Spanning Configurations and Representation Stability

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Abstract

Let $V_1, V_2, V_3, \ldots$ be a sequence of $\mathbb{Q}$-vector spaces where V_n carries an action of $\mathfrak{S}_n$. *Representation stability* and *multiplicity stability* are two related notions of when the sequence V_n has a limit. An important source of stability phenomena arises when V_n is the d^{th} homology group (for fixed d) of the configuration space of n distinct points in some fixed topological space X. We replace these configuration spaces with moduli spaces of tuples $(W_1, \ldots, W_n)$ of subspaces of a fixed complex vector space $\mathbb{C}^N$ such that $W_1 + \cdots + W_n = \mathbb{C}^N$. These include the varieties of *spanning line configurations* which are tied to the Delta Conjecture of symmetric function theory.

Mathematics Subject Classifications: 05E10

1 Introduction and Main Result

Let $(V_n)_{n \geqslant 1}$ be a sequence of finite-dimensional $\mathbb{Q}$-vector spaces where each V_n has an action of $\mathfrak{S}_n$. Introduced by Church [1] in a geometric context, *representation stability* gives a notion of the sequence V_n having a limit.

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Definition 1. *(Church [1])* Let $(V_n)_{n\geqslant 1}$ be a sequence of $\mathfrak{S}_n$ representations, and for each $n \geqslant 1$ let $f_n : V_n \to V_{n+1}$ be a linear map. Then we say that $(V_n)_{n\geqslant 1}$ is *(uniformly) representation stable* with respect to the maps $(f_n)_{n\geqslant 1}$ if for $n \gg 0$

- the map f_n is injective,
- we have $f_n(w \cdot v) = w \cdot f_n(v)$ for all $w \in \mathfrak{S}_n$ and all $v \in V_n$,
- the $\mathfrak{S}_{n+1}$ module generated by the image $f_n(V_n) \subseteq V_{n+1}$ is all of V_{n+1}, and
- the transposition $(n+1, n+2) \in \mathfrak{S}_{n+2}$ acts trivially on the image of the composition $\mathrm{im}(V_n \xrightarrow{f_n} V_{n+1} \xrightarrow{f_{n+1}} V_{n+2}) \subseteq V_{n+2}$.

The isotypic decompositions of a representation stable sequence V_n exhibit limiting properties. A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of positive integers which sum to n. We write $\lambda \vdash n$ to mean that λ is a partition of n and $|\lambda| = n$ for the sum of the parts of λ. The *(English) Ferrers diagram* of λ consists of λ_i left-justified boxes in row i; we identify partitions with their Ferrers diagrams.

Partitions of n are in bijective correspondence with irreducible representations of $\mathfrak{S}_n$; given $\lambda \vdash n$, let S^λ be the corresponding irreducible $\mathfrak{S}_n$-module.

If $\mu = (\mu_1, \mu_2, \dots)$ is a partition and $n \geqslant |\mu| + \mu_1$, the *padded partition* is $\mu[n] \vdash n$ is given by $\mu[n] := (n - |\mu|, \mu_1, \mu_2, \dots)$. Any partition $\lambda \vdash n$ may be expressed uniquely as $\lambda = \mu[n]$ for some partition μ. In fact, if $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ then $\lambda = \mu[n]$ where $\mu = (\lambda_2, \lambda_3, \dots)$.

For any $n \geqslant 1$, the $\mathfrak{S}_n$-module V_n decomposes into a direct sum of irreducibles. There exist unique multiplicities $m_{\mu,n}$ so that $V_n \cong \bigoplus_\mu m_{\mu,n} S^{\mu[n]}$ where the direct sum is over all partitions μ.

Definition 2. The sequence $(V_n)_{n\geqslant 1}$ is *uniformly multiplicity stable* if there exists N such that for any partition μ, we have $m_{\mu,n} = m_{\mu,n'}$ for all $n, n' \geqslant N$.

Church, Ellenberg, and Farb proved that multiplicity and representation stability are essentially equivalent.

Theorem 3. (Church-Ellenberg-Farb [2]) *Let $(V_n)_{n\geqslant 1}$ be a sequence of $\mathfrak{S}_n$-representations. Then $(V_n)_{n\geqslant 1}$ is uniformly multiplicity stable if and only if there exists some collection of linear maps $f_n : V_n \to V_{n+1}$ such that $(V_n)_{n\geqslant 1}$ is representation stable with respect to $(f_n)_{n\geqslant 1}$.*

Theorem 3 notwithstanding, writing down explicit maps $f_n : V_n \to V_{n+1}$ which realize the representation stability of a specific multiplicity stable sequence $(V_n)_{n\geqslant 1}$ can be difficult.

