Spectral radius conditions for the rigidity of graphs

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Abstract

Rigidity is the property of a structure that does not flex under an applied force. In the past several decades, the rigidity of graphs has been widely studied in discrete geometry and combinatorics. Laman (1970) obtained a combinatorial characterization of rigid graphs in \( \mathbb{R}^2 \). Lovász and Yemini (1982) proved that every 6-connected graph is rigid in \( \mathbb{R}^2 \). Jackson and Jordán (2005) strengthened this result, and showed that every 6-connected graph is globally rigid in \( \mathbb{R}^2 \). Thus every graph with algebraic connectivity greater than 5 is globally rigid in \( \mathbb{R}^2 \). In 2021, Cioabă, Dewar

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and Gu improved this bound, and proved that every graph with minimum degree at least 6 and algebraic connectivity greater than $2 + \frac{1}{\sqrt{3}}$ (resp., $2 + \frac{2}{\sqrt{3}}$) is rigid (resp., globally rigid) in $\mathbb{R}^2$. In this paper, we study the rigidity of graphs in $\mathbb{R}^2$ from the viewpoint of adjacency eigenvalues. Specifically, we provide a spectral radius condition for the rigidity (resp., globally rigidity) of 2-connected (resp., 3-connected) graphs with given minimum degree. Furthermore, we determine the unique graph attaining the maximum spectral radius among all minimally rigid graphs of order $n$.

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1 Introduction

Arising from mechanics, the rigidity of graphs is an important research object in discrete geometry and combinatorics, and has various applications in material science, engineering and biological science [5, 6, 9, 18].

A \textit{d-dimensional bar-and-joint framework} $(G, p)$ is the combination of an undirected simple graph $G = (V(G), E(G))$ and a map $p : V(G) \rightarrow \mathbb{R}^d$ that assigns a point in $\mathbb{R}^d$ to each vertex of $G$. Let $\| \cdot \|$ denote the Euclidean norm in $\mathbb{R}^d$. Two frameworks $(G, p)$ and $(G, q)$ are said to be \textit{equivalent} (resp., \textit{congruent}) if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all $uv \in E(G)$ (resp., for all $u, v \in V(G)$). A framework $(G, p)$ is \textit{generic} if the coordinates of its points are algebraically independent over $\mathbb{Q}$. The framework $(G, p)$ is \textit{rigid} in $\mathbb{R}^d$ if there exists an $\epsilon > 0$ such that every framework $(G, q)$ equivalent to $(G, p)$ satisfying $\|p(u) - q(u)\| < \epsilon$ for all $u \in V(G)$ is actually congruent to $(G, p)$. According to [1], a generic framework $(G, p)$ is rigid in $\mathbb{R}^d$ if and only if every generic framework of $G$ is rigid in $\mathbb{R}^d$. We say that a graph $G$ is \textit{rigid} in $\mathbb{R}^d$ if every/some generic framework of $G$ is rigid in $\mathbb{R}^d$, and is \textit{redundantly rigid} in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for every $e \in E(G)$. The framework $(G, p)$ is \textit{globally rigid} in $\mathbb{R}^d$ if every framework that is equivalent to $(G, p)$ is congruent to $(G, p)$. In [8], it was shown that if there exists a globally rigid generic framework $(G, p)$ in $\mathbb{R}^d$, then any other generic framework $(G, q)$ is also globally rigid in $\mathbb{R}^d$. For this reason, we say that a graph $G$ is \textit{globally rigid} in $\mathbb{R}^d$ if there exists a globally rigid generic framework $(G, p)$ in $\mathbb{R}^d$.

In 1970, Laman [20] provided a combinatorial characterization for rigid graphs in $\mathbb{R}^2$. Since then, some vertex- or edge-connectivity conditions for a graph to be rigid or globally rigid in $\mathbb{R}^2$ have been successively discovered. In 1982, Lovász and Yemini [21] constructed some 5-connected non-rigid graphs, and proved that every 6-connected graph is rigid. In 1992, Hendrickson [13] proved that every globally rigid graph with at least four vertices is 3-connected and redundantly rigid. In 2005, Jackson and Jordán [15] proved that every 6-connected graph is globally rigid. Later, they observed that a 6-edge-connected graph $G$ is globally rigid in $\mathbb{R}^2$, provided that $G - v$ is 4-edge-connected for all $v \in V(G)$ and $G - \{u, v\}$ is 2-edge-connected for all $u, v \in V(G)$ [16]. In 2007, Jackson, Servatius and Servatius [17] showed that every 4-connected essentially 6-connected graph (see [19] for the definition) is globally rigid. Very recently, Gu, Meng, Rolek, Wang and Yu[10] proved that every 3-connected essentially 9-connected graph is globally rigid. Naturally,
we consider the following problem:

**Problem 1.** Which spectral conditions can guarantee that a graph is rigid or globally rigid in $\mathbb{R}^2$?

For a graph $G$, let $D(G)$ denote the diagonal matrix of vertex degrees of $G$, and $A(G)$ denote the adjacency matrix of $G$. The Laplacian matrix of $G$ is defined as $L(G) = D(G) - A(G)$. The second least eigenvalue of $L(G)$, denoted by $\mu(G)$, is known as the algebraic connectivity of $G$. As the vertex-connectivity of $G$ is not less than $\mu(G)$, the results in [21, 15] imply that if $\mu(G) > 5$ then $G$ is globally rigid in $\mathbb{R}^2$. Based on some necessary conditions for packing rigid subgraphs, Cioabă, Dewar and Gu [3] strengthened this result, and proved that a graph $G$ with minimum degree $\delta \geq 6$ is rigid in $\mathbb{R}^2$ if $\mu(G) > 2 + \frac{1}{\delta - 1}$, and is globally rigid in $\mathbb{R}^2$ if $\mu(G) > 2 + \frac{2}{\delta - 1}$.

