Small Sets in Union-Closed Families

David Ellis

School of Mathematics
University of Bristol
Bristol, U.K.

david.ellis@bristol.ac.uk

Maria-Romina Ivan Imre Leader

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge, U.K.

{mri25,leader}@maths.cam.ac.uk

Submitted: Jan 29, 2022; Accepted: Nov 28, 2022; Published: Jan 13, 2023

Abstract

Our aim in this note is to show that, for any $\epsilon > 0$, there exists a union-closed family $\mathcal{F}$ with (unique) smallest set S such that no element of S belongs to more than a fraction ϵ of the sets in $\mathcal{F}$. More precisely, we give an example of a union-closed family with smallest set of size k such that no element of this set belongs to more than a fraction $(1 + o(1))\frac{\log_2 k}{2k}$ of the sets in $\mathcal{F}$.

We also give explicit examples of union-closed families containing 'small' sets for which we have been unable to verify the Union-Closed Conjecture.

Mathematics Subject Classifications: 05D05

1 Introduction

If X is a set, a family $\mathcal{F}$ of subsets of X is said to be *union-closed* if the union of any two sets in $\mathcal{F}$ is also in $\mathcal{F}$. The Union-Closed Conjecture (a conjecture of Frankl [5]) states that if X is a finite set and $\mathcal{F}$ is a union-closed family of subsets of X (with $\mathcal{F} \neq \{\emptyset\}$), then there exists an element $x \in X$ such that x is contained in at least half of the sets in $\mathcal{F}$. Despite the efforts of many researchers over the last forty-five years, and a recent Polymath project [7] aimed at resolving it, this conjecture remains wide open. It has only been proved under very strong constraints on the ground-set X or the family $\mathcal{F}$; for example, Balla, Bollobás and Eccles [3] proved it in the case where $|\mathcal{F}| \geqslant \frac{2}{3}2^{|X|}$; more recently, Karpas [6] proved it in the case where $|\mathcal{F}| \geqslant (\frac{1}{2} - c)2^{|X|}$ for a small absolute constant $c > 0$; and it is also known to hold whenever $|X| \leqslant 12$ or $|\mathcal{F}| \leqslant 50$, from work of Vučković and Živković [11] and of Roberts and Simpson [9]. Note that the Union-Closed Conjecture is not even known to hold in the weaker form where we replace the fraction $1/2$ by any other fixed $\epsilon > 0$.[1] For general background and a wealth of further information on the Union-Closed Conjecture see the survey of Bruhn and Schaudt [4].

As usual, if X is a set we write $\mathcal{P}(X)$ for its power-set. If X is a finite set and $\mathcal{F} \subset \mathcal{P}(X)$ with $\mathcal{F} \neq \emptyset$, we define the *frequency* of x (with respect to $\mathcal{F}$) to be $\gamma_x = |\{A \in \mathcal{F} : \ x \in A\}|/|\mathcal{F}|$, i.e., γ_x is the proportion of members of X that contain x. If a union-closed family contains a 'small' set, what can we say about the frequencies in that set?

If a union-closed family $\mathcal{F}$ contains a singleton, then that element clearly has frequency at least $1/2$, while if it contains a set S of size 2 then, as noted by Sarvate and Renaud [10], some element of S has frequency at least $1/2$. However, they also gave an example of a union-closed family $\mathcal{F}$ whose smallest set S has size 3 and yet where each element of S has frequency below $1/2$. Generalising a construction of Poonen [8], Bruhn and Schaudt [4] gave, for each $k \geqslant 3$, an example of a union-closed family with (unique) smallest set of size k and with every element of that set having frequency below $1/2$.

However, in these and all other known examples, there is always some element of a minimal-size set having frequency at least $1/3$. So it is natural to ask if there is really a constant lower bound for these frequencies.

Our aim in this note is to show that this is not the case.

Theorem 1. *For any positive integer k, there exists a union-closed family in which the (unique) smallest set has size k, but where each element of this set has frequency*

$$(1 + o(1))\frac{\log k}{2k}.$$

(All logarithms in this paper are to base 2. Also, as usual, the $o(1)$ denotes a function of k that tends to zero as k tends to infinity.)

Theorem 1 is proved by an explicit construction. It is asymptotically sharp, in view of results of Wójcik [12] and Balla [2]: Wójcik showed that if S is a set of size $k \geqslant 1$ in a finite union-closed family, then the average frequency of the elements in S is at least c_k, where $k \cdot c_k$ is defined to be the minimum average set-size over all union-closed families on the ground-set $[k]$, and Balla showed that $c_k = (1 + o(1))\frac{\log k}{2k}$, confirming a conjecture of Wójcik from [12].

