Proof of a Conjecture Involving Derangements and Roots of Unity

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Abstract

Let $n > 1$ be an odd integer, and let $\zeta$ be a primitive $n$th root of unity in the complex field. Via the Eigenvector-eigenvalue Identity, we show that

$$\sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 + \zeta^{j-\tau(j)}}{1 - \zeta^{j-\tau(j)}} = (-1)^{\frac{n-1}{2}} \frac{(n-2)!!}{n},$$

where $D(n-1)$ is the set of all derangements of $1, \ldots, n-1$. This confirms a previous conjecture of Z.-W. Sun. Moreover, for each $\delta = 0, 1$ we determine the value of $\det[x + m_{jk}]_{1\leq j,k \leq n-1}$ completely, where

$$m_{jk} = \begin{cases} 
(1 + \zeta^{j-k})/(1 - \zeta^{j-k}) & \text{if } j \neq k, \\
\delta & \text{if } j = k.
\end{cases}$$

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1 Introduction

For \( n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \), let \( S_n \) be the symmetric group of all permutations of \( \{1, \ldots, n\} \). A permutation \( \tau \in S_n \) is called a derangement of \( \{1, \ldots, n\} \) if \( \tau(j) \neq j \) for all \( j = 1, \ldots, n \). For convenience, we use \( D(n) \) to denote the set of all derangements of \( \{1, \ldots, n\} \). The derangement number \( D_n = |D(n)| \) plays important roles in enumerative combinatorics. It is well known that

\[
D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\]

(cf. (10.2) of [8, p. 90]).

Let \( n > 1 \) be an odd integer. Z.-W. Sun [5, Theorem 1.2] proved that

\[
\det \left[ \tan \frac{j - k}{n} \right]_{1 \leq j, k \leq n-1} = n^{-2}.
\]

As

\[
\tan \pi x = \frac{2 \sin \pi x}{2 \cos \pi x} = \frac{i}{1 + e^{2\pi i x}},
\]

we see that

\[
\det \left[ \tan \frac{j - k}{n} \right]_{1 \leq j, k \leq n-1} = i^{n-1} \det \left[ \frac{1 - \zeta^j}{1 + \zeta^j} \right]_{1 \leq j, k \leq n-1}
\]

\[
= (-1)^{(n-1)/2} \sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 - \zeta^{j - \tau(j)}}{1 + \zeta^{j - \tau(j)}},
\]

where \( \zeta = e^{2\pi i/n} \).

Z.-W. Sun ([6] and [7, Conj. 11.24]) conjectured that if \( n > 1 \) is odd and \( \zeta \) is a primitive \( n \)th root of unity in the complex field \( \mathbb{C} \) then

\[
\sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 + \zeta^{j - \tau(j)}}{1 - \zeta^{j - \tau(j)}} = (-1)^{n-1} \frac{((n-2)!!)^2}{n}.
\]

(1)

Our first goal is to prove an extension of this conjecture.

**Theorem 1.** Let \( n > 1 \) be an odd integer, and let \( \zeta \in \mathbb{C} \) be a primitive \( n \)th root of unity. For \( j, k = 1, \ldots, n \) define

\[
a_{jk} = \begin{cases} (1 + \zeta^{j-k})/(1 - \zeta^{j-k}) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}
\]

Then we have

\[
\det[x + a_{jk}]_{1 \leq j, k \leq n-1} = (-1)^{n-1} \frac{((n-2)!!)^2}{n}.
\]

(2)
Applying Theorem 1 with $x = 1$, we immediately obtain the following result.

**Corollary 2.** Let $n > 1$ be odd. Then, for any primitive $n$th root $\zeta \in \mathbb{C}$ of unity, we have
\[
\det[\tilde{a}_{jk}]_{1 \leq j, k \leq n-1} = (-1)^{\frac{n+1}{2}} \frac{(n-2)!!)^2}{n^{2n-1}},
\]
where
\[
\tilde{a}_{jk} = \begin{cases} 
1/(1 - \zeta^{j-k}) & \text{if } j \neq k, \\
1/2 & \text{if } j = k.
\end{cases}
\]

For any odd integer $n > 1$, Sun ([6] and [7, Conj. 11.22]) also conjectured that if $\zeta \in \mathbb{C}$ is a primitive $n$th root of unity then
\[
\sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1}{1 - \zeta^{j-\tau(j)}} = (n-1)^{n-1} \frac{(n-1)!^2}{n^{2n-1}}.
\]
(3)

Recently, X. Guo et al. [4] proved (3) via using the following result which dates back to Jacobi in 1834 (cf. P.B. Denton, S.J. Parke, T. Tao and X. Zhang [2, Theorem 1]).