In this paper, we focus on giving some answers to Problem 1 in terms of the (adjacency) spectral radius of graphs. The spectral radius of a graph $G$, denoted by $\rho(G)$, is the largest eigenvalue of its adjacency matrix $A(G)$. A graph is $k$-connected if removing fewer than $k$ vertices always leaves the remaining graph connected. Let $K_n$ denote the complete graph on $n$ vertices, and $B_{n,n_1}^i$ denote the graph obtained from $K_{n_1} \cup K_{n-n_1}$ by adding $i$ independent edges (with no common endvertex) between $K_{n_1}$ and $K_{n-n_1}$. The main results are as follows.

**Theorem 2.** Let $G$ be a $2$-connected graph with minimum degree $\delta \geq 6$ and order $n \geq 2\delta + 4$. If $\rho(G) \geq \rho(B_{n,\delta+1}^2)$, then $G$ is rigid unless $G \cong B_{n,\delta+1}^2$.

Hendrickson [13] proved that every globally rigid graph in $\mathbb{R}^d$ with at least $d + 2$ vertices is $(d + 1)$-connected and redundantly rigid. Thus it is necessary to assume that $G$ is $3$-connected when we consider the global rigidity of $G$ in $\mathbb{R}^2$.

**Theorem 3.** Let $G$ be a $3$-connected graph with minimum degree $\delta \geq 6$ and order $n \geq 2\delta + 4$. If $\rho(G) \geq \rho(B_{n,\delta+1}^3)$, then $G$ is globally rigid unless $G \cong B_{n,\delta+1}^3$.

A graph $G$ is minimally rigid if $G$ is rigid but $G - e$ is not rigid for all $e \in E(G)$. Note that a graph is rigid if and only if it has a minimally rigid spanning subgraph. In 1970, Leman [20] provided a characterization for minimally rigid graphs in $\mathbb{R}^2$ by using the edge count property, and proved that a graph $G$ with $n$ vertices and $m$ edges is a minimally rigid if and only if $m = 2n - 3$ and $e_G(X) \leq 2|X| - 3$ for all $X \subseteq V(G)$ with $|X| \geq 2$, where $e_G(X)$ is the number of edges of the subgraph $G[X]$ induced by $X$ in $G$. Minimally rigid graphs are also called *Leman graphs* in $\mathbb{R}^2$.

The join of two graphs $G$ and $H$, denoted by $G \join H$, is the graph obtained from $G \cup H$ by adding all possible edges between $G$ and $H$. Based on Leman’s characterization for minimally rigid graphs in $\mathbb{R}^2$, we determine the unique graph attaining the maximum spectral radius among all connected minimally rigid graphs of order $n$ in $\mathbb{R}^2$.

**Theorem 4.** Let $G$ be a connected minimally rigid graph of order $n \geq 3$. Then $\rho(G) \leq \rho(K_2 \join (n - 2)K_1)$, with equality if and only if $G \cong K_2 \join (n - 2)K_1$. 

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2 Preliminaries

In this section, we list some basic concepts and lemmas which will be used later.

Let $M$ be a real $n \times n$ matrix, and let $X = \{1, 2, \ldots, n\}$. Given a partition $\pi : X = X_1 \cup X_2 \cup \cdots \cup X_k$, the matrix $M$ can be correspondingly partitioned as

$$M = \begin{pmatrix}
M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\
M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
M_{k,1} & M_{k,2} & \cdots & M_{k,k}
\end{pmatrix}.$$

The quotient matrix of $M$ with respect to $\pi$ is defined as the $k \times k$ matrix $B_{\pi} = (b_{i,j})_{i,j=1}^k$ where $b_{i,j}$ is the average value of all row sums of $M_{i,j}$. The partition $\pi$ is equitable if each block $M_{i,j}$ of $M$ has constant row sum $b_{i,j}$. In this situation, the corresponding quotient matrix $B_{\pi}$ is also called equitable.

Lemma 5. (Brouwer and Haemers [2]; Godsil and Royle[7]) Let $M$ be a real symmetric matrix, and let $\lambda_1(M)$ be the largest eigenvalue of $M$. If $B_\pi$ is an equitable quotient matrix of $M$, then the eigenvalues of $B_\pi$ are also eigenvalues of $M$. Furthermore, if $M$ is nonnegative and irreducible, then $\lambda_1(M) = \lambda_1(B_\pi)$.

Recall that $B_{n,a}^i$ denotes the graph obtained from $K_{n1} \cup K_{n-n1}$ by adding $i$ independent edges between $K_{n1}$ and $K_{n-n1}$.

