Nonexistence of almost Moore digraphs of degrees 4 and 5 with self-repeats

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Abstract

An almost Moore (d,k)-digraph is a regular digraph of degree $d > 1$, diameter $k > 1$ and order $N(d,k) = d + d^2 + \cdots + d^k$. So far, their existence has only been shown for $k = 2$, whilst it is known that there are no such digraphs for $k = 3$, 4 and for $d = 2$, 3 when $k \geqslant 3$. Furthermore, under certain assumptions, the nonexistence for the remaining cases has also been shown. In this paper, we prove that $(4,k)$ and $(5,k)$-almost Moore digraphs with self-repeats do not exist for $k \geqslant 5$.

Mathematics Subject Classifications: 05C35, 05C20, 05C50

1 Introduction

Given two natural numbers d and k, the degree/diameter problem asks for the largest possible number of vertices in a [directed] graph with maximum [out-]degree d and diameter k (a survey is given by Miller and Širáň in [18]). Plesník and Znám in [19] and later Bridges and Toueg in [6] proved that the number of vertices in a digraph is less than the

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Moore bound, $M(d,k) = 1 + d + \cdots + d^k$ unless $d = 1$ or $k = 1$. Then, the question of finding for which values of $d > 1$ and $k > 1$ there exist digraphs of order

$$N(d,k) = M(d,k) - 1 = d + d^2 + \cdots + d^k$$

becomes an interesting problem. Regular digraphs of degree $d > 1$, diameter $k > 1$ and order $N(d,k)$ are called *almost Moore* (d,k)*-digraphs* (or (d,k)*-digraphs* for short). These digraphs turn out to be d-regular [17].

Concerning the existence of such (d,k)-digraphs, Fiol et al. showed in [12] that $(d,2)$-digraphs do exist for any degree $d > 1$ and Gimbert completed their classification for $k = 2$ in [14]. But so far, it seems that they do not exist for the remaining values of the diameter. Nevertheless, nonexistence has been proven only for a few cases. Conde et al. in [9, 10] showed the nonexistence of $(d,3)$ and $(d,4)$-digraphs. On the other hand, Miller and Fris in [16] proved that there are no $(2,k)$-digraphs with $k \geqslant 3$ and Baskoro et al. showed in [5] the nonexistence of $(3,k)$-digraphs for $k \geqslant 3$. In [11], Conde et al. proved that there are infinitely many primes k for which $(4,k)$-digraphs and $(5,k)$-digraphs do not exist.

Also we have to mention that there exist two conjectures such that, assuming that either of them is true, the nonexistence of (d,k)-digraphs for any $d \geqslant 4$ and $k \geqslant 5$ is proven. One of them is based on the structure of the out-neighbours of the k-type vertices, those whose distance to its repeat is k (see [1, 2]). From it Cholily in [7] proved the nonexistence. The other conjecture was given by Gimbert in [13] and it is related to the factorization in $\mathbb{Q}[x]$ of the polynomials $F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k)$, $\Phi_n(x)$ being the nth cyclotomic polynomial. In [8] the nonexistence is also proven assuming this conjecture.

In this paper, we prove that almost Moore digraphs of degree $d = 4$ and $d = 5$ with self-repeats do not exist for any diameter $k \geqslant 5$. To do this we take advantage of the cycle structure of the permutation of repeats given by Sillasen in [20] for such degrees.

2 Permutation cycle structures of $(4,k)$ and $(5,k)$-digraphs

Given a digraph G, we will denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its arcs. If u and v are vertices of G and (u,v) is an arc, it is said that u is *adjacent* to v. A *walk* of length ℓ from u to v is a sequence of vertices $u = w_0, w_1, \ldots, w_{\ell-1}, w_\ell = v$ such that each (w_{i-1}, w_i) is an arc. A digraph with maximum out-degree at most $d > 1$, diameter at most $k > 1$ and order $N = d + d^2 + \cdots + d^k$ must have all vertices with out-degree d and its diameter must be k (see [12]). Moreover, its in-degrees are also d (see [17]). Such a digraph is called (d,k)-digraph.

A (d,k)-digraph G has the property that for each vertex $v \in V(G)$ there exists only a vertex $u \in V(G)$, called the *repeat* of v and denoted by $r(v)$, such that there are exactly two walks from v to $r(v)$ of length at most k (one of them of length k). If $r(v) = v$, the vertex v is called a *self-repeat* of G. The map r, which assigns to each vertex $v \in V(G)$ the vertex $r(v)$, is an automorphism of G (see [3]). For any $t \geqslant 1$, we can define $r^t(v) = r(r^{t-1}(v))$, with $r^0(v) = v$. Then, the smallest integer $t \geqslant 1$ such that