**Theorem 3** (Eigenvector-eigenvalue Identity). Let $A$ be an $n \times n$ matrix over $\mathbb{C}$ which is Hermitian (i.e., the transpose $A^T$ of $A$ coincides with the conjugate of $A$), and let $\lambda_1, \ldots, \lambda_n$ be its $n$ real eigenvalues. Let $v_n = (v_{n,1}, \ldots, v_{n,n})^T$ be an eigenvector associated with the eigenvalue $\lambda_n$ of the matrix $A$ such that its norm $\|v_n\| = \sqrt{\sum_{j=1}^{n} |v_{n,j}|^2}$ equals 1. Let $j \in \{1, \ldots, n\}$ and let $A_j$ be the $(n-1) \times (n-1)$ Hermitian matrix formed by deleting the $j$th row and the $j$th column from $A$. Let $\lambda_{j,1}, \ldots, \lambda_{j,n-1}$ be all the real eigenvalues of $A_j$. Then we have
\[
|v_{n,j}|^2 \prod_{k=1}^{n-1} (\lambda_n - \lambda_k) = \prod_{k=1}^{n-1} (\lambda_n - \lambda_{j,k}).
\]

Motivated by Theorem 1, we also establish the following result.

**Theorem 4.** Let $n > 1$ be odd. Then, for any primitive $n$th root $\zeta \in \mathbb{C}$ of unity, we have
\[
\det[x + b_{jk}]_{1 \leq j, k \leq n-1} = (-1)^{\frac{n+1}{2}} (nx + 1) \frac{(n-1)!!)^2}{n(n-1)},
\]
(4)
where
\[
b_{jk} = \begin{cases} 
(1 + \zeta^{j-k})/(1 - \zeta^{j-k}) & \text{if } j \neq k, \\
1 & \text{if } j = k.
\end{cases}
\]

We are going to prove Theorems 1 and 4 in Sections 2 and 3, respectively.
2 Proof of Theorem 1

We need the following easy lemma.

Lemma 5. Let \( n \in \mathbb{Z}^+ \) and \( s \in \{0, \ldots, n-1\} \). For any primitive \( n \)th root \( \zeta \) of unity in a field \( F \), we have the identity

\[
\sum_{0 < r < n} \frac{\zeta^{-rs}}{1 - x\zeta^r} = \frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1}.
\]  

(5)

Proof. Clearly,

\[
\sum_{r=0}^{n-1} \frac{\zeta^{-rs}}{1 - x\zeta^r} = \sum_{r=0}^{n-1} \frac{\zeta^{-rs}}{1 - x^n} \sum_{k=0}^{n-1} (x\zeta^r)^k = \sum_{k=0}^{n-1} \frac{x^k}{1 - x^n} \sum_{r=0}^{n-1} \zeta^{r(k-s)} = \frac{nx^s}{1 - x^n}.
\]

Thus

\[
\sum_{r=1}^{n-1} \frac{\zeta^{-rs}}{1 - x\zeta^r} = \frac{nx^s}{1 - x^n} - \frac{1}{1 - x} = \frac{\sum_{j=0}^{n-1} x^j - nx^s}{x^n - 1}
\]

as desired. \( \square \)

Remark 6. Lemma 5 in the case \( F = \mathbb{C} \) is essentially equivalent to [3, Theorem 3.1].

Corollary 7. Let \( n \in \mathbb{Z}^+ \) and \( s \in \{0, \ldots, n-1\} \). Let \( \zeta \) be any primitive \( n \)th root of unity in the field \( \mathbb{C} \).

(i) If \( n \) is odd, then

\[
\sum_{0 < r < n} \frac{\zeta^{-rs}}{1 + \zeta^r} = \frac{(-1)^sn - 1}{2}.
\]

(6)

(ii) We have

\[
\sum_{0 < r < n} \frac{\zeta^{-rs}}{1 - \zeta^r} = \frac{n - 1}{2} - s.
\]

(7)

Proof. (i) When \( n \) is odd, putting \( x = -1 \) in (5) we immediately get (6).