Lemma 6. Let $i \geq 1$, $a \geq i + 1$ and $n \geq 2a + 2$. Then

$$\rho(B_{n,a+1}^i) < \rho(B_{n,a}^i).$$

Proof. Since $B_{n,a}^i$ contains $K_{n-a}$ as a proper subgraph, we have $\rho(B_{n,a}^i) > \rho(K_{n-a}) = n - a - 1$. Note that $A(B_{n,a}^i)$ has the equitable quotient matrix

$$C_\pi^a = \begin{bmatrix}
i - 1 & a - i & 1 & 0 \\
i & a - (i + 1) & 0 & 0 \\
1 & 0 & i - 1 & n - (a + i) \\
0 & 0 & i & n - (a + i + 1)
\end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of $C_\pi^a$ is

$$\varphi(C_\pi^a, x) = x^4 + (4-n)x^3 + (an-a^2-3n+5)x^2 + 2(an-a^2-i-n+1)x - i^2 + in - 2i.$$

Also note that $A(B_{n,a+1}^i)$ has the equitable quotient matrix $C_\pi^{a+1}$, which is obtained by replacing $a$ with $a + 1$ in $C_\pi^a$. As $n \geq 2a + 2$, we have

$$\varphi(C_\pi^{a+1}, x) - \varphi(C_\pi^a, x) = x(x + 2)(n - (2a + 1)) > 0$$

for all $x \geq n - a - 1$. This implies that $\lambda_1(C_\pi^{a+1}) < \lambda_1(C_\pi^a)$. Therefore, by Lemma 5, we have $\rho(B_{n,a+1}^i) < \rho(B_{n,a}^i)$, and the result follows. \hfill \Box
Lemma 7. (See [14, 22]) Let $G$ be a graph on $n$ vertices and $m$ edges with minimum degree $\delta \geq 1$. Then
\[
\rho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}},
\]
with equality if and only if $G$ is either a $\delta$-regular graph or a bidegreed graph in which each vertex is of degree either $\delta$ or $n - 1$.

Lemma 8. (See [14, 22]) For nonnegative integers $p$ and $q$ with $2q \leq p(p - 1)$ and $0 \leq x \leq p - 1$, the function $f(x) = (x - 1)/2 + \sqrt{2q - px + (1 + x)^2}/4$ is decreasing with respect to $x$.

Lemma 9. Let $a$ and $b$ be two positive integers. If $a \geq b$, then
\[
\binom{a}{2} + \binom{b}{2} < \binom{a + 1}{2} + \binom{b - 1}{2}.
\]

Proof. Note that $a \geq b$. Then
\[
\left(\frac{a + 1}{2}\right) + \left(\frac{b - 1}{2}\right) - \left(\frac{a}{2}\right) - \left(\frac{b}{2}\right) = a - b + 1 > 0.
\]
Thus the result follows. \hfill \Box

For $X \subseteq V(G)$, let $G[X]$ be the subgraph of $G$ induced by $X$, and let $e_G(X)$ be the number of edges in $G[X]$. Particularly, let $e(G) = e_G(V(G))$ denote the number of edges of $G$. For $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of edges with one endpoint in $X$ and one endpoint in $Y$, and $e_G(X, Y) = |E_G(X, Y)|$. In particular, let $\partial_G(X) = E_G(X, V(G) - X)$.

Lemma 10. (See [12]) Let $G$ be a graph with minimum degree $\delta$ and $U$ be a non-empty proper subset of $V(G)$. If $|\partial_G(U)| \leq \delta - 1$, then $|U| \geq \delta + 1$.

For any partition $\pi$ of $V(G)$, let $E_G(\pi)$ denote the set of edges in $G$ whose endpoints lie in different parts of $\pi$, and let $e_G(\pi) = |E_G(\pi)|$. A part is trivial if it contains a single vertex. Let $Z \subset V(G)$, and let $\pi$ be a partition of $V(G - Z)$ with $n_0$ trivial parts $v_1, v_2, \ldots, v_{n_0}$. Denote by $n_2(\pi) = \sum_{1 \leq i \leq n_0} |Z_i|$, where $Z_i$ is the set of vertices in $Z$ that are adjacent to $v_i$ for $1 \leq i \leq n_0$.

The following three lemmas about rigid graphs will play crucial roles in the proof of our main theorems.

Lemma 11. (See [11]) A graph $G$ contains $k$ edge-disjoint spanning rigid subgraphs if for every $Z \subset V(G)$ and every partition $\pi$ of $V(G - Z)$ with $n_0$ trivial parts and $n'_0$ nontrivial parts,
\[
e_{G-Z}(\pi) \geq k(3 - |Z|)n'_0 + 2kn_0 - 3k - n_Z(\pi).
\]

Lemma 12. (See [4, 16]) Let $G$ be a graph. Then $G$ is globally rigid if and only if either $G$ is a complete graph on at most three vertices or $G$ is 3-connected and redundantly rigid.
Lemma 13. (See [20]) A graph $G$ is a minimally rigid on $n$ vertices and $m$ edges if and only if $m = 2n - 3$ and

$$e_G(X) \leq 2|X| - 3$$

for $X \subseteq V(G)$ with $|X| \geq 2$.

3 Proof of the main theorems

In this section, we shall give the proofs of Theorems 2–4.

