + vw + wv_3 + uv_1,$$
$$E(T_2') = E(S_2) - xv_2 + wv_2 + uv,$$

as in Figure 10b. We claim that every vertex in $T_1' \sqcup T_2'$ is balanced. All degree differences are unchanged at big vertices except at v where a red edge has been added. By Lemma 13, there is a path in G from v_2 to v in T_1' that does not pass through v_3, so $d_{T_1'}(v) \geqslant 2$. As v is critical, it is balanced, as required.

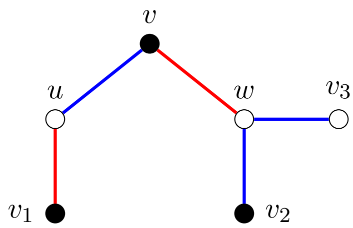
Case 2. $xv_2 \in E(S_1)$



(a) Configuration in $H = S_1 \sqcup S_2$.



(b) Configuration in $G = T_1^* \sqcup T_2^*$.



(c) Configuration in $G = T_1' \sqcup T_2'$ after swapping wv_2.

Figure 11: Reconstruction in Case 2.

Reverse the reductions to get $G = T_1^* \sqcup T_2^*$, where

$$E(T_1^*) = E(S_1) - xv_2 - vv_3 + uv + wv_3 + vw,$$
$$E(T_2^*) = E(S_2) - xv_1 + uv_1 + wv_2,$$

as in Figure 11b. After swapping wv_2 we obtain, by Lemma 13, the decomposition $G = T_1' \sqcup T_2'$ shown in Figure 11c. Since u and w are leaves in T_1', T_2' respectively with $vu \in E(T_1')$ and $vw \in E(T_2')$, we have $d_{T_i'}(v) \geqslant 2$ for $i = 1, 2$ and therefore v is balanced as it is critical. All other degree differences at big vertices remain unchanged. Hence, $G = T_1' \sqcup T_2'$ is balanced, a contradiction. $\square$

2.5 Discharging

In this section we conclude the proof of Theorem 2 by applying the lemmas above with $c = 4$.

Proof of Theorem 2. Let G be a counterexample minimising the number of vertices n. Define the initial charge function $f : V \to \mathbb{Q}$ by $f(v) = d(v)$ and the *discharging procedure* as follows. For each big vertex v and each edge vu it is incident to, send to the vertex u

- charge 1 if u is a 2-vertex,
- charge 1/2 if u is a poor 3-vertex,
- charge 1/2 if u is a bad 3-vertex (note that a bad 3-vertex receives a total charge of 1 from v because of the double edge),
- charge 1/3 if u is a rich 3-vertex.

Let $g\colon V \to \mathbb{Q}$ be the charge function after the discharging procedure has taken place. Then

$$\sum_{v \in V(G)} g(v) = \sum_{v \in V(G)} f(v) = 4n - 4.$$

We claim that every vertex v of G has $g(v) \geqslant 4$, which will give a contradiction.

Indeed, if $d(v) \geqslant 9$, then, by Lemma 9, v is adjacent to at most one 2-vertex and therefore

$$g(v) \geqslant 9 - (1 + 8 \cdot \tfrac{1}{2}) = 4.$$

If $d(v) = 8$, then, by Lemma 8, v cannot be adjacent to any 2-vertices and therefore

$$g(v) \geqslant 8 - 8 \cdot \tfrac{1}{2} = 4.$$

If $d(v) = 7$, we distinguish two cases.

1. If $\Gamma(v)$ contains only small vertices, then by Lemma 15.(i), v does not have bad 3-vertices in its neighbourhood, and by Lemmas 9, 15.(ii), 15.(iii), v has at most one neighbour that is a 2-vertex or a poor 3-vertex. Therefore, $g(v) \geqslant 7 - 1 - 6 \cdot 1/3 = 4$.
2. If $\Gamma(v)$ has a big vertex, using Lemmas 9, 14, 15.(ii), 15.(iii), we similarly get that $g(v) \geqslant 7 - 1 - 2 \cdot 1/2 - 3 \cdot 1/3 = 4$.

If $d(v) \in \{4, 5, 6\}$, then $g(v) = d(v) \geqslant 4$.

If $d(v) = 3$, there are two cases.

1. If v is a rich 3-vertex, it receives a charge of $1/3$ from each of its edges, thus $g(v) = 3 + 3 \cdot 1/3 = 4$.
2. If v is a poor or bad 3-vertex, it receives a charge of $1/2$ from two of its edges, thus $g(v) = 3 + 2 \cdot 1/2 = 4$.

If $d(v) = 2$, then, by Lemma 8, v receives a charge of 1 from both its neighbours and thus $g(v) = 4$, as required. □

3 General graphs

In this section we write $G = A + M$ to mean that $G = A \sqcup M$ where A is a spanning double tree and M a graph.

We deduce Theorem 3 from a slightly more general statement.

Theorem 16. *Let $G = A + M$. Then G admits a 4-balanced decomposition into subgraphs $G = G_1 \sqcup G_2$ such that $(A \cap G_1) \sqcup (A \cap G_2)$ is a double tree decomposition of A.*

Theorem 16 follows from similar arguments to those used for Theorem 2, with suitable modifications.

