Proof of a Conjecture Involving Derangements and Roots of Unity

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Submitted: Jul 6, 2022; Accepted: Mar 5, 2023; Published: Apr 7, 2023

Abstract

Let $n > 1$ be an odd integer, and let ζ be a primitive nth root of unity in the complex field. Via the Eigenvector-eigenvalue Identity, we show that

$$\sum_{\tau \in D(n-1)} \operatorname{sign}(\tau) \prod_{j=1}^{n-1} \frac{1+\zeta^{j-\tau(j)}}{1-\zeta^{j-\tau(j)}} = (-1)^{\frac{n-1}{2}} \frac{((n-2)!!)^2}{n},$$

where $D(n-1)$ is the set of all derangements of $1, \ldots, n-1$. This confirms a previous conjecture of Z.-W. Sun. Moreover, for each $\delta = 0, 1$ we determine the value of $\det[x + m_{jk}]_{1 \leqslant j,k \leqslant n-1}$ completely, where

$$m_{jk} = \begin{cases} (1+\zeta^{j-k})/(1-\zeta^{j-k}) & \text{if } j \neq k, \\ \delta & \text{if } j = k. \end{cases}$$

Mathematics Subject Classifications: 05A19, 11C20, 15A18, 15B57, 33B10.

*Corresponding author. Supported by the National Natural Science Foundation of China (grant no. 11971222).

1 Introduction

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, let S_n be the symmetric group of all permutations of $\{1, \ldots, n\}$. A permutation $\tau \in S_n$ is called a *derangement* of $1, \ldots, n$ if $\tau(j) \neq j$ for all $j = 1, \ldots, n$. For convenience, we use $D(n)$ to denote the set of all derangements of $1, \ldots, n$. The derangement number $D_n = |D(n)|$ plays important roles in enumerative combinatorics. It is well known that

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

(cf. (10.2) of [8, p. 90]).

Let $n > 1$ be an odd integer. Z.-W. Sun [5, Theorem 1.2] proved that

$$\det \left[\tan \pi \frac{j-k}{n} \right]_{1 \leqslant j,k \leqslant n-1} = n^{n-2}.$$

As

$$\tan \pi x = \frac{2 \sin \pi x}{2 \cos \pi x} = i \frac{1 - e^{2\pi i x}}{1 + e^{2\pi i x}},$$

we see that

$$\begin{aligned} \det \left[\tan \pi \frac{j-k}{n} \right]_{1 \leqslant j,k \leqslant n-1} &= i^{n-1} \det \left[\frac{1 - \zeta^{j-k}}{1 + \zeta^{j-k}} \right]_{1 \leqslant j,k \leqslant n-1} \\ &= (-1)^{(n-1)/2} \sum_{\tau \in D(n-1)} \operatorname{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 - \zeta^{j-\tau(j)}}{1 + \zeta^{j-\tau(j)}}, \end{aligned}$$

where $\zeta = e^{2\pi i/n}$.

Z.-W. Sun ([6] and [7, Conj. 11.24]) conjectured that if $n > 1$ is odd and ζ is a primitive nth root of unity in the complex field $\mathbb{C}$ then

$$\sum_{\tau \in D(n-1)} \operatorname{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 + \zeta^{j-\tau(j)}}{1 - \zeta^{j-\tau(j)}} = (-1)^{\frac{n-1}{2}} \frac{((n-2)!!)^2}{n}. \tag{1}$$

Our first goal is to prove an extension of this conjecture.

Theorem 1. *Let $n > 1$ be an odd integer, and let $\zeta \in \mathbb{C}$ be a primitive nth root of unity. For $j, k = 1, \ldots, n$ define*

$$a_{jk} = \begin{cases} (1 + \zeta^{j-k})/(1 - \zeta^{j-k}) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

Then we have

$$\det[x + a_{jk}]_{1 \leqslant j,k \leqslant n-1} = (-1)^{\frac{n-1}{2}} \frac{((n-2)!!)^2}{n}. \tag{2}$$

Applying Theorem 1 with $x = 1$, we immediately obtain the following result.