Proof of Theorem 2. Assume to the contrary that $G$ is not rigid. Then $G$ contains no spanning rigid subgraphs. By Lemma 11, there exist a subset $Z$ of $V(G)$ and a partition $\pi$ of $V(G - Z)$ with $n_0$ trivial parts $v_1, v_2, \ldots, v_{n_0}$ and $n'_0$ nontrivial parts $V_1, V_2, \ldots, V_{n'_0}$ such that

$$e_{G-Z}(\pi) \leq (3 - |Z|)n'_0 + 2n_0 - 4 - n_Z(\pi), \quad (1)$$

where $n_Z(\pi) = \sum_{1 \leq i \leq n_0} |Z_i|$, and $Z_i$ is the set of vertices in $Z$ that are adjacent to $v_i$ for $1 \leq i \leq n_0$. Note that $d_{G-Z}(v_i) \geq \delta - |Z_i|$. Then

$$e_{G-Z}(\pi) = \frac{1}{2} \left( \sum_{1 \leq i \leq n'_0} |\partial_{G-Z}(V_i)| + \sum_{1 \leq j \leq n_0} d_{G-Z}(v_j) \right)$$

$$\geq \frac{1}{2} \left( \sum_{1 \leq i \leq n'_0} |\partial_{G-Z}(V_i)| + \delta n_0 - \sum_{1 \leq j \leq n_0} |Z_j| \right) \quad (2)$$

$$\geq \frac{1}{2} \left( \sum_{1 \leq i \leq n'_0} |\partial_{G-Z}(V_i)| + 6n_0 - n_Z(\pi) \right) \quad (\text{since } \delta \geq 6),$$

and therefore,

$$e_{G-Z}(\pi) \geq 3n_0 - \frac{1}{2}n_Z(\pi). \quad (3)$$

We have the following two claims.

Claim 1. $|Z| \leq 2$.

Otherwise, $|Z| \geq 3$. By (1),

$$e_{G-Z}(\pi) \leq (3 - |Z|)n'_0 + 2n_0 - 4 - n_Z(\pi) \leq 2n_0 - 4 - n_Z(\pi).$$

Combining this with (3) yields that $n_0 + 4 + \frac{1}{2}n_Z(\pi) \leq 0$, which is impossible because $n_0 \geq 0$ and $n_Z(\pi) \geq 0$.

Claim 2. $n'_0 \geq 2$.

Otherwise, $n'_0 \leq 1$. By Claim 1, $0 \leq |Z| \leq 2$, and it follows from (1) that

$$e_{G-Z}(\pi) \leq (3 - |Z|)n'_0 + 2n_0 - 4 - n_Z(\pi) \leq 2n_0 - 1 - n_Z(\pi).$$
Combining this with (3), we have \( n_0 + 1 + \frac{1}{2}n_Z(\pi) \leq 0 \), which is also impossible.

Note that \( \rho(G) \geq \rho(B_{n, \delta+1}^2) > \rho(K_{n-\delta-1}) = n - \delta - 2 \). By Lemmas 7 and 8,

\[
e(G) > \frac{n^2}{2} - \frac{(2\delta + 3)n}{2} + (\delta + 1)^2.
\]

Since \( G \) is 2-connected,

\[
|\partial G - Z(V_i)| \geq 2 - |Z|
\]

for \( 1 \leq i \leq n_0' \). Recall that \( 0 \leq |Z| \leq 2 \) and \( n_0' \geq 2 \). We consider the following two situations.

**Case 1.** \( 0 \leq |Z| \leq 1 \).

First suppose that \( n_0' = 2 \). Then the partition \( \pi \) consists of two nontrivial parts \( V_1, V_2 \) and \( n_0 \) trivial parts. Putting (5) into (2), we get

\[
e_{G - Z}(\pi) \geq \frac{1}{2}(|\partial G - Z(V_1)| + |\partial G - Z(V_2)| + 6n_0 - n_Z(\pi)) \geq 2 - |Z| + 3n_0 - \frac{1}{2}n_Z(\pi).
\]

Combining this with (1) and \( n_0' = 2 \), we have

\[-n_0 - \frac{1}{2}n_Z(\pi) - |Z| \geq 0,
\]

and hence \( n_0 = 0, n_Z(\pi) = 0 \) and \( |Z| = 0 \) by the facts that \( n_0 \geq 0, n_Z(\pi) \geq 0 \) and \( |Z| \geq 0 \). This suggests that the partition \( \pi \) consists of two nontrivial parts \( V_1, V_2 \), and \( G - Z = G \).

Then \( V(G) = V_1 \cup V_2 \) and \( e_G(V_1, V_2) = e_G(\pi) \leq 2 \) by (1). Note that \( e_G(V_1, V_2) = \frac{1}{2}(|\partial_G(V_1)| + |\partial_G(V_2)|) \geq 2 \) by (5). Thus \( e_G(V_1, V_2) = 2 \). Let \( E_G(V_1, V_2) = \{f_1, f_2\} \). We assert that \( f_1 \) and \( f_2 \) are two independent edges. If not, suppose that \( f_1 \cap f_2 = \{u\} \). Then it is easy to see that \( u \) is a cut vertex of \( G \), which is impossible because \( G \) is 2-connected. Clearly, \( G \) is a spanning subgraph of \( B_{n, |V_1|}^2 \). Then

\[
\rho(G) \leq \rho(B_{n, |V_1|}^2),
\]

with equality if and only if \( G \cong B_{n, |V_1|}^2 \). Since \( \delta \geq 6 \) and \( |\partial_G(V_1)| = |\partial_G(V_2)| = 2 < \delta - 1 \), by Lemma 10, \( \min\{|V_1|, |V_2|\} \geq \delta + 1 \). Combining this with Lemma 6 and (6), we conclude that

\[
\rho(G) \leq \rho(B_{n, \delta+1}^2),
\]

with equality if and only if \( G \cong B_{n, \delta+1}^2 \). However, this is impossible because \( \rho(G) \geq \rho(B_{n, \delta+1}^2) \) and \( G \not\cong B_{n, \delta+1}^2 \).