Fix an integer $c \geqslant 2$ and suppose there are graphs $G = A + M$ with no c-balanced decomposition $G = G_1 \sqcup G_2$ such that $A \cap G_1$ and $A \cap G_2$ are spanning trees. We take G to be such a graph where

1. $e(M)$ is minimal,
2. subject to this, $|G|$ is minimal.

Again, define $v \in V(G)$ to be *big* if $d_G(v) \geqslant c+3$ and *small* otherwise. If $d_G(v) = c+3$ we again call v *critical.*

In figures, edges of M are dashed.

3.1 Edges of M

Lemma 17. *If $e \in E(M)$ is incident to a vertex v, then v is big and $d_G(v) \equiv c+1 \bmod 2$.*

Proof. Let $uv \in E(M)$. Remove uv, rebalance the resulting graph using minimality of G and add uv to the appropriate part so that u is balanced in the resulting decomposition $G = G_1 \sqcup G_2$. By construction we further have that $(A \cap G_1) \sqcup (A \cap G_2)$ is a double tree decomposition. All degree differences have been preserved at vertices of G other than u or v, and u is balanced in $G_1 \sqcup G_2$ by construction. Hence, the vertex v cannot be balanced in $G_1 \sqcup G_2$.

If v is small, then v is clearly balanced in $G = G_1 \sqcup G_2$. If $d_G(v) \equiv c \bmod 2$, then a parity argument similar to that of Lemma 8 shows that v is balanced. Thus, neither can occur. □

As a consequence, the edges of M are not incident to any 3-vertices. We will use the terminology of rich, poor and bad 3-vertices defined in Section 2.3. When we do edge swaps we will do them within the double tree A.

Note that all edges appearing in Lemmas 8, 9 and 11-15 are incident to a small vertex and so are in the double tree A by Lemma 17. Hence these lemmas all still hold in G. Indeed, reductions and reconstructions are unchanged when the vertices and edges involved are in A. For our purposes we require a slight strengthening of Lemma 15.(i) (this follows immediately from the proof of Lemma 15 when applied in this context).

Lemma 18. *Let $v \in V(G)$ be a critical vertex. If all neighbours of v in A are small, then v is not adjacent to any bad 3-vertex.*

Edges of M are subject to further constraints, which we will need in the discharging argument.

Lemma 19. *The subgraph M is a matching.*

Proof. Let $u, v, w \in V(G)$ and suppose that $uv, vw \in E(M)$. Then define $H = B + N$ by deleting edges uv, vw from M (to give N), and by adding a new 2-vertex x joined to both u and w in the double tree (to give B). Then $e(N) < e(M)$ and so by minimality we may find a balanced decomposition $H = H_1 \sqcup H_2$, where $(B \cap H_1) \sqcup (B \cap H_2)$ is a double tree. By symmetry we may assume $ux \in E(H_1)$, $xw \in E(H_2)$. Define $G = G_1 \sqcup G_2$ where $E(G_1) = E(H_1) - ux + uv$ and $E(G_2) = E(H_2) - xw + vw$. Then $G = G_1 \sqcup G_2$ is a balanced decomposition as degree differences have been preserved. Also, $(A \cap G_1) \sqcup (A \cap G_2)$ is a double tree as $(B \cap H_1) \sqcup (B \cap H_2)$ was, giving a contradiction. □

Lemma 20. *Let $v \in V(G)$. Then v cannot be both adjacent to a 2-vertex and incident to an edge of M.*

Proof. Suppose that $u, v \in V(G)$, $uv \in E(M)$ and v is adjacent to a 2-vertex w. Let $v' \neq v$ be the other neighbour of w. By Lemma 17, both u and v are big.

Case 1. $u = v'$.

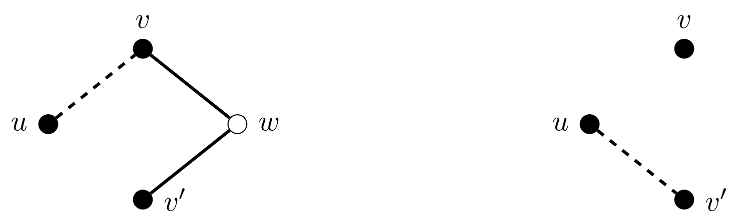
(a) Configuration in G. (b) Reduction to H.

Figure 12: Reduction step in Lemma 20 when $u = v'$.

Define $H = B+N$ by deleting w (so A becomes B) and removing uv (so M becomes N). Then $e(N) < e(M)$ and so, by minimality, there is a balanced decomposition $H = H_1 \sqcup H_2$ where $(B \cap H_1) \sqcup (B \cap H_2)$ is a double tree decomposition. Without loss of generality we may assume that $d_{H_1}(v) \leqslant d_{H_2}(v)$.

Define $G = G_1 \sqcup G_2$ where $E(G_1) = E(H_1) + uv + vw$ and $E(G_2) = E(H_2) + wu$. Then as $d_{H_1}(v) \leqslant d_{H_2}(v)$, the vertex v is balanced in $G_1 \sqcup G_2$. Since all other degree differences have been preserved, the decomposition $G = G_1 \sqcup G_2$ is balanced. Further, $(A \cap G_1) \sqcup (A \cap G_2)$ is a double tree, giving a contradiction.

Case 2. $u \neq v'$.



(a) Configuration in G. (b) Reduction to H.

Figure 13: Reduction step in Lemma 20 when $u \neq v'$.

