Spectral radius conditions for the rigidity of graphs

Dandan Fan*

School of Mathematics
East China University of Science and Technology
Shanghai 200237, China

College of Mathematics and Physics
Xinjiang Agricultural University
Urumqi, Xinjiang 830052, China

ddfan0526@163.com

Xueyi Huang

School of Mathematics
East China University of Science and Technology
Shanghai 200237, China

huangxymath@163.com

Huiqiu Lin†

School of Mathematics
East China University of Science and Technology
Shanghai 200237, China

huiqiulin@126.com

Submitted: Jun 8, 2022; Accepted: Apr 6, 2023; Published: Apr 21, 2023

Abstract

Rigidity is the property of a structure that does not flex under an applied force. In the past several decades, the rigidity of graphs has been widely studied in discrete geometry and combinatorics. Laman (1970) obtained a combinatorial characterization of rigid graphs in $\mathbb{R}^2$. Lovász and Yemini (1982) proved that every 6-connected graph is rigid in $\mathbb{R}^2$. Jackson and Jordán (2005) strengthened this result, and showed that every 6-connected graph is globally rigid in $\mathbb{R}^2$. Thus every graph with algebraic connectivity greater than 5 is globally rigid in $\mathbb{R}^2$. In 2021, Cioabă, Dewar

*Sponsored by Natural Science Foundation of Xinjiang Uygur Autonomous Region (Grant Nos. 2022D01B103).

†Corresponding author. Supported by the National Natural Science Foundation of China (Grant Nos. 12271162), Natural Science Foundation of Shanghai (No. 22ZR1416300).

and Gu improved this bound, and proved that every graph with minimum degree at least 6 and algebraic connectivity greater than $2+\frac{1}{\delta-1}$ (resp., $2+\frac{2}{\delta-1}$) is rigid (resp., globally rigid) in $\mathbb{R}^2$. In this paper, we study the rigidity of graphs in $\mathbb{R}^2$ from the viewpoint of adjacency eigenvalues. Specifically, we provide a spectral radius condition for the rigidity (resp., globally rigidity) of 2-connected (resp., 3-connected) graphs with given minimum degree. Furthermore, we determine the unique graph attaining the maximum spectral radius among all minimally rigid graphs of order n.

Mathematics Subject Classifications: 05C50

1 Introduction

Arising from mechanics, the rigidity of graphs is an important research object in discrete geometry and combinatorics, and has various applications in material science, engineering and biological science [5, 6, 9, 18].

A *d-dimensional bar-and-joint framework* (G, p) is the combination of an undirected simple graph $G = (V(G), E(G))$ and a map $p : V(G) \to \mathbb{R}^d$ that assigns a point in $\mathbb{R}^d$ to each vertex of G. Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are said to be *equivalent* (resp., *congruent*) if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ holds for all $uv \in E(G)$ (resp., for all $u, v \in V(G)$). A framework (G, p) is *generic* if the coordinates of its points are algebraically independent over $\mathbb{Q}$. The framework (G, p) is *rigid* in $\mathbb{R}^d$ if there exists an $\varepsilon > 0$ such that every framework (G, q) equivalent to (G, p) satisfying $\|p(u) - q(u)\| < \varepsilon$ for all $u \in V(G)$ is actually congruent to (G, p). According to [1], a generic framework (G, p) is rigid in $\mathbb{R}^d$ if and only if every generic framework of G is rigid in $\mathbb{R}^d$. We say that a graph G is *rigid* in $\mathbb{R}^d$ if every/some generic framework of G is rigid in $\mathbb{R}^d$, and is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for every $e \in E(G)$. The framework (G, p) is *globally rigid* in $\mathbb{R}^d$ if every framework that is equivalent to (G, p) is congruent to (G, p). In [8], it was shown that if there exists a globally rigid generic framework (G, p) in $\mathbb{R}^d$, then any other generic framework (G, q) is also globally rigid in $\mathbb{R}^d$. For this reason, we say that a graph G is *globally rigid* in $\mathbb{R}^d$ if there exists a globally rigid generic framework (G, p) in $\mathbb{R}^d$.