Corollary 2. *Let $n > 1$ be odd. Then, for any primitive nth root $\zeta \in \mathbb{C}$ of unity, we have*

$$\det[\tilde{a}_{jk}]_{1\leqslant j,k\leqslant n-1} = (-1)^{\frac{n-1}{2}} \frac{((n-2)!!)^2}{n2^{n-1}},$$

where

$$\tilde{a}_{jk} = \begin{cases} 1/(1-\zeta^{j-k}) & \text{if } j \neq k, \\ 1/2 & \text{if } j = k. \end{cases}$$

For any odd integer $n > 1$, Sun ([6] and [7, Conj. 11.22]) also conjectured that if $\zeta \in \mathbb{C}$ is a primitive nth root of unity then

$$\sum_{\tau\in D(n-1)} \operatorname{sign}(\tau) \prod_{j=1}^{n-1} \frac{1}{1-\zeta^{j-\tau(j)}} = \frac{(-1)^{\frac{n-1}{2}}}{n} \left(\frac{n-1}{2}!\right)^2. \tag{3}$$

Recently, X. Guo et al. [4] proved (3) via using the following result which dates back to Jacobi in 1834 (cf. P.B. Denton, S.J. Parke, T. Tao and X. Zhang [2, Theorem 1]).

Theorem 3 (Eigenvector-eigenvalue Identity)**.** *Let A be an $n \times n$ matrix over $\mathbb{C}$ which is Hermitian (i.e., the transpose A^T of A coincides with the conjugate of A), and let $\lambda_1, \ldots, \lambda_n$ be its n real eigenvalues. Let $v_n = (v_{n,1}, \ldots, v_{n,n})^T$ be an eigenvector associated with the eigenvalue λ_n of the matrix A such that its norm $\|v_n\| = \sqrt{\sum_{j=1}^n |v_{n,j}|^2}$ equals 1. Let $j \in \{1, \ldots, n\}$ and let A_j be the $(n-1)\times(n-1)$ Hermitian matrix formed by deleting the jth row and the jth column from A. Let $\lambda_{j,1}, \ldots, \lambda_{j,n-1}$ be all the real eigenvalues of A_j. Then we have*

$$|v_{n,j}|^2 \prod_{k=1}^{n-1}(\lambda_n - \lambda_k) = \prod_{k=1}^{n-1}(\lambda_n - \lambda_{j,k}).$$

Motivated by Theorem 1, we also establish the following result.

Theorem 4. *Let $n > 1$ be odd. Then, for any primitive nth root $\zeta \in \mathbb{C}$ of unity, we have*

$$\det[x + b_{jk}]_{1\leqslant j,k\leqslant n-1} = (-1)^{\frac{n+1}{2}} (nx+1) \frac{((n-1)!!)^2}{n(n-1)}, \tag{4}$$

where

$$b_{jk} = \begin{cases} (1+\zeta^{j-k})/(1-\zeta^{j-k}) & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

We are going to prove Theorems 1 and 4 in Sections 2 and 3, respectively.

2 Proof of Theorem 1

We need the following easy lemma.

Lemma 5. *Let $n \in \mathbb{Z}^+$ and $s \in \{0, \ldots, n-1\}$. For any primitive nth root ζ of unity in a field F, we have the identity*

$$\sum_{0<r<n} \frac{\zeta^{-rs}}{1 - x\zeta^r} = \frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1}. \tag{5}$$

Proof. Clearly,

$$\sum_{r=0}^{n-1} \frac{\zeta^{-rs}}{1 - x\zeta^r} = \sum_{r=0}^{n-1} \frac{\zeta^{-rs}}{1 - x^n} \sum_{k=0}^{n-1} (x\zeta^r)^k = \sum_{k=0}^{n-1} \frac{x^k}{1 - x^n} \sum_{r=0}^{n-1} \zeta^{r(k-s)} = \frac{nx^s}{1 - x^n}.$$

Thus

$$\sum_{r=1}^{n-1} \frac{\zeta^{-rs}}{1 - x\zeta^r} = \frac{nx^s}{1 - x^n} - \frac{1}{1 - x} = \frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1}$$

as desired. □

Remark 6. Lemma 5 in the case $F = \mathbb{C}$ is essentially equivalent to [3, Theorem 3.1].

Corollary 7. *Let $n \in \mathbb{Z}^+$ and $s \in \{0, \ldots, n-1\}$. Let ζ be any primitive nth root of unity in the field $\mathbb{C}$.*

(i) *If n is odd, then*

$$\sum_{0<r<n} \frac{\zeta^{-rs}}{1 + \zeta^r} = \frac{(-1)^s n - 1}{2}. \tag{6}$$

(ii) *We have*

$$\sum_{0<r<n} \frac{\zeta^{-rs}}{1 - \zeta^r} = \frac{n-1}{2} - s. \tag{7}$$

Proof. (i) When n is odd, putting $x = -1$ in (5) we immediately get (6).