Many geometric instances of representation stability arise from configuration spaces. If X is a topological space and $n \geqslant 1$, the n^{th} *configuration space* of X is the moduli space of n distinct points in X:

$$\mathrm{Conf}_n X := \{(x_1, \dots, x_n) \,:\, x_i \in X \text{ and } x_i \neq x_j \text{ for } i \neq j\}. \tag{1}$$

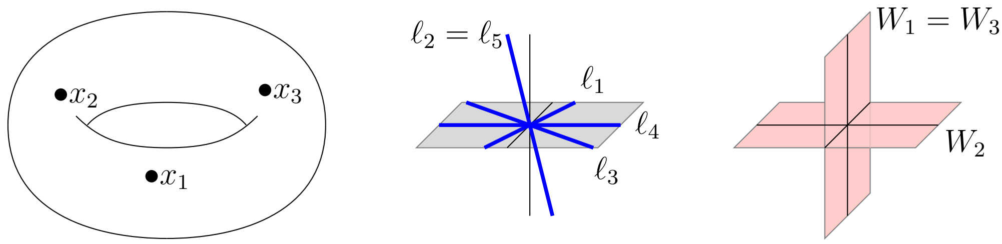

Figure 1: A point configuration, a line configuration, and a 2-plane configuration.

The left of Figure 1 shows a point in $\mathrm{Conf}_3(X)$ where X is the torus. The set $\mathrm{Conf}_n X$ has the subspace topology inherited from the n-fold product $X \times \cdots \times X$.

For $d \geqslant 0$, let $H_d(\mathrm{Conf}_n X)$ be the d^{th} homology group of the n^{th} configuration space of X.[1] The natural action of $\mathfrak{S}_n$ on $\mathrm{Conf}_n(X)$ induces an action on $H_d(\mathrm{Conf}_n X)$. There are many results stating that if the space X is 'nice', the sequence $(H_d(\mathrm{Conf}_n X))_{n \geqslant 1}$ is representation stable [1, 2]. Such stability results can be proven even when the homology of $\mathrm{Conf}_n(X)$ or its $\mathfrak{S}_n$-structure are unknown. In general, it is often easier to prove that a sequence V_n is representation stable than it is to find its $\mathfrak{S}_n$-structure.

We consider a matroidal variation on configuration spaces in which sequences of distinct points are replaced by spanning sequences of m-dimensional subspaces of $\mathbb{C}^k$. Let $\mathrm{Gr}(m, k)$ be the Grassmannian of m-planes in $\mathbb{C}^k$ and let $\mathrm{Gr}(m, k)^n$ be its n-fold self-product. We consider the open subvariety

$$X(m, k, n) := \{(W_1, \ldots, W_n) \in \mathrm{Gr}(m, k)^n \, : \, W_1 + \cdots + W_n = \mathbb{C}^k\} \tag{2}$$

of sequences $(W_1, \ldots, W_n)$ which span $\mathbb{C}^k$. Figure 1 shows a point in $X(1, 3, 5)$ (middle) and $X(2, 3, 3)$ (left).

The variety $X(m, k, n)$ is nonempty if and only if $k \leqslant mn$. There is a homotopy equivalence $X(1, n, n) \simeq \mathrm{Fl}(n)$ between $X(1, n, n)$ and the variety of flags in $\mathbb{C}^n$. Pawlowski and Rhoades [6] introduced the variety $X(1, k, n)$ of spanning line configurations $(\ell_1, \ldots, \ell_n)$ in $\mathbb{C}^k$ and presented its cohomology as

$$H^\bullet(X(1, k, n)) = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1}\rangle \tag{3}$$

where e_d is the degree d elementary symmetric polynomial. The above quotient rings were introduced earlier by Haglund, Rhoades, and Shimozono [4] in the context of Macdonald theory [3].

The symmetric group $\mathfrak{S}_n$ acts naturally on the variety $X(m, k, n)$ and on its homology. There are two natural ways to 'grow' a triple (m, k, n) preserving the condition $X(m, k, n) \neq \varnothing$:

$$(m, k, n) \rightsquigarrow (m, k, n + 1) \qquad \text{and} \qquad (m, k, n) \rightsquigarrow (m, k + m, n + 1).$$

We prove that both of these growth rules yield stability results.

