Min-Cost-Flow Preserving Bijection Between Subgraphs and Orientations

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Abstract

Consider an undirected graph $G = (V, E)$. A subgraph of G is a subset of its edges, while an orientation of G is an assignment of a direction to each of its edges. Provided with an integer circulation–demand $d : V \to \mathbb{Z}$, we show an explicit and efficiently computable bijection between subgraphs of G on which a d-flow exists and orientations on which a d-flow exists. Moreover, given a cost function $w : E \to (0, \infty)$ we can find such a bijection which preserves the w-min-cost-flow.

In 2013, Kozma and Moran [*Electron. J. Comb.* 20(3)] showed, using dimensional methods, that the number of subgraphs k-edge-connecting a vertex s to a vertex t is the same as the number of orientations k-edge-connecting s to t. An application of our result is an efficient, bijective proof of this fact.

Mathematics Subject Classifications: 05C20, 05C85

1 Introduction

Let $G = (V, E)$ be a simple graph with positive cost function $w : E \to (0, \infty)$. We regard G as a digraph, treating every undirected edge as a pair of directed edges in reverse direction. Denote the set of *subgraphs* of G by $\mathcal{S}(G) = \left\{K \subset E \ : \ \forall_{e \in E} |\{e, e^{\mathrm{R}}\} \cap K| \neq 1\right\}$, and the set of *orientations* of G by $\mathcal{O}(G) = \left\{L \subset E \ : \ \forall_{e \in E} |\{e, e^{\mathrm{R}}\} \cap L| = 1\right\}$, where $(u, v)^{\mathrm{R}} = (v, u)$.

A function $d : V \to \mathbb{Z}$ such that $\sum_{u \in V} d(u) = 0$ is called an *integer demand*. A d-flow on G is function $f : E \to [0, 1]$ such that for any $u \in V$ we have $\sum_{v \sim u} f((u, v)) - f((v, u)) = d(u)$. We say that f is a flow on a directed subgraph $D \subset E$ if $f(e) = 0$ for all

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$e \notin D$. Denote $\mathcal{S}_d$, $\mathcal{O}_d$, $\mathcal{D}_d$ for the set of subgraphs, orientations and directed subgraphs of G on which a d-flow exists. Given a cost function $w : E \to \mathbb{R}_+$ and a flow f, write $|f|_w = \sum_{e \in E} w(e)|f(e)|$ for the total w-cost of f. The w min-cost-flow satisfying d is the d-flow for which this cost is minimal.

Our main result is the following.

Theorem 1. *For any graph G, integer demand d and cost function w, there exists an explicit bijection between $\mathcal{S}_d$ and $\mathcal{O}_d$, computable in polynomial time, which preserves a w min-cost-flow.*

We call a path from s to t in G an (s,t)-path. A directed graph is said to *k-connect s and t* if there exist k edge-disjoint (s,t)-directed paths. Denote $\mathcal{S}_k$ and $\mathcal{O}_k$ the sets of subgraphs and orientations of G which k-connect s and t, respectively.

A collection of k disjoint (s,t)-directed paths is called *minimal* in a directed graph D if the total weight of edges participating in the paths is minimal. Recalling the classical Integrality Theorem, which guarantees that any integer-valued min-cost-flow problem in a graph (i.e. with capacity 1 for each edge) has an integer optimal solution, Theorem 1 implies the following.

Theorem 2. *For any weighted graph $G = (V, E, w)$ there exists an explicit bijection between $\mathcal{S}_k$ and $\mathcal{O}_k$, computable in polynomial time. Moreover, this bijection preserves a collection of k edge disjoint paths with minimal total weight with respect to w.*

The result could be easily generalized to vertex disjoint paths by introducing vertex capacities.

2 Background and motivation

The study of the relationship between subgraphs and orientations is a classical subject in combinatorics. In 1960 Nash-Williams [8], generalizing a 1939 result by Robbins [10], showed that every undirected graph G has a well-balanced orientation. Chvátal and Thomassen [2] proved that every undirected bridgeless (i.e. 2-edge-connected) graph of radius r admits an orientation of radius at most $r^2 + r$, and that this bound is best possible. In [1] Bernardi showed that evaluation of the Tutte polynomial counts both the spanning subgraphs and the orientations of G.

In 2013, Kozma and Moran [6], introduced Vapnik-Chervonenkis (VC) theory to the subject. They showed that there are several properties ϕ of graphs, for which number of subgraphs of a given graph G which satisfy ϕ is either the same, or dominates the number of orientations satisfying it. Their proof relies upon shattering extremal systems, using the sandwich theorem [9]. Recently Bucić, Janzer and Sudakov [3] used this method to count H-free orientations of a given graph G.