Define $H = B + N$ by deleting w (so A becomes B), removing uv from M and adding uv' (so M becomes N), as in Figure 13b. Then $e(N) = e(M)$ and $|H| < |G|$ so by minimality there is a balanced decomposition $H = H_1 \sqcup H_2$ where $(B \cap H_1) \sqcup (B \cap H_2)$ is a double tree decomposition. Without loss of generality, $uv' \in E(H_1)$.

Define $G = G_1 \sqcup G_2$ where $E(G_1) = E(H_1) - uv' + uv + wv'$, $E(G_2) = E(H_2) + vw$. Note that since $B \cap H_1$ and $B \cap H_2$ are both spanning trees, the two subgraphs $A \cap G_1$ and $A \cap G_2$ are as well. The decomposition is balanced as $H = H_1 \sqcup H_2$ is balanced and degree differences are preserved, a contradiction. □

Lemma 21. *Let $v \in V(G)$ be a critical vertex. Then v cannot be both adjacent to a poor 3-vertex and incident to an edge of M.*

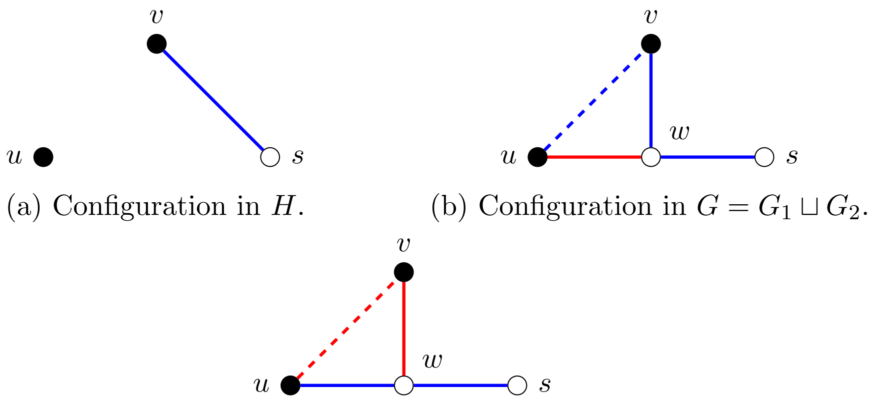
(a) Configuration in G. (b) Reduction to H.

Figure 14: Reduction step in Lemma 21 when $u = v'$.

Proof. Suppose that $u, v \in V(G)$ where v is critical, $uv \in E(M)$ and v is adjacent to a poor 3-vertex w. By Lemma 17, both u and v are big. Let s be the small neighbour of w and $v' \neq v$ be the other big neighbour of w.

First suppose that $u = v'$. By edge flipping and Lemma 13 we may assume that vw and ws are in the same tree. We carry out the standard reduction for 3-vertices at w so that A becomes a double tree B and delete uv from M to get N– see Figure 14b. Let $H = B + N$.

Now $e(N) < e(M)$, so by minimality there is a balanced decomposition $H = H_1 \sqcup H_2$ where $(B \cap H_1) \sqcup (B \cap H_2)$ is a double tree decomposition. Without loss of generality, $vs \in E(B) \cap E(H_1)$.



(a) Configuration in H. (b) Configuration in $G = G_1 \sqcup G_2$.

(c) Configuration in $G = G_1' \sqcup G_2'$.

Figure 15: Reconstruction step in Lemma 21 when $u = v'$.

Define $G = G_1 \sqcup G_2$ by

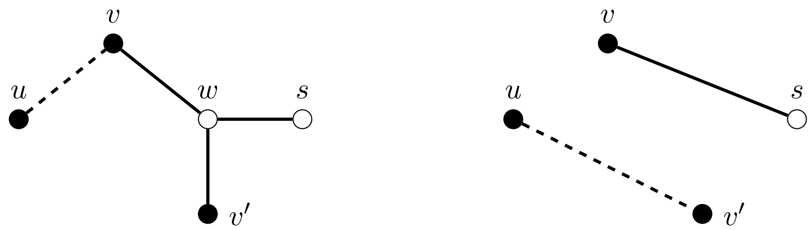
$$E(G_1) = E(H_1) - vs + uv + vw + ws,$$
$$E(G_2) = E(H_2) + uw,$$

as in Figure 15b. Degree differences at all big vertices of G except v have been preserved and $(A \cap G_1) \sqcup (A \cap G_2)$ is a double tree. Hence, if v is balanced we have a contradiction. If $d_{G_2}(v) \geqslant 2$ then v is balanced. Otherwise, swap the edge uw in the double tree decomposition $A = (A \cap G_1) \sqcup (A \cap G_2)$ to obtain $A = T_1 \sqcup T_2$. Define $G = G_1' \sqcup G_2'$ where

$$E(G_1') = E(T_1) \sqcup (E(M) \cap G_1) - uv$$
$$E(G_2') = E(T_2) \sqcup (E(M) \cap G_2) + uv,$$

as in Figure 15c. Then degree differences at all vertices of G except v have been preserved and $(A \cap G_1') \sqcup (A \cap G_2')$ is a double tree, but now $d_{G_2}(v) \geqslant 2$, giving a contradiction.