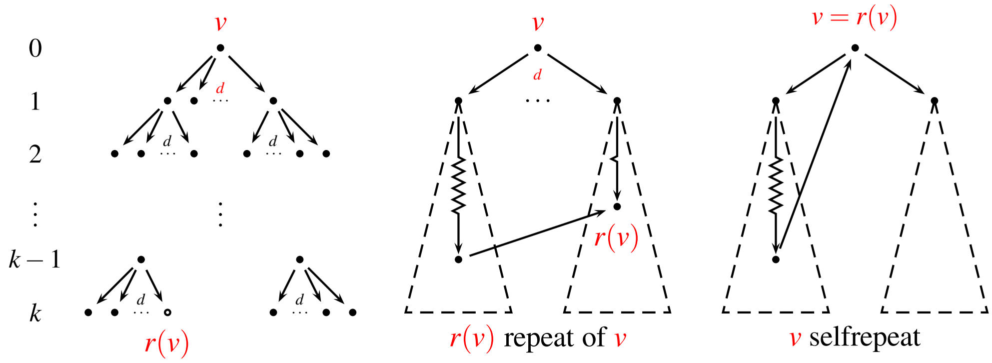

Figure 1: Repeat of a vertex in a (d, k)-digraph

$r^t(v) = v$ is called the *order* of v. In Figure 1, we can see graphically the notion of repeat of a vertex v, showing the different possibilities for the level in which $r(v)$ belongs.

Note that a (d, k)-digraph does not contain cycles of length less than k and in case that v is a vertex belonging in a cycle of length k then v is a self-repeat vertex.

Given a (d, k)-digraph G, its adjacency matrix $\boldsymbol{A}$ satisfies the equation

$$\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^k = \boldsymbol{J} + \boldsymbol{P} \tag{1}$$

where $\boldsymbol{J}$ denotes the all-one matrix and $\boldsymbol{P} = (p_{ij})$ is the (0,1)-matrix associated with the map r, which is equivalent to saying $p_{ij} = 1$ iff $r(i) = j$. The map r, which is a permutation of the set of vertices $V(G) = \{1, \ldots, N\}$, has a cycle structure which corresponds to its unique decomposition into disjoint cycles. The number of permutation cycles of r of each length $i \leqslant N$, will be denoted by m_i and the vector

$$(m_1, m_2, \ldots, m_N)$$

will be referred as the permutation cycle structure of G. It means that there are m_1 self-repeats, $2m_2$ vertices of order 2 under the permutation r and so on. Hence

$$\sum_{i=1}^{N} i m_i = N.$$

We will consider $(4, k)$ and $(5, k)$-digraphs G with diameter $k \geqslant 5$. For these cases, the possible cycle structures of the permutation of repeats are given by Sillasen [20]. The corresponding structures containing self-repeats have also been deduced in [2] by Baskoro et al.

Proposition 1. *Let G be a (d, k)-digraph, $d = 4$ or $d = 5$, with order $N = d + d^2 + \cdots + d^k$. The permutation cycle structure of G must be one of these forms:*

- *If $d = 4$:*

$$\begin{array}{ll} (k, 0, m_3, 0, \ldots, 0), & k + 3m_3 = N, \\ (0, \ldots, 0, m_i, 0, \ldots, 0), & i m_i = N,\ i \geqslant 2. \end{array}$$

- *If $d = 5$:*

$$\begin{array}{ll}
(k, m_2, 0, \ldots, 0), & k + 2m_2 = N, \\
(k, 0, 0, m_4, 0, \ldots, 0) & k + 4m_4 = N, \\
(0, \ldots, 0, m_i, 0, \ldots, 0), & im_i = N,\ i \geqslant 2, \\
(0, \ldots, 0, m_j, 0, \ldots, m_{2j}, 0, \ldots, 0), & jm_j + 2jm_{2j} = N,\ j \geqslant 2,\ \textit{with either}
\end{array}$$

$k+2$ *vertices of order* j *and* $N-k-2$ *of order* $2j$, *or*
$M(3,k)+1$ *vertices of order* j *and* $N-M(3,k)-1$ *of order* $2j$.

We will see that (d,k)-digraphs, $k \geqslant 5$, with these permutation cycle structures with $m_1 = k$ do not exist either when $d = 4$ or $d = 5$.

Proposition 2. *The adjacency matrix $\boldsymbol{A}$ of a (d,k)-digraph, $d = 4$ or $d = 5$, with permutation cycle structure with $m_1 = k$ satisfies*

$$\operatorname{Tr} \boldsymbol{A}^i = 0,\ 1 \leqslant i \leqslant k-1, \quad \operatorname{Tr} \boldsymbol{A}^k = k, \quad \operatorname{Tr} \boldsymbol{PA} = 0, \quad \operatorname{Tr} \boldsymbol{A}^{k+1} = dN - k.$$

Proof. Since G has no cycles of length less than k, its adjacency matrix $\boldsymbol{A}$ satisfies

$$\operatorname{Tr} \boldsymbol{A}^i = 0 \text{ for } i = 1, 2, \ldots, k-1.$$

Since $\operatorname{Tr} \boldsymbol{P} = m_1 = k$, we have in our case $\operatorname{Tr} \boldsymbol{A}^k = \operatorname{Tr} \boldsymbol{P} = k$. From (1), we have that $\boldsymbol{A} + \boldsymbol{A}^2 + \cdots + \boldsymbol{A}^{k+1} = \boldsymbol{JA} + \boldsymbol{PA}$. Then, taking into account $\boldsymbol{JA} = d\boldsymbol{J}$ because G is diregular, we deduce

$$\operatorname{Tr} \boldsymbol{A}^k + \operatorname{Tr} \boldsymbol{A}^{k+1} = dN + \operatorname{Tr} \boldsymbol{PA}. \tag{2}$$