(ii) Letting $x \to 1$ in (5) we obtain (7) since

$$\begin{aligned}\lim_{x \to 1} \frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1} &= \lim_{x \to 1} \frac{(\sum_{j=0}^{n-1} x^j - nx^s)'}{(x^n - 1)'} = \lim_{x \to 1} \frac{\sum_{0<j<n} jx^{j-1} - nsx^{s-1}}{nx^{n-1}} \\ &= \frac{\sum_{j=0}^{n-1} j - ns}{n} = \frac{1}{n} \sum_{j=0}^{n-1} j - s = \frac{n-1}{2} - s\end{aligned}$$

by L'Hospital's rule.

Combining the above, we have completed the proof of Corollary 7. □

Remark 8. It seems that the identity (7) should be known long time ago. We note that it essentially appeared as [3, (3.5)] though $(n-1)/2$ in [3, (3.5)] should be corrected as $(n+1)/2$.

Now we give an auxiliary proposition.

Proposition 9. *Let $n \in \mathbb{Z}^+$, $k \in \{1, \ldots, n\}$ and $s \in \{0, \ldots, n-1\}$. For any primitive nth root ζ of unity in a field F, we have*

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 + x\zeta^{j-k}}{1 - x\zeta^{j-k}} \zeta^{s(k-j)} = 1 + 2\frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1} - n\delta_{s0}, \tag{8}$$

where the Kronecker symbol δ_{st} is 1 or 0 according as $s = t$ or not. Consequently, if ζ is a primitive nth root of unity in $\mathbb{C}$, then

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{s(k-j)} = \begin{cases} n - 2s & \text{if } 0 < s < n, \\ 0 & \text{if } s = 0. \end{cases} \tag{9}$$

Proof. In view of Lemma 5, we have

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 + x\zeta^{j-k}}{1 - x\zeta^{j-k}} \zeta^{s(k-j)} &= \sum_{r=1}^{n-1} \frac{1 + x\zeta^r}{1 - x\zeta^r} \zeta^{-sr} = 2\sum_{r=1}^{n-1} \frac{\zeta^{-rs}}{1 - x\zeta^r} - \sum_{r=1}^{n-1} \zeta^{-rs} \\ &= 2\frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1} + 1 - \sum_{r=0}^{n-1} \zeta^{-rs} = 2\frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1} + 1 - n\delta_{s0}. \end{aligned}$$

This proves (8).

When $F = \mathbb{C}$, letting $x \to 1$ in (8) or using the identity (7), we get (9). □

We also need another lemma.

Lemma 10 (Sun [5]). *For any matrix $M = [m_{jk}]_{0 \leqslant j,k \leqslant n}$ over $\mathbb{C}$, we have*

$$\det[x + m_{jk}]_{0 \leqslant j,k \leqslant n} = \det(M) + x \det(M'),$$

where $M' = |m'_{jk}|_{1 \leqslant j,k \leqslant n}$ with $m'_{jk} = m_{jk} - m_{j0} - m_{0k} + m_{00}$.

Proof of Theorem 1. Obviously $A = [a_{jk}]_{1 \leqslant k,j \leqslant n}$ is a Hermitian matrix. For each $k = 1, \ldots, n$, by Proposition 9 we have

$$\sum_{j=1}^{n} a_{jk} \zeta^{-js} = \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{-js} = \begin{cases} (n - 2s)\zeta^{-ks} & \text{if } s \in \{1, \ldots, n-1\}, \\ 0 & \text{if } s = n. \end{cases}$$

Thus $\lambda_s = n - 2s$ $(s = 1, \ldots, n-1)$ and $\lambda_n = 0$ are all the eigenvalues of A; moreover, for each $s = 1, \ldots, n$, the column vector

$$v^{(s)} = \frac{1}{\sqrt{n}} (\zeta^{-s}, \zeta^{-2s}, \ldots, \zeta^{-ns})^T$$

is an eigenvector of norm 1 associated with the eigenvalue λ_s.