Now suppose that \( n_0' \geq 3 \). Let \( \delta' \) denote the minimum degree of \( G - Z \). Then \( \delta' \geq \delta - |Z| \). If the partition \( \pi \) contains at most one nontrivial part, say \( V_j \) (\( 1 \leq j \leq n_0' \)), such that \( |\partial G - Z(V_j)| \leq \delta' - 1 \), then \( |\partial G - Z(V_i)| \geq \delta' \) for all \( i \in \{1, \ldots, n_0'\} \setminus \{j\} \). It follows that

\[
2e_{G - Z}(\pi) = \sum_{1 \leq i \leq n_0'} |\partial_G - Z(V_i)| + \sum_{1 \leq j \leq n_0} d_{G - Z}(v_j)
\]

\[
\geq (n_0' - 1)\delta' + 2 - |Z| + \delta n_0 - n_Z(\pi) \quad \text{(since } |\partial_G - Z(V_j)| \geq 2 - |Z|)\)
\[
(n'_0 - 1)(\delta - |Z|) + 2 - |Z| + \delta n_0 - n_Z(\pi) \quad (\text{since } \delta' \geq \delta - |Z|) \\
= 2(3 - |Z|)n'_0 + 4n_0 - 8 - 2n_Z(\pi) + (\delta - 6 + |Z|)n'_0 + (\delta - 4)n_0 - \delta + 10 + n_Z(\pi) \\
\geq 2(3 - |Z|)n'_0 + 4n_0 - 8 - 2n_Z(\pi) + 2\delta - 8 + 3|Z| + n_Z(\pi) \quad (\text{since } n'_0 \geq 3 \text{ and } n_0 \geq 0) \\
> 2(3 - |Z|)n'_0 + 4n_0 - 8 - 2n_Z(\pi) \quad (\text{since } \delta \geq 6, \text{ } n_Z(\pi) \geq 0 \text{ and } 0 \leq |Z| \leq 1),
\]

which contradicts (1). Therefore, the partition \( \pi \) contains at least two nontrivial parts, say \( V_1, V_2 \), such that \( |\partial G - Z(V_i)| \leq \delta' - 1 \) for \( i = 1, 2 \). Furthermore, by Lemma 10, we obtain \( |V_i| \geq \delta' + 1 \) for \( i = 1, 2 \). We first consider \( |Z| = 0 \). Then \( \delta' = \delta \), and \( |V_i| \geq \delta + 1 \) for \( i = 1, 2 \). If \( |V_i| = \max\{|V_1|, |V_2|, \ldots, |V_{n'_0}|\} \) or \( |V_2| = \max\{|V_1|, |V_2|, \ldots, |V_{n'_0}|\} \), since \( |V_i| \geq \delta + 1 \) and \( |V_j| \geq 2 \) for \( i = 1, 2 \) and \( 3 \leq j \leq n'_0 \), by Lemma 9,

\[
\sum_{1 \leq i \leq n'_0} e_G(V_i) \leq \left( \frac{\delta + 1}{2} \right) + \left( \frac{n - \delta - 3}{2} \right) + \left( \frac{2}{2} \right).
\]

If there exists a nontrivial part, say \( V_j \), such that \( |V_j| = \max\{|V_1|, |V_2|, \ldots, |V_{n'_0}|\} \) for some \( 3 \leq j \leq n'_0 \). Similarly,

\[
\sum_{1 \leq i \leq n'_0} e_G(V_i) \leq 2 \left( \frac{\delta + 1}{2} \right) + \left( \frac{n - 2\delta - 2}{2} \right).
\]

Since \( |V_i| \geq \delta + 1 \) for \( i = 1, 2 \) and \( V_3 \geq 2 \), we have \( n_0 \leq n - \sum_{1 \leq i \leq 3} |V_i| \leq n - 2\delta - 4 \) and \( n'_0 \leq \frac{n - (2\delta + 4) - n_0}{2} + 3 \). Note that \( G - Z = G \) and \( n_Z(\pi) = 0 \). Then

\[
e_G(\pi) \leq 3n'_0 + 2n_0 - 4 \leq \frac{3n}{2} - 3\delta - 1 + \frac{n_0}{2} \leq 2n - 4\delta - 3
\]

by (1). Thus,

\[
e(G) = \sum_{1 \leq i \leq n'_0} e_G(V_i) + \sum_{1 \leq i \leq n_0} e_G(v_i) + e_G(\pi) \\
\leq \max \left\{ \left( \frac{\delta + 1}{2} \right) + \left( \frac{n - \delta - 3}{2} \right) + \left( \frac{2}{2} \right), 2 \left( \frac{\delta + 1}{2} \right) + \left( \frac{n - 2\delta - 2}{2} \right) \right\} + e_G(\pi) \\
\leq \left( \frac{\delta + 1}{2} \right) + \left( \frac{n - \delta - 3}{2} \right) + \left( \frac{2}{2} \right) + e_G(\pi) \quad (\text{since } \delta \geq 6 \text{ and } n \geq 2\delta + 4) \\
\leq \frac{n^2}{2} - \frac{(2\delta + 3)n}{2} + \delta^2 + 4.
\]