We may therefore assume $u \neq v'$. By edge flipping and Lemma 13 we may assume that



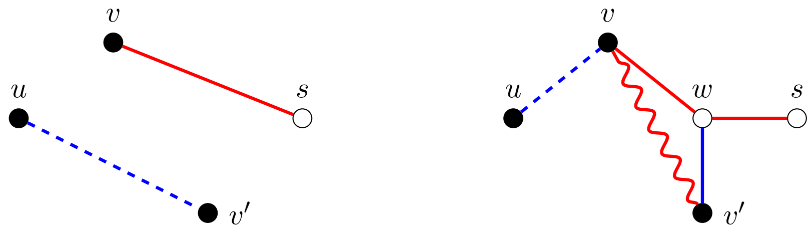
(a) Configuration in G. (b) Reduction to H.

Figure 16: Reduction step in Lemma 21 when $u \neq v'$.

edges vw and ws are in the same tree. We carry out the standard reduction for 3-vertices at w so that A becomes a double tree B. We let $N = M - uv + uv'$. See Figure 16b. Let $H = B + N$.

Then $e(N) = e(M)$ and $|H| < |G|$ so, by minimality, there is a balanced decomposition $H = H_1 \sqcup H_2$ where $(B \cap H_1) \sqcup (B \cap H_2)$ is a double tree decomposition. Without loss of generality, $uv' \in E(N) \cap E(H_1)$.

Case 1. $vs \in E(B) \cap E(H_2)$.



(a) Configuration in H. (b) Reconstruction in G.

Figure 17: Reconstruction step in Case 1.

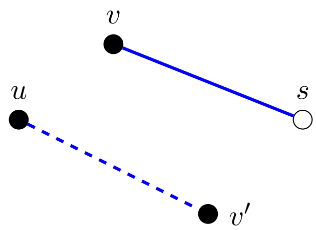
Reverse the reductions to give a decomposition $G = G_1 \sqcup G_2$ defined by

$$E(G_1) = E(H_1) - uv' + uv + wv',$$
$$E(G_2) = E(H_2) - vs + vw + ws,$$

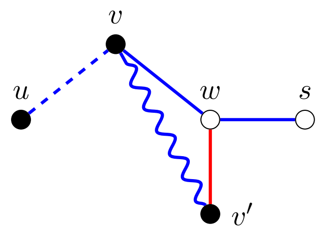
as in Figure 17b. Since $B \cap H_1$ and $B \cap H_2$ are both connected, $T_1 \coloneqq A \cap G_1$ and $T_2 \coloneqq A \cap G_2$ are as well and form a double tree decomposition for A.

All degree differences at big vertices have been preserved except at v where an extra blue edge is present. But, by Lemma 13, the path from v to v' in T_2 does not contain s, so $d_{G_2}(v) \geqslant d_{T_2}(v) \geqslant 2$ and so v is balanced in $G = G_1 \sqcup G_2$ as it is critical. Hence, $G = G_1 \sqcup G_2$ is balanced, a contradiction.

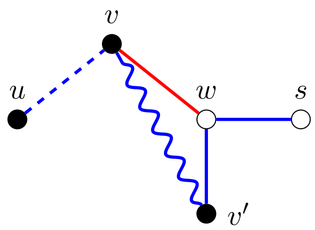
Case 2. $vs \in E(B) \cap E(H_1)$.



(a) Configuration in H.



(b) Configuration in $G = G_1^* \sqcup G_2^*$.



(c) Configuration in $G = G_1 \sqcup G_2$.

Figure 18: Reconstruction step in Case 2.

Reverse the reductions to give a decomposition $G = G_1^* \sqcup G_2^*$ where

$$E(G_1^*) = E(H_1) - uv' - vs + uv + vw + ws,$$
$$E(G_2^*) = E(H_2) + wv',$$

as in Figure 18b. Since $B \cap H_1$ and $B \cap H_2$ are both connected, $T_1 \coloneqq A \cap G_1^*$ and $T_2 \coloneqq A \cap G_2^*$ are as well and form a double tree decomposition for A. Let $A = S_1 \sqcup S_2$ be the double tree decomposition obtained by swapping edge wv' in A (this swaps with wv by Lemma 13). Let $G = G_1 \sqcup G_2$ be the decomposition where

$$E(G_1) = E(M \cap G_1^*) \sqcup E(S_1),$$
$$E(G_2) = E(M \cap G_2^*) \sqcup E(S_2).$$

Then $S_1 = A \cap G_1$ and $S_2 = A \cap G_2$ are both spanning trees. All degree differences at big vertices have been preserved with respect to $H_1 \sqcup H_2$ except at v where an extra red edge is present. Furthermore, $d_{G_1}(v) \geqslant d_{S_1}(v)+1 \geqslant 2$ and so v is balanced in $G = G_1 \sqcup G_2$ as it is critical. Hence, $G = G_1 \sqcup G_2$ is balanced, a contradiction. □

3.2 Discharging

Proof of Theorem 16. Let $G = A + M$ be a counterexample to the bound $c = 4$ such that

1. $e(M)$ is minimal,
2. subject to this, $|G|$ is minimal.

Define the charge function $f\colon V \to \mathbb{Q}$ to be the degree of v in the double tree A: $f(v) = d_A(v)$. Define the *discharging procedure* similarly to the proof of Theorem 2. For each edge $uv \in E(A)$, a big vertex v sends to its neighbour u

- charge 1 if u is a 2-vertex,
- charge 1/2 if u is a poor 3-vertex,
- charge 1/2 if u is a bad 3-vertex,
- charge 1/3 if u is a rich 3-vertex.