Let A_n be the Hermitian matrix $[a_{jk}]_{1\leqslant k,j\leqslant n-1}$, and let $\lambda_{n,1},\ldots,\lambda_{n,n-1}$ be all the eigenvalues of A_n. Note that $v^{(n)}=(1,\ldots,1)^T/\sqrt{n}$. Applying Theorem 3 with $j=n$, we obtain that

$$(-1)^{n-1}\det(A_n)=\prod_{k=1}^{n-1}(0-\lambda_{n,k})=\left|\frac{1}{\sqrt{n}}\right|^2\prod_{k=1}^{n-1}(0-\lambda_k)=\frac{(-1)^{n-1}}{n}\prod_{k=1}^{n-1}(n-2k)$$

and hence

$$\begin{aligned}\det(A_n)&=\frac{1}{n}\prod_{k=1}^{(n-1)/2}(n-2k)(n-2(n-k))\\&=\frac{(-1)^{(n-1)/2}}{n}\prod_{k=1}^{(n-1)/2}(n-2k)^2=\frac{(-1)^{(n-1)/2}}{n}((n-2)!!)^2.\end{aligned}$$

On the other hand,

$$\det(A_n)=\det(A_n^T)=\det[a_{jk}]_{1\leqslant j,k\leqslant n-1}=\sum_{\tau\in D(n-1)}\operatorname{sign}(\tau)\prod_{j=1}^{n-1}\frac{1+\zeta^{j-\tau(j)}}{1-\zeta^{j-\tau(j)}}.$$

Combining the last two equalities, we immediately get (2) for $x=0$.

By Lemma 10, we have

$$\det[x+a_{jk}]_{1\leqslant j,k\leqslant n-1}=\det(A_n^T)+x\det(A_n'),$$

where $A_n'=[a_{jk}']_{2\leqslant j,k\leqslant n-1}$ with

$$a_{jk}'=a_{jk}-a_{j1}-a_{1k}+a_{11}=a_{jk}-a_{j1}-a_{1k}.$$

It is easy to see that $a_{kj}'=-a_{jk}'$ for all $j,k=2,\ldots,n-1$. So we have

$$\det(A_n')=\det(-A_n')=(-1)^{n-2}\det(A_n')=-\det(A_n')$$

and hence

$$\det[x+a_{jk}]_{1\leqslant j,k\leqslant n-1}=\det(A_n)+x\det(A_n')=\det(A_n)=\frac{(-1)^{(n-1)/2}}{n}((n-2)!!)^2.$$

This ends our proof. □

3 Proof of Theorem 4

Lemma 11. *Let $n\in\{2,3,4,\ldots\}$, and let ζ be a primitive nth root of unity. For $j,k=1,\ldots,n$, define*

$$c_{jk}=\begin{cases}1/(1-\zeta^{j-k}) & \text{if } j\neq k,\\ 0 & \text{if } j=k.\end{cases}$$

(i) *The n eigenvalues of* $[c_{jk}+\delta_{jk}]_{1\leqslant j,k\leqslant n}$ *are* $s-(n-1)/2$ $(s=1,\ldots,n)$.

(ii) *If n is odd, then*

$$\det[c_{jk}+\delta_{jk}]_{1\leqslant j,k\leqslant n-1}=(-1)^{\frac{n+1}{2}}\frac{(n+1)((n-1)!!)^2}{n(n-1)2^{n-1}}. \tag{10}$$

Proof. (i) For $j,k=1,\ldots,n$, let

$$t_{jk}=\begin{cases}1+i\cot\pi\frac{j-k}{n} & \text{if } j\neq k,\\ 0 & \text{if } j=k.\end{cases}$$

By F. Calogero and A. M. Perelomov [1, Theorem 1], the n numbers $2s-n-1$ $(s=1,\ldots,n)$ are all the eigenvalues of the matrix $[t_{jk}]_{1\leqslant j,k\leqslant n}$. Thus

$$\det[x\delta_{jk}-t_{jk}]_{1\leqslant k\leqslant n}=\prod_{s=1}^{n}(x-(2s-n-1)). \tag{11}$$

For $j,k=1,\ldots,n$ with $j\neq k$, clearly

$$t_{jk}=1-\frac{2\cos\pi\frac{j-k}{n}}{2i\sin\pi\frac{j-k}{n}}=1-\frac{e^{2\pi i\frac{j-k}{n}}+1}{e^{2\pi i\frac{j-k}{n}}-1}=\frac{2}{1-e^{2\pi i\frac{j-k}{n}}}.$$

