Impartial Hypergraph Games

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Abstract

We study two building games and two removing games played on a finite hypergraph. In each game two players take turns selecting vertices of the hypergraph until the set of jointly selected vertices satisfies a condition related to the edges of the hypergraph. The winner is the last player able to move. The building achievement game ends as soon as the set of selected vertices contains an edge. In the building avoidance game the players are not allowed to select a set that contains an edge. The removing achievement game ends as soon as the complement of the set of selected vertices no longer contains an edge. In the removing avoidance game the players are not allowed to select a set whose complement does not contain an edge. We develop some generic tools for finding the nim-value of these games and show that the nim-value can be an arbitrary nonnegative integer. The outcome of many of these games were previously determined for several special cases in algebraic and combinatorial settings. We provide several examples and show how our tools can be used to refine these results by finding nim-values.

Mathematics Subject Classifications: 91A46, 05C65, 05C35, 06A15

1 Introduction

Avoidance and achievement games, sometimes called positional games, are combinatorial games that are extensively studied in both impartial [20, 21] and partizan [5] settings. The focus of this paper is a class of impartial games that provides a common framework for many of the special cases considered in the literature and listed in Section 8. This class of games is played on a finite hypergraph, so we call them impartial hypergraph games. Two players take turns selecting previously unselected vertices until the set of jointly selected vertices satisfies a condition related to the edges of the hypergraph. We consider four different games. Two of them are building games [1, 3, 13, 14, 36]. In these games, the players try to achieve or avoid having the set of selected vertices contain an edge. The other two games are removing games [23]. In these games, the players try to achieve or avoid having the complement of the set of selected vertices no longer contain an edge. In the removing games, the players essentially delete vertices until the remaining vertices no longer contain an edge. As is standard in normal play, the last player to move is the winner.

Every impartial game [2, 32] has an associated nim-value or Sprague-Grundy number. The nim-value contains a lot of information about the game. It encodes the outcome of the game and describes the game's behavior with respect to game sums. Because of this, games with the same nim-value are considered equivalent.

Many hypergraph games can be effectively analyzed using structure theory. This theory uses an equivalence relation called structure equivalence on the game positions. Structure equivalence is compatible with the option structure of a hypergraph game. This means that the quotient digraph of the game digraph contains enough information to determine the nim-value of the game. This quotient digraph is often much smaller than the original digraph. So it provides a practical algorithm for finding the nim-value. It also provides a visualization of the game that can create useful insights for proving results about families of games. Structure equivalence is used for group and convex geometry games in [7, 8, 9, 10, 11, 17, 28].

After some background information on impartial games, hypergraphs, and closure systems in Section 2, we develop the basic general theory of impartial hypergraph games in Section 3. Structure theory is developed in Sections 4–6 as a generalization of results in [17]. In Section 7, we find the nim-values of games played on hypergaphs with relatively simple edge structures. Section 8 connects our theory to the existing literature on avoidance and achievement games. In Section 9, we show that the nim-value of a hypergraph game can be any nonnegative integer. We finish with a few open questions in Section 10.

2 Preliminaries

The *parity* of a non-negative integer n is denoted by $\text{pty}(n) := n \bmod 2$. The *parity of a set* A is the parity of the size of the set, that is, $\text{pty}(A) := \text{pty}(|A|)$. For $f : X \to Y$ and $A \subseteq X$, we write $f(A) := \{f(a) \mid a \in A\}$ for the image of the subset A. The complement of a set A is denoted by A^{C}. We use the notation $\mathsf{C}_V(\mathcal{H}) = \{V \setminus A \mid A \in \mathcal{H}\}$ or simply $\mathsf{C}(\mathcal{H}) = \{A^{\mathsf{C}} \mid A \in \mathcal{H}\}$ for a family $\mathcal{H}$ of subsets of V.

2.1 Impartial games

We recall the basic terminology of impartial combinatorial games. See [2, 32] for further details. An *impartial game* G consists of a finite set $\mathcal{P}$ of *positions*, a starting position, and a function $\text{Opt} : \mathcal{P} \to 2^{\mathcal{P}}$ that provides the set of *options* $\text{Opt}(P)$ for all $P \in \mathcal{P}$. At the beginning of the game the starting position becomes the current position. Two players take turns picking an option of the current position to become the new current position. A position P is called *terminal* if it has no options, that is, $\text{Opt}(P) = \emptyset$. The game ends when the current position becomes a terminal position. The winner of the game is the last player to move. Every game must finish after finitely many turns. An impartial game is essentially an acyclic digraph called the *game digraph*. The vertices are the positions of the game and the arrows connect positions to their options. Game play consists of moving a token along the arrows starting at the initial position until a sink is reached.

The *minimum excludent* $\mathrm{mex}(A)$ of a set A of nonnegative integers is the smallest nonnegative integer missing from A. The nim-value $\mathrm{nim}(P) := \mathrm{mex}(\mathrm{nim}(\mathrm{Opt}(P)))$ of a position P is defined recursively as the minimum excludent of the nim-values of the options of P. The nim-value of a game is the nim-value of its starting position. A position P is *losing* (P-position) for the player about to move if $\mathrm{nim}(P) = 0$. A position P is *winning* (N-position) for the player about to move if $\mathrm{nim}(P) \neq 0$. The winning strategy is to move into a position with nim-value 0 if possible.

The sum of two impartial games G_1 and G_2 is an impartial game $\mathsf{G}_1 + \mathsf{G}_2$, where in each turn the players make a valid move in exactly one of the two games. The positions of $\mathsf{G}_1 + \mathsf{G}_2$ are of the form (P, Q), where P is a position of G_1 and Q is a position of G_2, and $\mathrm{Opt}(P, Q) = \{P\} \times \mathrm{Opt}(Q) \cup \mathrm{Opt}(P) \times \{Q\}$.

The *nimber* $*n$ is the game with options $\mathrm{Opt}(*n) = \{*0, \dots, *(n-1)\}$. Induction shows that $\mathrm{nim}(*n) = n$. The following is a well-known technique for finding nim-values.

Proposition 2.1. For all impartial games G, $\mathrm{nim}(\mathsf{G}) = n$ if and only if $\mathsf{G} + *n$ is won by the second player.

2.2 Hypergraphs

A *hypergraph* $H = (V, \mathcal{H})$ consist of a finite set V of vertices and a family $\mathcal{H} \subseteq 2^V$ of *edges*. Our general reference for hypergraphs is [12]. We do allow $\mathcal{H}$ to be empty or to contain the empty set. We say H is *simple* if $\mathcal{H}$ is a *Sperner family*. That is, no edge is contained in another edge.

A *transversal* of $\mathcal{H}$ is a subset T of V that intersects every edge. Transversals are also called *hitting* sets or *vertex covers*. The family $\mathrm{Tr}(\mathcal{H})$ of minimal transversals of $\mathcal{H}$ is a Sperner family. Finding $\mathrm{Tr}(\mathcal{H})$ is an important NP-complete problem [25, 29]. If $\mathcal{H}$ and $\mathcal{K}$ are Sperner families, then $\mathrm{Tr}(\mathcal{H}) = \mathcal{K}$ if and only if $\mathrm{Tr}(\mathcal{K}) = \mathcal{H}$. In particular, $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H})) = \mathcal{H}$ if $\mathcal{H}$ is a Sperner family. The transversal of the hypergraph H is the hypergraph $\mathrm{Tr}(H) = (V, \mathrm{Tr}(\mathcal{H}))$.

Example 2.2. If $\mathcal{H} = \{\{1\}, \{1, 2\}, \{3\}\}$ then $\mathrm{Tr}(\mathcal{H}) = \{\{1, 3\}\}$ and $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H})) = \{\{1\}, \{3\}\}$.

Example 2.3. If $\mathcal{H} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ then $\mathrm{Tr}(\mathcal{H}) = \mathcal{H}$.

Example 2.4. The empty set is a minimal transversal of the empty family, so $\mathrm{Tr}(\emptyset) = \{\emptyset\}$. No set intersects the empty set, so $\mathrm{Tr}(\{\emptyset\}) = \emptyset$.

Example 2.5. [12, Example 2] The *complete r-uniform hypergraph* K_n^r has vertex set $V = \{1, \dots, n\}$ and edge family $\mathcal{H} = \binom{V}{r}$. The minimal transversals of $\mathcal{H}$ are the sets with size $n - r + 1$. That is $\mathrm{Tr}(\mathcal{H}) = \binom{V}{n-r+1}$. Note that $\mathrm{Tr}(\mathcal{H}) = \mathcal{H}$ for $n = 2r - 1$.

Proposition 2.6. If $\mathcal{H}$ is a Sperner family, then $\bigcup \mathrm{Tr}(\mathcal{H}) = \bigcup \mathcal{H}$.

Proof. It is clear from the definition that $\bigcup \mathrm{Tr}(\mathcal{H}) \subseteq \bigcup \mathcal{H}$. Replacing $\mathcal{H}$ by $\mathrm{Tr}(\mathcal{H})$ gives $\bigcup \mathcal{H} \subseteq \bigcup \mathrm{Tr}(\mathcal{H})$. ☐

A set S of vertices of H is *stable* if S contains no edge of H. Stable sets are also called *independent* sets. We denote the family of maximal stable sets of H by $\mathcal{S}_H$. We also use the simpler notation $\mathcal{S}$ if the hypergraph H is clear from context. The following well-known fact plays an important role in our development.

Lemma 2.7. A subset P of V is stable in H if and only if $P^{\complement}$ is a transversal of $\mathcal{H}$.

This implies that P is maximal stable if and only if $P^{\complement}$ is a minimal transversal of $\mathcal{H}$. That is,

$$\mathcal{S}_H = \complement(\mathrm{Tr}(\mathcal{H})).$$

If H is simple, then we also have $\mathcal{H} = \mathrm{Tr}(\complement(\mathcal{S}_H))$, so the relationships can be summarized with the following diagram:

$$\mathcal{S}_{\mathrm{Tr}(H)} \xleftrightarrow{\complement} \mathcal{H} \xleftrightarrow{\mathrm{Tr}} \mathrm{Tr}(\mathcal{H}) \xleftrightarrow{\complement} \mathcal{S}_H$$

Note that a subset P of V is stable in H if and only if P is contained in a maximal stable set, that is, $P \subseteq S$ for some $S \in \mathcal{S}_H$

Lemma 2.8. Let H be a simple hypergraph. A subset P of V is stable in $\mathrm{Tr}(H)$ if and only if $P^{\complement}$ contains an edge of H.

Proof. A subset P is stable in $\mathrm{Tr}(H)$ if an only if P is contained in a maximal stable set of $\mathrm{Tr}(H)$, that is, $P \subseteq S$ for some $S \in \mathcal{S}_{\mathrm{Tr}(H)} = \complement(\mathcal{H})$. This happens exactly when $P^{\complement}$ contains an edge of H. ☐

Example 2.9. If $V = \{1, 2, 3\}$, $H = (V, \mathcal{H})$ with $\mathcal{H} = \{\{1, 2\}, \{3\}\}$, then $\mathcal{S}_H = \complement(\mathrm{Tr}(\mathcal{H})) = \complement(\{\{1, 3\}, \{2, 3\}\}) = \{\{2\}, \{1\}\}$.

Example 2.10. The family of maximal stable sets for the complete r-uniform hypergraph $H = K_n^r$ is $\mathcal{S}_H = \complement\binom{V}{n-r+1} = \binom{V}{r-1}$.

2.3 Closure systems

A *closure operator* on a set S is a function $\mathrm{cls} : 2^S \to 2^S$ that satisfies the following conditions for all subsets P, Q of S:

1. $P \subseteq \mathrm{cls}(P)$ (extensive);
2. $P \subseteq Q$ implies $\mathrm{cls}(P) \subseteq \mathrm{cls}(Q)$ (increasing);
3. $\mathrm{cls}(\mathrm{cls}(P)) = \mathrm{cls}(P)$ (idempotent).

A *pre-closure operator* on S is a function $\mathrm{cl} : 2^S \to 2^S$ that is extensive and increasing. A subset P of S is *dense* or *generating* under a pre-closure operator cl if $\mathrm{cl}(P) = S$.

A *closure system* on a set S is a nonempty collection $\mathcal{C}$ of subsets of S that is closed under intersections. The empty intersection is allowed, so $S = \bigcap \emptyset \in \mathcal{C}$. We say that the pair $(S, \mathcal{C})$ or simply S is a *closure space*. We call the elements of $\mathcal{C}$ *closed* sets. A comprehensive reference for closure systems is the survey article [16].

Closure operators and closure systems are two sides of a coin. Given a closure operator $\mathrm{cls} : 2^S \to 2^S$, the range $\mathcal{C} = \{\mathrm{cls}(P) \mid P \subseteq S\}$ of the closure operator forms a closure system on S. Given a closure system $\mathcal{C}$, the corresponding closure operator $\mathrm{cls} : 2^S \to 2^S$ is defined by $\mathrm{cls}(P) := \bigcap\{C \in \mathcal{C} \mid P \subseteq C\}$.

Example 2.11. Consider the closure system $\mathcal{C} = \{\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4,5\}\}$ on $S = \{1,2,3,4,5\}$. We have $\mathrm{cls}(\emptyset) = \mathrm{cls}(\{1\}) = \{1,2\}$. The singleton set $\{5\}$ is dense.

Example 2.12. Let S be a subset of $\mathbb{R}^n$. The collection $\mathcal{K}$ containing the intersections of S with convex subsets of $\mathbb{R}^n$ forms a closure system on S. This closure system is called the *affine convex geometry* on S. The corresponding closure operator is usually denoted by τ. For example, if $S = \{1,2,3\} \subseteq \mathbb{R}$ then

$$\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}.$$

Note that $\tau(\{1,3\}) = \{1,2,3\}$, so $\{1,3\}$ is a dense subset.

Example 2.13. Let G be a finite group. The collection $\mathcal{C}$ of subgroups of G is a closure system on G. The corresponding closure operator $P \mapsto \langle P \rangle : 2^G \to 2^G$ outputs the subgroup $\langle P \rangle$ generated by the subset P.

3 Impartial hypergraph games

Let $H = (V, \mathcal{H})$ be a hypergraph. In an *impartial hypergraph game* on H, two players alternately select previously unselected vertices in V until the game ends at a *terminal position*. The last player to make a move wins the game. In all games the set P of jointly selected elements is the current position of the game. Observe that an option of a position always has the opposite parity.

We consider two *building games*. The *achieve* game $\mathrm{ACV}(H)$ ends as soon as P contains one of the edges of H. In the *avoid* game $\mathrm{AVD}(H)$, the players are not allowed to select an element if the resulting P would contain one of the edges of H.

We also consider two *removing games*. The *destroy* game $\mathrm{DST}(H)$ ends as soon as $P^\complement$ is a stable set of H. In the *preserve* game $\mathrm{PRV}(H)$, the players are not allowed to select an element if the resulting $P^\complement$ would be a stable set of H.