Combining this with (4), we have \( \delta < \frac{3}{2} \), which is impossible because \( \delta \geq 6 \). Now assume that \( |Z| = 1 \). Note that \( \delta' \geq \delta - 1 \). Then \( |V_i| \geq \delta' + 1 \geq \delta \) for \( i = 1, 2 \). Since \( |V_3| \geq 2 \), we have \( n'_0 \leq \frac{n - |Z| - n_0 - \sum_{1 \leq i \leq 3} |V_i|}{2} + 3 \leq \frac{n - n_0 - 2\delta + 3}{2} \). Let \( Z = \{w\} \). Then \( d_G(w) - n_Z(\pi) \leq n - n_0 - 1 \), and it follows from (1) that

\[
e_{G - Z}(\pi) + d_G(\pi) \leq 2n'_0 + 2n_0 - 4 - n_Z(\pi) + d_G(w) \\
\leq 2n'_0 + n_0 + n - 5 \\
\leq 2n - 2\delta - 2.
\]
Again by Lemma 9, we obtain
\[
e(G) \leq \max \left\{ \left( \frac{\delta}{2} + \frac{n-|Z|}{2} \right) + \left( \frac{\delta}{2} \right), 2\left( \frac{\delta}{2} \right) + \left( \frac{n-|Z|-2\delta}{2} \right) \right\} + e_{G-Z}(\pi) + d_G(w) \\
\leq \left( \frac{\delta}{2} + \frac{n-|Z|}{2} \right) + \left( \frac{2}{2} \right) + e_{G-Z}(\pi) + d_G(w) (\text{since } \delta \geq 6 \text{ and } n \geq 2\delta + 4) \\
\leq \frac{n^2}{2} - \frac{(2\delta + 3)n}{2} + \delta^2 + \delta + 5.
\]

Combining this with (4), we have \( \delta < 4 \), which is also impossible.

**Case 2.** \(|Z| = 2\).

By (1), we have
\[
e_{G-Z}(\pi) \leq n'_0 + 2n_0 - 4 - n_Z(\pi). \tag{7}
\]

If \( 2 \leq n'_0 \leq 3 \), combining (2), (5) and (7), we have

\[
0 \leq \sum_{1 \leq i \leq n'_0} |\partial_{G-Z}(V_i)| \leq 2n'_0 - 8 - 2n_0 - n_Z(\pi) \leq -2,
\]

a contradiction. Thus \( n'_0 \geq 4 \). Let \( \delta' \) denote the minimum degree of \( G - Z \). Then \( \delta' \geq \delta - 2 \). If the partition \( \pi \) contains at most one nontrivial part, say \( V_j \) \((1 \leq j \leq n'_0)\), such that \( |\partial_{G-Z}(V_j)| \leq \delta' - 1 \), then \( |\partial_{G-Z}(V_i)| \geq \delta' \) for all \( i \in \{1, \ldots, n'_0\} \setminus \{j\} \). It follows that
\[
2e_{G-Z}(\pi) = \sum_{1 \leq i \leq n'_0} |\partial_{G-Z}(V_i)| + \sum_{1 \leq j \leq n_0} d_{G-Z}(v_j) \\
\geq (n'_0 - 1)\delta' + \delta n_0 - n_Z(\pi) \\
\geq (n'_0 - 1)(\delta - 2) + \delta n_0 - n_Z(\pi) (\text{since } \delta' \geq \delta - 2) \\
= (2n'_0 + 4n_0 - 8 - 2n_Z(\pi)) + (\delta - 4)n'_0 - \delta + (\delta - 4)n_0 + n_Z(\pi) + 10 \\
\geq 2n'_0 + 4n_0 - 8 - 2n_Z(\pi) + 3\delta - 6 (\text{since } n'_0 \geq 4, n_0 \geq 0 \text{ and } n_Z(\pi) \geq 0) \\
> 2n'_0 + 4n_0 - 8 - 2n_Z(\pi) (\text{since } \delta \geq 6),
\]

contrary to (7). Therefore, the partition \( \pi \) contains at least two nontrivial parts, say \( V_1, V_2 \), such that \( |\partial_{G-Z}(V_i)| \leq \delta' - 1 \) for \( i = 1, 2 \). Furthermore, by Lemma 10, \( |V_i| \geq \delta' + 1 \geq \delta - 1 \) for \( i = 1, 2 \), and hence \( n'_0 \leq \frac{n - |Z| - 2(\delta - 1)}{2} + 2 = \frac{n}{2} - \delta + 2 \). Since \(|Z| = 2\), we have \( |\partial_G(Z)| + e_G(Z) - n_Z(\pi) \leq 2(n - 2 - n_0) + 1 \), and it follows from (7) that
\[
e_{G-Z}(\pi) + |\partial_G(Z)| + e_G(Z) \leq n'_0 + 2n - 7 \leq \frac{5n}{2} - \delta - 5.
\]

Recall that \( \delta \geq 6 \) and \( n \geq 2\delta + 4 \). By Lemma 9,
\[
e(G) \leq \max \left\{ \left( \frac{\delta - 1}{2} + \frac{2}{2} + \frac{n - |Z| - \delta - 3}{2} \right), 2\left( \frac{\delta - 1}{2} \right) + \left( \frac{2}{2} + \frac{n - |Z| - 2\delta}{2} \right) \right\} + e_{G-Z}(\pi) + |\partial_G(Z)| + e_G(Z)
\]

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\[
\leq \left( \frac{\delta-1}{2} \right) + 2\left( \frac{2}{2} \right) + \frac{n-|Z|-\delta-3}{2} + e_{G-Z}(\pi) + |\partial_{G}(Z)| + e_{G}(Z)
\]
\[
\leq \frac{n^2}{2} - \frac{(2\delta+6)n}{2} + \delta^2 + 3\delta + 13.
\]
Combining this with (4), we obtain \( n < \frac{2}{3}\delta + 8 \), which is impossible because \( n \geq 2\delta + 4 \) and \( \delta \geq 6 \).