Let $g\colon V \to \mathbb{Q}$ be the charge function after the discharging procedure has taken place. Then

$$\sum_{v\in V(G)} g(v) = \sum_{v\in V(G)} f(v) = 2e(A) = 4n - 4.$$

We claim that every vertex v of G has $g(v) \geqslant 4$, which will give a contradiction. As in the proof of Theorem 2, the claim holds if $v \in V(G)$ is not incident to any edge of M. If v is incident to at least one edge of M, then, by Lemmas 17, 19 and 20, v is big, has odd degree, is incident to exactly one $e \in E(M)$, and is not adjacent to any 2-vertices. There are two cases remaining:

1. $d(v) = k \geqslant 9$. Then $g(v) \geqslant (k-1) - (k-1)/2 \geqslant 4$.

2. $d(v) = 7$. By Lemma 21, v is not adjacent to any poor 3-vertex. If all neighbours of v in A are small, then, by Lemma 18, $g(v) \geqslant 6 - 6 \cdot 1/3 \geqslant 4$. Otherwise v has a big neighbour in A, so by Lemma 14, $g(v) \geqslant 6 - 2 \cdot 1/2 - 3 \cdot 1/3 \geqslant 4$. □

4 Balancing infinite graphs

Let $G = (V, E)$ be an undirected graph. Recall that a spanning subgraph $H \subset G$ is called *semiconnected* if it contains an edge of every finite cut of G. We note that this notion depends on the ambient graph G and that for finite graphs, the notions of spanning connected and semiconnected subgraphs coincide.

The following are the main results of this section, which, together with Theorem 2 and Theorem 3 respectively, can be combined to yield Theorem 4.

Theorem 22. *Let c be minimal such that any finite double tree has a c-balanced decomposition into spanning trees. Then if G is a countable infinite double tree, it admits a c-balanced decomposition $G = S_1 \sqcup S_2$ where S_1, S_2 are semiconnected and acyclic.*

Theorem 23. *Let c be minimal such that any finite graph containing a spanning double tree has a c-balanced decomposition into spanning connected graphs. Then if G is a countable infinite graph containing a spanning double tree, it admits a c-balanced decomposition $G = S_1 \sqcup S_2$ where S_1, S_2 are semiconnected.*

As the proof of both of these theorems is virtually the same, we only spell out a proof of the first.

Proof of Theorem 22. Fix a double tree decomposition $G = T_1 \sqcup T_2$ for G. Without loss of generality we may assume that G is locally finite. Indeed, if $v \in V(G)$ is a vertex of infinite degree with neighbours $x_1, x_2, \dots$, we may replace it with a path of double edges $v_0v_1v_2\dots$ where v_i is connected to every vertex in $\{x_{(c+1)i+k} : k \in [c+1]\}$. Within each T_i, the vertex v becomes a countably infinite path. Note that under this operation, $G = T_1 \sqcup T_2$ remains a countably infinite graph and each T_i remains a tree.

Applying this reduction to every vertex of infinite degree we obtain a locally finite countable double tree H. Moreover, any finite cut in G is a finite cut in H, so if $S \subset H$ is semiconnected, then S produces a semiconnected subgraph of G after contracting paths of H corresponding to infinite degree vertices of G. If H has a c-balanced decomposition $T_1 \sqcup T_2$ into semiconnected spanning graphs, every vertex v_i must have at least one edge of both T_1 and T_2 that is not an edge of the path. Hence we may reconstruct a balanced decomposition into semiconnected spanning graphs for G by merging the double paths we created, as degrees with infinite degree have infinite degree in both trees after merging.

Let $V = \{v_1, v_2, \dots\}$ and $V_i = \{v_1, \dots, v_i\}$ for $i \in \mathbb{N}$. For each n we define G_n to be the graph obtained by contracting each connected component C of $G - V_n$ to a vertex v_C, referred to as *auxiliary* vertices of G_n. Each graph G_n is finite as $e(V_n, G - V_n)$ is finite, since V_n is finite and G is locally finite. Further, each G_n contains a double tree H_n such that $H_n[V_n] = G[V_n]$. Indeed, let $T_1 \sqcup T_2$ be a double tree decomposition for G. Contracting the connected components of $G - V_n$ may create cycles. Since T_1, T_2 restricted to V_n are both acyclic, each such cycle necessarily contains some v_C, for some connected component C of $G - V_n$. Hence, we may remove edges incident to auxiliary vertices until we obtain a double tree H_n.

By assumption, each H_n has a c-balanced decomposition $T_1^{(n)} \sqcup T_2^{(n)}$. By a standard compactness argument we may pass to a subsequence $(n_k)_k$ such that for every $k \geqslant \ell$ the decompositions agree on V_ℓ, *i.e.*

$$T_1^{(n_k)}[V_\ell] = T_1^{(n_\ell)}[V_\ell],$$
$$T_2^{(n_k)}[V_\ell] = T_2^{(n_\ell)}[V_\ell].$$

Take S_1 and S_2 to be the unions of $(T_1^{(n_k)}[V_k])_k$ and $(T_2^{(n_k)}[V_k])_k$, respectively. Clearly, S_1 and S_2 are spanning subgraphs of G. Since G is locally finite, for any $v \in V(G)$ there is some K such that for $k \geqslant K$, we have $\Gamma(v) \subset V_k$ and thus $\{e \in E(G) : v \in e\} \subset E(T_1^{(n_k)}) \sqcup E(T_2^{(n_k)})$. This implies that S_1 and S_2 partition the edges of G and since every decomposition $T_1^{(n_k)} \sqcup T_2^{(n_k)}$ is c-balanced, we conclude that $S_1 \sqcup S_2$ is c-balanced. Moreover, S_1 and S_2 are acyclic as any cycle would be contained in $T_1^{(n_\ell)}[V_\ell]$ or $T_2^{(n_\ell)}[V_\ell]$ for some ℓ, a contradition.