Note that $\zeta=e^{2\pi ia/n}$ for some $1\leqslant a\leqslant n$ with $\gcd(a,n)=1$. Applying the Galois automorphism σ_a in the Galois group $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ with $\sigma_a(e^{2\pi i/n})=e^{2\pi ia/n}$, we obtain from (11) the polynomial identity

$$\det[x\delta_{jk}-2c_{jk}]_{1\leqslant k\leqslant n}=\prod_{s=1}^{n}(x-(2s-n-1)). \tag{12}$$

Thus

$$\det[x\delta_{jk}-c_{jk}]_{1\leqslant j,k\leqslant n}=\prod_{s=1}^{n}\left(x-s+\frac{n+1}{2}\right),$$

and hence

$$\begin{aligned}\det[x\delta_{jk}-c_{jk}-\delta_{jk}]_{1\leqslant j,k\leqslant n}&=\det[(x-1)\delta_{jk}-c_{jk}]_{1\leqslant j,k\leqslant n}\\&=\prod_{s=1}^{n}\left(x-1-s+\frac{n+1}{2}\right)=\prod_{s=1}^{n}\left(x-\left(s-\frac{n-1}{2}\right)\right).\end{aligned}$$

So the numbers $s-(n-1)/2$ $(s=1,\ldots,n)$ are all the eigenvalues of $[c_{jk}+\delta_{jk}]_{1\leqslant j,k\leqslant n}$.

(ii) Now assume that n is odd. Let

$$\{\lambda_1,\ldots,\lambda_n\}=\left\{\frac{3-n}{2},\frac{5-n}{2},\ldots,\frac{n+1}{2}\right\}$$

with $\lambda_n = 0$. Then the column vector

$$v^{(n)} = \frac{1}{\sqrt{n}}(\zeta^{-\frac{n-1}{2}}, \zeta^{-2\frac{n-1}{2}}, \dots, \zeta^{-n\frac{n-1}{2}})^T$$

is an eigenvector of norm 1 associated with the eigenvalue λ_n of $C = [c_{jk} + \delta_{jk}]_{1 \leqslant j,k \leqslant n}$; in fact, for each $j = 1, \dots, n$, clearly

$$\begin{aligned} \zeta^{j\frac{n-1}{2}} \sum_{k=1}^{n} (c_{jk} + \delta_{jk}) \zeta^{-k\frac{n-1}{2}} &= \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{\zeta^{(j-k)\frac{n-1}{2}}}{1 - \zeta^{j-k}} + 1 \\ &= \sum_{r=1}^{n-1} \frac{\zeta^{-r\frac{n+1}{2}}}{1 - \zeta^{r}} + 1 = \frac{n-1}{2} - \frac{n+1}{2} + 1 = 0 \end{aligned}$$

by applying (7) with $s = (n+1)/2$.

Let C_n be the Hermitian matrix $[c_{jk} + \delta_{jk}]_{1 \leqslant j,k \leqslant n-1}$, and let $\lambda_{n,1}, \dots, \lambda_{n,n-1}$ be all the eigenvalues of C_n. Note that $v^{(n)} = (\zeta^{-\frac{n-1}{2}}, \dots, \zeta^{-\frac{n(n-1)}{2}})^T / \sqrt{n}$. Applying Theorem 3 with $j = n$, we obtain that

$$(-1)^{n-1} \det(C_n) = \prod_{k=1}^{n-1} (0 - \lambda_{n,k}) = \left| \frac{\zeta^{-\frac{n(n-1)}{2}}}{\sqrt{n}} \right|^2 \prod_{k=1}^{n-1} (0 - \lambda_k) = \frac{(-1)^{n-1}}{n} \prod_{\substack{k=1 \\ k \neq \frac{n-1}{2}}}^{n} \left(k - \frac{n-1}{2} \right)$$

and hence

$$\begin{aligned} \det(C_n) &= \frac{(n-1)(n+1)}{2^{n-1} n} \prod_{k=1}^{(n-3)/2} (n-1-2k)(n-1-2(n-1-k)) \\ &= (-1)^{\frac{n+1}{2}} \frac{(n-1)(n+1)}{2^{n-1} n} \prod_{k=1}^{(n-3)/2} (n-1-2k)^2 \\ &= (-1)^{\frac{n+1}{2}} \frac{(n+1)((n-1)!!)^2}{2^{n-1} n (n-1)}. \end{aligned}$$