[1] We use singular homology with rational coefficients.

Theorem 4. *Fix integers $m, k, r \geqslant 0$. Both sequences*

$$H_d(X(m,k,n)) \qquad H_d(X(m,mn-r,n)) \qquad (n \geqslant 0) \tag{4}$$

of homology $\mathfrak{S}_n$-modules are representation stable.

Theorem 4 is a 'matroidal analogue' of configuration space stability results. The cohomology ring $H^\bullet(X(m,k,n))$ was presented by Rhoades [7], but its graded $\mathfrak{S}_n$-isomorphism type – like the $\mathfrak{S}_n$-structures of the homology representations of Theorem 4 – is unknown. Our proof of Theorem 4 does not use the cohomology presentation in [7], but rather goes through the theory of FI-modules and a geometric result on the inclusion $X(m,k,n) \subseteq \mathrm{Gr}(m,k)^n$. Our methods illustrate that a sequence of $\mathfrak{S}_n$-modules can be shown to exhibit stability even in the presence of relatively little information about these modules.

The remainder of the paper is organized as follows. In **Section 2** we give background material on affine pavings and the category of FI-modules. In **Section 3** we prove Theorem 4. This paper is an abridged and generalized version of the FPSAC 2020 extended abstract [5].

2 Affine Pavings and FI-modules

Let Z be a complex variety. An *affine paving* of Z is a chain

$$\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_r = Z$$

of closed subvarieties of Z such that each difference $Z_i - Z_{i-1}$ is isomorphic to a disjoint union of affine spaces. If $\varnothing = Z_0 \subset Z_1 \subset \cdots \subset Z_r = Z$ is an affine paving and $0 \leqslant j \leqslant m$, let $U := Z - Z_j$. The inclusion $\iota : U \hookrightarrow Z$ induce a map $\iota_* : H_\bullet(U) \to H_\bullet(Z)$ on homology. When U arises from an affine paving of Z as above, the map ι_* is injective.

The standard affine paving of complex projective space $\mathbb{P}^{k-1}$ is given in coordinates by

$$\varnothing \subset [\star : 0 : \cdots : 0 : 0] \subset [\star : \star : \cdots : 0 : 0] \subset \cdots \subset [\star : \star : \cdots : \star : 0] \subset [\star : \star : \cdots : \star : \star] = \mathbb{P}^{k-1}$$

where the stars represent free complex entries. More generally, the Schubert decomposition of the Grassmannian $\mathrm{Gr}(m,k)$ induces an affine paving of this space. Taking the n-fold product of this paving with itself yields a paving of $\mathrm{Gr}(m,k)^n$, but this naïve paving interacts poorly with the inclusion $X(m,k,n) \subseteq \mathrm{Gr}(m,k)^n$. A nonstandard affine paving of $\mathrm{Gr}(m,k)^n$ was crucial to the presentation of the cohomology of $X(m,k,n)$ in [7].

Theorem 5. (Rhoades [7]) *There exists an affine paving $Z_0 \subset Z_1 \subset \cdots \subset Z_r$ of the Grassmann product $\mathrm{Gr}(m,k)^n$ such that $X(m,k,n) = \mathrm{Gr}(m,k)^n - Z_j$ for some j.*

In particular, if $\iota : X(m,k,n) \hookrightarrow \mathrm{Gr}(m,k)^n$ is inclusion, then $\iota_* : H_\bullet(X(m,k,n)) \hookrightarrow H_\bullet(\mathrm{Gr}(m,k)^n)$ is an injection.

Theorem 5 was a component of the presentation of $H^\bullet(X(m,k,n))$ in [7]. It will be a key tool in our proof, as well. The other ingredient we need is the category of FI-modules recalled below.

Let FI be the category whose objects are the finite sets $[n] := \{1, 2, \ldots, n\}$ for $n \geqslant 0$ and whose morphisms are injective functions $f : [n] \to [p]$. Let Vect be the category of $\mathbb{Q}$-vector spaces with morphisms given by linear maps.