It remains to check that S_1 and S_2 intersect every finite cut of G. Let (A, B) be a finite cut of G. Since (A, B) is finite, there is some k such that $E(A, B) \subset G[V_k]$. Let $x \in A \cap V_k$ and $y \in B \cap V_k$. Since $T_1^{(n_k)}$ is connected and contains V_k, it contains a path

P from x to y. We claim that $P \cap E(A, B) \neq \emptyset$, finishing the proof. Indeed, the path P may be extended to a path P' between x and y in G such that P' and P coincide on $G[V_k]$, and whose only additional edges have endpoints outside of V_k. Since (A, B) is a cut of G, $P' \cap E(A, B) \neq \emptyset$. But $E(A, B) \subset E(G[V_k])$ so $P \cap E(A, B) \neq \emptyset$. Hence, S_1 is semiconnected. Similarly, S_2 is semiconnected. □

This compactness argument can easily be modified to yield Theorem 23 by applying Theorem 3 instead of Theorem 2 in the proof.

5 Digraphs

Arborescences are the natural analogue for trees in digraphs and so Hörsch's Question 5 asks whether the digraph analogue of Theorem 1 holds. A natural analogue of connectedness for digraphs is strong connectedness. The following question is then the digraph analogue of Theorem 3: does any union of two strongly connected digraphs allow a balanced decomposition into two strongly connected digraphs? We answer both this and Question 5 in the negative. In fact, our counterexamples have unique decompositions and these decompositions are not balanced.

5.1 Arborescences

In this subsection we answer Question 5 in the negative. More precisely, we show the following.

Theorem 24. *Let $k \geqslant 2$. For any $c > 0$, there is a digraph $D = (V, A)$ that is the disjoint union of k arborescences (all rooted at the same vertex), such that, in any decomposition $D = A_1 \sqcup \cdots \sqcup A_k$ into arborescences, there is some vertex v and some i, j with*

$$|d^{\text{out}}_{A_i}(v) - d^{\text{out}}_{A_j}(v)| > c.$$

Proof. For $k = 2$, we construct an example on vertex set $V = \{v_1, \ldots, v_n\}$ as in Figure 19.

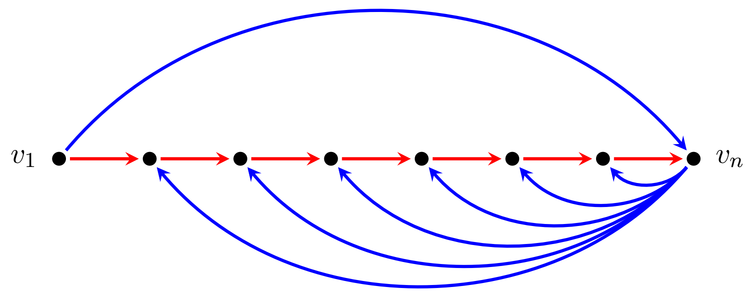


Figure 19: Construction in Theorem 24 when $k = 2$.

Let B_1 be the arborescence in blue and B_2 the directed path in red. We claim that this is the unique double arborescence decomposition of the resulting digraph D_n, up to reordering, thus proving the result as

$$d_{B_1}^{\text{out}}(v_n) - d_{B_2}^{\text{out}}(v_n) = n - 2.$$

Indeed, let $D_n = C_1 \sqcup C_2$ be an arbitrary double arborescence decomposition of D_n. Vertex v_1 has in-degree zero so must be the root of both C_1, C_2. Without loss of generality, $\overrightarrow{v_1 v_n} \in A(C_1)$. Since C_2 is rooted at v_1, it contains a directed path from v_1 to v_n. But the only such path that does not use the arc $\overrightarrow{v_1 v_n}$ is the path B_2. Hence, $C_2 = B_2$ and $C_1 = B_1$, as claimed.

This example can easily be generalised to show Theorem 24 for general k, for example by adding $k-2$ copies of the directed path B_2. $\square$

5.2 Strongly connected digraphs

We now give the counterexample for the second question mentioned above.

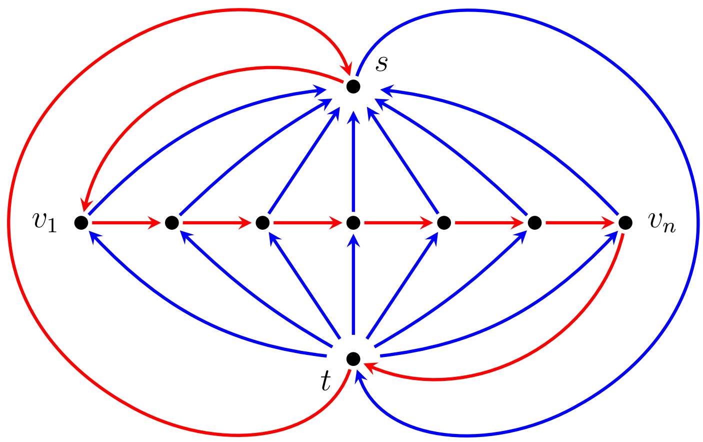


Figure 20: Construction in Theorem 25 when $k = 2$.