It is known that $\operatorname{Tr} \boldsymbol{PA} = |R(G)|$ (see [13], Section 3), where

$$R(G) = \{v \in V(G) \mid (r(v), v) \in E(G)\}. \tag{3}$$

Besides, in [13], Proposition 3, Gimbert showed that there exists a partition of the set $R(G)$, $R(G) = C_1 \cup C_2 \cup \ldots \cup C_h$, such that each $C_i = \{v_i, r(v_i), \ldots, r^{t_i - 1}(v_i)\}$, where $v_i \in R(G)$ has order $t_i \geqslant k+1$ as an element of the permutation r. Nevertheless, in our case, taking into account the permutation cycle structures with $m_1 = k$ given in Proposition 1, we get a contradiction. Indeed, if $d = 4$ all vertices have order $\leqslant 3$ (since $m_i = 0$, $\forall i \geqslant 4$) whereas if $d = 5$ all vertices have order $\leqslant 4$ (since $m_i = 0$, $\forall i \geqslant 5$). Thus, since we are considering diameter $k \geqslant 5$, we have $R(G) = \emptyset$ and hence $\operatorname{Tr} \boldsymbol{PA} = 0$. Therefore $\operatorname{Tr} \boldsymbol{A}^{k+1} = dN - k$. □

2.1 Computing some traces of $\boldsymbol{PA}^\ell$

In order to compute the traces of $\boldsymbol{PA}^\ell$, $\ell \geqslant 1$, we generalize the set $R(G)$ defined in (3). Note that

$$(\boldsymbol{PA}^\ell)_{ii} = \sum_{j=1}^{N} \boldsymbol{P}_{ij} \boldsymbol{A}^\ell_{ji} = \boldsymbol{A}^\ell_{r(i)i}.$$

Taking into account that the entry uv of the matrix $\boldsymbol{A}^\ell$ is precisely the number of walks of length ℓ from u to v, then $\operatorname{Tr}\boldsymbol{PA}^\ell$ is the number of vertices u such that there is a walk of length ℓ from $r(u)$ to u, where each vertex u is counted according to the number of $r(u) \to u$ walks of length ℓ. This is precisely the cardinality of the multiset

$$R_\ell(G) = \{u \in V(G) \mid \text{ there is a } r(u) \to u \text{ walk of length } \ell\}.$$

Note that $R_1(G)$ is the set $R(G)$ defined above.

Proposition 3. *Let $(m_1, m_2, \dots, m_s, 0, \dots, 0)$ be the permutation cycle structure of a (d,k)-digraph. If $R_\ell(G) \neq \emptyset$, then $\ell \geqslant \frac{k+1}{s}$.*

Proof. Let $u \in R_\ell(G)$ and let us denote by $r(u), w_1, \dots, w_{\ell-1}, u$ a walk of length ℓ from $r(u)$ to u in G. Let t be the order of u as an element of the permutation r. Since r is an automorphism of G, we have that the sequences $r^{t'+1}(u), r^{t'}(w_1), \dots, r^{t'}(w_{\ell-1}), r^{t'}(u)$ for all $1 \leqslant t' < t$, are sequences of arcs in G. Finally, the sequence

$$u = r^t(u), r^{t-1}(w_1), \dots, r^{t-1}(w_{\ell-1}), r^{t-1}(u), \dots, r(u), \dots, u$$

is a cycle of length ℓt if $r^{t'}(w_i) \neq r^{t''}(w_j)$ for all $i \neq j$ and $t' \neq t''$. Otherwise, shorter cycles appear inside this sequence. Taking into account that a (d,k)-digraph contains no cycles of length less than k and contains at most a cycle of length k consisting of its self-repeats, then $\ell s \geqslant \ell t \geqslant k+1$ and the result follows. □

Recall that $\operatorname{Tr}\boldsymbol{PA}^\ell = |R_\ell(G)|$. Then we have the following result:

Corollary 4. *Let $\boldsymbol{A}$ be the adjacency matrix of a (d,k)-digraph with permutation matrix $\boldsymbol{P}$ and $(m_1, m_2, \dots, m_s, 0, \dots, 0)$ being the permutation cycle structure. Then*

$$\operatorname{Tr}\boldsymbol{PA}^\ell = 0, \quad 1 \leqslant \ell < \frac{k+1}{s}.$$

Considering our permutation cycle structures for degree $d = 4$ and diameter $k \geqslant 5$ given in Proposition 1 we have:

Corollary 5. *The adjacency matrix $\boldsymbol{A}$ of a $(4,k)$-digraph with permutation matrix $\boldsymbol{P}$ and permutation cycle structure with $m_1 = k$ satisfies*

$$\operatorname{Tr}\boldsymbol{PA}^\ell = 0, \quad \operatorname{Tr}\boldsymbol{A}^{k+\ell} = d^{k+\ell} - d^\ell, \quad 1 < \ell < \frac{k+1}{3}.$$