Remarkably, there are union-closed families containing small sets, even sets of size 3, for which we have been unable to verify the Union-Closed Conjecture. We give some examples at the end of the paper.

[1] *Note added in proof:* shortly before the acceptance of this manuscript, Gilmer [arXiv:2211.09055] obtained a breakthrough on the Union-Closed Conjecture, showing that it holds in the weaker form with the fraction $1/2$ replaced by $1/100$.

2 Proof of main result

For our construction, we need the following 'design-theoretic' lemma.

Lemma 2. *For any positive integers $k > t$ there exist infinitely many positive integers d such that t divides dk and the following holds. If X is a set of size dk/t, then there exists a family $\mathcal{A} = \{A_1, \ldots, A_k\}$ of k d-element subsets of X, such that each element of X is contained in exactly t sets in $\mathcal{A}$, and for $2 \leqslant r \leqslant t$, any r distinct sets in $\mathcal{A}$ have intersection of size*

$$d\frac{(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)},$$

i.e.

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}| = d\frac{(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)}$$

for any $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant k$.

Proof. Let q be a positive integer, and set $d = \binom{k-1}{t-1}q^t$; we will take $|X| = \binom{k}{t}q^t$. Partition $[qk]$ into k sets, $B_1, B_2, \ldots, B_k$ say, each of size q; we call these sets 'blocks'. We let X be the set of all t-element subsets of $[qk]$ that contain at most one element from each block. For each $i \in [k]$ we let A_i be the family of all sets in X that contain an element from the block B_i. Clearly, $|A_i| = \binom{k-1}{t-1}q^t = d$ for each $i \in [k]$, and each element of X appears in exactly t of the A_i. Also, for example $A_i \cap A_j$ consists of all sets in X that contain both an element from the block B_i and an element from the block B_j, so

$$|A_i \cap A_j| = \binom{k-2}{t-2}q^t = \binom{k-1}{t-1}q^t\frac{t-1}{k-1} = d\frac{t-1}{k-1}.$$

It is easy to check that the other intersections also have the claimed sizes. □

We remark that, in what follows, it is vital that the integer d in Lemma 2 can be taken to be arbitrarily large as a function of k and t.

Proof of Theorem 1. We define $n = dk/t + k$, we take $d \in \mathbb{N}$ as in the above lemma, and we let $X = [dk/t]$; the claim yields a family $\mathcal{A} = \{A_1, \ldots, A_k\}$ of k d-element subsets of $X = [dk/t]$ such that each element of $[dk/t]$ is contained in exactly t of the sets in $\mathcal{A}$, and for any $2 \leqslant r \leqslant t$, any r distinct sets in $\mathcal{A}$ have intersection of size

$$d\frac{(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)}.$$

Write $m = dk/t$. We take $\mathcal{F} \subset \mathcal{P}([n])$ to be the smallest union-closed family containing the k-element set $\{m+1, \ldots, m+k\}$ and all sets of the form $\{m+i\} \cup (X \setminus \{x\})$ where $i \in [k]$ and $x \in A_i$.

For brevity, we write $S_0 = \{m+1, m+2, \ldots, m+k\}$. We will show that each element of S_0 has frequency

$$(1+o(1))\frac{\log k}{2k},$$

provided t and d are chosen to be appropriate functions of k; moreover, with these choices, S_0 will be the smallest set in $\mathcal{F}$.

Clearly, $\mathcal{F}$ contains S_0, all sets of the form $S_0 \cup (X \setminus \{x\})$ for $x \in X$, all sets of the form $R \cup X$ where R is a nonempty subset of S_0, and finally all sets of the form $R \cup (X \setminus \{x\})$, where $R = \{m+i_1, \ldots, m+i_r\}$ is a nonempty r-element subset of S_0 and $x \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}$, for $1 \leqslant r \leqslant t$. It is easy to see that the family $\mathcal{F}$ contains no other sets.

It follows that

$$\begin{aligned}|\mathcal{F}| &= 1 + dk/t + (2^k - 1) + \sum_{r=1}^{t} \binom{k}{r} d \frac{(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)} \\ &= dk/t + 2^k + \frac{dk}{t} \sum_{r=1}^{t} \binom{t}{r} \\ &= dk/t + 2^k + \frac{dk}{t}(2^t - 1) \\ &= 2^k + \frac{dk2^t}{t}.\end{aligned}$$

On the other hand, the number of sets in $\mathcal{F}$ that contain the element $m+1$ is equal to

$$\begin{aligned}&1 + dk/t + 2^{k-1} + \sum_{r=1}^{t} \binom{k-1}{r-1} d \frac{(t-1)(t-2)\cdots(t-r+1)}{(k-1)(k-2)\cdots(k-r+1)} \\ &= 1 + dk/t + 2^{k-1} + d \sum_{r=1}^{t} \binom{t-1}{r-1} \\ &= 1 + dk/t + 2^{k-1} + 2^{t-1} d.\end{aligned}$$