(ii) Letting \( x \to 1 \) in (5) we obtain (7) since

\[
\lim_{x \to 1} \sum_{j=0}^{n-1} \frac{x^j - nx^s}{x^n - 1} = \lim_{x \to 1} \sum_{j=0}^{n-1} \frac{x^j - nx^s}{(x^n - 1)'} = \lim_{x \to 1} \frac{\sum_{0 < j < n} jx^{j-1} - nx^{s-1}}{nx^{n-1}}
\]

\[
= \sum_{j=0}^{n-1} \frac{j - ns}{n} = \frac{1}{n} \sum_{j=0}^{n-1} j - s = \frac{n - 1}{2} - s
\]

by L’Hospital’s rule.

Combining the above, we have completed the proof of Corollary 7. \( \square \)
**Remark 8.** It seems that the identity (7) should be known long time ago. We note that it essentially appeared as [3, (3.5)] though \((n-1)/2\) in [3, (3.5)] should be corrected as \((n+1)/2\).

Now we give an auxiliary proposition.

**Proposition 9.** Let \(n \in \mathbb{Z}^+, k \in \{1, \ldots, n\}\) and \(s \in \{0, \ldots, n-1\}\). For any primitive \(n\)th root \(\zeta\) of unity in a field \(F\), we have

\[
\sum_{j=1}^{n} \frac{1 + x\zeta^{j-k}}{1 - x\zeta^{j-k}} \zeta^{s(k-j)} = 1 + 2 \sum_{j=0}^{n-1} \frac{x^j - nx^s}{x^n - 1} - n\delta_{s0},
\]

where the Kronecker symbol \(\delta_{st}\) is 1 or 0 according as \(s = t\) or not. Consequently, if \(\zeta\) is a primitive \(n\)th root of unity in \(\mathbb{C}\), then

\[
\sum_{j=1}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{s(k-j)} = \begin{cases} n - 2s & \text{if } 0 < s < n, \\ 0 & \text{if } s = 0. \end{cases}
\]

**Proof.** In view of Lemma 5, we have

\[
\sum_{j=1}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{s(k-j)} = \sum_{r=1}^{n-1} \frac{1 + \zeta^r}{1 - \zeta^r} \zeta^{-sr} = 2 \sum_{r=1}^{n-1} \frac{\zeta^{-rs}}{1 - x\zeta^r} - \sum_{r=1}^{n-1} \zeta^{-rs}
\]

\[
= 2 \sum_{j=0}^{n-1} \frac{x^j - nx^s}{x^n - 1} + 1 - \sum_{r=0}^{n-1} \zeta^{-rs} = 2 \sum_{j=0}^{n-1} \frac{x^j - nx^s}{x^n - 1} + 1 - n\delta_{s0}.
\]

This proves (8).

When \(F = \mathbb{C}\), letting \(x \to 1\) in (8) or using the identity (7), we get (9). \(\square\)

We also need another lemma.

**Lemma 10 (Sun [5]).** For any matrix \(M = [m_{jk}]_{0 \leq j, k \leq n}\) over \(\mathbb{C}\), we have

\[
\det[x + m_{jk}]_{0 \leq j, k \leq n} = \det(M) + x \det(M'),
\]

where \(M' = [m'_{jk}]_{1 \leq j, k \leq n}\) with \(m'_{jk} = m_{jk} - m_{j0} - m_{0k} + m_{00}\).

**Proof of Theorem 1.** Obviously \(A = [a_{jk}]_{1 \leq j, k \leq n}\) is a Hermitian matrix. For each \(k = 1, \ldots, n\), by Proposition 9 we have

\[
\sum_{j=1}^{n} a_{jk} \zeta^{-js} = \sum_{j=1}^{n} \frac{1 + \zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{-js} = \begin{cases} (n - 2s)\zeta^{-ks} & \text{if } s \in \{1, \ldots, n-1\}, \\ 0 & \text{if } s = n. \end{cases}
\]

Thus \(\lambda_s = n - 2s\ (s = 1, \ldots, n-1)\) and \(\lambda_n = 0\) are all the eigenvalues of \(A\); moreover, for each \(s = 1, \ldots, n\), the column vector

\[
\nu^{(s)} = \frac{1}{\sqrt{n}}(\zeta^{-s}, \zeta^{-2s}, \ldots, \zeta^{-ns})^T
\]
is an eigenvector of norm 1 associated with the eigenvalue $\lambda_s$.