Proof. Since for degree $d = 4$ the unique permutation cycle structure with $m_1 = k$ is $(k, 0, m_3, 0, \dots, 0)$, from Corollary 4 we have $\operatorname{Tr}\boldsymbol{PA}^\ell = 0$, for $1 \leqslant \ell < \frac{k+1}{s}$ with $s = 3$. Concerning $\operatorname{Tr}\boldsymbol{A}^{k+\ell}$, note that for $\ell = 2$ (in which case $k \geqslant 6$) we have from (1) that

$$\operatorname{Tr}\boldsymbol{A}^k + \operatorname{Tr}\boldsymbol{A}^{k+1} + \operatorname{Tr}\boldsymbol{A}^{k+2} = \operatorname{Tr}\boldsymbol{JA}^2 + \operatorname{Tr}\boldsymbol{PA}^2 = d^2N.$$

Then, from Proposition 2, it turns out that $\operatorname{Tr}\boldsymbol{A}^{k+2} = d^2N - \operatorname{Tr}\boldsymbol{A}^k - \operatorname{Tr}\boldsymbol{A}^{k+1} = d^2N - dN = d^{k+2} - d^2$. Now we can derive the claim for $2 < \ell < \frac{k+1}{3}$ by strong induction on ℓ. Indeed, assuming $\operatorname{Tr}\boldsymbol{A}^{k+i} = d^{k+i} - d^i$ holds for $2 \leqslant i < \ell$, it turns out that $\operatorname{Tr}\boldsymbol{A}^{k+\ell} = d^\ell N - \sum_{i=0}^{\ell-1} \operatorname{Tr}\boldsymbol{A}^{k+i} = d^\ell N - d^{\ell-1}N = d^{k+\ell} - d^\ell$. □

Moreover, taking into account that for the cycle structure $(k, 0, m_3, 0, \ldots, 0)$ the permutation matrix $\boldsymbol{P}$ and the automorphism r satisfy, respectively, $\boldsymbol{P}^2 = \boldsymbol{P}^{-1}$ and $r^2 = r^{-1}$, we can extend the previous result as follows:

Corollary 6. *The adjacency matrix $\boldsymbol{A}$ of a $(4, k)$-digraph with permutation matrix $\boldsymbol{P}$ and permutation cycle structure with $m_1 = k$ satisfies*

$$\operatorname{Tr} \boldsymbol{P}^2 \boldsymbol{A}^\ell = 0, \quad 1 \leqslant \ell < \frac{k+1}{3}.$$

Proof. In this case,

$$(\boldsymbol{P}^2 \boldsymbol{A}^\ell)_{ii} = \sum_{j=1}^{N} \boldsymbol{P}^2_{ij} \boldsymbol{A}^\ell_{ji} = \sum_{j=1}^{N} \boldsymbol{P}^{-1}_{ij} \boldsymbol{A}^\ell_{ji} = \boldsymbol{A}^\ell_{r^{-1}(i)i},$$

which coincides with the cardinality of

$$R'_\ell(G) = \{u \in V(G) \mid \text{ there is a } r^{-1}(u) \to u \text{ walk of length } \ell\}.$$

As in the proof of Proposition 3, the order t of $u \in R'_\ell(G)$ satisfies $\ell t \geqslant k+1$, that is $\ell \geqslant (k+1)/t$. Since the order t of each vertex is $\leqslant 3$, it turns out $R'_\ell(G) = \emptyset$ when $\ell < (k+1)/3$ and hence $\operatorname{Tr} \boldsymbol{P}^2 \boldsymbol{A}^\ell = 0$. □

3 On the characteristic polynomial of $(4, k)$ and $(5, k)$-digraphs

Given a permutation matrix $\boldsymbol{P}$ of order N and the all-one matrix $\boldsymbol{J}$, the characteristic polynomial of $\boldsymbol{J} + \boldsymbol{P}$ is (see[4])

$$\phi(\boldsymbol{J} + \boldsymbol{P}, x) = \det(x\boldsymbol{I} - (\boldsymbol{J} + \boldsymbol{P})) = (x - (N+1))(x-1)^{m_1 - 1} \prod_{i=2}^{N} (x^i - 1)^{m_i},$$

where $(m_1, m_2, \ldots, m_N)$ is the permutation cycle structure of $\boldsymbol{P}$. Its factorization in $\mathbb{Q}[x]$ in terms of cyclotomic polynomials $\Phi_n(x)$ is given by:

$$\phi(\boldsymbol{J} + \boldsymbol{P}, x) = (x - (N+1))(x-1)^{m(1)-1} \prod_{n=2}^{N} \Phi_n(x)^{m(n)}, \tag{4}$$

where $m(n) = \sum_{n|i} m_i$ represents the total number of permutation cycles of order multiple of n. Notice that $\boldsymbol{J} + \boldsymbol{P}$ is a diagonalizable matrix in $\mathbb{C}$ and its minimal polynomial is

$$m(\boldsymbol{J} + \boldsymbol{P}, x) = (x - (N+1))(x-1) \prod_{m(n) \neq 0} \Phi_n(x). \tag{5}$$