In particular, it was shown in [6] that $|\mathcal{S}_k| = |\mathcal{O}_k|$. Their method, however, is non-constructive, and its naïve algorithmic application is of exponential complexity in $|E|$. In Theorem 2 we obtain an explicit, natural and efficiently computable bijection between $\mathcal{S}_k$ and $\mathcal{O}_k$, which preserves a particular collection of k-disjoint paths.

2.1 Notation and conventions

Throughout $G = (V, E)$, the base graph, $w : E \to \mathbb{R}_+$, the weight function, $d : V \to \mathbb{Z}$, the demand function, $E' \in \mathcal{O}(E)$ an arbitrary orientation of G and an *a priori* order $e_1, \ldots, e_{|E|}$ on the edges of E are all fixed. We also define

$$\chi(e) = \begin{cases} e & \text{for } e \in E', \\ e^{\mathrm{R}} & \text{for } e^{\mathrm{R}} \in E'. \end{cases}$$

Given a directed subgraph $D \subset E$ we write $w(D) = \sum_{e \in D} w(e)$.

For simplicity we assume that E has no two distinct subsets of equal total weight so that for any subgraph $D \subset E$ and any demand function d, the min-cost-flow (i.e. minimum w-cost d-flow) on D is unique. To extend our results to the general case, extend the partial order induced by w to a complete order, by breaking ties lexicographically in our *a-priori* order on $e_1, \ldots, e_{|E|}$. Namely, by treating $w(f)$ as smaller than $w(f')$ also when both weights are equal and $f'(e_i) \geqslant f(e_i)$ for the minimum i for which $f'(e_i) \neq f(e_i)$.

We denote the **unique** solution to the min-cost-flow problem in the directed subgraph $D \subset E$ by $A(D)$, whenever such a solution exists. Using the integrality theorem, we treat $A(D)$ both as a set of directed edges and as a flow.

To simplify addition and subtraction of edges from a directed subgraph we employ the orientation operation $D \oplus e := \{D \cup \{e\}\} \setminus e^{\mathrm{R}}$, the symmetric inclusion operation, $D + e := D \cup \{e, e^{\mathrm{R}}\}$ and the symmetric exclusion operation $D - e := D \setminus \{e, e^{\mathrm{R}}\}$.

3 The bijection

Our bijection relies on the following lemma.

Lemma 3. *Let $D \in \mathcal{D}_d$ and $e \in G$. Then at least one of the following holds:*

- $A(D \oplus e) = A(D)$,
- $A(D \oplus e^{\mathrm{R}}) = A(D)$.

Proof. If either $e \in D$, $e^{\mathrm{R}} \in D$ or both, the lemma is straightforward, as $A(D)$ cannot include both e and e^{R}. We may therefore assume that $\{e, e^{\mathrm{R}}\} \cap D = \emptyset$.

The classical Integrality Theorem, guarantees that any min-cost-flow problem in a graph (i.e. with capacity 1 for each edge) has an integer optimal solution. Write F_0 for the min-cost d-flow in D, F_1 for the min-cost d-flow in $D \cup \{e\}$, and F_2 for the min-cost d-flow in $D \cup \{e^{\mathrm{R}}\}$.

Assume for the sake of obtaining a contradiction that these three flows are distinct, so that, by monotonicity, $|F_1|_w, |F_2|_w < |F_0|_w$. In particular, this implies, by the integrality theorem, that F_1 assigns flow 1 to e and F_2 assigns flow 1 to e^{R}, as otherwise one of these flows would be valid also on D. This implies, however, that $\frac{F_1+F_2}{2}$, which is also a d-flow, assigns a total of 0 flow to e, so that it is a proper flow on D. Clearly, the total weight of this flow is less than the maximum among $|F_1|_w$ and $|F_2|_w$, a contradiction to the minimality of F_0. By our assumption that $|F|_w$ uniquely characterizes F, we deduce that either $F_0 = F_1$ or $F_0 = F_2$. $\square$

We also require the following observation.

Lemma 4. *Let $D \in \mathcal{D}_d$ and $e \in A(D)$, then $A(D) = A(D + e)$.*

Proof. Assume to the contrary that $A(D + e) \neq A(D) = A(D \oplus e)$ and denote by F_0 the flow corresponding to $A(D)$ and F_1 for the flow corresponding to $A(D+e)$, as in the proof of lemma 3. By monotonicity, we have $|F_1|_w < |F_0|_w$. By minimality this implies that $F_1(e^{\mathrm{R}}) = 1$ while by our assumption $F_0(e) = 1$. Hence $\frac{F_0+F_1}{2}$ is a flow on D satisfying $|\frac{F_0+F_1}{2}| < |F_1|$, a contradiction. Therefore $A(D) = A(D + e)$. $\square$

Equipped with Lemma 3, the orientation E' and our order $(e_1, \dots, e_{|E|})$, we are ready to present our bijection in the next couple of sections.