The games $\mathrm{ACV}(H)$ and $\mathrm{DST}(H)$ are called *achievement games*, while $\mathrm{AVD}(H)$ and $\mathrm{PRV}(H)$ are called *avoidance games*. The terminology is summarized in the following table:

	achievement	avoidance
building	achieve ACV	avoid AVD
removing	destroy DST	preserve PRV

For a subset $\mathcal{A}$ of 2^V we define

$$\mathrm{Min}(\mathcal{A}) := \{P \in \mathcal{A} \mid (\forall Q \in \mathcal{A})\, Q \subseteq P \Rightarrow Q = P\}$$

be the set of minimal elements of $\mathcal{A}$ with respect to inclusion. We also define

$$\mathrm{Upp}(\mathcal{A}) := \{P \in 2^V \mid (\exists Q \in \mathcal{A})\, Q \subseteq P\}.$$

For a fixed hypergaph game type, the three games played on $(V, \mathrm{Min}(\mathcal{H}))$, $(V, \mathcal{H})$, and $(V, \mathrm{Upp}(\mathcal{H}))$ are the same. So we assume that a game is always played on a simple hypergraph H, so that $\mathcal{H} = \mathrm{Min}(\mathrm{Upp}(\mathcal{H}))$.

Remark 3.1. The positions of $\mathrm{AVD}(H)$ are the stable sets of H. The positions of $\mathrm{ACV}(H)$ are harder to describe. The empty set is always the starting position. A nonempty subset P of V is a position of $\mathrm{ACV}(H)$ if and only if $P \setminus \{v\}$ is stable for some v.

Example 3.2. Let $H = K_n^r$ be the complete r-uniform hypergraph. The terminal positions in all four games have the same size. So the outcome only depends on the parity of the size of the terminal positions. In fact, it is easy to verify using Proposition 2.1 that $\mathrm{nim}(\mathrm{ACV}(H)) = \mathrm{pty}(r)$ and $\mathrm{nim}(\mathrm{AVD}(H)) = \mathrm{pty}(r-1)$ while $\mathrm{nim}(\mathrm{DST}(H)) = \mathrm{pty}(n-r+1)$ and $\mathrm{nim}(\mathrm{PRV}(H)) = \mathrm{pty}(n-r)$. The winning strategy is simply random play.

Example 3.3. Let $H = (V, \mathcal{H})$ with $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$. The family of maximal stable sets is

$$\mathcal{S}_H = \mathsf{C}(\mathrm{Tr}(\mathcal{H})) = \mathsf{C}(\{\{b\}, \{a, c\}\}) = \{\{a, c\}, \{b\}\}.$$

Figure 3.1 shows the game digraphs of all four impartial hypergraph games. The second player has a winning strategy for $\mathrm{ACV}(H)$ since $\mathrm{nim}(\mathrm{ACV}(H)) = 0$. The first player has a winning strategy for the other three games since $\mathrm{nim}(\mathrm{AVD}(H)) = 2 = \mathrm{nim}(\mathrm{DST}(H))$ and $\mathrm{nim}(\mathrm{PRV}(H)) = 1$. In fact, the first player wins these three games after one move. The hypergraph games for $\mathrm{Tr}(\mathcal{H})$ are the same. We will see later that this is no accident.

Note that the digraph of an avoidance game is a subdigraph of the corresponding achievement game. The positions missing from the avoidance game are exactly the terminal positions of the corresponding achievement game.

Also note that the nested sets $\{a, b\}$ and $\{a, b, c\}$ are both terminal positions of the achievement game $\mathrm{ACV}(H) = \mathrm{DST}(\mathrm{Tr}(H))$. This can never happen for an avoidance game.

The option relationship for avoidance games is very simple.

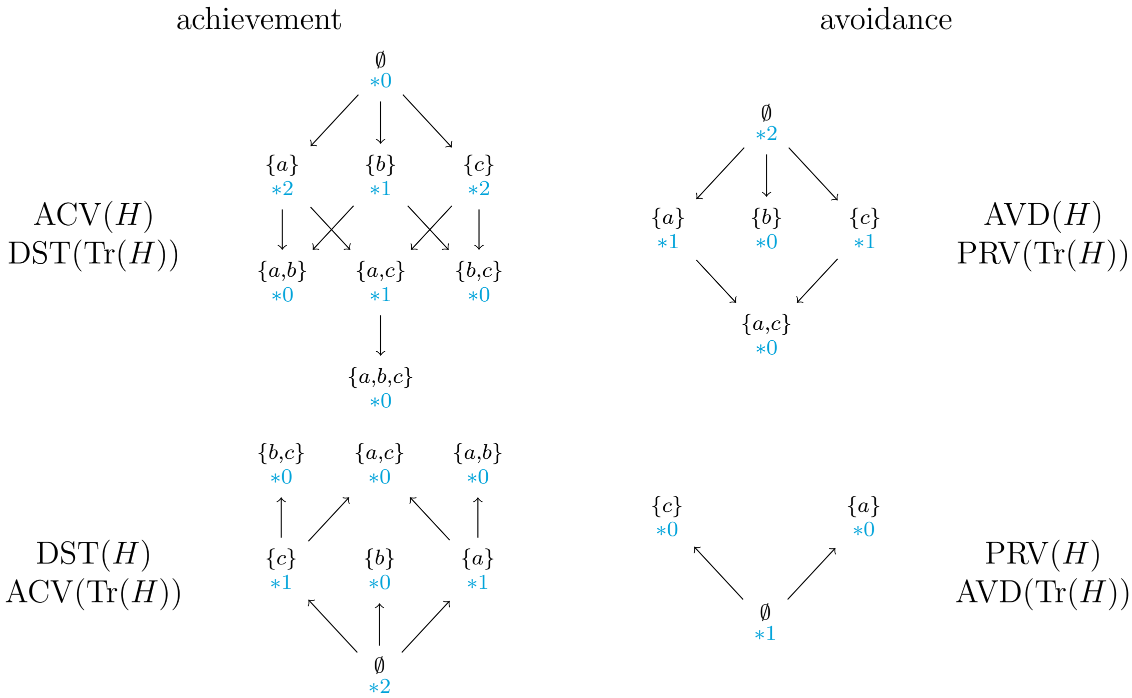


Figure 3.1: Game digraphs for H and $\mathrm{Tr}(H)$ with $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$. Note that $\mathrm{Tr}(\mathcal{H}) = \{\{b\}, \{a, c\}\}$.

Remark 3.4. Let P and Q be positions of an avoidance game. Then $Q \in \mathrm{Opt}(P)$ if and only if $Q = P \cup \{v\}$ for some $v \in Q \setminus P$.

The option relationship is slightly more complicated for the achievement games because terminal positions can be nested.

Remark 3.5. Let P and Q be positions of an achievement game. Then $Q \in \mathrm{Opt}(P)$ if and only if $Q = P \cup \{v\}$ for some $v \in Q \setminus P$ and P is a position of the corresponding avoidance game. If P is not a terminal position, then $P \cup \{v\} \in \mathrm{Opt}(P)$ for all $v \in P^{\complement}$.

Now we prove the equality of hypergraph games on H and $\mathrm{Tr}(H)$ suggested by Example 3.3.

Proposition 3.6. If H is a simple hypergraph, then $\mathrm{PRV}(H) = \mathrm{AVD}(\mathrm{Tr}(H))$ and $\mathrm{DST}(H) = \mathrm{ACV}(\mathrm{Tr}(H))$.

Proof. A subset P of V is a position of $\mathrm{PRV}(H)$ if and only if $P^{\complement}$ contains an edge of H. Lemma 2.8 implies that this happens exactly when P is stable in $\mathrm{Tr}(H)$. So the positions of $\mathrm{PRV}(H)$ and $\mathrm{AVD}(\mathrm{Tr}(H))$ are the same. Positions P and Q of any avoidance game satisfy $Q \in \mathrm{Opt}(P)$ if and only $Q = P \cup \{v\}$ for some $v \in Q \setminus P$. So the two games also have the same option relationships.

A nonempty subset P of V is a position of $\mathrm{DST}(H)$ if and only if $(P \setminus \{v\})^{\complement}$ contains an edge of H for some v. Lemma 2.8 implies that this happens exactly when $P \setminus \{v\}$ is stable in $\mathrm{Tr}(H)$. So the positions of $\mathrm{DST}(H)$ and $\mathrm{ACV}(\mathrm{Tr}(H))$ are the same by Remark 3.1.

$$
\begin{tikzcd}
 & \emptyset \arrow[dl] \arrow[d] \arrow[dr] & \\
\{1\} \arrow[d] & \{2\} & \{3\} \\
\{1,2\} & &
\end{tikzcd}
$$

Figure 3.2: An impartial game that is not a hypergraph game.

The options relations of the two games are also the same by Remark 3.5 since we already saw that the corresponding avoidance games are the same. $\square$

Remark 3.7. The underlying graph of the game digraph of an impartial hypergraph game is a subgraph of the graph constructed from the Hasse diagram of the Boolean lattice of subsets of V. The initial position is always the empty set.

Example 3.8. Not every seemingly reasonable subgraph of the Hasse diagram of the Boolean lattice is a hypergraph game. Figure 3.2 shows such a digraph. It is the digraph of an impartial game but not of a hypergraph game on the vertex set $\{1, 2, 3\}$. Since $\{1, 2\} \notin \mathrm{Opt}(\{2\})$, it is not an avoidance game. To see that it is not an achievement game, note that $\{1\}$ is not a terminal position. So $\{1, 3\}$ should be an option of $\{1\}$ by Remark 3.5 but $\{1, 3\}$ is not a game position.

Example 3.9. Every game must have a starting position. If $H = \{V, \emptyset\}$ then $\mathrm{ACV}(H)$ and $\mathrm{PRV}(H)$ are not defined since V is a stable set. If $H = \{V, \{\emptyset\}\}$ then $\mathrm{DST}(H)$ and $\mathrm{AVD}(H)$ are not defined since $\emptyset$ is not a stable set.

The next result shows that the game digraphs of $\mathrm{ACV}(H)$ and $\mathrm{PRV}(H)$ are reversed complementary.

Proposition 3.10. Let P and $Q = P \cup \{v\}$ be subsets of V and $H = (V, \mathcal{H})$. Then $Q \in \mathrm{Opt}(P)$ in $\mathrm{ACV}(H)$ if and only if $P^{\complement} \notin \mathrm{Opt}(Q^{\complement})$ in $\mathrm{PRV}(H)$.

Proof. First assume $Q \in \mathrm{Opt}(P)$ in $\mathrm{ACV}(H)$. Then P is not a terminal position of $\mathrm{ACV}(H)$, and so P is a stable set of H. So $P^{\complement}$ is not a game position of $\mathrm{PRV}(H)$. This means $P^{\complement}$ cannot be an option of $Q^{\complement}$ in $\mathrm{PRV}(H)$.

Now assume $P^{\complement} \notin \mathrm{Opt}(Q^{\complement})$ in $\mathrm{PRV}(H)$. Then P is a stable set of H. Hence P is a position of $\mathrm{ACV}(H)$ and P is not a terminal position. Thus $Q \in \mathrm{Opt}(P)$ in $\mathrm{ACV}(H)$ by Remark 3.5. $\square$

The following is an easy consequence.

Corollary 3.11. The game digraphs of $\mathrm{DST}(H)$ and $\mathrm{AVD}(H)$ are reversed complementary.

Proof. Replacing H with $\mathrm{Tr}(H)$ in Proposition 3.10 shows that $\mathrm{DST}(H) = \mathrm{ACV}(\mathrm{Tr}(H))$ and $\mathrm{AVD}(H) = \mathrm{PRV}(\mathrm{Tr}(H))$ are complementary. $\square$

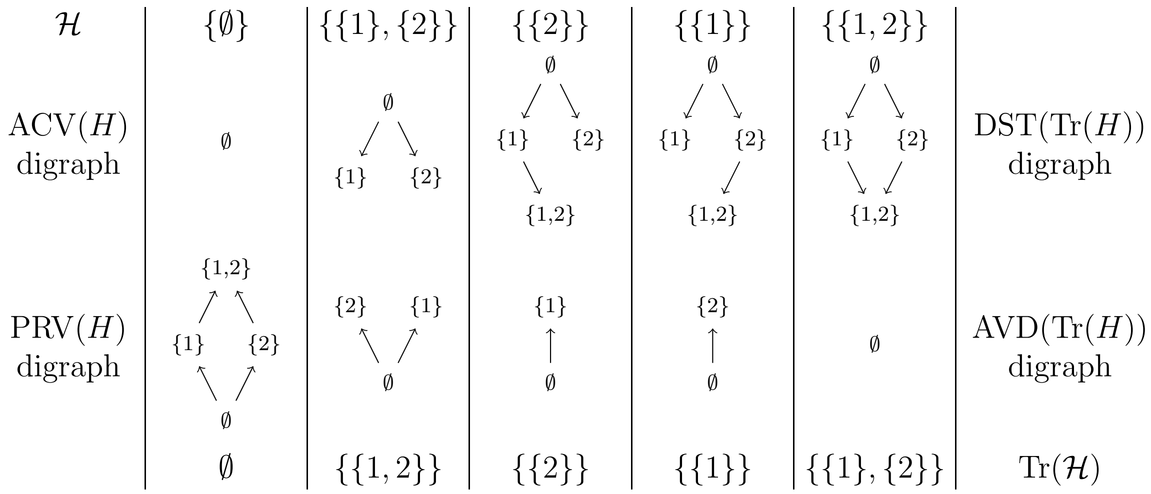


Figure 3.3: All hypergraph games with vertex set $V = \{1, 2\}$.

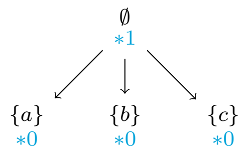


Figure 3.4: Game digraph for ACV(H) and AVD(K).

Figure 3.1 demonstrates the complementary nature of the game digraphs of the hypergraph games. We provide another example.

Example 3.12. Figure 3.3 shows all the hypergraph games with vertex set $V = \{1, 2\}$. Each columns shows a pair of complementary digraphs.

The next example shows that a game can be both an achivement and an avoidance game.

Example 3.13. Let $H = (V, \mathcal{H})$ with $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a\}, \{b\}, \{c\}\}$, and $K = (V, \mathcal{K})$ with $\mathcal{K} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. The games ACV($H$) and AVD($K$) are the same since they have the same game digraph, as shown in Figure 3.4. The first player has a winning strategy since the nim-value is 1.

Proposition 3.14. Let $H = (V, \mathcal{H})$ and $K = (V, \mathcal{K})$ be simple hypergraphs. The games ACV(H) and AVD(K) are the same if and only if $\mathcal{H} = \binom{V}{k}$ and $\mathcal{K} = \binom{V}{k+1}$ for some $k \in \{0, |V|\}$.

Proof. The backward direction clearly holds. Assume ACV(H) = AVD(K) and consider a nonempty position P of this game. Since P is stable in K, $P \setminus \{v\}$ is also stable in K and hence a game position for all $v \in P$. Since $P \setminus \{v\}$ is not a terminal position of

$\mathrm{ACV}(H)$, $P \setminus \{v\} \cup \{w\}$ is also a game position for all $w \in P^{\complement}$. This shows that if Q is a subset of V satisfying $|Q| \leqslant |P|$, then Q is a game position. Let k be the size of the largest game position. Then every subset of V with size at most k is a game position and the terminal positions are the elements of $\binom{V}{k}$. Thus $\mathcal{H} = \binom{V}{k}$ and $\mathcal{K} = \binom{V}{k+1}$. $\square$

The following result often provides a simple way to find the nim-value of an avoidance game.

Proposition 3.15. If every set in $\mathcal{S}_H$ has the same parity r, then $\mathrm{nim}(\mathrm{AVD}(H)) = r$.