Lemma 7. *The adjacency matrix $\boldsymbol{A}$ of a (d, k)-digraph G is a diagonalizable matrix in $\mathbb{C}$.*

Proof. If G has permutation matrix $\boldsymbol{P}$, taking into account the adjacency matrix $\boldsymbol{A}$ satisfies the identity $\boldsymbol{I} + \boldsymbol{A} + \cdots + \boldsymbol{A}^k = \boldsymbol{J} + \boldsymbol{P}$ and substituting x by $1 + x + \cdots + x^k$ in $m(\boldsymbol{J} + \boldsymbol{P}, x)$ we get a new polynomial $p(x)$, which vanishes at $\boldsymbol{A}$. Since the factors $x(1+x+\cdots+x^{k-1})$ and $\Phi_n(1+x+\cdots+x^k)$ have no multiple roots, the claim follows. □

From equations (1) and (4), the problem of the factorization of the characteristic polynomial of G, $\phi(G, x) = \det(x\boldsymbol{I} - \boldsymbol{A})$ in $\mathbb{Q}[x]$ is related to the study of factorization in $\mathbb{Q}[x]$ of the polynomial:

$$F_{n,k}(x) = \Phi_n(1 + x + \cdots + x^k),\ n \geqslant 2.$$

If $F_{n,k}(x)$ is irreducible in $\mathbb{Q}[x]$, then $F_{n,k}(x)$ is a factor of $\phi(G, x)$ and its multiplicity is $m(n)/k$ (see [13]). More than this, the "cyclotomic conjecture" proposed by Gimbert gives the factorization in $\mathbb{Q}[x]$ of the polynomials $F_{n,k}(x)$. Assuming this conjecture, the nonexistence of (d, k)-digraphs is proven in [8].

From (5) we derive the following result:

Lemma 8. *The adjacency matrix $\boldsymbol{A}$ of a (d, k)-digraph, $d = 4, 5$, satisfies $p(\boldsymbol{A}) = 0$, where*

- *if $d = 4$ with permutation cycle structure $(k, 0, m_3, 0, \ldots, 0)$, $N = k + 3m_3$,*

$$p(x) = (x - d)x(x^{k-1} + \cdots + x + 1)F_{3,k}(x), \tag{6}$$

  with $F_{3,k}(x) = (x^k + \cdots + x + 1)^2 + (x^k + \cdots + x + 1) + 1$.

- *if $d = 5$ with permutation cycle structure $(k, m_2, 0, \ldots, 0)$, $N = k + 2m_2$,*

$$p(x) = (x - d)x(x^{k-1} + \cdots + x + 1)F_{2,k}(x), \tag{7}$$

  with $F_{2,k}(x) = x^k + \cdots + x + 2$.

- *if $d = 5$ with permutation cycle structure $(k, 0, 0, m_4, 0, \ldots, 0)$, $N = k + 4m_4$,*

$$p(x) = (x - d)x(x^{k-1} + \cdots + x + 1)F_{2,k}(x)F_{4,k}(x), \tag{8}$$

  with $F_{4,k}(x) = (x^k + \cdots + x + 1)^2 + 1$.

4 Nonexistence of $(4, k)$-digraphs with self-repeats

In this section we consider (d, k)-digraphs with $d = 4$ and $k \geqslant 5$ containing self-repeats, that is, whose permutation cycle structure is $(k, 0, m_3, 0, \ldots, 0)$.

Proposition 9. *Almost Moore digraphs of degree $d = 4$ and diameter k with self-repeats do not exist in the following cases:*

- *$k \geqslant 5$ is an odd number.*
- *$k \geqslant 6$ is an even number of the form $k = 2(p - 1)$ where p is a prime number.*

Proof. Notice that $4N$ is precisely the number of arcs in a $(4,k)$-digraph G, hence Equation (2) together with the condition $\operatorname{Tr} \boldsymbol{PA} = 0$ shows that each arc of the digraph G belongs to exactly one cycle of G of length k or $k+1$. This means that there exists a positive integer $t \in \mathbb{Z}^+$ such that

$$4N = k + t(k+1). \tag{9}$$

Clearly this is impossible for any odd number $k \geqslant 5$. More in general, since $N = 4 + 4^2 + \cdots + 4^k = \frac{4}{3}(4^k - 1)$, we have from (9) that

$$t = \frac{4^{k+2} - 13}{3(k+1)} - 1$$

and consequently, a necessary condition for the existence of G is

$$4^{k+2} \equiv 13 \pmod{3(k+1)} \tag{10}$$

Let $k = 2s$. We show next that $4^{2s} \equiv 1 \pmod{3(2s+1)}$ whenever $s = p - 1$ being p a prime number. Indeed, clearly $4^{p-1} \equiv 1 \pmod 3$ and since $4^{p-1} \equiv 1 \pmod p$ we have that $4^{p-1} \equiv 1 \pmod{3p}$. Any prime number $p > 2$ is an odd number $p = 2s+1$, hence $4^{2s} \equiv 1 \pmod{3(2s+1)}$. □

Remark 10. We performed an exhaustive computer search for all values of k with $5 \leqslant k < 10^6$ satisfying Equation (10) and we found that there are none satisfying this condition. Hence $(4,k)$-digraphs do not exist for this range of values of k.