An FI-*module* is a covariant functor $V : \mathsf{FI} \to \mathsf{Vect}$. We write $V(n)$ instead of $V([n])$ for the vector space corresponding to $[n]$. More explicitly, an FI-module consists of

- a $\mathbb{Q}$-vector space $V(n)$ for each $n \geqslant 0$, and
- a $\mathbb{Q}$-linear map $V(f) : V(n) \to V(p)$ for each injection $f : [n] \to [p]$

such that $V(f \circ g) = V(f) \circ V(g)$ for any two injections $f : [n] \to [p]$ and $g : [p] \to [q]$ and $V(\mathrm{id}_{[n]}) = \mathrm{id}_{V(n)}$. Submodules and quotients of FI-modules are defined in the natural way. If V is an FI-module, then $V(n)$ is naturally an $\mathfrak{S}_n$-module for each n.

An FI-module V is *finitely generated* if there is a finite subset S of the disjoint union $\bigsqcup_{n \geqslant 0} V(n)$ such that no proper FI-submodule W of V satisfies $S \subseteq \bigsqcup_{n \geqslant 0} W(n)$. Finite generation of FI-modules and representation stability are equivalent notions.

Theorem 6. (Church-Ellenberg-Farb [2]) *Let $\iota_n : [n] \hookrightarrow [n+1]$ denote the standard injection, and let V be a finitely generated FI-module. Then the sequence $(V(n))_{n \geqslant 1}$ is representation stable with respect to the maps $V(\iota_n)$. Every representation stable sequence arises in this way.*

One reason Theorem 6 is useful is that finite generation in the category of FI-modules is inherited by submodules (and quotient modules, although we will not use this). Said differently, the category of FI-modules is Noetherian.

Theorem 7. (Snowden [8]; see also Church-Ellenberg-Farb [2]) *Any submodule of a finitely generated FI-module is finitely generated.*

3 Proof of Theorem 4

We need to prove the stability of two sequences of homology representations of $\mathfrak{S}_n$, namely

$$H_d(X(m,k,n)) \qquad \text{and} \qquad H_d(X(m, mn-r, n))$$

for fixed integers m, k, and r. The stability of the sequence $H_d(X(m,k,n))$ is easier to establish; we handle this case first. For any subset $I \subseteq [k]$ we let $E_I := \mathrm{span}\{e_i : i \in I\}$ be the coordinate subspace of $\mathbb{C}^k$ spanned by the corresponding basis vectors.

For any injection $f : [n] \hookrightarrow [p]$, we define an associated map $\mu_f : \mathrm{Gr}(m,k)^n \to \mathrm{Gr}(m,k)^p$ by the rule

$$\iota_f : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_p) \tag{5}$$

where $W'_{f(i)} = W_i$ for $1 \leqslant i \leqslant n$ and $W'_j = E_{[m]}$ whenever $1 \leqslant j \leqslant p$ is not in the image of f. As an example, if $m = 3$, $k = 6$, and $f : [3] \to [5]$ is $f(1) = 4, f(2) = 2, f(3) = 1$ then $\mu_f : \mathrm{Gr}(3,6)^3 \to \mathrm{Gr}(3,6)^5$ is

$$\mu_f : (W_1, W_2, W_3) \mapsto (W_3, W_2, E_{[3]}, W_1, E_{[3]})$$

for all $(W_1, W_2, W_3) \in \mathrm{Gr}(3,6)^3$.

For any two injections $f : [n] \hookrightarrow [p]$ and $f : [p] \hookrightarrow [q]$ we have

$$\mu_{g \circ f} = \mu_g \circ \mu_f. \tag{6}$$

Furthermore, the map μ_f preserves the spanning condition; abusing notation, we denote the restricted map on X-spaces by the same symbol $\mu_f : X(m,k,n) \to X(m,k,p)$.

For any injection $f : [n] \hookrightarrow [p]$, the induced maps $(\mu_f)_*$ on homology fit into a commutative square

$$\begin{array}{ccc} H_\bullet(X(m,k,n)) & \xrightarrow{\iota_*} & H_\bullet(\mathrm{Gr}(m,k)^n) \\ {\scriptstyle (\mu_f)_*}\downarrow & & \downarrow{\scriptstyle (\mu_f)_*} \\ H_\bullet(X(m,k,p)) & \xrightarrow[\iota_*]{} & H_\bullet(\mathrm{Gr}(m,k)^p) \end{array}$$