Proof. The sets in $\mathcal{S}_H$ are the terminal positions of the game. The second player wins $\mathrm{AVD}(H) + *r$ using random play. $\square$

A version of this result is true for achievement games but it is not very useful since finding the terminal positions of an achievement game is often difficult.

4 Structure theory for the building games AVD and ACV

In this section we develop structure theory for hypergraph games. This is our main tool to find the nim-value of a hypergraph game. The idea of structure equivalence originates in [17]. Structure theory is successfully used in [7, 8, 9, 10, 11, 28] to analyze group and convex geometry generating games.

4.1 Structure equivalence

Consider a simple hypergraph $H = (V, \mathcal{H})$. For a subset P of V define $\phi_H(P) := \{S \in \mathcal{S}_H \mid P \subseteq S\}$. We use this to define an equivalence relation on 2^V.

Definition 4.1. Two subsets P and Q of V are *structure equivalent* in H if $\phi_H(P) = \phi_H(Q)$. In this case we write $P \sim Q$.

Let

$$\mathcal{I}_H = \{\bigcap \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}_H\}$$

be the closure system generated by the family $\mathcal{S}_H$ of stable sets. The closure of a subset P of V in this closure system is denoted by $\lceil P \rceil := \bigcap \phi_H(P)$. Note that $\lceil I \rceil = I$ for all $I \in \mathcal{I}_H$. In this closure system, $\mathcal{H}$ is the family of minimal generating sets and $\mathcal{S}_H$ is the family of maximal non-generating sets.

The smallest set $\Phi_H := \bigcap \mathcal{S}_H$ in $\mathcal{I}_H$ is called the *Frattini subset*. The Frattini subset is structure equivalent to the empty set. In fact, $\lceil \emptyset \rceil = \Phi_H$.

The following is a generalization of [28, Proposition 4.9]. It shows that structure equivalence is the *cospanning relation* of the closure operator $P \mapsto \lceil P \rceil$ as defined in [27, 37].

Proposition 4.2. Two subsets P and Q of V are structure equivalent if and only if $\lceil P \rceil = \lceil Q \rceil$.

Proof. The forward direction is clear from the definitions. For a contradiction suppose that $\lceil P \rceil = \lceil Q \rceil$ but there is an $S \in \mathcal{S}_H$ such that $P \subseteq S$ and $Q \not\subseteq S$. Then there is a $v \in Q \setminus S$. This is impossible since $v \in Q \subseteq \lceil Q \rceil = \lceil P \rceil = \bigcap \phi_H(P) \subseteq S$. □

The following result is easy to verify from the definitions but it also follows [37, Theorem 2.15, Proposition 3.4] from the fact that structure equivalence is the cospanning relation of a closure operator.

Proposition 4.3. Structure equivalence in H satisfies the following properties

1. $P \sim Q$ implies $P \sim P \cup Q$;
2. $P \subseteq Q \subseteq R$ and $P \sim R$ imply $P \sim Q$;
3. $P \in \mathcal{I}_H$ and $P \setminus \{v\} \notin \mathcal{I}_H$ imply $P \setminus \{v\} \sim P$;

for all $P, Q, R \subseteq V$ and $v \in P$.

Consider a building hypergraph game on $H = (V, \mathcal{H})$. We restrict structure equivalence to the set of game positions. The *structure class* X_I for $I \in \mathcal{I}_H$ consists of the game positions that are structure equivalent to I. Note that I is the largest element of X_I for all $I \in \mathcal{I}_H \setminus \{V\}$. If P is a game position, then $P \sim \lceil P \rceil$ and $P \in X_{\lceil P \rceil}$.

For $\mathrm{ACV}(H)$, the structure class X_V contains all the terminal game positions. Note that V might not be a game position in which case $V \notin X_V$. The mapping $I \mapsto X_I$ is a bijection from $\mathcal{I}_H$ to the set of structure classes.

For $\mathrm{AVD}(H)$, no game position is structure equivalent to V so $X_V = \emptyset$. The mapping $I \mapsto X_I$ is a bijection from $\mathcal{I}_H \setminus \{V\}$ to the set of nonempty structure classes.

A structure class is called *terminal* if it contains a terminal position. The only terminal structure class is X_V for $\mathrm{ACV}(H)$. The terminal structure classes are X_I with $I \in \mathcal{S}_H$ for $\mathrm{AVD}(H)$.

Example 4.4. Let $H = (V, \mathcal{H})$ with $V = \{a, b, c\}$, $\mathcal{H} = \{\{a, b\}\}$ so that $\mathcal{S}_H = \mathsf{C}(\mathrm{Tr}(\mathcal{H})) = \mathsf{C}(\{\{a\}, \{b\}\}) = \{\{a, c\}, \{b, c\}\}$, $\mathcal{I}_H = \{\{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, and $\Phi_H = \{c\}$. The structure classes for $\mathrm{AVD}(H)$ are $X_{\{c\}} = \{\emptyset, \{c\}\}$, $X_{\{a,c\}} = \{\{a\}, \{a, c\}\}$, and $X_{\{b,c\}} = \{\{b\}, \{b, c\}\}$. There is one additional structure class $X_{\{a,b,c\}} = \{\{a, b\}, \{a, b, c\}\}$ for $\mathrm{ACV}(H)$.

4.2 Compatibility of game options with structure equivalence

Our goal is to show that structure equivalence is compatible with the option structure of the building hypergraph games. The following is a generalization of [17, Corollary 3.11].

Proposition 4.5. Consider a building hypergraph game G on $H = (V, \mathcal{H})$. Let P and Q be game positions such that $P, Q \in X_I \neq X_J$. If $\mathrm{Opt}(P) \cap X_J \neq \emptyset$, then $\mathrm{Opt}(Q) \cap X_J \neq \emptyset$.

Proof. Assume $\mathrm{Opt}(P) \cap X_J \neq \emptyset$. Then there is a $v \in V \setminus P$ such that $P \cup \{v\} \in$

$\mathrm{Opt}(P) \cap X_J$. Then $Q \cup \{v\} \in X_J$ since

$$\begin{aligned}\phi_H(Q \cup \{v\}) &= \{S \in \mathcal{S}_H \mid Q \cup \{v\} \subseteq S\} \\ &= \{S \in \phi_H(Q) \mid v \in S\} \\ &= \{S \in \phi_H(P) \mid v \in S\} \\ &= \{S \in \mathcal{S}_H \mid Q \cup \{v\} \subseteq S\} \\ &= \phi_H(P \cup \{v\}).\end{aligned}$$

It remains to show that $Q \cup \{v\}$ is a game position and an option of Q. First consider the case when G is an achievement game. Since P is not a terminal position, P must be stable. So P is contained in a maximal stable set S. Position Q is also contained in S, and so Q is also stable. Hence $Q \cup \{v\}$ must be an option of Q by Remark 3.5.

Now consider the case when G is an avoidance game. Since $P \cup \{r\}$ is stable, the structure equivalent $Q \cup \{r\}$ is also stable. Hence $Q \cup \{r\}$ is an option of Q by Remark 3.4. □

Definition 4.6. We say X_J is an *option* of X_I if $X_J \cap \mathrm{Opt}(I) \neq \emptyset$. The set of options of X_I is denoted by $\mathrm{Opt}(X_I)$.

Proposition 4.5 implies that if $X_J \in \mathrm{Opt}(X_I)$ then $X_J \cap \mathrm{Opt}(P) \neq \emptyset$ for all $P \in X_I$. The following is a generalization of [17, Lemma 3.14].

Lemma 4.7. If A and B_k are sets containing non-negative integers such that $\mathrm{mex}(A) \in B_k$ for all $k \in K$, then $\mathrm{mex}(A \cup \{\mathrm{mex}(B_k) \mid k \in K\}) = \mathrm{mex}(A)$.

Proof. Since $\mathrm{mex}(A) \in B_k$, $\mathrm{mex}(B_k) \neq \mathrm{mex}(A)$ for all $k \in K$. □

The following is a generalization of [17, Proposition 3.15].

Proposition 4.8. Let $\mathcal{P}$ be the set of positions of a building hypergraph game G on $H = (V, \mathcal{H})$. If $P, Q \in \mathcal{P}$ such that $P \sim Q$ and $\mathrm{pty}(P) = \mathrm{pty}(Q)$, then $\mathrm{nim}(P) = \mathrm{nim}(Q)$.

Proof. Let

$$Z := \{(P, Q) \in \mathcal{P} \times \mathcal{P} \mid P \sim Q \text{ and } \mathrm{pty}(P) = \mathrm{pty}(Q)\}.$$

We say $(P, Q) \succeq (M, N)$ exactly when $P \subseteq M$ and $Q \subseteq N$. Then $(Z, \succeq)$ is a finite partially ordered set. We proceed by structural induction on Z. Consider a minimal element (P, Q) of Z. If $\mathsf{G} = \mathrm{AVD}(H)$, then we must have $P = Q$. If $\mathsf{G} = \mathrm{ACV}(H)$ then P and Q might be different. For both games P and Q are terminal positions and so $\mathrm{nim}(P) = 0 = \mathrm{nim}(Q)$. So the claim holds for these minimal elements.

Now let (P, Q) be an element of Z that is not minimal and let $I := \lceil P \rceil = \lceil Q \rceil$. Then $I \neq V$. We consider several cases. The claim clearly holds if $P = I = Q$.

Next, assume $P \neq I \neq Q$. Then both P and Q have options in X_I. In fact, $P \cup \{u\} \in \mathrm{Opt}(P) \cap X_I$ for each $u \in I \setminus P$ and $Q \cup \{v\} \in \mathrm{Opt}(Q) \cap X_I$ for each $v \in I \setminus Q$. If M and N are options of P and Q in X_I respectively, then $\mathrm{pty}(M) = \mathrm{pty}(N)$. Hence $\mathrm{nim}(M) = \mathrm{nim}(N)$ by induction since $(P, Q) \succ (M, N)$. If P has an option M in some $X_J \neq X_I$, then Q also has an option N in X_J by Proposition 4.5. Since $M, N \in X_J$ and

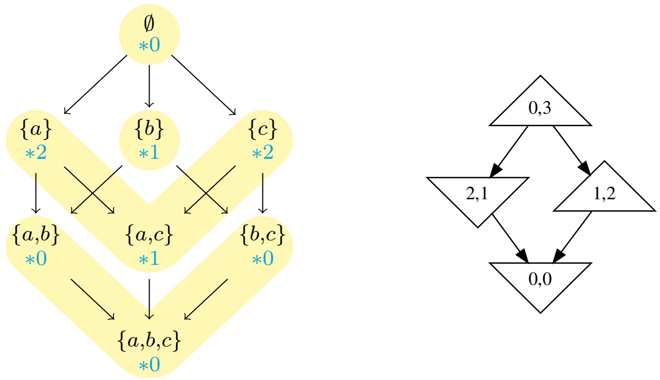


Figure 4.1: Game digraph indicating the structure classes and structure diagram for ACV(H) with $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$.

$\text{pty}(M) = \text{pty}(N)$, we have $(P, Q) \succ (M, N)$. Hence $\text{nim}(M) = \text{nim}(N)$ by induction. This proves that $\text{nim}(\text{Opt}(P)) = \text{nim}(\text{Opt}(Q))$, and so $\text{nim}(P) = \text{nim}(Q)$.

Finally, assume $P \neq I = Q$. In this case $\text{Opt}(Q) \cap X_I = \emptyset$ but $\text{Opt}(P) \cap X_I \neq \emptyset$. For each $M \in \text{Opt}(P) \cap X_I$ there is an $R_M \in \text{Opt}(M) \cap X_I$ since $\text{pty}(M) \neq \text{pty}(I)$. We have $\text{nim}(R_M) = \text{nim}(Q)$ by induction since $(P, Q) \succ (R_M, Q)$. With $A := \text{nim}(\text{Opt}(Q))$ and $B_M := \text{nim}(\text{Opt}(M)$, we have

$$\text{mex}(A) = \text{nim}(Q) = \text{nim}(N) \in B_M.$$

Just like in the previous case, Proposition 4.5 implies that the options of P and Q outside of X_I have the same set of nim-values. Lemma 4.7 now implies that

$$\begin{aligned}\text{nim}(P) &= \text{mex}(\text{nim}(\text{Opt}(P))) \\ &= \text{mex}(\text{nim}(\text{Opt}(Q)) \cup \text{nim}(\text{Opt}(P) \cap X_I)) \\ &= \text{mex}(A \cup \{\text{mex}(B_M) \mid M \in \text{Opt}(P) \cap X_I\}) \\ &= \text{mex}(A) = \text{mex}(\text{nim}(\text{Opt}(Q)) \\ &= \text{nim}(Q).\end{aligned}$$

□

Example 4.9. Let $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$ as in Example 3.3, so that $\mathcal{S}_H = \{\{a, c\}, \{b\}\}$. Figure 4.1 shows the structure classes of ACV(H). Note that $\{a\}$ and $\{c\}$ are structure equivalent and have the same parity. So they are guaranteed to have the same nim-value of 2.

4.3 Type calculus

Consider a building game on the simple hypergraph $H = (V, \mathcal{H})$.

Definition 4.10. The *type* of the structure class X_I is

$$\text{type}(X_I) := (\text{pty}(I), \text{nim}_0(X_I), \text{nim}_1(X_I)).$$

If $I = V$ then $\mathrm{nim}_0(X_V) := 0$ and $\mathrm{nim}_1(X_V) := 0$. If $I \neq V$ then

$$\begin{aligned}\mathrm{nim}_{\mathrm{pty}(I)}(X_I) &:= \mathrm{mex}(\mathrm{nim}_{1-\mathrm{pty}(I)}(\mathrm{Opt}(X_I)),\\ \mathrm{nim}_{1-\mathrm{pty}(I)}(X_I) &:= \mathrm{mex}(\mathrm{nim}_{\mathrm{pty}(I)}(\mathrm{Opt}(X_I)) \cup \{\mathrm{nim}_{\mathrm{pty}(I)}(X_I)\}).\end{aligned}$$

We call the recursive computation of types using the options of structure classes *type calculus*.

Note that X_V is special since it only contains the terminal positions of the achieve game.

Example 4.11. Assume that $\mathrm{Opt}(X_I) = \{X_J, X_K\}$ satisfying $\mathrm{type}(X_J) = (0, 0, 1)$ and $\mathrm{type}(X_K) = (1, 1, 0)$. If $\mathrm{pty}(I) = 0$ then

$$\begin{aligned}\mathrm{nim}_0(X_I) &= \mathrm{mex}(\{\mathrm{nim}_1(X_J), \mathrm{nim}_1(X_K)\}) = \mathrm{mex}(\{1, 0\}) = 2,\\ \mathrm{nim}_1(X_I) &= \mathrm{mex}(\{\mathrm{nim}_0(X_J), \mathrm{nim}_0(X_K), \mathrm{nim}_0(X_I)\}) = \mathrm{mex}(\{0, 1, 2\}) = 3\end{aligned}$$

and $\mathrm{type}(X_I) = (0, 2, 3)$. If $\mathrm{pty}(I) = 1$ then

$$\begin{aligned}\mathrm{nim}_1(X_I) &= \mathrm{mex}(\{\mathrm{nim}_0(X_J), \mathrm{nim}_0(X_K)\}) = \mathrm{mex}(\{0, 1\}) = 2,\\ \mathrm{nim}_0(X_I) &= \mathrm{mex}(\{\mathrm{nim}_1(X_J), \mathrm{nim}_1(X_K), \mathrm{nim}_1(X_I)\}) = \mathrm{mex}(\{1, 0, 2\}) = 3\end{aligned}$$

and $\mathrm{type}(X_I) = (1, 3, 2)$.