4.1 Matrix approach

Let $\boldsymbol{A}$ be the adjacency matrix of a $(4,k)$-digraph, $k \geqslant 5$, whose permutation cycle structure is $(k, 0, m_3, 0, \ldots, 0)$. Since $\boldsymbol{A}$ is a diagonalizable matrix (see Lemma 7), $\boldsymbol{A}$ can be expressed in a basis of eigenvectors as a diagonal matrix with eigenvalues (see [13]),

- d with multiplicity 1;
- λ_i, $1 \leqslant i \leqslant m_3 + k - 1$, roots of the factor $x^k + \cdots + x^2 + x$;
- α_i, $1 \leqslant i \leqslant m_3$, roots of the factor $x^k + \cdots + x^2 + x + 1 - \rho$, ρ being a primitive cubic root of unity; and
- β_i, $1 \leqslant i \leqslant m_3$, conjugates of α_i, that is, roots of the factor $x^k + \cdots + x^2 + x + 1 - \rho^2$.

That is, in such a basis,

$$\boldsymbol{A} = \begin{pmatrix} d & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{n_3} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \alpha_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \alpha_{m_3} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \beta_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \beta_{m_3} \end{pmatrix},$$

with $n_3 = m_3 + k - 1$. In the same basis, the matrices of $\boldsymbol{J}$ and $\boldsymbol{P}$ are:

$$\boldsymbol{J} = \begin{pmatrix} N & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{P} = \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \rho & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \rho & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \rho^2 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \rho^2 \end{pmatrix}.$$

From this, $\operatorname{Tr} \boldsymbol{J} = N$, $\operatorname{Tr} \boldsymbol{P} = k + m_3 + m_3(\rho + \rho^2) = k$, and the trace of $\boldsymbol{A}$, which is the sum of the roots of its characteristic polynomial, can be written as follows

$$\operatorname{Tr}(\boldsymbol{A}) = d + \sum_{i=1}^{m_3+k-1} \lambda_i + \sum_{i=1}^{m_3} \alpha_i + \sum_{i=1}^{m_3} \beta_i = 0.$$

Since $\boldsymbol{A}$ is a diagonalizable matrix (see Lemma 7), so is $\boldsymbol{A}^\ell$ in the same basis of eigenvectors. Thus,

$$\operatorname{Tr}(\boldsymbol{A}^\ell) = d^\ell + \sum_{i=1}^{m_3+k-1} \lambda_i^\ell + \sum_{i=1}^{m_3} \alpha_i^\ell + \sum_{i=1}^{m_3} \beta_i^\ell, \qquad 1 \leqslant \ell < k. \tag{11}$$

Note that we can express

$$\sum_{i=1}^{m_3} \alpha_i^\ell = a_\ell + b_\ell \rho \quad \text{and} \quad \sum_{i=1}^{m_3} \beta_i^\ell = a_\ell + b_\ell \rho^2, \qquad a_\ell, b_\ell \in \mathbb{Z}. \tag{12}$$

Indeed, $\sum_{i=1}^{m_3} \alpha_i^\ell$ corresponds to the sum of the ℓth powers of all roots of some irreducible factors of $x^k + \cdots + x^2 + x + 1 - \rho$ in $\mathbb{Q}(\rho)$. Then according to Newton-Girard formulas these sums only depend on the coefficients of their terms. The sum $\sum_{i=1}^{m_3} \beta_i^\ell$ is the conjugate of $\sum_{i=1}^{m_3} \alpha_i^\ell$.

Proposition 11. *Let G be a $(4,k)$-digraph with self-repeats. Then*

$$0 = \operatorname{Tr}(\boldsymbol{A}^\ell) = d^\ell + \sum_{i=1}^{m_3+k-1} \lambda_i^\ell, \quad 1 \leqslant \ell < \frac{k+1}{3}. \tag{13}$$

Proof. Taking into account identities (11) and (12) we have

$$0 = \operatorname{Tr}(\boldsymbol{A}^\ell) = d^\ell + \sum_{i=1}^{m_3+k-1} \lambda_i^\ell + (a_\ell + b_\ell \rho) + (a_\ell + b_\ell \rho^2).$$

From Corollary 5 we also have

$$0 = \operatorname{Tr}(\boldsymbol{P}\boldsymbol{A}^\ell) = d^\ell + \sum_{i=1}^{m_3+k-1} \lambda_i^\ell + (a_\ell \rho + b_\ell \rho^2) + (a_\ell \rho^2 + b_\ell \rho).$$