It follows that the frequency of $m+1$ (or, by symmetry, of any other element of S_0) equals

$$\frac{1 + kd/t + 2^{k-1} + 2^{t-1}d}{2^k + dk2^t/t} = \frac{(1 + 2^{k-1})/d + k/t + 2^{t-1}}{2^k/d + k2^t/t}.$$

To (asymptotically) minimise this expression, we take $t = \lfloor \log k \rfloor$ and $d \to \infty$ (for fixed k); this yields a union-closed family in which the (unique) smallest set (namely S_0) has size k, and every element of that set has frequency

$$(1 + o(1)) \frac{\log k}{2k},$$

proving the theorem. □

3 An open problem

We now turn to some explicit examples of union-closed families containing small sets for which we have been unable to establish the Union-Closed Conjecture. For simplicity, we concentrate on the most striking case, when the family contains a set of size 3, and indeed is generated by sets of size 3.

Our families live on ground-set $\mathbb{Z}_n^2$, the $n \times n$ torus.

Question 3. Let $n \in \mathbb{N}$ and let $R \subset \mathbb{Z}_n$ with $|R| = 3$. Does the Union-Closed Conjecture hold for the union-closed family $\mathcal{F}$ of subsets of $\mathbb{Z}_n^2$ generated by all the translates of $R \times \{0\}$ and of $\{0\} \times R$?

(Here, as usual, we say a union-closed family $\mathcal{F}$ is *generated by* a family $\mathcal{G}$ if it consists of all unions of sets in $\mathcal{G}$.)

Perhaps the most interesting case is when n is prime. In that case we may assume that $R = \{0, 1, r\}$ for some r, and so one feels that the verification of the Union-Closed Conjecture should be a triviality, but it seems not to be. Note that all the families in Question 3 are transitive families, in the sense that all points 'look the same', so that the Union-Closed Conjecture is equivalent to the assertion that the average size of the sets in the family is at least $n^2/2$.

We mention that the corresponding result in $\mathbb{Z}_n$ (in other words, the special case of the Union-Closed Conjecture for the union-closed family on ground-set $\mathbb{Z}_n$ generated by all translates of R) is known to hold: this is proved in [1].

We have verified the special case of Question 3 where $R = \{0, 1, 2\}$. A sketch of the proof is as follows. Assume that $n \geqslant 6$, and let $\mathcal{F} \subset \mathcal{P}(\mathbb{Z}_n^2)$ be the union-closed family generated by all translates of $\{0, 1, 2\} \times \{0\}$ and of $\{0\} \times \{0, 1, 2\}$ (we call these translates *3-tiles*, for brevity). Let $C = \{0, 1, 2, 3\}^2$, a 4×4 square. Consider the bipartite graph $H = (\mathcal{X}, \mathcal{Y})$ with vertex-classes $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{X}$ consists of all subsets of C with size less than 8 that are intersections with C of sets in $\mathcal{F}$, $\mathcal{Y}$ consists of all subsets of C with size greater than 8 that are intersections with C of sets in $\mathcal{F}$, and we join $S \in \mathcal{X}$ to $S' \in \mathcal{Y}$ if $|S'| + |S| \geqslant 16$ and $S' = S \cup U$ for some union U of 3-tiles that are contained within C. It can be verified (by computer) that H has a matching $m : \mathcal{X} \to \mathcal{Y}$ of size $|\mathcal{X}| = 16520$. Such a matching m gives rise to an injection

$$f : \{S \in \mathcal{F} : \ |S \cap C| < |C|/2\} \to \{S \in \mathcal{F} : \ |S \cap C| > |C|/2\}$$

given by

$$f(S) = (S \setminus C) \cup m(S \cap C)$$

with the property that $|S \cap C| + |f(S) \cap C| \geqslant |C|$ for all $S \in \mathcal{F}$ with $|S \cap C| < |C|/2$. It follows that a uniformly random subset of $\mathcal{F}$ has intersection with $|C|$ of expected size at least $|C|/2$, which in turn implies that there is an element of C with frequency at least $1/2$ (and in fact, since $\mathcal{F}$ is transitive, every element has frequency at least $1/2$).

We remark that this proof does not work if one tries to replace $C = \{0, 1, 2, 3\}^2$ by $\{0, 1, 2\}^2$, as the resulting bipartite graph $H' = (\mathcal{X}', \mathcal{Y}')$ does not contain a matching of size $|\mathcal{X}'|$.

We remark also that it would be nice to find a non-computer proof of the above result.

Acknowledgements

We are very grateful to Igor Balla for bringing the papers [2] and [12] to our attention.

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