The type of a structure class X_I encodes the parity of I and the nim numbers of the positions in X_I.

Proposition 4.12. If $P \in X_I$ then $\mathrm{nim}(P) = \mathrm{nim}_{\mathrm{pty}(P)}(X_I)$.

Proof. We use structural induction on the positions, together with Propositions 4.5 and 4.8. The statement is clearly true for $I = V$. Now assume that $I \neq V$.

Any option Q of position I is in X_J for some $X_J \in \mathrm{Opt}(X_I)$. On the other hand, if $X_J \in \mathrm{Opt}(X_I)$ then X_J contains an option Q of I. Hence

$$\mathrm{nim}(I) = \mathrm{mex}(\mathrm{nim}_{1-\mathrm{pty}(I)}(\mathrm{Opt}(I))) = \mathrm{nim}_{\mathrm{pty}(I)}(X_I)$$

by induction.

First assume that P is a position in X_I such that $\mathrm{pty}(P) = \mathrm{pty}(I)$. Then $\mathrm{nim}(P) = \mathrm{nim}(I)$ by Proposition 4.8. Thus $\mathrm{nim}(P) = \mathrm{nim}(I) = \mathrm{nim}_{\mathrm{pty}(I)}(X_I) = \mathrm{nim}_{\mathrm{pty}(P)}(X_I)$.

Now assume that P is a position in X_I such that $\mathrm{pty}(P) = 1-\mathrm{pty}(I)$. Since P is strictly smaller than I and $I \neq V$, P must have an option in X_I. Every Q in $\mathrm{Opt}(P) \cap X_I$ satisfies $\mathrm{nim}(Q) = \mathrm{nim}_{\mathrm{pty}(Q)}(X_I) = \mathrm{nim}_{\mathrm{pty}(I)}(X_I)$ by induction. Proposition 4.5 implies that every option of P that is not in X_I must be a position in X_J for some $X_J \in \mathrm{Opt}(X_I)$. On the other hand, Proposition 4.5 also implies that if $X_J \in \mathrm{Opt}(X_I)$ then X_J contains an option of P. Every Q in $\mathrm{Opt}(P) \cap X_J$ with $X_J \in \mathrm{Opt}(X_I)$ satisfies $\mathrm{nim}(Q) = \mathrm{nim}_{\mathrm{pty}(Q)}(X_J) = \mathrm{nim}_{\mathrm{pty}(I)}(X_J)$ by induction. Thus

$$\begin{aligned}\mathrm{nim}(P) &= \mathrm{mex}(\mathrm{nim}(\mathrm{Opt}(P)))\\ &= \mathrm{mex}(\mathrm{nim}((\mathrm{Opt}(P) \cap \bigcup \mathrm{Opt}(X_I)) \cup (\mathrm{Opt}(P) \cap X_I))\\ &= \mathrm{mex}(\mathrm{nim}_{\mathrm{pty}(I)}(\mathrm{Opt}(X_I)) \cup \{\mathrm{nim}_{\mathrm{pty}(I)}(X_I)\})\\ &= \mathrm{nim}_{1-\mathrm{pty}(I)}(X_I) = \mathrm{nim}_{\mathrm{pty}(P)}(X_I).\end{aligned}$$

□

Note that $X_I = \{I\}$ is possible. In this case X_I contains no position with parity $1 - \text{pty}(I)$. Also note that the nim number of the game is the nim number $\text{nim}(\emptyset) = \text{mex}_0(X_{\Phi_H})$ of the starting position $\emptyset$.

4.4 Structure diagram

The *structure digraph* of a building game on the simple hypergraph H has vertex set $\{X_I \mid I \in \mathcal{I}_H\}$ and arrow set $\{(X_I, X_J) \mid X_J \in \text{Opt}(X_I)\}$. We visualize the structure digraph with a *structure diagram* that also shows the type of each structure class. Within a structure diagram, a vertex X_I is represented by a triangle pointing up or down depending on the parity of I. The triangle points down when $\text{pty}(I) = 1$ and points up when $\text{pty}(I) = 0$. The numbers within each triangle represent the nim numbers of the positions within the structure class. The first number is the common nim number $\text{nim}_0(X_I)$ of all even positions in X_I, while the second number is the common nim number $\text{nim}_1(X_I)$ of all odd positions in X_I.

Algorithm 4.13. The structure diagram is our primary tool to find the nim-value of a building game on the simple hypergraph H. We use the following steps:

1. Compute the family $\mathcal{S}_H = \complement(\text{Tr}(\mathcal{H}))$ of maximal stable sets.
2. Build the structure diagram.
   (a) Build the structure digraph.
      i. Compute the elements of the closure system $\mathcal{I}_H = \{\bigcap \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}_H\}$. This provides the vertex set of the structure digraph
      ii. Find $\text{Opt}(X_I)$ for each $I \in \mathcal{I}_H$ by determining the structure classes containing the options of position I. This provides the arrows of the structure digraph.
   (b) Use type calculus to recursively compute $\text{type}(X_I)$ for each $I \in \mathcal{I}_H$.
3. The nim-value of the game is the second component $\text{mex}_0(X_{\Phi_H})$ of $\text{type}(X_{\Phi_H})$.

The algorithm is useful because instead of processing the potentially huge full game digraph we only need to process the structure digraph which is a hopefully smaller quotient digraph. Of course, the larger the structure equivalence classes are, the better the algorithm works.

Example 4.14. Figure 4.1 shows the structure diagram of $\text{ACV}(H)$ introduced in Example 4.9. The game digraph contains 8 vertices, while the structure diagram contains only 4. Note that the type of the structure class containing the Frattini subset $\Phi = \emptyset$ is $\text{type}(X_\Phi) = (0, 0, 3)$. This type indicates that any position in this structure class with odd parity has a nim-value of 3. Since $X_\Phi = \{\emptyset\}$, there is no such position in this structure class. Similarly, $\text{type}(X_{\{b\}}) = (1, 2, 1)$ and $X_{\{b\}} = \{\{b\}\}$ contains no position with even parity and a nim-value of 2.

Example 4.15. Figure 4.2 shows the structure diagram for each building game on the vertex set $V = \{1, 2\}$. Note that the structure classes can be singleton sets, in which case the structure diagrams has no advantage over the full game digraph. This happens for example for the achieve game with $\mathcal{H} = \{\{1, 2\}\}$.

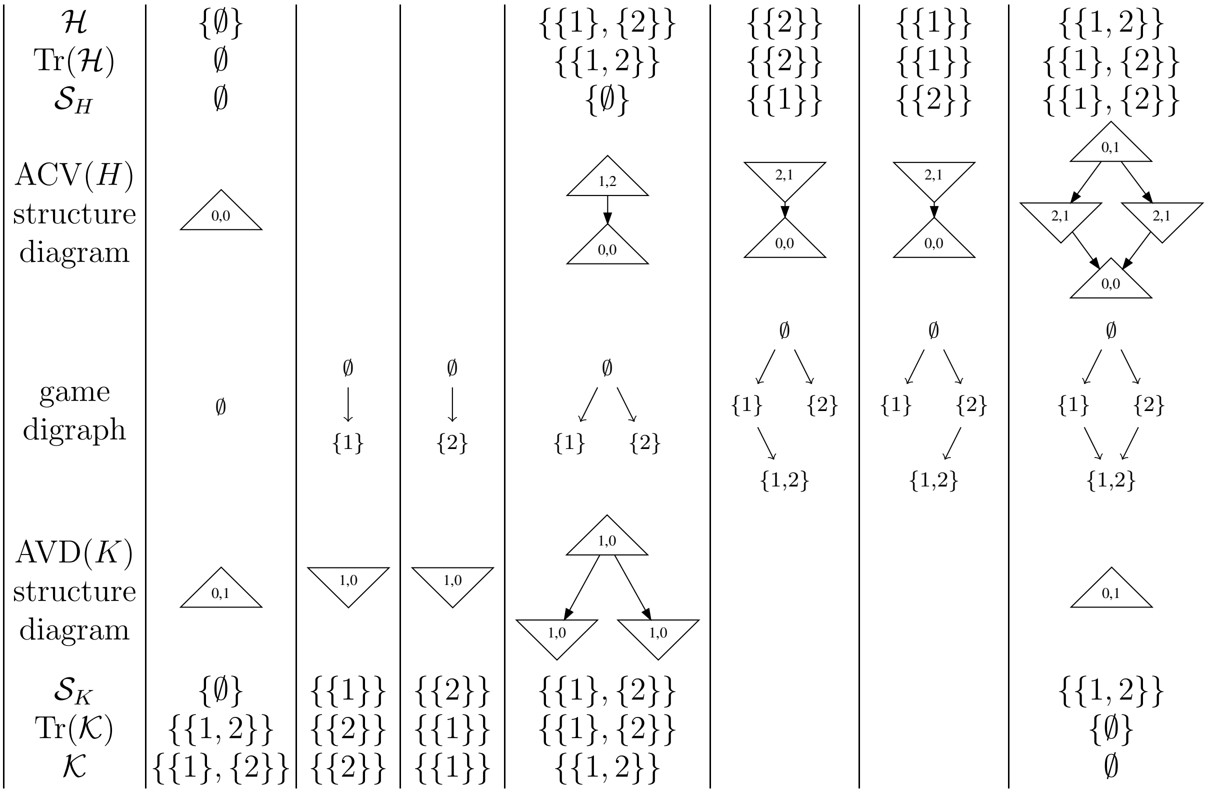


Figure 4.2: All non-isomorphic building hypergraph games on $V = \{1, 2\}$.

5 Structure theory for the removing games PRV and DST

A removing game on a simple hypergraph H is equal to a building game on $\mathrm{Tr}(H)$. So for these games the appropriate structure equivalence is with respect to $\mathrm{Tr}(H)$. Note that $\mathcal{S}_{\mathrm{Tr}(H)} = \mathsf{C}(\mathrm{Tr}(\mathrm{Tr}(\mathcal{H}))) = \mathsf{C}(\mathcal{H})$.

Example 5.1. Let $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$ as in Example 3.3, so that $\mathcal{S}_{\mathrm{Tr}(H)} = \mathsf{C}(\mathcal{H}) = \{\{c\}, \{a\}\}$ and $\mathcal{I}_{\mathrm{Tr}(H)} = \{\emptyset, \{a\}, \{c\}, \{a, b, c\}\}$. Figure 5.1 shows the structure classes of $\mathrm{DST}(H) = \mathrm{ACV}(\mathrm{Tr}(H))$.

The following is a generalization of a similar result of [6].

Proposition 5.2. If H is a simple hypergraph, then $\Phi_H = \Phi_{\mathrm{Tr}(H)}$.

Proof. Propositon 2.6 implies that

$$\begin{aligned}\Phi_H &= \bigcap \mathcal{S}_H = \bigcap \mathsf{C}(\mathrm{Tr}(\mathcal{H}))\\ &= \mathsf{C}(\bigcup \mathrm{Tr}(\mathcal{H})) = \mathsf{C}(\bigcup \mathcal{H})\\ &= \bigcap \mathsf{C}(\mathcal{H}) = \bigcap \mathcal{S}_{\mathrm{Tr}(H)} = \Phi_{\mathrm{Tr}(H)}.\end{aligned}$$

□

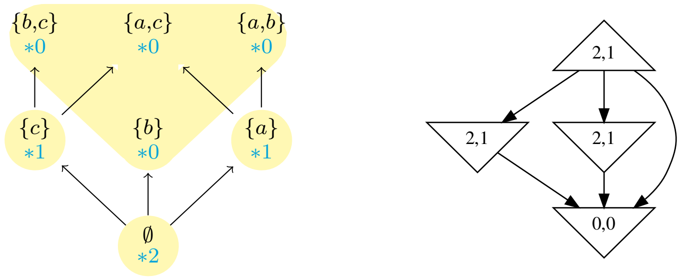


Figure 5.1: Game digraph indicating the structure classes and structure diagram for $\text{DST}(H) = \text{ACV}(\text{Tr}(H))$ with $V = \{a, b, c\}$ and $\mathcal{H} = \{\{a, b\}, \{b, c\}\}$.

6 Simplified structure diagram

The structure diagram can be relatively easily found by a computer but it can be too large to provide any intuition about the game. So it is often useful to make further identifications in the structure diagram to build a simplified structure diagram. We want to make enough identifications to create a manageable diagram but too many identifications results in a simple but meaningless diagram. This is a delicate balance and the best approach depends on the hypergraph. The automorphism group of the hypergraph or of the structure diagram provides good opportunities for identifications but these automorphism groups can be difficult to compute. In this paper we use easy to compute conditions. We identify X_I and X_J if the following conditions hold.

Condition 6.1.

1. $\text{pty}(I) = \text{pty}(J)$;
2. $\text{type}(\text{Opt}(X_I)) = \text{type}(\text{Opt}(X_J))$;
3. The lengths of the longest directed paths starting at X_I and at X_J are the same.

The first two conditions guarantee that $\text{type}(X_I) = \text{type}(X_J)$. The third condition avoids vertical collapsing. These conditions rely on the outgoing arrows of the structure digraph and ignore the incoming arrows. In the simplified structure diagram we use shaded triangles if they represent several structure classes.

Example 6.2. Figure 6.1 shows the structure diagram and the simplified structure diagram of $\text{ACV}(H)$ for a hypergraph H with vertex set $\{1, 2, 3, 4\}$ satisfying

$$\mathcal{S}_H = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}\}.$$

Note that we identified all the structure classes corresponding to the maximal stable sets even though $X_{\{2,4\}}$ is the only structure class that is an option of X_{Φ_H}.

7 Special hypergraphs

In this section we find criteria for the nim-value of games played on hypergraphs with a relatively simple structure. We are mainly concerned about the building games but many

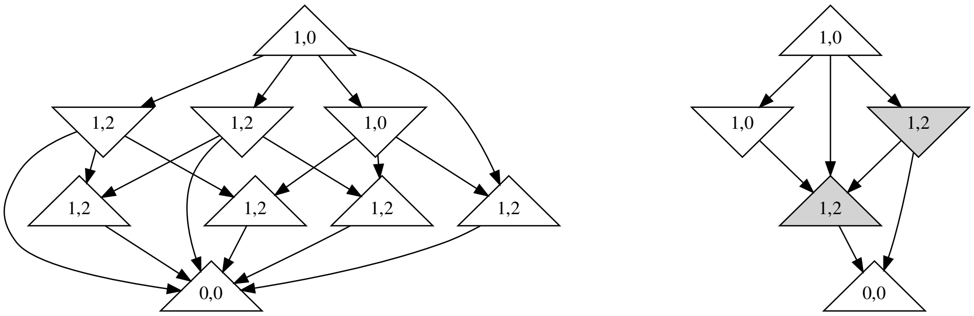


Figure 6.1: Structure diagram and simplified structure diagram.

of our results have versions for the removing games since those are building games on the transversal hypergraph.

We warm up with a very simple case.

7.1 Single edge

In this subsection we study games played on a hypergraph with a single edge. The removing games are easy to analyze.

Proposition 7.1. If $H = (V, \mathcal{H})$ is a hypergraph containing only one nonempty edge A, then $\operatorname{nim}(\mathrm{PRV}(H)) = \operatorname{pty}(A^{\complement})$ and $\operatorname{nim}(\mathrm{DST}(H)) = 1 + \operatorname{pty}(A^{\complement})$.