3.1 The $\mathcal{S}_d \to \mathcal{O}_d$ bijection

The bijection $\phi : \mathcal{S}_d \to \mathcal{O}_d$ is iteratively obtained by firstly orienting e_1, then e_2 and so forth. This is done by applying a sequence of maps $\phi_i : \mathcal{D}_d \to \mathcal{D}_d$ for $i \in \{1, \dots, |E|\}$, such that for all $i \leqslant |E|$ we have $\phi_i(D) \setminus \{e_i, e_i^{\mathrm{R}}\} = D \setminus \{e_i, e_i^{\mathrm{R}}\}$ and $A(D) = A(\phi_i(D))$. Define $\phi_i(D)$ as follows.

Firstly, we make sure that the orientation of e_i will not alter the min-cost d-flow.

1. if $A(D) \neq A\big(D \oplus e_i\big)$ we set $\phi_i(D) := D \oplus e_i^{\mathrm{R}}$,
2. if $A(D) \neq A\big(D \oplus e_i^{\mathrm{R}}\big)$ we set $\phi_i(D) := D \oplus e_i$.

In the remaining case, where $A(D) = A(D \oplus \{e_i\}) = A\big(D \oplus \{e_i^{\mathrm{R}}\}\big)$ we do the following:

3. if $e_i \in D$ we set $\phi_i(D) := D \oplus \chi(e_i)$,
4. if $e_i \notin D$ we set $\phi_i(D) := D \oplus \chi(e_i)^{\mathrm{R}}$.

Observe that, by Lemma 3, rules (1.) and (2.) are mutually exclusive so that ϕ_i is well defined, $A(D) = A(\phi_i(D))$ and $|\phi_i(D) \cap \{e_i, e_i^{\mathrm{R}}\}| = 1$.

We then set $\phi(K) := \phi_{|E|} \circ \phi_{|E-1|} \circ \cdots \circ \phi_1(K)$ so that ϕ maps $\mathcal{S}_d$ to $\mathcal{O}_d$ and

$$A(K) = A(\phi(K)). \tag{1}$$

3.2 The $\mathcal{O}_d \to \mathcal{S}_d$ bijection

The reverse bijection $\psi : \mathcal{O}_d \to \mathcal{S}_d$ is obtained similarly. This time we iterate by first deciding whether to include both $e_{|E|}$ & $e_{|E|}^{\mathrm{R}}$ or neither of them, then $e_{|E|-1}$ & $e_{|E|-1}^{\mathrm{R}}$ and so forth. This is done by applying a sequence of maps $\psi_i : \mathcal{D}_d \to \mathcal{D}_d$ for $i \in \{1, \dots, |E|\}$, such that for all $i \leqslant |E|$ we have $\psi_i(D) \setminus \{e_i, e_i^{\mathrm{R}}\} = D \setminus \{e_i, e_i^{\mathrm{R}}\}$.

Firstly, we verify that the decision to include or exclude e_i & e_i^{R} will not alter the min-cost d-flow.

1. if $A(D) \neq A\big(D + e_i\big)$ we set $\psi_i(D) := D - e_i$,
2. if $A(D) \neq A\big(D - e_i\big)$ we set $\psi_i(D) := D + e_i$.

In the remaining case, where $A(D) = A(D + e_i) = A(D - e_i)$, we do the following:

3. if $\chi(e_i) \in D$ we set $\psi_i(D) := D + e_i$,
4. if $\chi(e_i) \notin D$ we set $\psi_i(D) := D - e_i$.

Observe that, by Lemma 4 rules (1.) and (2.) are mutually exclusive, as the former's condition is impossible if $\{e_i, e_i^{\mathrm{R}}\} \cap A(D) \neq \emptyset$ and the latter's is impossible otherwise. Hence that ψ_i is well defined, $A(D) = A(\psi_i(D))$ and $|\psi_i(D) \cap \{e_i, e_i^{\mathrm{R}}\}| \neq 1$.

We then set $\psi(L) := \psi_1 \circ \psi_2 \circ \cdots \circ \psi_{|E|}(L)$ so that ψ maps $\mathcal{O}_d$ to $\mathcal{S}_d$ and

$$A(L) = A(\psi(L)). \tag{2}$$

4 Proof of bijectivity

In this section we establish the fact that ψ is the inverse function of ϕ and, as a consequence, Theorem 1.