This completes the proof. \( \square \)

Recall that, for any partition \( \pi \) of \( V(G) \), \( E_{G}(\pi) \) denotes the set of edges in \( G \) whose ends lie in different parts of \( \pi \), and \( e_{G}(\pi) = |E_{G}(\pi)| \).

**Proof of Theorem 3.** Assume to the contrary that \( G \) is not globally rigid. Since \( G \) is a 3-connected graph with minimum degree \( \delta \geq 6 \) and order \( n \geq 2\delta + 4 \), by Lemma 12, we see that \( G \) is not redundantly rigid. This suggests that there exists an edge \( f \) of \( G \) such that \( G - f \) is not rigid. Furthermore, by Lemma 11, there exist a subset \( Z \) of \( V(G) \) and a partition \( \pi \) of \( V(G - f - Z) \) with \( n_0 \) trivial parts \( v_1, v_2, \ldots, v_{n_0} \) and \( n_0' \) nontrivial parts \( V_1, V_2, \ldots, V_{n_0'} \) such that

\[
e_{G-f-Z}(\pi) \leq (3 - |Z|)n_0' + 2n_0 - 3 - n_Z(\pi). \tag{8}
\]

First we assume that \( f \in E_{G-Z}(\pi) \). Then \( e_{G-f-Z}(\pi) = e_{G-Z}(\pi) - 1 \). By (8),

\[
e_{G-Z}(\pi) \leq (3 - |Z|)n_0' + 2n_0 - 3 - n_Z(\pi). \tag{9}
\]

Recall that \( n_Z(\pi) = \sum_{1 \leq i \leq n_0} |Z_i| \), where \( Z_i \) is the set of vertices in \( Z \) that are adjacent to \( v_i \) for \( 1 \leq i \leq n_0 \). Note that \( d_{G-Z}(v_i) \geq \delta - |Z_i| \). Then

\[
e_{G-Z}(\pi) = \frac{1}{2} \left( \sum_{1 \leq i \leq n_0'} |\partial_{G-Z}(V_i)| + \sum_{1 \leq j \leq n_0} d_{G-Z}(v_j) \right)
\]
\[
\geq \frac{1}{2} \left( \sum_{1 \leq i \leq n_0'} |\partial_{G-Z}(V_i)| + 6n_0 - n_Z(\pi) \right) \quad \text{(since \( \delta \geq 6 \)),}
\]

and hence

\[
e_{G-Z}(\pi) \geq 3n_0 - \frac{1}{2}n_Z(\pi). \tag{11}
\]

We have the following two claims.

**Claim 1.** \( |Z| \leq 2 \).

Otherwise, \( |Z| \geq 3 \). By (9),

\[
e_{G-Z}(\pi) \leq (3 - |Z|)n_0' + 2n_0 - 3 - n_Z(\pi) \leq 2n_0 - 3 - n_Z(\pi).
\]

Combining this with (11), we have \( n_0 + \frac{1}{2}n_Z(\pi) + 3 \leq 0 \), which is impossible because \( n_0 \geq 0 \) and \( n_Z(\pi) \geq 0 \).
Claim 2. \( n'_0 \geq 2 \).

Otherwise, \( n'_0 \leq 1 \). By Claim 1, \( 0 \leq |Z| \leq 2 \), and it follows from (9) that

\[
e_G-Z(\pi) \leq (3 - |Z|)n'_0 + 2n_0 - 3 - n_Z(\pi) \leq 2n_0 - n_Z(\pi).
\]

(12)

Combining this with (11), we have

\[
n_0 + \frac{1}{2} n_Z(\pi) \leq 0.
\]

This implies that all equalities hold in (11) and (12), and hence \( n'_0 = 1 \), \( n_0 = 0 \), \( n_Z(\pi) = 0 \) and \( |Z| = 0 \). Then from (8) we deduce that \( e_{G-f-Z}(\pi) \leq -1 \), a contradiction.

Note that \( \rho(G) \geq \rho(B^3_{n,\delta+1}) > \rho(K_{n-\delta-1}) = n - \delta - 2 \). By Lemmas 7 and 8,

\[
e(G) > \frac{n^2}{2} - \frac{(2\delta + 3)n}{2} + (\delta + 1)^2.
\]

(13)

Since \( G \) is 3-connected,

\[
|\partial_{G-Z}(V_i)| \geq 3 - |Z|.
\]

(14)

Recall that \( 0 \leq |Z| \leq 2 \) and \( n'_0 \geq 2 \). We consider the following two situations.

Case 1. \( 0 \leq |Z| \leq 1 \).