Proof. Since $\mathcal{S}_{\mathrm{Tr}(H)} = \complement(\mathcal{H}) = \{A^{\complement}\}$, $\operatorname{nim}(\mathrm{PRV}(H)) = \operatorname{pty}(A^{\complement})$ by Proposition 3.15. In fact, there is only one structure class $X_{A^{\complement}}$ whose type is $\operatorname{type}(X_{A^{\complement}}) = (\operatorname{pty}(A^{\complement}), \operatorname{pty}(A^{\complement}), 1 - \operatorname{pty}(A^{\complement}))$. The achievement game $\mathrm{DST}(H)$ has two structure classes X_V and $X_{A^{\complement}}$. Their types are $\operatorname{type}(X_V) = (\operatorname{pty}(V), 0, 0)$ and $\operatorname{type}(X_{A^{\complement}}) = (\operatorname{pty}(A^{\complement}), 1 + \operatorname{pty}(A^{\complement}), 2 - \operatorname{pty}(A^{\complement}))$ by type calculus. □

The building games require a bit more effort.

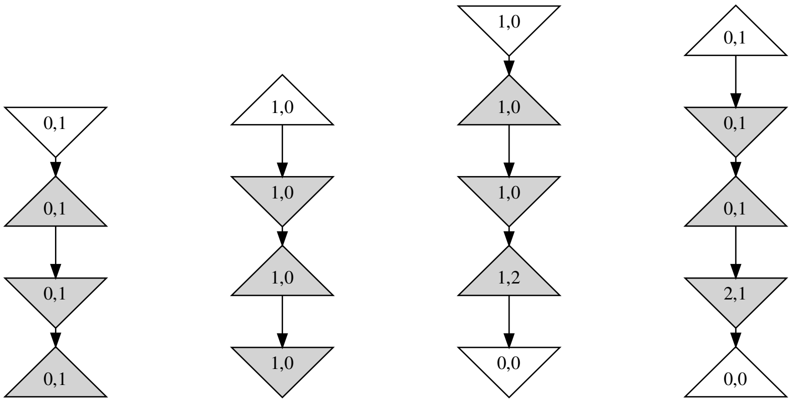
Proposition 7.2. If $H = (V, \mathcal{H})$ is a hypergraph containing only one nonempty edge A, then $\operatorname{nim}(\mathrm{AVD}(H)) = 1 - \operatorname{pty}(V)$ and

$$\operatorname{nim}(\mathrm{ACV}(H)) = \begin{cases} 2, & |A| = 1 \text{ and } \operatorname{pty}(V) = 0 \\ \operatorname{pty}(V), & \text{otherwise.} \end{cases}$$

Proof. Since $\mathcal{S}_H = \complement(\mathrm{Tr}(\mathcal{H})) = \{\{a\}^{\complement} \mid a \in A\}$, $\operatorname{nim}(\mathrm{AVD}(H)) = \operatorname{pty}(|V| - 1)$ by Proposition 3.15.

To prove the claim about the achievement game, first note that $\mathcal{I}_{\mathcal{H}} = \{B^{\complement} \mid B \subseteq A\}$. It is easy to see that $\operatorname{Opt}(X_{B^{\complement}}) = \{X_{(B \setminus \{b\})^{\complement}} \mid b \in B\}$. Induction on the size of B together with type calculus shows that

$$\operatorname{type}(X_{B^{\complement}}) = \begin{cases} (0,0,0), & |B| = 0 \\ (1,2,1), & |B| = 1 \\ (\operatorname{pty}(B), 1, 0), & |B| \geqslant 2 \end{cases}$$



AVD(H)	AVD(H)	ACV(H)	ACV(H)
pty(V) $= 1$	pty(V) $= 0$	pty(V) $= 1$	pty(V) $= 0$
$\lvert A\rvert = 4$	$\lvert A\rvert = 4$	$\lvert A\rvert = 4$	$\lvert A\rvert = 4$

Figure 7.1: Simplified structure diagrams for the building games on a hypergraph with a single edge. Note that X_{Φ_H} in AVD(H) is not identified with any other structure class because Condition 6.1(3) is not satisfied.

if $\operatorname{pty}(V) = 1$, and

$$\operatorname{type}(X_{B^\complement}) = \begin{cases} (1,0,0), & |B| = 0 \\ (0,1,2), & |B| = 1 \\ (\operatorname{pty}(B),0,1), & |B| \geqslant 2 \end{cases}$$

if $\operatorname{pty}(V) = 0$. Figure 7.1 provides a visual aid for the induction. Note that the number of triangles in the simplified structure diagram for the achievement games is $|A| + 1$. $\square$

7.2 Pairwise disjoint edges

In this subsection we consider hypergraphs for which $\mathcal{H}$ contains pairwise disjoint sets.

Proposition 7.3. If the edges of a hypergraph $H = (V, \mathcal{H})$ are pairwise disjoint, then

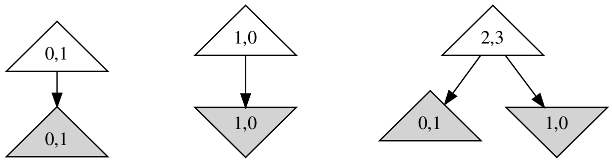
$$\operatorname{nim}(\operatorname{AVD}(H)) = \operatorname{pty}(|V| - |\mathcal{H}|).$$

Proof. A minimal transversal of $\mathcal{H}$ contains a single vertex from each edge. So every set in $\operatorname{Tr}(\mathcal{H})$ contains $|\mathcal{H}|$ elements. Hence every set in $\mathcal{S}_H = \complement(\operatorname{Tr}(\mathcal{H}))$ contains $|V| - |\mathcal{H}|$ elements. The result now follows from Proposition 3.15. $\square$

The achieve game can be quite complicated even in this simple case.

7.3 Pairwise disjoint maximal stable sets

In this subsection we consider hypergraphs for which $\mathcal{S}_H$ contains pairwise disjoint sets.



$Y = \{0\}$ $Y = \{1\}$ $Y = \{0,1\}$

Figure 7.2: Simplified structure diagrams for AVD(H) for a hypergraph H with pairwise disjoint maximal stable sets.

Proposition 7.4. If $H = (V, \mathcal{H})$ is a hypergraph such that $\mathcal{S}_H$ is a family of pairwise disjoint sets and $Y := \{\text{pty}(A) \mid A \in \mathcal{S}_H\}$, then

$$\text{nim}(\text{AVD}(H)) = \begin{cases} 0, & Y = \{0\} \\ 1, & Y = \{1\} \\ 2, & Y = \{0,1\}. \end{cases}$$

Proof. The intersection of any two maximal stable set is $\Phi_H = \emptyset$. The simplified structure diagram for each case is shown in Figure 7.2. □

Proposition 7.5. If $H = (V, \mathcal{H})$ is a hypergraph such that $\mathcal{S}_H$ is a family of pairwise disjoint sets and $Y := \{\text{pty}(A) \mid A \in \mathcal{S}_H\}$, then

$$\text{nim}(\text{ACV}(H)) = \begin{cases} 0, & \mathcal{S}_H \text{ covers } V \\ 1, & Y = \{0\} \text{ and } \mathcal{S}_H \text{ does not cover } V \\ 2, & Y = \{1\} \text{ and } \mathcal{S}_H \text{ does not cover } V \\ 3, & Y = \{0,1\} \text{ and } \mathcal{S}_H \text{ does not cover } V. \end{cases}$$

Proof. If $\mathcal{S}_H$ covers V, then $X_V \notin \text{Opt}(X_{\Phi_H})$. If $\mathcal{S}_H$ does not cover V, then there is a $v \in V \setminus \bigcup \mathcal{S}_H$. So $X_V \in \text{Opt}(X_{\Phi_H})$ since $\{v\} \in X_V$. One can now easily verify the claim by drawing the simplified structure diagrams. These diagrams are extensions of the diagrams with the additional structure class X_V in Figure 7.2. □

7.4 Pairwise disjoint minimal transversals

We saw that if $\mathcal{H}$ contains a single edge A, then $\text{Tr}(\mathcal{H}) = \complement(\mathcal{S}_H) = \{\{a\} \mid a \in A\}$ contains pairwise disjoint sets. Proposition 7.2 can be generalized for the case when $\text{Tr}(\mathcal{H})$ is a family of pairwise disjoint sets. The nim-values can be easily determined in terms of the *signature* $\sigma(\mathcal{H}) = (e, o)$, where $e := |\{B \in \text{Tr}(\mathcal{H}) \mid \text{pty}(B) = 0\}|$ and $o := |\{B \in \text{Tr}(\mathcal{H}) \mid \text{pty}(B) = 1\}|$.

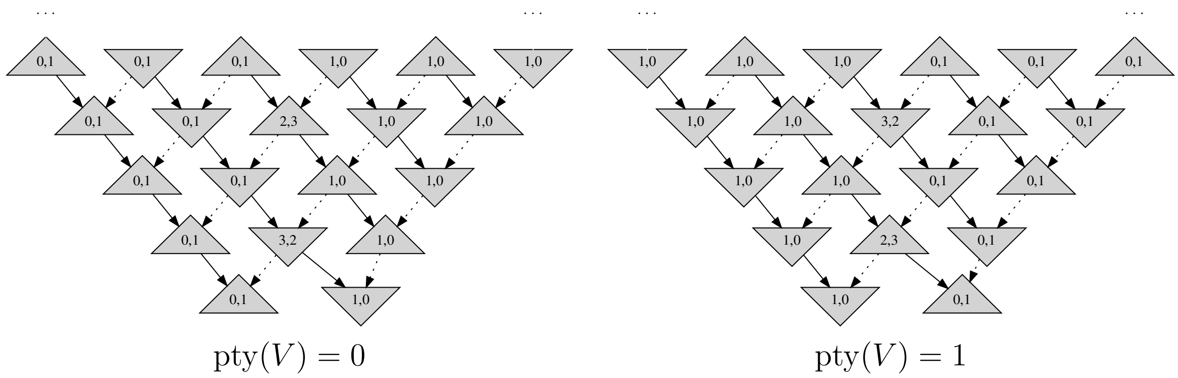


Figure 7.3: Type calculus computation for $\delta(X_I) \leqslant 5$. The pattern of the rows repeat with a period of 4.

Proposition 7.6. Let $H = (V, \mathcal{H})$ be a hypergraph such that $\mathrm{Tr}(\mathcal{H})$ is a family of pairwise disjoint sets. If the signature is $\sigma(\mathcal{H}) = (e, o)$, then

$$\mathrm{nim}(\mathrm{AVD}(H)) = \begin{cases} \mathrm{pty}(V), & e > o \\ 1 - \mathrm{pty}(V), & e < o \\ 3 - |\,\mathrm{pty}(e) - \mathrm{pty}(V)|, & e = o. \end{cases}$$

Proof. Since $\mathrm{Tr}(\mathcal{H})$ is a family of pairwise disjoint sets, $\mathcal{T} \mapsto \bigcap \mathcal{T} : 2^{\mathcal{S}_H} \setminus \{\emptyset\} \to \mathcal{I}_H$ is a bijection. For $I = \bigcap \mathcal{T} \in \mathcal{I}_H$ we define the signature of X_I to be $\sigma(X_I) := (e, o)$, where $e = |\{S \in \mathcal{T} \mid \mathrm{pty}(S^{\complement}) = 0\}|$ and $o = |\{S \in \mathcal{T} \mid \mathrm{pty}(S^{\complement}) = 1\}|$. We also define $\delta(X_I) := |\mathcal{T}| = e + o$. Note that $\sigma(X_{\Phi_H}) = \sigma(\mathcal{H})$, $\delta(X_{\Phi_H}) = |\mathcal{S}_H|$, and $\delta(S) = 1$ for all $S \in \mathcal{S}_H$.

Let $I = \bigcap \mathcal{T} \in \mathcal{I}_H$ and $\sigma(X_I) = (e, o)$. If $X_J \in \mathrm{Opt}(X_I)$ then $I \cup \{v\} \in X_J$ for some $v \in J \setminus I$ and there is a unique $S \in \mathcal{S}_H$ such that $v \in S^{\complement}$. Hence $J = \bigcap(\mathcal{T} \setminus \{S\})$ and $\delta(X_I) = \delta(X_J) + 1$. If $e \geqslant 1$ then there is an $S \in \mathcal{T}$ such that $\mathrm{pty}(S^{\complement}) = 0$. If $o \geqslant 1$ then there is an $S \in \mathcal{T}$ such that $\mathrm{pty}(S^{\complement}) = 1$. In either case $X_J \in \mathrm{Opt}(X_I)$ for $J = \bigcap(\mathcal{T} \setminus \{S\})$. Hence $\mathrm{Opt}(X_I) = \{\bigcap(\mathcal{T} \setminus \{S\}) \mid S \in \mathcal{T}\}$ and

$$\sigma(\mathrm{Opt}(X_I)) = \begin{cases} \{(e-1, 0)\}, & o = 0 \\ \{(0, o-1)\}, & e = 0 \\ \{(e-1, o), (e, o-1)\}, & e, o \geqslant 1. \end{cases}$$

The result now follows from type calculus and structural induction on the structure classes. The details of the type calculus computation are shown in Figure 7.3. The signature (e, o) of a structure class becomes $(e-1, o)$ as we move backwards along a solid arrow, while it becomes $(e, o-1)$ as we move backwards along a dotted arrow. $\square$

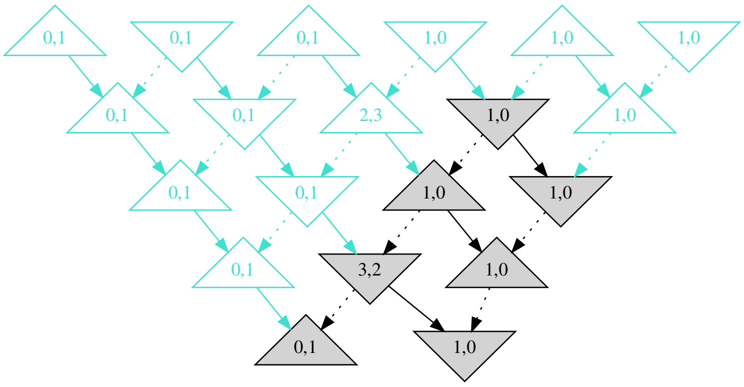


Figure 7.4: Simplified structure diagram for a hypergraph with pairwise disjoint $\mathrm{Tr}(\mathcal{H})$, $\mathrm{pty}(V) = 0$, and signature $\sigma(\mathcal{H}) = (1, 3)$.

Example 7.7. Consider the hypergraph H with $V = \{1, \ldots, 6\}$ and

$$\mathrm{Tr}(\mathcal{H}) = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}.$$

The signature is $\sigma(\mathcal{H}) = (1, 3)$. Figure 7.4 shows the simplified structure diagram of $\mathrm{AVD}(H)$ as a subdiagram of the infinite diagram of Figure 7.3. The nim-value of the game is 1 as guaranteed by Proposition 7.6.

The $\mathrm{ACV}(H)$ can be handled similarly. The proof of the next result is essentially that of [28, Proposition 6.5].