Let $A_n$ be the Hermitian matrix $[a_{jk}]_{1 \leq j, k \leq n-1}$, and let $\lambda_{n, 1}, \ldots, \lambda_{n, n-1}$ be all the eigenvalues of $A_n$. Note that $v^{(n)} = (1, \ldots, 1)^T / \sqrt{n}$. Applying Theorem 3 with $j = n$, we obtain that

$$(-1)^{n-1} \det(A_n) = \prod_{k=1}^{n-1} (0 - \lambda_{n,k}) = \frac{1}{\sqrt{n}} \prod_{k=1}^{n-1} (0 - \lambda_k) = \frac{(-1)^{n-1}}{n} \prod_{k=1}^{n-1} (n - 2k)$$

and hence

$$\det(A_n) = \frac{1}{n} \prod_{k=1}^{(n-1)/2} (n - 2k)(n - 2(n - k)) = \frac{(-1)^{(n-1)/2}}{n} \prod_{k=1}^{(n-1)/2} (n - 2k)^2 = \frac{(-1)^{(n-1)/2}}{n} ((n - 2)!!)^2.$$

On the other hand,

$$\det(A_n) = \det(A_n^T) = \det[a_{jk}]_{1 \leq j, k \leq n-1} = \sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 + \zeta^{j-\tau(j)}}{1 - \zeta^{j-\tau(j)}}.$$

Combining the last two equalities, we immediately get (2) for $x = 0$.

By Lemma 10, we have

$$\det[x + a_{jk}]_{1 \leq j, k \leq n-1} = \det(A_n^T) + x \det(A'_n),$$

where $A'_n = [a'_{jk}]_{2 \leq j, k \leq n-1}$ with

$$a'_{jk} = a_{jk} - a_{j1} - a_{1k} + a_{11} = a_{jk} - a_{j1} - a_{1k}.$$

It is easy to see that $a'_{kj} = -a'_{jk}$ for all $j, k = 2, \ldots, n - 1$. So we have

$$\det(A'_n) = \det(-A'_n) = (-1)^{n-2} \det(A'_n) = -\det(A'_n)$$

and hence

$$\det[x + a_{jk}]_{1 \leq j, k \leq n-1} = \det(A_n) + x \det(A'_n) = \det(A_n) = \frac{(-1)^{(n-1)/2}}{n} ((n - 2)!!)^2.$$

This ends our proof.

\section{Proof of Theorem 4}

\textbf{Lemma 11.} Let $n \in \{2, 3, 4, \ldots\}$, and let $\zeta$ be a primitive $n$th root of unity. For $j, k = 1, \ldots, n$, define

$$c_{jk} = \begin{cases} 1/(1 - \zeta^{j-k}) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

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The $n$ eigenvalues of $[c_{jk} + \delta_{jk}]_{1 \leq j,k \leq n}$ are $s - (n - 1)/2$ ($s = 1, \ldots, n$).

(ii) If $n$ is odd, then

$$\det[c_{jk} + \delta_{jk}]_{1 \leq j,k \leq n-1} = (-1)^{n+1} \frac{(n+1)((n-1)!!)^2}{n(n-1)2^{n-1}}.$$  \hspace{1cm} (10)

Proof. (i) For $j,k = 1, \ldots, n$, let

$$t_{jk} = \begin{cases} 1 + i \cot \frac{\pi j-k}{n} & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

By F. Calogero and A. M. Perelomov [1, Theorem 1], the $n$ numbers $2s - n - 1$ ($s = 1, \ldots, n$) are all the eigenvalues of the matrix $[t_{jk}]_{1 \leq j,k \leq n}$. Thus

$$\det[x \delta_{jk} - t_{jk}]_{1 \leq k \leq n} = \prod_{s=1}^{n} (x - (2s - n)).$$ \hspace{1cm} (11)

For $j,k = 1, \ldots, n$ with $j \neq k$, clearly

$$t_{jk} = 1 - \frac{2 \cos \frac{\pi j-k}{n}}{2i \sin \frac{\pi j-k}{n}} = 1 - \frac{e^{2\pi i \frac{j-k}{n}} + 1}{e^{2\pi i \frac{j-k}{n}} - 1} = \frac{2}{1 - e^{2\pi i \frac{j-k}{n}}}.$$