Theorem 25. *Let $k \geqslant 2$. For any $c > 0$, there is a digraph $D = (V, A)$ that is the union of k strongly connected digraphs, such that in any decomposition $D = S_1 \sqcup \cdots \sqcup S_k$ into strongly connected digraphs, there is a vertex v and some i, j with*

$$|d_{S_i}^{\text{out}}(v) - d_{S_j}^{\text{out}}(v)| > c.$$

Proof. For $k = 2$ we construct a family (D_n) of examples. The digraph D_n has vertex set $V = \{s, t, v_1, \ldots, v_n\}$ as in Figure 20. Let S_1 and S_2 be the digraphs in blue and red, respectively. It is sufficient to show that this is the unique decomposition of D_n into strongly connected digraphs, as

$$|d_{S_1}^{\text{out}}(t) - d_{S_2}^{\text{out}}(t)| = n - 1.$$

Let $D_n = R_1 \sqcup R_2$ be a decomposition of D_n into strongly connected digraphs. Without loss of generality, $\overrightarrow{st} \in A(R_1)$. Since R_2 is strongly connected, there is a path from s to t in R_2. The only such path that does not use $\overrightarrow{st}$ is the path P with arcs $\{\overrightarrow{sv_1}, \overrightarrow{v_1v_2}, \dots, \overrightarrow{v_{n-1}v_n}, \overrightarrow{v_nt}\}$. Hence, all arcs of P are in R_2. Since R_1 is strongly connected, there are paths from t to v_i and from v_i to s in R_1, for each $i \in [n]$. The only such paths disjoint from P are the arcs $\overrightarrow{tv_i}$ and $\overrightarrow{v_is}$, respectively. Hence, $R_1 = S_1$ and finally $R_2 = S_2$, as claimed.

Similarly as for Theorem 24, these examples can be generalised to arbitrary $k \geqslant 2$ by adding $k-2$ copies of S_1. $\square$

6 Complexity

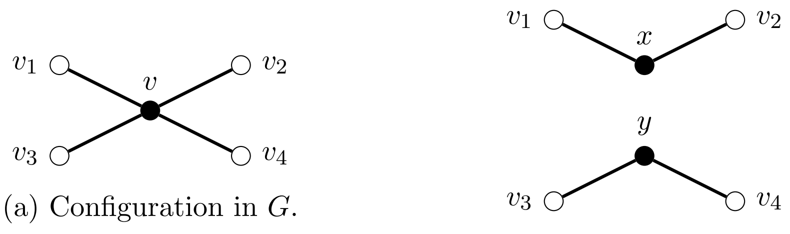
In this section we will show that the decision problem "Given an Eulerian double tree, does it have a perfectly balanced double tree decomposition?" is NP-hard, addressing Question 6. We will refer to this problem as PBDT. We need the following results.

1. Péroche [9]: the decision problem "Given a graph with maximum degree 4, does it contain two edge-disjoint Hamiltonian cycles?" is NP-complete.[1]

2. Roskind, Tarjan [10]: there is an algorithm which, given a graph G, decides in polynomial time whether G is a double tree, and outputs a double tree decomposition if it is.

Note that if a graph contains two edge-disjoint Hamiltonian cycles, then every vertex has degree at least 4. So we immediately deduce from the result of Péroche that the decision problem "given a 4-regular graph, does it contain two edge-disjoint Hamiltonian cycles?" is NP-complete. It suffices to reduce this problem to PBDT. Let $\mathcal{A}$ be an algorithm solving PBDT.

Given a 4-regular graph G, fix a vertex v and let its neighbours be v_1, v_2, v_3, v_4. We perform the following reductions: for $i = 1, 2, 3$, let G_i be the graphs obtained by removing v, adding vertices x, y and adding edges from x to v_1, v_{i+1} and connecting y to the other two v_j.



(a) Configuration in G.

(b) Configuration in G_1.

For $i = 1, 2, 3$, run the algorithm of Roskind and Tarjan on G_i. If it outputs a double tree decomposition for G_i, run $\mathcal{A}$ on it.

[1] This problem is referred to as '2-PAR' in the paper of Péroche.

Claim 26. *The graph G contains two disjoint Hamiltonian cycles if and only if one of G_1, G_2, G_3 has a perfectly balanced double tree decomposition.*

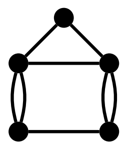
Proof. Note that for $i = 1, 2, 3$, if G_i has a perfectly balanced decomposition $T_1 \sqcup T_2$ then T_1, T_2 are two edge-disjoint Hamiltonian paths with endpoints x and y. Indeed, every $v \in V(G_i) \setminus \{x, y\}$ must have degree 2 in each tree and x, y must be leaves.

Therefore, a perfectly balanced double tree decomposition in G_i corresponds to two edge-disjoint Hamiltonian cycles in G by merging vertices x and y. Conversely, two edge-disjoint Hamiltonian cycles in G yield a perfectly balanced decomposition in at least one of the three splittings of v into x and y described above. □

Hence, the above algorithm is a valid polynomial time reduction of finding two edge-disjoint Hamiltonian cycles in a 4-regular graph to PBDT.