Proposition 7.8. Let $H = (V, \mathcal{H})$ be a hypergraph such that $\mathrm{Tr}(\mathcal{H})$ is a family of pairwise disjoint sets. If the signature is $\sigma(\mathcal{H}) = (e, o)$, then

$$\mathrm{nim}(\mathrm{ACV}(S, W)) = \begin{cases} 1, & (e, o) = (1, 0) \\ 2, & (e, o) \in \{(0, 1), (1, 2)\} \\ 3, & (e, o) \in \{(1, 1), (2, 1)\} \\ 0, & \text{otherwise} \end{cases}$$

when $\mathrm{pty}(V) = 0$, and

$$\mathrm{nim}(\mathrm{ACV}(S, W)) = \begin{cases} 0, & (e, o) \in \{(0, 0), (1, 1)\} \\ 2, & (e, o) \in \{(1, 0), (2, 0)\} \\ 1, & \text{otherwise} \end{cases}$$

when $\mathrm{pty}(V) = 1$.

8 Examples

There is an extensive literature about games that are impartial hypergraph games. Many of these games [35] start with an empty graph on a set of vertices and the players build a graph by adding edges until the graph has a certain property. These include diameter avoidance games [13], triangle avoidance games [15, 31], saturation games [33], path achievement games [36], connect-it games [23], minimum degree games [19], and degree games [22].

Other games start with a graph and the players select vertices until the set of selected vertices has a certain property. These include geodetic closure games [6, 14, 24, 30] and general position games [26, 34].

Games on other mathematical objects like groups [3, 4, 7, 8, 9, 10, 17], matroids [1], convex geometries [28], and Cayley graphs [18] are also studied.

In this section we provide some detailed examples to show how these games can be considered as hypergraph games.

8.1 Generate and do not generate games

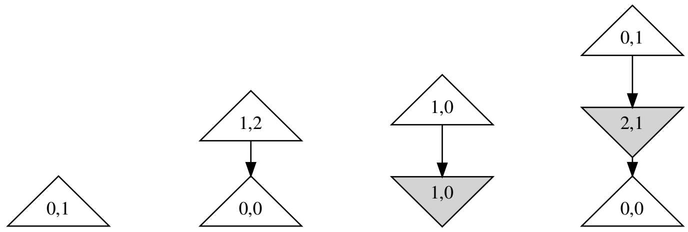
Some examples in this section rely on the notion of a generating set with respect to a pre-closure operator. Given a pre-closure operator $\mathrm{cl} : 2^S \to 2^S$ and a subset $\mathcal{W}$ of 2^S of *winning sets*. Let $\mathcal{H} := \mathrm{Min}(\{P \subseteq S \mid W \subseteq \mathrm{cl}(P) \text{ for some } W \in \mathcal{W}\})$. The achievement game $\mathrm{ACV}(H)$ and avoidance game $\mathrm{AVD}(H)$ on the hypergraph $H := (S, \mathcal{H})$ are often called *generate* and *do not generate* and denoted by $\mathrm{GEN}(\mathrm{cls}, \mathcal{W})$ and $\mathrm{DNG}(\mathrm{cls}, \mathcal{W})$. We can think of the elements of $\mathcal{H}$ as the minimal generating sets while the stable sets as maximal non-generating sets.

The family $\mathcal{W}$ of winning sets is most often only contains S. In this case $\mathcal{H}$ is the set of dense subsets of S and the simplified notations $\mathrm{GEN}(\mathrm{cl})$ and $\mathrm{DNG}(\mathrm{cl})$ can be used. We even use the notations $\mathrm{GEN}(S)$ and $\mathrm{DNG}(S)$ if S has a default standard closure operator. We call removing games in this context *terminate* $\mathrm{TER}(\mathrm{cl})$ and *do not terminate* $\mathrm{DNT}(\mathrm{cl})$. They are studied in [6] and in [23] under the name *disconnect-it games*.

8.2 Group generating games

The generate and do not generate games on groups are studied in [3, 4, 17, 8, 9, 10, 7, 11]. The closure operator is $P \mapsto \langle P \rangle : 2^G \to 2^G$ is the generated subgroup operator on a finite group G. The family of winning sets is $\mathcal{W} = \{G\}$.

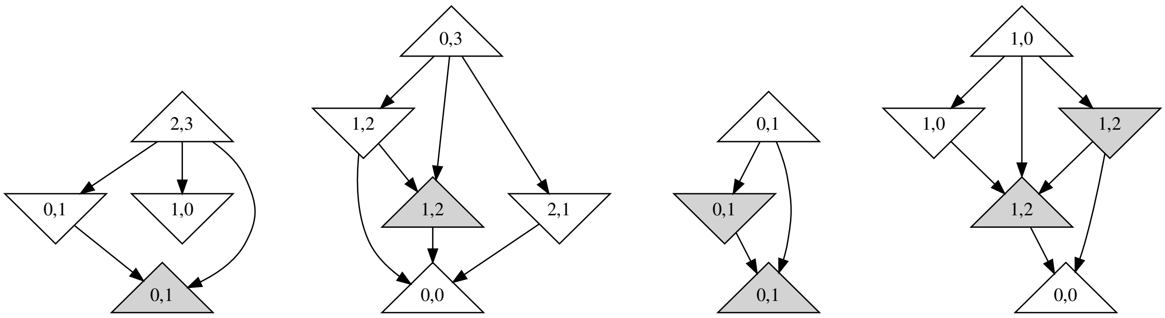
Example 8.1. Let $G = \mathbb{Z}_4$ be the cyclic group with elements $\{0, 1, 2, 3\}$. Consider the hypergraph $H = (G, \mathcal{H})$ with the family $\mathcal{H} = \{\{1\}, \{3\}\}$ of minimal generating sets. Then $\mathrm{GEN}(G) = \mathrm{ACV}(H)$ and $\mathrm{DNG}(G) = \mathrm{AVD}(H)$. The maximal stable sets are the maximal subgroups of G, so $\mathcal{S}_H = \{\{0, 2\}\}$. We can also analyze the removing games as building games on $\mathrm{Tr}(H)$. We have $\mathrm{Tr}(\mathcal{H}) = \{\{1, 3\}\}$ and $\mathcal{S}_{\mathrm{Tr}(H)} = \{\{0, 1, 2\}, \{0, 2, 3\}\}$. So $\mathrm{TER}(G) = \mathrm{ACV}(\mathrm{Tr}(H))$ and $\mathrm{DNT}(G) = \mathrm{AVD}(\mathrm{Tr}(H))$. Note that H has pairwise disjoint edges and the families $\mathcal{S}_H$ and $\mathrm{Tr}(\mathcal{H})$ are singletons, so several results in Section 7 can be applied. The structure diagrams are shown in Figure 8.1.



$\mathrm{DNG}(\mathbb{Z}_4)$ $\mathrm{GEN}(\mathbb{Z}_4)$ $\mathrm{DNT}(\mathbb{Z}_4)$ $\mathrm{TER}(\mathbb{Z}_4)$

$\mathrm{AVD}(H)$ $\mathrm{ACV}(H)$ $\mathrm{PRV}(H)$ $\mathrm{DST}(H)$

Figure 8.1: Simplified structure diagrams for the group generating games on $\mathbb{Z}_4$.



$\mathrm{DNG}(S, \mathcal{W})$ $\mathrm{GEN}(S, \mathcal{W})$ $\mathrm{DNT}(S, \mathcal{W})$ $\mathrm{TER}(S, \mathcal{W})$

$\mathrm{AVD}(H)$ $\mathrm{ACV}(H)$ $\mathrm{PRV}(H)$ $\mathrm{DST}(H)$

Figure 8.2: Simplified structure diagrams for convex closure games.

8.3 Convex closure games

The generate game on convex geometries is studied in [28] using the abstract convex closure operator $\tau : 2^S \to 2^S$ on a finite point set S.

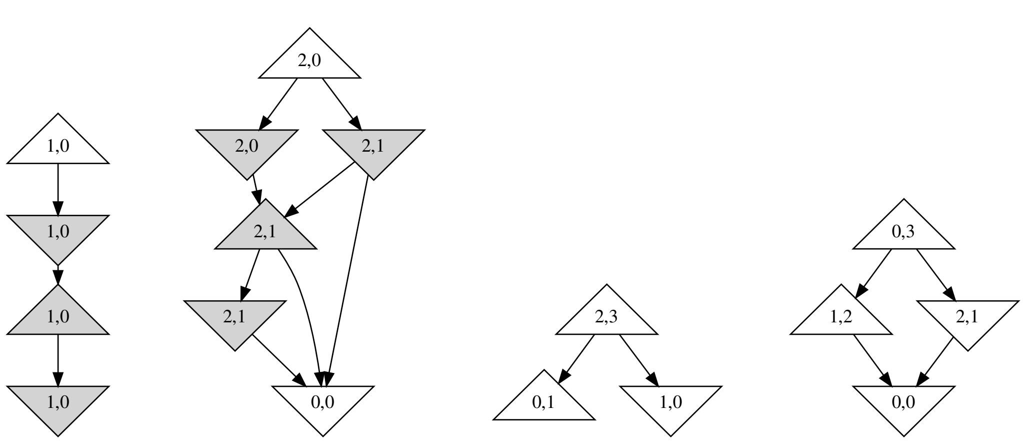
Example 8.2. Consider the affine convex geometry of $S = \{1, 2, 3, 4\} \subseteq \mathbb{R}$ with $\mathcal{W} = \{\{1, 2, 3\}, \{3, 4\}\}$. Let $H = (G, \mathcal{H})$ be the hypergraph with the family

$$\mathcal{H} = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$$

of minimal sets whose convex closure contain one of the winning sets. Then $\mathrm{DNG}(S, \mathcal{W}) = \mathrm{AVD}(H)$, $\mathrm{GEN}(S, \mathcal{W}) = \mathrm{ACV}(H)$, $\mathrm{DNT}(S, \mathcal{W}) = \mathrm{PRV}(H)$, and $\mathrm{TER}(S, \mathcal{W}) = \mathrm{DST}(H)$. We have

$$\begin{aligned} \mathrm{Tr}(\mathcal{H}) &= \{\{1, 2, 3\}, \{1, 4\}, \{3, 4\}\}, \\ \mathcal{S}_H &= \{\{1, 2\}, \{2, 3\}, \{4\}\}, \\ \mathcal{S}_{\mathrm{Tr}(H)} &= \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}\}. \end{aligned}$$

The structure diagrams are shown in Figure 8.2.



AVD(H) ACV(H) PRV(H) DST(H)

Figure 8.3: Simplified structure diagrams for the geodetic games on the graph $K_{2,3}$.

8.4 Geodetic closure games

Let $G = (V, E)$ be a connected graph. A *geodesic* is a shortest path between two vertices. The *geodetic closure* of a subset P of V is the set (P) of vertices that are contained on a geodesic between two vertices of P. The mapping $P \mapsto (P) : 2^V \to 2^V$ is a pre-closure operator. A set P of vertices is called a *geodetic cover* if $(P) = V$. A *geodetic basis* is a geodetic cover with minimum size.

The outcome for the achievement and avoidance games for this pre-closure operator are studied in [14, 24, 30].

Example 8.3. Consider the complete bipartite graph $G = K_{2,3}$ with bipartition $V = \{1,2\} \cup \{3,4,5\}$. The family of minimal generating sets is $\mathcal{H} = \{\{1,2\},\{3,4,5\}\}$. So the avoidance game is AVD(H) and the achievement game is ACV(H) on the hypergraph $H = (V, \mathcal{H})$ with

$$\mathcal{S}_H = \{\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}.$$

Since every maximal non-generating set has an odd size, $\mathrm{nim}(\mathrm{AVD}(H)) = 1$ by Proposition 3.15. The structure diagram in Figure 8.3 shows that $\mathrm{nim}(\mathrm{ACV}(H)) = 2$. So both building games are won by the first player as proved in [14, Theorems 9 and 5].

We can also analyze the removing games as building games on $\mathrm{Tr}(H)$. We have $\mathcal{S}_{\mathrm{Tr}(H)} = \mathsf{C}(\mathcal{H}) = \mathcal{H}$. The structure diagrams in Figure 8.3 show that $\mathrm{nim}(\mathrm{PRV}(H)) = 2$ and $\mathrm{nim}(\mathrm{DST}(H)) = 0$.

We can easily recover the following results of [24].

Proposition 8.4. [24, Proposition 6 and 14] Let G be a graph of order n. Assume G has a unique geodetic basis S such that $|S| \geqslant 2$ and for every geodetic cover S', $S \subseteq S'$. The geodetic achievement game is won by the first player if and only if n is odd. The geodetic avoidance game is won by the first player if and only if n is even.

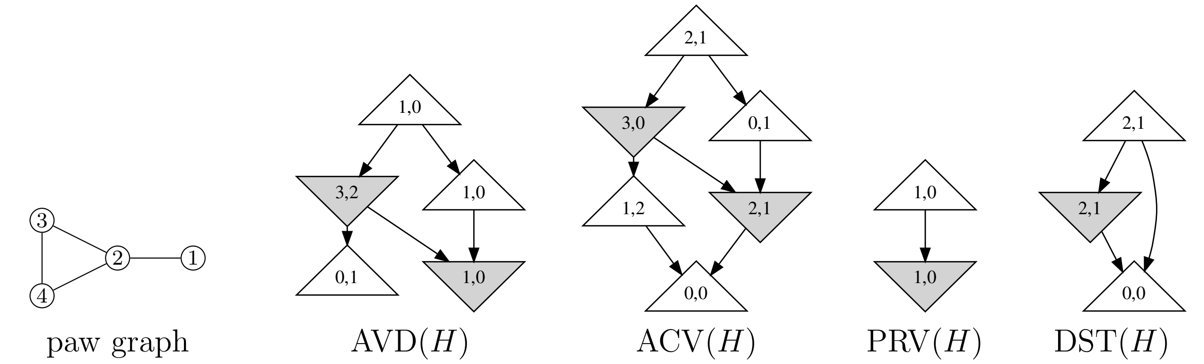


Figure 8.4: Simplified structure diagrams for the general position games on the paw graph.

Proof. The conditions imply that there is a unique minimal geodetic cover S. So we are playing building games on the hypergraph $H = (V, \mathcal{H})$ satisfying $|V| = n$ and $\mathcal{H} = \{S\}$. So the results are special cases of Proposition 7.2. $\square$

8.5 General position games on graphs

Let $G = (V, E)$ be a connected graph. A set of vertices is in general position if no three elements of the set lie on a shortest path of G. The general position achievement game studied in [26, 34] requires the players to keep the set of jointly chosen vertices in general position. This is actually an avoidance game in our setting on the hypergraph $H = (V, \mathcal{H})$ with $\mathcal{H}$ consisting of the minimal sets of vertices that are not in general position.

Example 8.5. Let G be the paw graph shown in Figure 8.4 so that $V = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$. Then $\mathrm{Tr}(\mathcal{H}) = \{\{1\}, \{2\}, \{3, 4\}\}$, and

$$\mathcal{S}_H = \{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \quad \mathcal{S}_{\mathrm{Tr}(H)} = \{\{3\}, \{4\}\}.$$

The structure diagrams for the general position games are also shown in Figure 8.4. Note that $\mathrm{Tr}(\mathcal{H})$ contains pairwise disjoint subsets, so $\mathrm{nim}(\mathrm{AVD}(H)) = 1$ Proposition 7.6 and $\mathrm{nim}(\mathrm{ACV}(H)) = 2$ by Proposition 7.8 since $\sigma(\mathcal{H}) = (1, 2)$ and $\mathrm{pty}(V) = 0$. Also note that $\mathcal{S}_{\mathrm{Tr}(H)}$ contains pairwise disjoint sets, so $\mathrm{nim}(\mathrm{PRV}(H)) = 1$ by Proposition 7.4 and $\mathrm{nim}(\mathrm{DST}(H)) = 2$ by Proposition 7.5 since $\{\mathrm{pty}(A) \mid A \in \mathcal{S}_{\mathrm{Tr}(H)}\} = \{1\}$ and $\mathcal{S}_{\mathrm{Tr}(H)}$ does not cover V.