where the horizontal maps are those induced by the injections $X(m,k,n) \hookrightarrow \mathrm{Gr}(m,k)^n$ and $X(m,k,p) \hookrightarrow \mathrm{Gr}(m,k)^p$. Theorem 5 implies that the horizontal arrows are injections, so that $[n] \mapsto H_d(X(m,k,n))$ is a sub-FI-module of $[n] \mapsto H_d(\mathrm{Gr}(m,k)^n)$ for each degree d. The Künneth Theorem implies

$$H_d(\mathrm{Gr}(m,k)^n) = \bigoplus_{d_1 + \cdots + d_n = d} H_{d_1}(\mathrm{Gr}(m,k)) \otimes \cdots \otimes H_{d_n}(\mathrm{Gr}(m,k)) \tag{7}$$

as graded vector spaces. Since $H_{d_i}(\mathrm{Gr}(m,k))$ is a finite-dimensional vector space for each d_i, the FI-module $[n] \mapsto H_d(\mathrm{Gr}(m,k)^n)$ is finitely generated for d fixed. Theorem 7 implies that $[n] \mapsto H_d(X(m,k,n))$ is also a finitely generated FI-module, and Theorem 6 shows that $H_d(X(m,k,n))$ is representation stable.

Now fix integers m, r and consider the sequence $X(m, mn-r, n)$ of spaces for $n \geqslant 0$. One would like to put an FI-structure on the spaces $X(m, mn-r, n)$ which is compatible with their inclusions $X(m, mn-r, n) \hookrightarrow \mathrm{Gr}(m, mn-r)^n$ into Grassmann products. However, complications arise since the ambient dimension $mn - r$ increases with n, resulting in an FI-structure which only holds when one passes to homology.

More precisely, if $f : [n] \hookrightarrow [p]$ is an injection, define $\nu_f : \mathrm{Gr}(m, mn-r)^n \hookrightarrow \mathrm{Gr}(m, mp-r)^p$ by the formula

$$\nu_f : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_p) \tag{8}$$

where the spaces W'_j are determined as follows. If $f(i) = j$, we set $W'_j := W_i$, where we view W_i as a subspace of $\mathbb{C}^{mp-r}$ by means of the embedding $\mathbb{C}^{mn-r} \subseteq \mathbb{C}^{mp-r}$ along the first $mn - r$ coordinates. For the $p - n$ elements $1 \leqslant j_1 < \cdots < j_{p-n} \leqslant p$ of $[p]$ which are not in the image of f, we let $W'_{j_\ell} \subseteq \mathbb{C}^{mp-n}$ be the coordinate subspace

$$W'_{j_\ell} := E_{[m(n+\ell-1)-r+1, m(n+\ell)-r]} = \mathrm{span}\{e_i : m(n+\ell-1) - r + 1 \leqslant i \leqslant m(n+\ell) - r\}.$$

An example should clarify the definition of ν_f. Suppose $m = 2$, $n = 3$, $r = 1$, and $p = 6$. If $f : [3] \hookrightarrow [6]$ is given by $f(1) = 3$, $f(2) = 1$, $f(3) = 6$ then $\nu_f : \mathrm{Gr}(2,5)^3 \to \mathrm{Gr}(2,11)^3$ is given by

$$\nu_f : (W_1, W_2, W_3) \mapsto (W_2, E_{\{6,7\}}, W_1, E_{\{8,9\}}, E_{\{10,11\}}, W_3).$$

where we regard $W_1, W_2, W_3 \subset \mathbb{C}^{11}$ by means of the inclusion $\mathbb{C}^5 \subseteq \mathbb{C}^{11}$ along the first five coordinates. In general, whenever $\mu_f : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_p)$ and $W_1 + \cdots + W_n = \mathbb{C}^{mn-r}$, we have $W'_1 + \cdots + W'_p = \mathbb{C}^{mp-r}$. The map ν_f therefore restricts to a map $X(m, mn-r, n) \hookrightarrow X(m, mp-r, p)$; we use the same symbol ν_f to refer to this restriction.