7 Conclusion

We have shown that every double tree has a partition into two trees such that the degrees at each vertex differ by at most four (improving on Hörsch's [5] bound of five). Can this be further improved? There are examples of double trees that admit a 2-balanced double tree decomposition, but no 1-balanced double tree decomposition. The only such examples known to the authors involve taking an odd cycle, whose edges cannot be colored blue/red without creating a vertex with degree difference 2, and making it into a double tree while preserving degree differences. See below for example. In any double tree decomposition, one of the vertices of the triangle has degree difference 2.



It seems natural to conjecture that this lower bound is tight.

Conjecture 27. Any double tree has a 2-balanced double tree decomposition.

The question of balancing double trees can naturally be generalised to balancing k-trees, as well as graphs containing k edge-disjoint trees.

Question 28. Let $k \geqslant 2$. What are the smallest constants $c_k, d_k > 0$ such that the following hold?

- Any finite graph which is the union of k edge-disjoint spanning trees has a k-tree decomposition that is c_k-balanced.
- Any finite graph containing k edge-disjoint spanning trees has a d_k-balanced decomposition into connected spanning subgraphs.

By repeatedly applying Theorem 1, Hörsch [5] obtained the bound $c_k \leqslant 22 \log k$. We could similarly derive improved bounds on c_k and d_k by repeatedly applying Theorem 3. When the requirement that each graph in the decomposition is a tree is dropped (so any k-edge colouring of the original graph G is allowed), a uniform bound on the colour-degree differences is attainable. Indeed, let $\mathcal{H}$ be the hypergraph whose vertices are the edges of G and whose hyperedges are the stars centred at each vertex of G. Then $\mathcal{H}$ has maximum degree 2 and so bounded discrepancy – see, for example, the paper of Doerr and Srivastav [4, Theorem 3.7].

In particular, given a finite graph G containing k edge-disjoint spanning trees, we can apply any bound on c_k to any k edge-disjoint spanning trees in G, then balance the remaining edges using bounded discrepancy. Consequently, upper bounds on c_k are bounds on d_k with a constant error term. It would be particularly interesting to resolve Hörsch's conjecture [5] of whether there is a uniform upper bound on the c_k.

The digraphs used for the proofs of Theorems 24 and 25 in Section 5 rely on the uniqueness of the decompositions into arborescences/strongly connected digraphs. It is natural to ask what happens if our starting digraph is less restricted. If the starting digraph is a union of more arborescences than colours in the edge-colouring, then there will be many possible decompositions.

Question 29. Are there constants c, t such that if D is a disjoint union of t spanning arborescences sharing a root, then the edges of D can be coloured blue/red such that the out-degrees are c-balanced and both graphs contain arborescences?

The same question is also interesting for strongly connected digraphs. The hypothesis that D is a disjoint union of t strongly connected spanning digraphs is slightly cumbersome and it would seem natural to replace it with some high connectivity condition. As far as we are aware, the following question is open and would be interesting to resolve.

Question 30. For each positive integer t is there a constant k such that the edges of any k-strongly connected digraph can be partitioned into t parts each of which is spanning and strongly connected?

For undirected graphs the corresponding statement follows from the Tutte-Nash-Williams characterisation with $k = 4t$.

Acknowledgements

The authors would like to thank Lex Schrijver for helpful comments and the referee for a very careful reading and excellent suggestions for improving the presentation.

References

[1] J. Bang-Jensen, F. Havet and A. Yeo. The complexity of finding arc-disjoint branching flows. *Discrete Applied Mathematics*, **209**:16–26, 2016.

[2] S. Bessy, F. Hörsch, A. K. Maia, D. Rautenbach and I. Sau. FPT algorithms for packing k-safe spanning rooted sub(di)graphs. arXiv:2105.01582, 2021.

[3] J. Chuzhoy, M. Parter and Z. Tan. On packing low-diameter spanning trees. In *47th International Colloquium on Automata, Languages, and Programming*, volume 168 of *LIPIcs. Leibniz Int. Proc. Inform.,* Art. No. 33, 18. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020.

[4] B. Doerr and A. Srivastav. Multicolour discrepancies. *Combinatorics, Probability and Computing*, **12**:365–399, 2003.

[5] F. Hörsch. Globally balancing spanning trees. *European Journal of Combinatorics*, **109**:Paper No. 103644, 15, 2023.

[6] M. Kriesell. Balancing two spanning trees. *Networks*, **57**(4):351–353, 2011.

[7] C. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society*, **1**(1):445–450, 1961.

[8] J. G. Oxley. On a packing problem for infinite graphs and independence spaces. *Journal of Combinatorial Theory, Series B*, **26**(2):123–130, 1979.

[9] B. Péroche. NP-completeness of some problems of partitioning and covering in graphs. *Discrete Applied Mathematics*, **8**:195–208, 1984.

[10] J. Roskind and R. E. Tarjan. A note on finding minimum-cost edge-disjoint spanning trees. *Mathematics of Operations Research*, **10**:701–708, 1985.

[11] W. T. Tutte. On the problem of decomposing a graph into n connected factors. *Journal of the London Mathematical Society*, **1**(1):221–230, 1961.