Proposition 8.6. If $G = (V, E)$ is a graph such that V is in not in general position, then the nim number of the general position preserve game is $1 - \mathrm{pty}(V)$.

Proof. The minimal sets that are not in general position contain 3 vertices. So every edge in $\mathcal{H}$ has size 3 and the sets in $\mathcal{S}_{\mathrm{Tr}(H)}$ have size $|V| - 3$. Thus $\mathrm{nim}(\mathrm{PRV}(H)) = 1 - \mathrm{pty}(V)$ by Proposition 3.15. $\square$

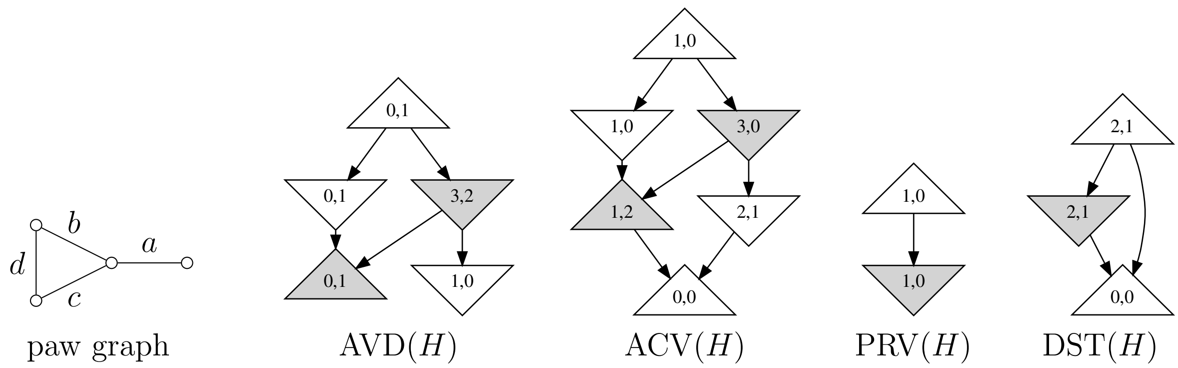


Figure 8.5: Simplified structure diagrams for the connect-it games on the paw graph.

8.6 Connect-it games

Let $G = (V, E)$ be a connected graph. Consider the hypergraph $H = (E, \mathcal{H})$, where $\mathcal{H}$ is the family consisting of the edge sets of the spanning trees of G. The hypergraph games AVD(H), ACV(H), PRV(H), DST(H) are studied in [23] under the names DON'T Connect-it, DO Connect-it, DON'T Disconnect-it, DO Disconnect-it.

Example 8.7. Let G be the paw graph shown in Figure 8.5 so that

$$\mathcal{H} = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}.$$

Then $\mathcal{S}_H = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c, d\}\}$ and $\mathcal{S}_{\mathrm{Tr}(H)} = \{\{b\}, \{c\}, \{d\}\}$. The structure diagrams for the connect-it games are also shown in Figure 8.5. Note that $\mathcal{S}_{\mathrm{Tr}(H)}$ contains pairwise disjoint sets, so Propositions 7.4 and 7.5 apply for the removing games.

8.7 Minimum degree games for graphs

Avoidance and achievement games with the purpose of making a graph with n vertices and a minimum degree δ are studied in [19]. We can generalize these games for a given connected graph $G = (V, E)$. Consider the hypergraph $H = (E, \mathcal{H})$ where $\mathcal{H}$ is the family consisting of each minimal set A of edges of G for which each vertex of G is incident to at least δ edges in A.

Example 8.8. Let G be the paw graph shown in Figure 8.6 and $\delta = 1$, so that

$$\mathcal{H} = \{\{a, b, c\}, \{a, d\}\}.$$

Then $\mathcal{S}_H = \{\{a, b\}, \{a, c\}, \{b, c, d\}\}$ and $\mathcal{S}_{\mathrm{Tr}(H)} = \{\{b, c\}, \{d\}\}$. The structure diagrams for the connect-it games are also shown in Figure 8.6.

8.8 Degree achievement and avoidance games for graphs

Avoidance and achievement games with the purpose of making a graph with n vertices and a vertex with a given degree d are studied in [22]. We demonstrate these games on an example.

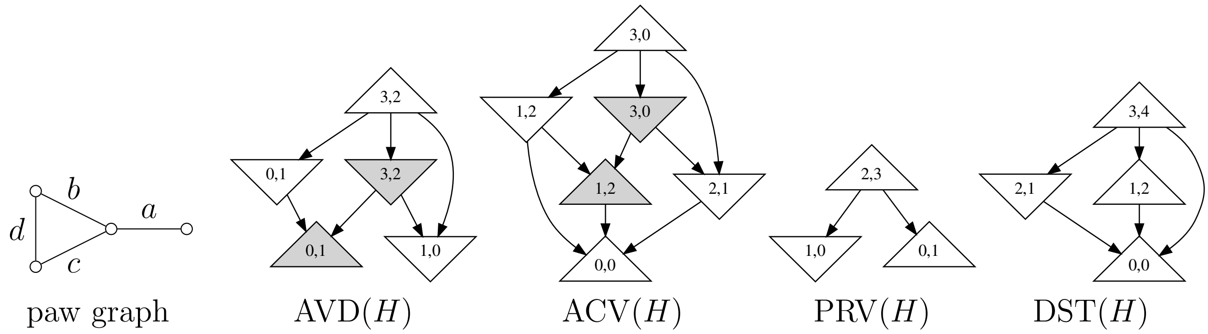


Figure 8.6: Simplified structure diagrams for the minimum degree games on the paw graph with $\delta = 1$.

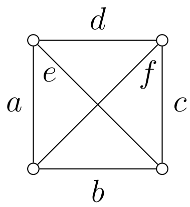


Figure 8.7: Edge labeling for the graph K_4.

Example 8.9. We consider the $n = 4$ and $d = 3$ case. Let $H = (E, \mathcal{H})$ where E is the edge set of the complete graph K_4 shown in Figure 8.7 and

$$\mathcal{H} = \{\{a, d, e\}, \{a, b, f\}, \{b, c, e\}, \{c, d, f\}\}.$$

Easy computation of the structure diagrams show that the nim numbers of ACV(H), AVD(H), and DST(H) are 0 and nim(PRV(H)) = 1.

9 Spectrum of nim-values

Our goal in this section is to show that the nim-value of AVD(H) and ACV(H) can be any nonnegative integer. To do this, we need to develop some tools to construct structure diagrams for games created from simpler games. These tools are interesting in their own right.

9.1 Nim-values of AVD

For $\mathcal{H}, \mathcal{K} \subseteq 2^V$ we define $\mathcal{H} \vee \mathcal{K} := \{A \cup B \mid A \in \mathcal{H} \text{ and } B \in \mathcal{K}\}$. For $\mathcal{S}, \mathcal{T} \subseteq 2^{\mathbb{N}}$, we define the *extension* $\mathcal{S}^{\vee} := \mathcal{S} \vee \{\{w + 1\}\}$ and disjoint union $\mathcal{S} \uplus \mathcal{T} := \mathcal{S} \cup \{A + w \mid A \in \mathcal{T}\}$, where $w := \max(\bigcup \mathcal{S})$ and $A + w := \{a + w \mid a \in A\}$.

Example 9.1. If $\mathcal{S} = \{\{1\}, \{2, 3\}\}$ and $\mathcal{T} = \{\{1, 2\}, \{3\}\}$, then $w = 3$ with $\mathcal{S}^{\vee} = \{\{1, 4\}, \{2, 3, 4\}\}$ and $\mathcal{S} \uplus \mathcal{T} = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}\}$.

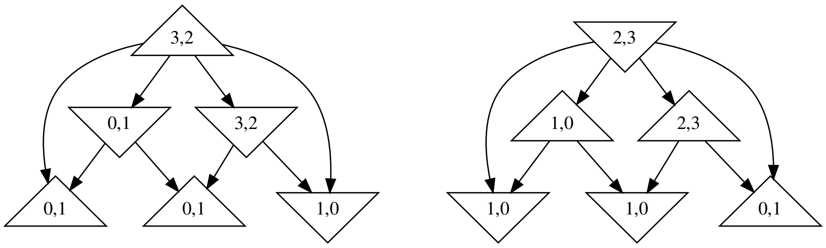


Figure 9.1: Structure diagrams for AVD(H) and AVD(K) with $H = (V, \mathcal{H})$ and $K = (V \cup \{w\}, \mathcal{H})$.

Since we need to deal with games on several hypergraphs, We will use the notations Opt_H, type_H, X_I^H to indicate the hypergraph H.

Proposition 9.2. Consider the avoid game on the simple hypergraphs $H = (V, \mathcal{H})$ and $K = (V \cup \{w\}, \mathcal{H})$ satisfying $w \notin V$. Then

1. $\mathcal{S}_K = \mathcal{S}_H \vee \{\{w\}\}$;
2. $\mathcal{I}_K = \mathcal{I}_H \vee \{\{w\}\}$;
3. $\phi_K(P) = \phi_H \vee \{\{w\}\}$;
4. $X^K_{J\cup\{w\}} \in \mathrm{Opt}_K(X^K_{I\cup\{w\}})$ if and only if $X^H_J \in \mathrm{Opt}_H(X^H_I)$;
5. $\mathrm{type}_H(I) = (p, a, b)$ implies $\mathrm{type}_K(I \cup \{w\}) = (1 - p, b, a)$.

Proof. Part (1) follows from the computation

$$\mathcal{S}_K = \{V \cup \{w\} \setminus A \mid A \in \mathrm{Tr}(\mathcal{H})\} = \{(V \setminus A) \cup \{w\} \mid A \in \mathrm{Tr}(\mathcal{H})\} = \mathcal{S}_H \vee \{\{w\}\}.$$

Parts (2–4) are immediate consequences. It is clear that $\mathrm{pty}(I \cup \{w\}) = 1 - \mathrm{pty}(I)$. The rest of Part (5) follows by structural induction on the structure classes. □

The result tells us how to construct the structure diagram of AVD(K) from the structure diagram of AVD(H). We need to flip the parity and swap the nim-values for each structure class. We demonstrate this on an example.

Example 9.3. Let $H = (V, \mathcal{H})$ with $V = \{1, \ldots, 5\}$ and

$$\mathcal{H} = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}\}.$$

Figure 9.1 shows the structure diagrams for AVD(H) and AVD(K) with and $K = (V \cup \{6\}, \mathcal{H})$. Note that $\mathcal{S}_H = \{\{1, 2\}, \{2, 3\}, \{3, 4, 5\}$ and

$$\mathcal{S}_K = \mathcal{S}_H^{\vee} = \{\{1, 2, 6\}, \{2, 3, 6\}, \{3, 4, 5, 6\}\}.$$

Proposition 9.4. Consider the avoid games on the hypergraphs $H = (V, \mathcal{H})$ and $K = (W, \mathcal{K})$ satisfying $V \cap W = \emptyset$ and $\Phi_H \neq \emptyset \neq \Phi_K$. If $G = (U, \mathcal{G})$ is a hypergraph satisfying $U = V \cup W$ and $\mathcal{S}_G = \mathcal{S}_H \cup \mathcal{S}_K$, then

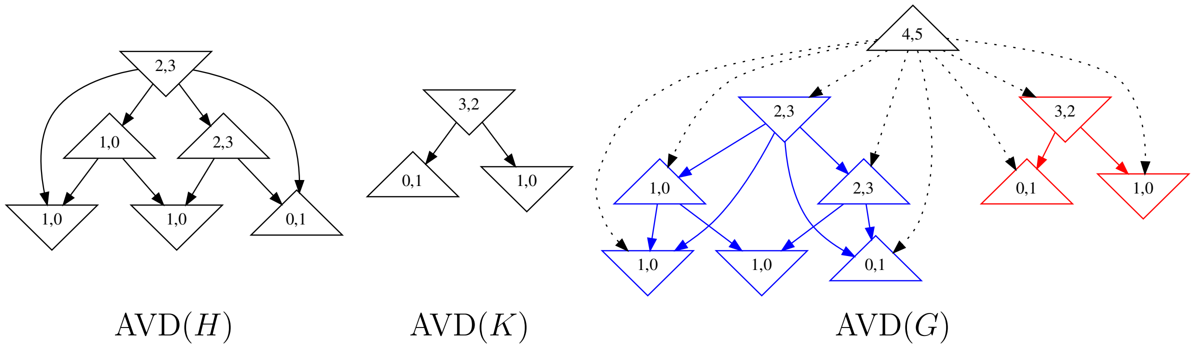


Figure 9.2: Structure diagrams for $\mathrm{AVD}(H)$, $\mathrm{AVD}(K)$, and $\mathrm{AVD}(G)$ with $\mathcal{S}_G = \mathcal{S}_H \cup \mathcal{S}_K$.

1. $\mathcal{I}_G = \mathcal{I}_H \cup \mathcal{I}_K \cup \{\emptyset\}$;
2. $X_I^G = X_I^H$ for $I \in \mathcal{I}_H \setminus \{\Phi_H\}$;
3. $X_I^G = X_I^K$ for $I \in \mathcal{I}_K \setminus \{\Phi_K\}$;
4. $X_{\Phi_H}^G = X_{\Phi_H}^H \setminus \{\emptyset\}$;
5. $X_{\Phi_K}^G = X_{\Phi_K}^K \setminus \{\emptyset\}$;
6. $X_{\Phi_G}^G = \{\emptyset\}$ and $\Phi_G = \emptyset$;
7. $\mathrm{Opt}_G(X_I^G) = \mathrm{Opt}_H(X_I^H)$ for $I \in \mathcal{I}_H$;
8. $\mathrm{Opt}_G(X_I^G) = \mathrm{Opt}_K(X_I^K)$ for $I \in \mathcal{I}_K$;
9. $\mathrm{Opt}_G(X_\emptyset^G) = \{X_{\Phi_H}^G, X_{\Phi_K}^G\} \cup \mathrm{Opt}_H(X_{\Phi_H}^H) \cup \mathrm{Opt}_K(X_{\Phi_K}^K)$.

Proof. Parts (1–6) are clear from the definitions. Parts (7,8) follow from the fact that a subset P of $V \cup W$ is not stable in G unless $P \subseteq V$ or $P \subseteq W$.