First suppose that \( n'_0 = 2 \). Then the partition \( \pi \) consists of two nontrivial parts \( V_1, V_2 \) and \( n_0 \) trivial parts. Putting (14) into (10), we get

\[
e_{G-Z}(\pi) \geq \frac{1}{2}(|\partial_{G-Z}(V_1)| + |\partial_{G-Z}(V_2)| + 6n_0 - n_Z(\pi)) \geq 3 - |Z| + 3n_0 - \frac{1}{2} n_Z(\pi).
\]

Combining this with (9) and \( n'_0 = 2 \), we have

\[
-n_0 - \frac{1}{2} n_Z(\pi) - |Z| \geq 0,
\]

and hence \( n_0 = 0 \), \( n_Z(\pi) = 0 \) and \( |Z| = 0 \) by the facts \( n_0 \geq 0 \), \( n_Z(\pi) \geq 0 \) and \( |Z| \geq 0 \). This suggests that the partition \( \pi \) consists of two nontrivial parts \( V_1, V_2 \), and \( G-Z = G \). Then \( V(G) = V_1 \cup V_2 \) and \( e_G(V_1, V_2) = e_G(\pi) \leq 3 \) by (9). Note that \( e_G(V_1, V_2) = \frac{1}{2}(|\partial_G(V_1)| + |\partial_G(V_2)|) \geq 3 \) by (14). Thus \( e_G(V_1, V_2) = 3 \). Let \( E_G(V_1, V_2) = \{f_1, f_2, f\} \). We assert that \( f_1, f_2, f \) are three independent edges. If not, then \( G \) cannot be 3-connected, a contradiction. Observe that \( G \) is a spanning subgraph of \( B^3_{n,|V_1|} \). Then

\[
\rho(G) \leq \rho(B^3_{n,|V_1|}),
\]

(15)

with equality if and only if \( G \cong B^3_{n,|V_1|} \). Since \( \delta \geq 6 \) and \( |\partial_G(V_1)| = |\partial_G(V_2)| = 3 < \delta - 1 \), by Lemma 10, \( \min\{|V_1|, |V_2|\} \geq \delta + 1 \). Combining this with Lemma 6 and (15), we have

\[
\rho(G) \leq \rho(B^3_{n,\delta+1}),
\]

with equality if and only if \( G \cong B^3_{n,\delta+1} \). However, this is impossible because \( \rho(G) \geq \rho(B^3_{n,\delta+1}) \) and \( G \cong B^3_{n,\delta+1} \). If \( n'_0 \geq 3 \), by using (13) and a similar analysis as in Case 1 of Theorem 2, we also can deduce a contradiction.
Case 2. $|Z| = 2$.

In this case, the proof is similar as in Case 2 of Theorem 2, and we omit it.

Now we assume that $f \notin E_{G-Z}(\pi)$. Then

$$e_{G-Z}(\pi) = e_{G-f-Z}(\pi) \leq (3 - |Z|)n_0 + 2n_0 - 4 - n_Z(\pi).$$

By similar arguments as above, we also can deduce a contradiction.

This completes the proof.

\[\square\]

Proof of Theorem 4. Suppose that $G$ has the maximum spectral radius among all minimally rigid graphs of order $n \geq 3$. By Lemma 13, we have $e(G) = 2n - 3$ and $e_G(X) \leq 2|X| - 3$ for all $X \subseteq V(G)$ with $|X| \geq 2$. Note that $K_2 \nabla(n - 2)K_1$ is a minimally rigid graph. Then

$$\rho(G) \geq \rho(K_2 \nabla(n - 2)K_1) = \frac{1 + \sqrt{8n - 15}}{2}. \hspace{1cm} (16)$$

Let $\delta$ denote the minimum degree of $G$. We assert that $\delta \geq 2$. In fact, if there exists some vertex $u \in V(G)$ such that $d_G(u) = 1$, then $e_G(V(G) \setminus \{u\}) = 2n - 4$. However, since $V(G) \setminus \{u\} = n - 1 \geq 2$, we have $e_G(V(G) \setminus \{u\}) \leq 2|V(G) \setminus \{u\}| - 3 = 2n - 5$ by the above argument, a contradiction. Then, by Lemmas 7 and 8,

$$\rho(G) \leq \frac{1}{2} + \sqrt{2e(G) - 2n + \frac{9}{4}} = \frac{1 + \sqrt{8n - 15}}{2}. \hspace{1cm} (17)$$

Thus the equalities hold in (16) and (17). It follows that $\delta = 2$ and $G$ is either a 2-regular graph, or a bigraphed graph in which each vertex is of degree 2 or $n - 1$ by Lemma 7. If $n = 3$, then $G \cong K_3$, as required. Now suppose that $n \geq 4$. Let $t = |\{v \in V(G) | d_G(v) = n - 1\}|$. If $0 \leq t \leq 1$, then $e(G) < 2n - 3$, and if $t \geq 3$ then $e(G) > 2n - 3$, both are impossible. Thus $t = 2$, and $G \cong K_2 \nabla(n - 2)K_1$.

This completes the proof. \[\square\]

4 Concluding remarks

In this paper, we provide a spectral radius condition for the rigidity (resp., globally rigidity) of 2-connected (resp., 3-connected) graphs with given minimum degree in $\mathbb{R}^2$. In particular, we give the answers to Problem 1 for $k = 2, 3$. Note that every 6-connected graph is rigid (resp., globally rigid). Thus, the Problem 1 becomes more involved for $k = 4, 5$. When $k = 4, 5$, by using similar analysis as Theorems 2 and 3, we can obtain that a $k$-connected graph $G$ is rigid (resp., globally rigid) if $\rho(G) > \rho(B^k_{n,\delta+1})$. As $B^k_{n,\delta+1}$ is both rigid and globally rigid for $k = 4, 5$, we end the paper by proposing the following problem for further research.

Problem 14. Let $k \in \{4, 5\}$, and let $G$ be a $k$-connected graph with minimum degree $\delta \geq 6$ and order $n \geq 2\delta + 4$. Is it true that $G$ is rigid (resp. globally rigid) when $\rho(G) \geq \rho(B^k_{n,\delta+1})$?
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