This concludes the proof. □

Proof of Theorem 4. Let B be the $n \times n$ matrix $[b_{jk}]_{1 \leqslant k,j \leqslant n}$. With the aid of (9),

$$1 + \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{s(k-j)} = \begin{cases} n + 1 - 2s & \text{if } 0 < s < n, \\ 1 & \text{if } s = 0. \end{cases} \tag{13}$$

Thus, for each $k = 1, \dots, n$, we have

$$\sum_{j=1}^{n} b_{jk} \zeta^{-js} = \zeta^{-ks} + \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{-js} = \begin{cases} (n + 1 - 2s)\zeta^{-ks} & \text{if } s \in \{1, \dots, n-1\}, \\ 1 & \text{if } s = n. \end{cases}$$

Recall that n is odd. Let

$$\{\mu_1, \ldots, \mu_n\} = \{n-1, n-3, \ldots, 2, 1, 0, -2, \ldots, -(n-3)\}$$

with $\mu_n = 0$. By the above, the column vector

$$u^{(n)} = \frac{1}{\sqrt{n}}(\zeta^{-\frac{n+1}{2}}, \zeta^{-2\frac{n+1}{2}}, \ldots, \zeta^{-n\frac{n+1}{2}})^T$$

is an eigenvector of norm 1 associated with the eigenvalue μ_n of the matrix B.

Let B_n be the Hermitian matrix $[b_{jk}]_{1\leqslant k,j\leqslant n-1}$, and let $\mu_{n,1}, \ldots, \mu_{n,n-1}$ be all the eigenvalues of B_n. Note that $u^{(n)} = (\zeta^{-\frac{n+1}{2}}, \ldots, \zeta^{-\frac{n(n+1)}{2}})^T/\sqrt{n}$. Applying Theorem 3 with $j = n$, we obtain that

$$(-1)^{n-1}\det(B_n) = \prod_{k=1}^{n-1}(0-\mu_{n,k}) = \left|\frac{\zeta^{-\frac{n(n+1)}{2}}}{\sqrt{n}}\right|^2 \prod_{k=1}^{n-1}(0-\mu_k) = \frac{(-1)^{n-1}}{n}\prod_{\substack{k=1\\ k\neq\frac{n+1}{2}}}^{n-1}(n+1-2k)$$

and hence

$$\begin{aligned}\det(B_n) &= \frac{n-1}{n}\prod_{k=2}^{(n-1)/2}(n+1-2k)(n+1-2(n+1-k))\\ &= (-1)^{(n+1)/2}\frac{n-1}{n}\prod_{k=1}^{(n-1)/2}(n+1-2k)^2 = (-1)^{(n+1)/2}\frac{((n-1)!!)^2}{n(n-1)}.\end{aligned}$$

This proves (4) for $x = 0$.

By Lemma 10 we have

$$\det[x+b_{jk}]_{1\leqslant j,k\leqslant n-1} = \det(B_n^T) + x\det(B_n')$$

for certain $(n-2)\times(n-2)$ matrix B_n' over $\mathbb{C}$ not depending on x. As $1+b_{jk} = 2(c_{jk}+\delta_{jk})$ (with c_{jk} given by Lemma 11) for all $j, k = 1, \ldots, n-1$, we have

$$\begin{aligned}\det(B_n) + \det(B_n') &= \det[1+b_{jk}]_{1\leqslant j,k\leqslant n} = 2^{n-1}\det[c_{jk}+\delta_{jk}]_{1\leqslant j,k\leqslant n-1}\\ &= (n+1)(-1)^{(n+1)/2}\frac{((n-1)!!)^2}{n(n-1)} = (n+1)\det(B_n)\end{aligned}$$

with the aid of Lemma 11. Therefore

$$\begin{aligned}\det[x+b_{jk}]_{1\leqslant j,k\leqslant n-1} &= \det(B_n) + x(n\det(B_n)) = (1+nx)\det(B_n)\\ &= (-1)^{\frac{n+1}{2}}(1+nx)\frac{((n-1)!!)^2}{n(n-1)}\end{aligned}$$

as desired. This ends our proof of Theorem 4. $\square$

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