Now we show Part (9). Let $v \in \Phi_H$. Then $\{v\} \in \mathrm{Opt}_G(\emptyset) \cap X_{\Phi_H}^G$, and so $X_{\Phi_H}^G \in \mathrm{Opt}_G(X_\emptyset)$. If $J \in \mathcal{I}_H \setminus \{\Phi_H\}$ then

$$\begin{aligned} X_J^H \in \mathrm{Opt}_H(X_{\Phi_H}^H) &\Leftrightarrow \Phi_H \cup \{v\} \in X_J^H \text{ for some } v \in V \\ &\Leftrightarrow \emptyset \cup \{v\} \in X_J^H \text{ for some } v \in V \\ &\Leftrightarrow X_J^G = X_J^H \in \mathrm{Opt}_H(X_\emptyset^G) \end{aligned}$$

since $\emptyset \in X_{\Phi_H}^H$. Similar argument work for K. $\square$

The result tells us how to construct the structure diagram of $\mathrm{AVD}(G)$ from the structure diagrams of $\mathrm{AVD}(H)$ and $\mathrm{AVD}(K)$. Roughly speaking, we need to add the structure class $X_\emptyset^G$ and connect this class to the Frattini structure classes X_{Φ_H} and X_{Φ_K} of $\mathrm{AVD}(H)$ and $\mathrm{AVD}(K)$ and to the options of these Frattini structure classes. We demonstrate this on an example.

Example 9.5. Let $H = (\{1, \ldots, 6\}, \mathcal{H})$, $K = (\{7, \ldots, 10\}, \mathcal{K})$, and $G = (\{1, \ldots, 10\}, \mathcal{G})$ such that $\mathcal{S}_H = \{\{1,2,6\}, \{2,3,6\}, \{3,4,5,6\}\}$, $\mathcal{S}_K = \{\{7,8\}, \{8,9,10\}\}$, and $\mathcal{S}_G = \mathcal{S}_H \cup \mathcal{S}_K$. Figure 9.2 shows the structure diagrams for $\mathrm{AVD}(H)$, $\mathrm{AVD}(K)$, and $\mathrm{AVD}(G)$.

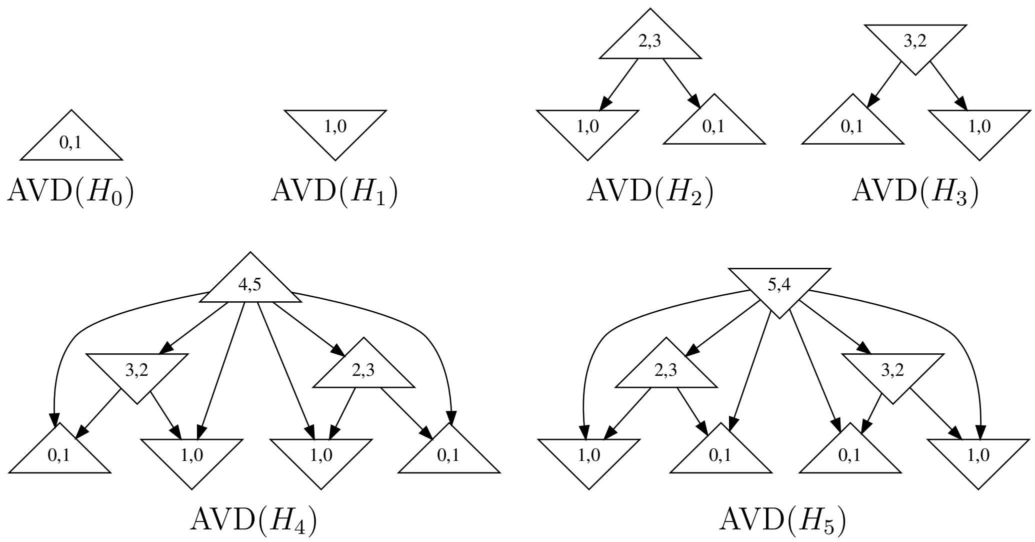


Figure 9.3: Structure diagrams for AVD(H_k).

We are now ready to construct a family H_k of hypergraphs satisfying $\operatorname{nim}(\operatorname{AVD}(H_K)) = k$ for each nonnegative integer k. The construction is done through the family of maximal stable sets.

Definition 9.6. We define $\mathcal{S}_0 := \{\emptyset\}$ and

$$\mathcal{S}_k := \begin{cases} \mathcal{S}_{k-1}^{\vee}, & k \text{ is odd} \\ \mathcal{S}_{k-2}^{\vee} \uplus \mathcal{S}_{k-1}^{\vee}, & k \text{ is even} \end{cases}$$

recursively. We also let $H_k := (V_k, \mathcal{H}_k)$ with $V_k := \bigcup \mathcal{S}_k$ and $\mathcal{H}_k := \operatorname{Tr}(\complement_{V_k}(\mathcal{S}_k))$.

Example 9.7. The table shows the construction of the first few H_k:

k	$\mathcal{S}_k$	V_k	$\mathcal{H}_k$
0	$\{\emptyset\}$	$\emptyset$	$\emptyset$
1	$\{\{1\}\}$	$\{1\}$	$\emptyset$
2	$\{\{1\},\{2,3\}\}$	$\{1,2,3\}$	$\{\{1,2\},\{1,3\}\}$
3	$\{\{1,4\},\{2,3,4\}\}$	$\{1,2,3,4\}$	$\{\{1,2\},\{1,3\}\}$
4	$\{\{1,4\},\{2,3,4\},\{5,8,9\},\{6,7,8,9\}\}$	$\{1,\dots,9\}$	$\{\{1,2\},\dots,\{5,7\}\}$

Note 9.8. We construct H_{2l+1} as the extension of H_{2l}. These two hypergraphs satisfy the assumptions of Proposition 9.2. To construct H_{2l+2}, we use the extension of H_{2l} and a shift of the extension of H_{2l+1}. These hypergraphs satisfy the assumptions of Proposition 9.4.

Example 9.9. Figure 9.3 shows the structure diagrams of AVD(H_k) for $k \in \{0,\dots,5\}$. Note that in every structure diagram X_Φ has every other structure class as an option.

Proposition 9.10. If k is a nonnegative integer, then $\operatorname{nim}(\mathrm{AVD}(H_K)) = k$.

Proof. We are going to use induction on k to show the following:

1. $X^{H_k}_{\Phi_{H_k}}$ has every other structure class as an option;
2. $\begin{aligned}\{\operatorname{type}(X^{H_k}_I) \mid I \in \mathcal{I}_{H_k}\} = {}&\{(0, 2l, 2l+1) \mid 0 \leqslant 2l \leqslant k - 2\operatorname{pty}(k)\}\cup\\ &\{(1, 2l+1, 2l) \mid 0 \leqslant 2l \leqslant k - 1 + \operatorname{pty}(k)\};\end{aligned}$
3. $\operatorname{type}(X^{H_k}_{\Phi_{H_k}}) = (\operatorname{pty}(k), k, k+1-2\operatorname{pty}(k))$

All claims are true for $k \in \{0, 1\}$ by Figure 9.3. For the inductive step we consider two cases based on the parity of k.

First assume that k is even. Then H_{k+1} is constructed from H_k by an extension. We use Proposition 9.2 to verify all three claims for H_{k+1}. Claim (1) is true for H_{k+1} since it is true for H_k by induction and the construction does not change the option structure of the diagram. Claim (2) follows from the computation

$$\begin{aligned}\{\operatorname{type}(X^{H_{k+1}}_I) \mid I \in \mathcal{I}_{H_{k+1}}\} &= \{(1, 2l+1, 2l) \mid 0 \leqslant 2l \leqslant k\}\cup\\ &\quad\ \{(0, 2l, 2l+1) \mid 0 \leqslant 2l \leqslant k-1\}\\ &= \{(0, 2l, 2l+1) \mid 0 \leqslant 2l \leqslant (k+1) - 2\operatorname{pty}(k+1)\}\cup\\ &\quad\ \{(1, 2l+1, 2l) \mid 0 \leqslant 2l \leqslant k - 1 + \operatorname{pty}(k+1)\}\end{aligned}$$

using Proposition 9.2(5). Claim (3) also follows from Proposition 9.2(5) since

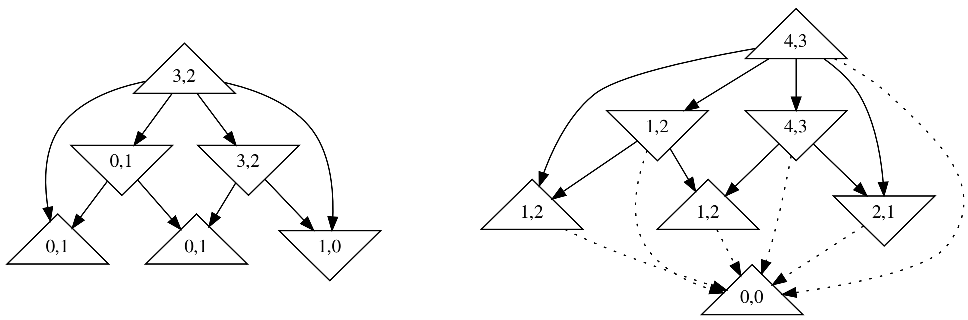
$$\operatorname{type}(X^{H_{k+1}}_{\Phi_{H_{k+1}}}) = (1, k+1, k)$$

by flipping and swapping $\operatorname{type}(X^{H_k}_{\Phi_{H_k}}) = (0, k, k+1)$.

Now assume that k is odd. Then H_{k+1} is constructed from H_{k-1} and H_k by extensions and a disjoint union. We use Propositions 9.2 and 9.4 to verify all three claims for H_{k+1}. Claim (1) is true for H_{k+1} by Proposition 9.4(9) since it is true for both H_{k-1} and H_k by induction. To verify Claim (3) let $(0, a, b) := \operatorname{type}(X^{H_{k+1}}_{\Phi_{H_{k+1}}})$. Then a and b can be computed from $\{\operatorname{type}(X^{H_{k-1}}_I) \mid I \in \mathcal{I}_{H_{k-1}}\}$ and $\{\operatorname{type}(X^{H_k}_I) \mid I \in \mathcal{I}_{H_k}\}$ using Proposition 9.4(9) and type calculus:

$$\begin{aligned}a &= \operatorname{mex}(\{2l \mid 0 \leqslant 2l \leqslant k-1\} \cup \{2l+1 \mid 0 \leqslant 2l \leqslant k-2\}\cup\\ &\qquad\quad \{2l \mid 0 \leqslant 2l \leqslant k-2\} \cup \{2l+1 \mid 0 \leqslant 2l \leqslant k\})\\ &= \operatorname{mex}(\{0, \ldots, k\}) = k+1,\end{aligned}$$

$$\begin{aligned}b &= \operatorname{mex}(\{2l+1 \mid 0 \leqslant 2l \leqslant k-1\} \cup \{2l \mid 0 \leqslant 2l \leqslant k-2\}\cup\\ &\qquad\quad \{2l+1 \mid 0 \leqslant 2l \leqslant k-2\} \cup \{2l \mid 0 \leqslant 2l \leqslant k\} \cup \{a\})\\ &= \operatorname{mex}(\{0, \ldots, k+1\}) = k+2.\end{aligned}$$



AVD(H) ACV(H)

Figure 9.4: Structure diagrams for AVD(H) and ACV(H).

Claim (2) follows from the computation

$$\begin{aligned}\{\text{type}(X_I^{H_{k+1}}) \mid I \in \mathcal{I}_{H_{k+1}}\} &= \{(0,2l,2l+1) \mid 0 \leqslant 2l \leqslant k-1\} \cup \\ &\quad \{(1,2l+1,2l) \mid 0 \leqslant 2l \leqslant k-2\} \cup \\ &\quad \{(0,2l,2l+1) \mid 0 \leqslant 2l \leqslant k-2\} \cup \\ &\quad \{(1,2l+1,2l) \mid 0 \leqslant 2l \leqslant k\} \cup \\ &\quad \{(0,k+1,k+2)\} \\ &= \{(0,2l,2l+1) \mid 0 \leqslant 2l \leqslant k+1\} \cup \\ &\quad \{(1,2l+1,2l) \mid 0 \leqslant 2l \leqslant (k+1)-1\}. \end{aligned}$$ □

Corollary 9.11. The nim-value of AVD(H) can be any nonnegative integer.

9.2 Nim-values of ACV

Our goal in this section is to show that the nim-value of ACV(H) can also be any nonnegative integer.

Example 9.12. Let $H = (V, \mathcal{H})$ with $V = \{1, \ldots, 6\}$ and $\mathcal{S}_H = \{\{1,2\},\{2,3\},\{3,4,5\}\}$. Figure 9.4 shows the structure diagrams for AVD(H) and ACV(H). Since $6 \notin \bigcup \mathcal{S}_H$, $\{6\} \in \mathcal{H}$. So $X_V \in \text{Opt}(X_I)$ for all $I \in \mathcal{I}_H \setminus V$. If $\text{type}(X_I) = (p, a, b)$ in AVD(H), then $\text{type}(X_I) = (p, a+1, b+1)$ in ACV(H).

The following result is a generalization of [17, Proposition 6.8].

Proposition 9.13. If $H = (V, \mathcal{H})$ is a hypergraph such that $V \setminus \bigcup \mathcal{S}_H \neq \emptyset$, then $\text{nim}(\text{ACV}(H)) = \text{nim}(\text{AVD}(H)) + 1$.

Proof. Let $w \in V \setminus \bigcup \mathcal{S}_H$. Then $\{w\} \in X_V$ for ACV(H). Hence $X_V \in \text{Opt}(X_I)$ for all $I \in \mathcal{I}_H \setminus V$ since $I \cup \{w\} \in X_V$. Thus the structure digraph of ACV(H) can be constructed from the structure digraph of AVD(H) by connecting every structure class to the additional structure class X_V. Structural induction on the structure classes combined with type calculus shows that the nim-values in $\text{type}(X_I)$ are one larger for ACV(H) than they were for AVD(H). □

Corollary 9.14. The nim-value of ACV(H) can be any nonnegative integer.

Proof. Figure 4.1 shows that the nim-value of ACV(H) can be 0. We insert an additional vertex w into V_k without altering $\mathcal{S}_k$ in Definition 9.6. This creates the hypergraph $\tilde{H}_k = (\tilde{V}_k, \mathrm{Tr}(\complement_{\tilde{V}_k}(\mathcal{S}_k)))$ with $\tilde{V}_k = V_k \cup \{w\}$. The avoidance game remains the same since marking this vertex is never allowed in AVD$(\tilde{H}_k)$. Thus $\mathrm{nim}(\mathrm{ACV}(\tilde{H}_k)) = \mathrm{nim}(\mathrm{AVD}(\tilde{H}_k)) + 1 = k + 1$. □

9.3 Nim-values of PRV and DST

Since the removing games are actually building games in disguise, we have the following.

Corollary 9.15. The nim-value of PRV(H) and the nim-value of DST(H) can be any non-negative integer.

10 Further questions

We finish with a few unresolved questions for further study.

1. We saw in Example 3.8 that not every impartial game can be realized as a hypergraph game. What properties must a digraph have to be the game digraph of a hypergraph game?
2. How to use information about the automorphism group of the hypergraph together with the structure diagram? A simplified structure diagram takes advantage of certain symmetries of the hypergraph but only after the structure diagram is built. Using the automorphism group in Algorithm 4.13 might speed up the computations.
3. Proposition 3.10 shows that the digraphs of ACV(H) and PRV(H) are complementary. Does this connection force any restriction on the nim-values of these games?
4. Hypergraphs can have many interesting properties. How do these properties translate into results about the nim-value of the corresponding hypergraph games?
5. The nim-value of a group generating game cannot be an arbitrary nonnegative integer [11]. This is a consequence of Lagrange's Theorem, which restricts the allowed families of maximal subgroups. Can we determine the spectrum of nim-values for the other games mentioned in Section 8.

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