Subtracting one equation from the other we get $a_\ell = 0$. Besides,

$$0 = \operatorname{Tr}(\boldsymbol{P}^2\boldsymbol{A}^\ell) = d^\ell + \sum_{i=1}^{m_3+k-1} \lambda_i^\ell + b_\ell \rho^3 + b_\ell \rho^3,$$

from where, it turns out $b_\ell = 0$ and the claim follows. □

Concerning the sums $\sum_{i=1}^{m_3+k-1} \lambda_i^\ell$, we know the eigenvalues λ_i, $1 \leqslant i \leqslant m_3 + k - 1$, are roots of the factor

$$x^k + \cdots + x^2 + x = x \prod_{n \neq 1\, n|k} \Phi_n(x).$$

Since the cyclotomic polynomials $\Phi_n(x)$ are irreducible in $\mathbb{Q}[x]$, it follows that there exist nonnegative integers a_n such that

$$\sum_{i=1}^{m_3+k-1} \lambda_i^\ell = \sum_{n \neq 1,\, n|k} a_n S_\ell(\Phi_n(x)), \tag{14}$$

where $S_\ell(a(x))$ denotes the sum of the ℓth powers of all roots of $a(x)$.

The sums $S_\ell(\Phi_n(x))$ are known as *Ramanujan sums* and can be computed as follows (see [15]):

Lemma 12. *Let n and ℓ be two positive integers. Then*

$$S_\ell(\Phi_n(x)) = \sum_{j|\gcd(n,\ell)} \mu\left(\frac{n}{j}\right) j,$$

where $\mu(n)$ denotes the Möbius function.

Theorem 13. *Almost Moore digraphs of degree $d = 4$ with self-repeats do not exist for diameter $k \geqslant 5$.*

Proof. Let G be an $(4,k)$-digraph with self-repeats. From (13) and (14), its adjacency matrix $\boldsymbol{A}$ satisfies

$$0 = \operatorname{Tr} \boldsymbol{A}^\ell = d^\ell + \sum_{n\neq 1,\, n|k} a_n S_\ell(\Phi_n(x)), \quad 1 \leqslant \ell < \frac{k+1}{3}.$$

Note if ℓ and k are relatively prime then for every $n \mid k$ we have

$$S_\ell(\Phi_n(x)) = \mu(n). \tag{15}$$

In particular, if there exists an integer ℓ such that

$$\gcd(\ell, k) = 1 \quad \text{and} \quad 1 < \ell < \frac{k+1}{3}, \tag{16}$$

then $S_\ell(\Phi_n(x)) = S_1(\Phi_n(x))$ for all n with $n \mid k$, which would imply that

$$\operatorname{Tr} \boldsymbol{A}^\ell - \operatorname{Tr} \boldsymbol{A} = d^k - d = 0,$$

which is impossible unless $d = 1$ or $k = 1$.

Now, we will prove that there exists an integer ℓ satisfying (16) if $k \geqslant 20$ (see Remark 10 for the remaining values of k). More precisely, we show that if $k \geqslant 20$ then there exists a positive integer ℓ with $1 < \ell < (k+1)/3$ such that $\gcd(k, \ell) = 1$. Consider the distinct consecutive prime numbers until $\frac{k+1}{3}$:

$$2 = p_1 < p_2 < \cdots < p_r < \frac{k+1}{3} \leqslant p_{r+1}.$$

If for the contrary, $\gcd(k, \ell) > 1$ for every positive integer ℓ with $1 < \ell < \frac{k+1}{3}$, then it means that

$$k = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, \quad \alpha_i \geqslant 1.$$

If $k \geqslant 20$ then $(k+1)/3 \geqslant 7$ and therefore $r \geqslant 3$. Hence

$$\lfloor (k+1)/3 \rfloor = p_1^{\alpha_1} p_2^{\alpha_2 - 1} p_3^{\alpha_3} \ldots p_r^{\alpha_r} \geqslant 2p_r. \tag{17}$$

Recall now that Ramanujan primes are the smallest integers R_n for which there are at least n primes between $x/2$ and x, for all $x \geqslant R_n$. Then, since 2 is the 1st Ramanujan prime, there exists a prime number between p_r and $2p_r$. Thus, $p_{r+1} < 2p_r$ and it turns out

$$\lfloor (k+1)/3 \rfloor \leqslant p_{r+1} < 2p_r,$$

which is a contradiction with (17). ◻

5 Nonexistence of $(5, k)$-digraphs with self-repeats

We will consider the permutation cycle structures given in Proposition 1 for $(5, k)$-digraphs, $k \geqslant 5$, with $m_1 = k$ self-repeats, that is,

$$(k, m_2, 0, \ldots, 0) \quad \text{and} \quad (k, 0, 0, m_4, 0, \ldots, 0).$$

Theorem 14. *Almost Moore digraphs of degree $d = 5$ with permutation cycle structure $(k, m_2, 0, \ldots, 0)$ do not exist for diameter $k \geqslant 5$.*

Proof. Let G be a $(5, k)$-digraph with structure $(k, m_2, 0, \ldots, 0)$. Note that such a structure is not possible. Indeed, in this case the unique factor $F_{n,k}(x)$ appearing in the characteristic polynomial $\phi(G, x)$ is according to (7),

$$F_{2,k}(x) = \Phi_2(1 + x + \cdots + x^k) = 2 + x + x^2 + \cdots + x^k,$$

which is irreducible in $\mathbb{Q}[x]$ [13]. Hence the cyclotomic conjecture holds in this particular case (see [13]). Therefore such a digraph does not exist (see [8], Theorem 2). Indeed, the characteristic polynomial factorizes as

$$\phi(G, x) = (x - 5)x^{a_0} \prod_{n|k,\, n \neq 1} \Phi_n(x)^{a_n} F_{2,k}(x)^{m_2/k}, \qquad a_0 + \sum_{n|k,\, n \neq 1} \varphi(n) a_n = k + m_2 - 1.$$