Note that $\zeta = e^{2\pi i a/n}$ for some $1 \leq a \leq n$ with $\gcd(a,n) = 1$. Applying the Galois automorphism $\sigma_a$ in the Galois group $\text{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ with $\sigma_a(e^{2\pi i/n}) = e^{2\pi i a/n}$, we obtain from (11) the polynomial identity

$$\det[x \delta_{jk} - 2c_{jk}]_{1 \leq k \leq n} = \prod_{s=1}^{n} (x - (2s - n)).$$ \hspace{1cm} (12)

Thus

$$\det[x \delta_{jk} - c_{jk}]_{1 \leq j,k \leq n} = \prod_{s=1}^{n} \left(x - s + \frac{n+1}{2}\right),$$

and hence

$$\det[x \delta_{jk} - c_{jk} - \delta_{jk}]_{1 \leq j,k \leq n} = \det[(x-1) \delta_{jk} - c_{jk}]_{1 \leq j,k \leq n}$$

$$= \prod_{s=1}^{n} \left(x - 1 - s + \frac{n+1}{2}\right) = \prod_{s=1}^{n} \left(x - \left(s - \frac{n-1}{2}\right)\right).$$

So the numbers $s - (n - 1)/2$ ($s = 1, \ldots, n$) are all the eigenvalues of $[c_{jk} + \delta_{jk}]_{1 \leq j,k \leq n}$.

(ii) Now assume that $n$ is odd. Let

$$\{\lambda_1, \ldots, \lambda_n\} = \left\{\frac{3-n}{2}, \frac{5-n}{2}, \ldots, \frac{n+1}{2}\right\}$$

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with \( \lambda_n = 0 \). Then the column vector

\[
v^{(n)} = \frac{1}{\sqrt{n}} (\zeta^{-\frac{n+1}{2}}, \zeta^{-\frac{n+3}{2}}, \ldots, \zeta^{-\frac{n(n-1)}{2}})^T
\]

is an eigenvector of norm 1 associated with the eigenvalue \( \lambda_n \) of \( C = [c_{jk} + \delta_{jk}]_{1 \leq j, k \leq n} \); in fact, for each \( j = 1, \ldots, n \), clearly

\[
\zeta^{j\frac{n+1}{2}} \sum_{k=1}^{n} (c_{jk} + \delta_{jk}) \zeta^{-k\frac{n+1}{2}} = \sum_{k=1, k \neq j}^{n} \frac{\zeta^{(j-k)\frac{n+1}{2}}}{1 - \zeta^{j-k}} + 1
\]

\[
= \sum_{r=1}^{n-1} \frac{\zeta^{-r\frac{n+1}{2}}}{1 - \zeta^{-r}} + 1 = \frac{n-1}{2} - \frac{n+1}{2} + 1 = 0
\]

by applying (7) with \( s = (n+1)/2 \).

Let \( C_n \) be the Hermitian matrix \([c_{jk} + \delta_{jk}]_{1 \leq j, k \leq n-1}\), and let \( \lambda_n, \ldots, \lambda_{n-1} \) be all the eigenvalues of \( C_n \). Note that \( v^{(n)} = (\zeta^{-\frac{n+1}{2}}, \ldots, \zeta^{-\frac{n(n-1)}{2}})^T / \sqrt{n} \). Applying Theorem 3 with \( j = n \), we obtain that

\[
(-1)^{n-1} \det(C_n) = \prod_{k=1}^{n-1} (0 - \lambda_{n,k}) = \left| \frac{\zeta^{-\frac{n(n-1)}{2}}}{\sqrt{n}} \right|^2 \prod_{k=1}^{n-1} (0 - \lambda_k) = \frac{(-1)^{n-1}}{n} \prod_{k=1}^{n} \left( k - \frac{n-1}{2} \right)
\]

and hence

\[
\det(C_n) = \frac{(n-1)(n+1)}{2^{n-1} n} \prod_{k=1}^{(n-3)/2} (n - 1 - 2k)(n - 1 - 2(n - 1 - k))
\]

\[
= (-1)^{n+1} \frac{(n-1)(n+1)}{2^{n-1} n} \prod_{k=1}^{(n-3)/2} (n - 1 - 2k)^2
\]

\[
= (-1)^{n+1} \frac{(n+1)((n-1)!!)^2}{2^{n-1} n(n-1)}.
\]