If $f : [n] \hookrightarrow [p]$ and $g : [p] \hookrightarrow [q]$ are two injections, we do **not** typically have the equality of maps $\nu_{g \circ f} = \nu_g \circ \nu_f$. For example, suppose $m = 2, r = 1$, $(n, p, q) = (3, 6, 7)$, and $f : [3] \hookrightarrow [6]$ is as above. Define $g : [6] \hookrightarrow [7]$ by $g(1) = 5, g(2) = 1, g(3) = 3, g(4) = 6, g(5) = 2, g(6) = 7$. Then

$$\nu_g \circ \nu_f : (W_1, W_2, W_3) \mapsto (E_{\{6,7\}}, E_{\{10,11\}}, W_1, E_{\{12,13\}}, W_2, E_{\{8,9\}}, W_3)$$

whereas

$$\nu_{g \circ f} : (W_1, W_2, W_3) \mapsto (E_{\{6,7\}}, E_{\{8,9\}}, W_1, E_{\{10,11\}}, W_2, E_{\{12,13\}}, W_3).$$

In general, the spaces $W_1, \ldots, W_n$ will appear in the same positions (and in the same order) in the images $(W'_1, \ldots, W'_q)$ of the tuple $(W_1, \ldots, W_n)$ under either $\nu_g \circ \nu_f$ or $\nu_{g \circ f}$, but the E's will usually appear in a different order.

For fixed injections $g : [n] \hookrightarrow [p]$ and $f : [p] \hookrightarrow [q]$, there is an invertible linear transformation $A \in \mathrm{GL}_{mq-r}(\mathbb{C})$ such that

$$\nu_g \circ \nu_f : (W_1, \ldots, W_n) \mapsto (A \cdot W'_1, \ldots, A \cdot W'_q) \text{ where } \nu_{g \circ f} : (W_1, \ldots, W_n) \mapsto (W'_1, \ldots, W'_q) \tag{9}$$

for all $(W_1, \ldots, W_n) \in \mathrm{Gr}(m, mn - r)^n$. Recall that the group $\mathrm{GL}_{mq-r}(\mathbb{C})$ is path connected. Any path from I to A in $\mathrm{GL}_{mq-r}(\mathbb{C})$ provides a homotopy between $\nu_g \circ \nu_f$ and $\nu_{g \circ f}$. In summary, we have

$$\nu_{g \circ f} \simeq \nu_g \circ \nu_f \tag{10}$$

as maps on spaces so that the induced maps $(\nu_{g \circ f})_* = (\nu_g)_* \circ (\nu_f)_*$ on homology coincide.

As before, if $f : [n] \hookrightarrow [p]$ is an injection, we have a commutative square of maps on homology

$$\begin{array}{ccc} H_\bullet(X(m,mn-r,n)) & \xrightarrow{\iota_*} & H_\bullet(\mathrm{Gr}(m,mn-r)^n) \\ {\scriptstyle (\nu_r)_*}\downarrow & & \downarrow{\scriptstyle (\nu_f)_*} \\ H_\bullet(X(m,mp-r,p)) & \xrightarrow[\iota_*]{} & H_\bullet(\mathrm{Gr}(m,mp-r)^p) \end{array}$$

where Theorem 5 guarantees that the horizontal maps are injective. The last paragraph shows that, for a fixed degree d, we have an FI-module $[n] \mapsto H_d(\mathrm{Gr}(m,mn-r)^n)$ with submodule $[n] \mapsto H_d(X(m,mn-r,n))$. The Künneth formula implies

$$H_d(\mathrm{Gr}(m,mn-r)^n) = \bigoplus_{d_1+\cdots+d_n=d} H_{d_1}(\mathrm{Gr}(m,mn-r)) \otimes \cdots \otimes H_{d_n}(\mathrm{Gr}(m,mn-r)). \quad (11)$$

The space $\mathrm{Gr}(m,mn-r)$ admits an affine paving whose cells (the *Schubert cells*) are indexed by partitions λ whose Young diagrams fit inside a $m \times [m(n-1)-r]$ box. The (complex) dimension of the Schubert cell corresponding to λ is the number of boxes $|\lambda|$ in λ. Consequently, the homology of $\mathrm{Gr}(m,mn-r)$ is concentrated in even degrees and for d_i even the Schubert cells induce a basis

$$\left\{\sigma_\lambda \,:\, |\lambda| = \frac{d_i}{2},\ \lambda \subseteq (m(n-1)-r)^m\right\}$$

of $H_{d_i}(\mathrm{Gr}(m,mn-r))$. Since the Schubert cells are compatible with the embeddings $\mathbb{C}^{mn-r} \subseteq \mathbb{C}^{mp-r}$, we conclude that $[n] \mapsto H_d(\mathrm{Gr}(m,mn-r)^n)$ is a finitely-generated FI-module. Theorems 6 and 7 complete the proof.

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