Since the trace of $\boldsymbol{A}^\ell$, whith $\boldsymbol{A}$ the adjacency matrix of G and $1 \leqslant \ell \leqslant k$, is the sum of the ℓth powers of all roots of $\phi(G, x)$, we have

$$0 = \operatorname{Tr} \boldsymbol{A}^\ell = 5^\ell + \sum_{n|k,\, n \neq 1} a_n S_\ell(\Phi_n(x)) + \frac{m_2}{k} S_\ell(F_{2,k}(x)).$$

Taking $\ell = 1$ and another value for ℓ less than k and relatively prime with k, it follows from (15) that $S_1(\Phi_n(x)) = S_\ell(\Phi_n(x)) = \mu(n)$ and from Lemma 3 in [8] that $S_1(F_{2,k}(x)) = S_\ell(F_{2,k}(x)) = -\varphi(n)$. Therefore

$$0 = \operatorname{Tr} \boldsymbol{A}^\ell - \operatorname{Tr} \boldsymbol{A} = 5^\ell - 5,$$

which is impossible. □

Theorem 15. *Almost Moore digraphs of degree $d = 5$ with permutation cycle structure $(k, 0, 0, m_4, 0, \ldots, 0)$ do not exist for diameter $k \geqslant 5$.*

Proof. Concerning the structure $(k, 0, 0, m_4, 0, \ldots, 0)$, we have $m(2) = m_4$. Then, since as before the factor $F_{2,k}(x)$ (which appears in the characteristic polynomial, see (8)) is irreducible, we get $k \mid m_4$. Therefore, taking $h = m_4/k$, it turns out $k(1 + 4h) = 5 + 5^2 + \cdots + 5^k = 5(5^k - 1)/4$, that is,

$$5^k \equiv 1 \pmod{4k}. \tag{18}$$

If $k = p > 2$ is a prime number, since $5^p \equiv 5 \pmod{p}$, we have $5 \equiv 1 \pmod{k}$, which is not possible. If k is an odd composite number with prime factorization

$$k = p_1^{r_1} p_2^{r_2} \dots p_s^{r_s}, \quad 2 < p_1 < p_2 < \dots < p_s,$$

from (18) we also derive $5^k \equiv 1 \pmod{p_1}$. Using Fermat's Little Theorem

$$1 = 5^k = 5^{\frac{k}{p_1} p_1} = 5^{\frac{k}{p_1}} \equiv 5^{k/p_1^{r_1}} \pmod{p_1}.$$

Consider $d = \gcd(p_1 - 1, k/p_1^{r_1})$. Since $p_1 < p_2, \dots, p_s$ it turns out $d = 1$. Thus, there exist two integers x, y such that

$$(p_1 - 1)x + (k/p_1^{r_1})y = 1.$$

Hence

$$5 = 5^{(p_1-1)x + (k/p_1^{r_1})y} = (5^{p_1-1})^x \cdot (5^{k/p_1^{r_1}})^y \equiv 1 \pmod{p_1}.$$

Therefore $5 \equiv 1 \pmod{p_1}$ so that $p_1 = 2$, which is a contradiction in the case k odd.

In the case k even with $v_2(k) = \alpha \geqslant 1$, we can see by induction that $v_2(5^k - 1) = \alpha + 2$. Now, we will prove there is no even integer k satisfying

$$5(5^k - 1) = 4k(1 + 4h). \tag{19}$$

Assume that first $k = 2^\alpha$. By induction we can prove

$$(5^{2^\alpha} - 1)/2^{\alpha+2} \equiv 3 \pmod{4}, \tag{20}$$

which is a contradiction with (19). Indeed, for $\alpha = 1$ we get $(5^2 - 1)/2^3 = 3$. Assuming true for α, for $\alpha + 1$ we have

$$(5^{2^{\alpha+1}} - 1)/2^{\alpha+3} = ((5^{2^\alpha} - 1)/2^{\alpha+2})((5^{2^\alpha} + 1)/2) \equiv 3 \pmod{4}.$$

Note that congruence (20) can be extended to an integer $k = 2^\alpha k'$, with $\alpha \geqslant 1$ and $2 \nmid k'$, as follows

$$(5^{2^\alpha k'} - 1)/2^{\alpha+2} \equiv k' + 2 \pmod{4},$$

which contradicts equality (19). □

We have seen $(5, k)$-digraphs with permutation cycle structures $(k, m_2, 0, \dots, 0)$ and $(k, 0, 0, m_4, 0, \dots, 0)$ do not exist for diameter $k \geqslant 5$. Since they are the unique structures containing selfrepeteats for $d = 5$, the nonexistence of them can be concluded.

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