This is an immediate consequence of the following

Proposition 5. *For all $D \in \mathcal{D}_d$ it holds that*

- *if $|D \cap \{e_i, e_i^{\mathrm{R}}\}| \neq 1$ then $D = \psi_i \circ \phi_i(D)$*
- *if $|D \cap \{e_i, e_i^{\mathrm{R}}\}| = 1$ then $D = \phi_i \circ \psi_i(D)$*

Proof. Firstly, we show three statements,

$$\begin{gathered}\text{either } A(D) = A(D \oplus e_i) \text{ or } A(D) = A(D \oplus e_i^{\mathrm{R}}),\\ \text{either } A(D) = A(D + e_i) \text{ or } A(D) = A(D - e_i),\\ A(D) = A(D + e_i) = A(D - e_i) \text{ if and only if } A(D) = A(D \oplus e_i) = A(D \oplus e_i^{\mathrm{R}}).\end{gathered} \tag{3}$$

The first two observations are immediate from Lemma 3 and Lemma 4, respectively. To see the last equivalence, observe that the same two lemmata imply that the statements $A(D + e_i) \neq A(D - e_i)$ and $A(D \oplus e_i) \neq A(D \oplus e_i^{\mathrm{R}})$ are both equivalent to the fact that $e_i \in A(D + e_i)$ or $e_i^{\mathrm{R}} \in A(D + e_i)$.

Using (3) we deduce that $\psi_i(D)$,$\phi_i(D)$,$\psi_i \circ \phi_i(D)$ and $\phi_i \circ \psi_i(D)$, are either all produced by rules (1.) and (2.) of their respective definition, or all produced by rules (3.) and (4.).

We first consider the case that they are all produced by rules (1.) and (2.). In this case

$$A(D \oplus e_i) \neq A(D) \text{ or } A(D \oplus e_i) \neq A(D), \tag{4}$$

$$A(D + e_i) \neq A(D) \text{ or } A(D - e_i) \neq A(D). \tag{5}$$

Observe that if $|\psi_i(D) \cap \{e_i, e_i^{\mathrm{R}}\}| \neq 1$ then $\{\psi_i(\phi_i(D)), D\} \subset \{D + e_i, D - e_i\}$. so that the first item of the proposition follows from (5) together with the fact that, by (1) and (2), $A(\psi_i(\phi_i(D))) = A(D)$. Similarly, if $|\psi_i(D) \cap \{e_i, e_i^{\mathrm{R}}\}| = 1$ then $\phi_i(\psi_i(D)), D \in \{D \oplus e_i, D \oplus e_i^{\mathrm{R}}\}$, so that the first item of the proposition follows from (4) together with the fact that, by (1) and (2), $A(\phi_i(\psi_i(D))) = A(D)$.

We are left with the case that $\psi_i(D)$,$\phi_i(D)$,$\psi_i \circ \phi_i(D)$ and $\phi_i \circ \psi_i(D)$ are all produced by rules (3.) and (4.), whence

- If $\{e_i, e_i^{\mathrm{R}}\} \subset D$ then $\phi(D) = D \oplus \chi(e_i)$ and $\psi_i \circ \phi_i(D) = (D \oplus \chi(e_i)) + e_i = D$.
- If $\{e_i, e_i^{\mathrm{R}}\} \cap D = \emptyset$ then $\phi(D) = D \oplus \chi(e_i)^{\mathrm{R}}$ and $\psi_i \circ \phi_i(D) = (D \oplus \chi(e_i)^{\mathrm{R}}) - e_i = D$.
- If $\{e_i, e_i^{\mathrm{R}}\} \cap D = \{\chi(e_i)\}$ then $\psi(D) = D + e_i$ and $\phi_i \circ \psi_i(D) = (D + e_i) \oplus \chi(e_i) = D$.
- If $\{e_i, e_i^{\mathrm{R}}\} \cap D = \{\chi(e_i)^{\mathrm{R}}\}$ then $\psi(D) = D - e_i$ and $\phi_i \circ \psi_i(D) = (D - e_i) \oplus \chi(e_i)^{\mathrm{R}} = D$.

□

5 Complexity

As for Theorem 2, finding the minimal k disjoint (s, t)-directed path could be done efficiently using the Suurballe algorithm [11], an extension of the Dijkstra algorithm [4]. The worst case complexity of this algorithm is $O(k|E| + k|V| \log |V|)$. The general case of Theorem 1, has the complexity of solving the min-cost-flow problem, solvable via linear programming. For a survey on the complexity of the problem in various settings see [5].

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