This concludes the proof.

\[ \square \]

**Proof of Theorem 4.** Let \( B \) be the \( n \times n \) matrix \([b_{jk}]_{1 \leq j, k \leq n}\). With the aid of (9),

\[
1 + \sum_{j=1}^{n} 1 + \frac{\zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{s(k-j)} = \begin{cases} n + 1 - 2s & \text{if } 0 < s < n, \\ 1 & \text{if } s = 0. \end{cases}
\]

(13)

Thus, for each \( k = 1, \ldots, n \), we have

\[
\sum_{j=1}^{n} b_{jk} \zeta^{-js} = \zeta^{-ks} + \sum_{j=1}^{n} 1 + \frac{\zeta^{j-k}}{1 - \zeta^{j-k}} \zeta^{-js} = \begin{cases} (n + 1 - 2s)\zeta^{-ks} & \text{if } s \in \{1, \ldots, n-1\}, \\ 1 & \text{if } s = n. \end{cases}
\]

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Recall that \( n \) is odd. Let 
\[
\{\mu_1, \ldots, \mu_n\} = \{n-1, n-3, \ldots, 2, 1, 0, -2, \ldots, -(n-3)\}
\]
with \( \mu_n = 0 \). By the above, the column vector 
\[
u^{(n)} = \frac{1}{\sqrt{n}}(\zeta^{-\frac{n+1}{2}}, \zeta^{-\frac{n+1}{2}}, \ldots, \zeta^{-\frac{n+1}{2}})^T
\]
is an eigenvector of norm 1 associated with the eigenvalue \( \mu_n \) of the matrix \( B \).

Let \( B_n \) be the Hermitian matrix \([b_{jk}]_{1 \leq j, k \leq n-1}\), and let \( \mu_{n,1}, \ldots, \mu_{n,n-1} \) be all the eigenvalues of \( B_n \). Note that \( u^{(n)} = (\zeta^{-\frac{n+1}{2}}, \ldots, \zeta^{-\frac{n+1}{2}})^T/\sqrt{n} \). Applying Theorem 3 with \( j = n \), we obtain that 
\[
(-1)^{n-1} \det(B_n) = \prod_{k=1}^{n-1} (0 - \mu_{n,k}) = \left| \frac{\zeta^{-\frac{n+1}{2}}}{\sqrt{n}} \right| \prod_{k=1}^{n-1} (0 - \mu_k) = \frac{(-1)^{n-1}}{n} \prod_{k=1}^{n-1} (n + 1 - 2k)
\]
and hence
\[
\det(B_n) = \frac{n-1}{n} \prod_{k=2}^{(n-1)/2} (n + 1 - 2k)(n + 1 - 2(n + 1 - k)) = (-1)^{(n+1)/2} \frac{n-1}{n} \prod_{k=1}^{(n-1)/2} (n + 1 - 2k)^2 = (-1)^{(n+1)/2} \left( \frac{(n-1)!!}{n(n-1)} \right)^2.
\]
This proves (4) for \( x = 0 \).

By Lemma 10 we have
\[
\det[x + b_{jk}]_{1 \leq j, k \leq n-1} = \det(B_n^T) + x \det(B_n')
\]
for certain \((n-2) \times (n-2)\) matrix \( B_n' \) over \( \mathbb{C} \) not depending on \( x \). As \( 1 + b_{jk} = 2(c_{jk} + \delta_{jk}) \) (with \( c_{jk} \) given by Lemma 11) for all \( j, k = 1, \ldots, n-1 \), we have
\[
\det(B_n) + \det(B_n') = \det[1 + b_{jk}]_{1 \leq j, k \leq n} = 2^{n-1} \det[c_{jk} + \delta_{jk}]_{1 \leq j, k \leq n-1} = (n+1)(-1)^{(n+1)/2} \left( \frac{(n-1)!!}{n(n-1)} \right)^2 = (n+1) \det(B_n)
\]
with the aid of Lemma 11. Therefore
\[
\det[x + b_{jk}]_{1 \leq j, k \leq n-1} = \det(B_n) + x(n \det(B_n)) = (1 + nx) \det(B_n)
\]
\[
= (-1)^{\frac{n+1}{2}} (1 + nx) \left( \frac{(n-1)!!}{n(n-1)} \right)^2
\]
as desired. This ends our proof of Theorem 4. \(\square\)
References