In 1970, Laman [20] provided a combinatorial characterization for rigid graphs in $\mathbb{R}^2$. Since then, some vertex- or edge-connectivity conditions for a graph to be rigid or globally rigid in $\mathbb{R}^2$ have been successively discovered. In 1982, Lovász and Yemini [21] constructed some 5-connected non-rigid graphs, and proved that every 6-connected graph is rigid. In 1992, Hendrickson [13] proved that every globally rigid graph with at least four vertices is 3-connected and redundantly rigid. In 2005, Jackson and Jordán [15] proved that every 6-connected graph is globally rigid. Later, they observed that a 6-edge-connected graph G is globally rigid in $\mathbb{R}^2$, provided that $G - v$ is 4-edge-connected for all $v \in V(G)$ and $G - \{u, v\}$ is 2-edge-connected for all $u, v \in V(G)$ [16]. In 2007, Jackson, Servatius and Servatius [17] showed that every 4-connected essentially 6-connected graph (see [19] for the definition) is globally rigid. Very recently, Gu, Meng, Rolek, Wang and Yu[10] proved that every 3-connected essentially 9-connected graph is globally rigid. Naturally, we consider the following problem:

Problem 1. Which spectral conditions can guarantee that a graph is rigid or globally rigid in $\mathbb{R}^2$?

For a graph G, let $D(G)$ denote the diagonal matrix of vertex degrees of G, and $A(G)$ denote the adjacency matrix of G. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$. The second least eigenvalue of $L(G)$, denoted by $\mu(G)$, is known as the *algebraic connectivity* of G. As the vertex-connectivity of G is not less than $\mu(G)$, the results in [21, 15] imply that if $\mu(G) > 5$ then G is globally rigid in $\mathbb{R}^2$. Based on some necessary conditions for packing rigid subgraphs, Cioabă, Dewar and Gu [3] strengthened this result, and proved that a graph G with minimum degree $\delta \geqslant 6$ is rigid in $\mathbb{R}^2$ if $\mu(G) > 2 + \frac{1}{\delta-1}$, and is globally rigid in $\mathbb{R}^2$ if $\mu(G) > 2 + \frac{2}{\delta-1}$.

In this paper, we focus on giving some answers to Problem 1 in terms of the (adjacency) spectral radius of graphs. The *spectral radius* of a graph G, denoted by $\rho(G)$, is the largest eigenvalue of its adjacency matrix $A(G)$. A graph is *k-connected* if removing fewer than k vertices always leaves the remaining graph connected. Let K_n denote the complete graph on n vertices, and B^i_{n,n_1} denote the graph obtained from $K_{n_1} \cup K_{n-n_1}$ by adding i independent edges (with no common endvertex) between K_{n_1} and K_{n-n_1}. The main results are as follows.

Theorem 2. *Let G be a 2-connected graph with minimum degree $\delta \geqslant 6$ and order $n \geqslant 2\delta + 4$. If $\rho(G) \geqslant \rho(B^2_{n,\delta+1})$, then G is rigid unless $G \cong B^2_{n,\delta+1}$.*

Hendrickson [13] proved that every globally rigid graph in $\mathbb{R}^d$ with at least $d + 2$ vertices is $(d + 1)$-connected and redundantly rigid. Thus it is necessary to assume that G is 3-connected when we consider the global rigidity of G in $\mathbb{R}^2$.

Theorem 3. *Let G be a 3-connected graph with minimum degree $\delta \geqslant 6$ and order $n \geqslant 2\delta + 4$. If $\rho(G) \geqslant \rho(B^3_{n,\delta+1})$, then G is globally rigid unless $G \cong B^3_{n,\delta+1}$.*

A graph G is *minimally rigid* if G is rigid but $G - e$ is not rigid for all $e \in E(G)$. Note that a graph is rigid if and only if it has a minimally rigid spanning subgraph. In 1970, Leman [20] provided a characterization for minimally rigid graphs in $\mathbb{R}^2$ by using the edge count property, and proved that a graph G with n vertices and m edges is a minimally rigid if and only if $m = 2n - 3$ and $e_G(X) \leqslant 2|X| - 3$ for all $X \subseteq V(G)$ with $|X| \geqslant 2$, where $e_G(X)$ is the number of edges of the subgraph $G[X]$ induced by X in G. Minimally rigid graphs are also called *Leman graphs* in $\mathbb{R}^2$.

The *join* of two graphs G and H, denoted by $G\nabla H$, is the graph obtained from $G \cup H$ by adding all possible edges between G and H. Based on Leman's characterization for minimally rigid graphs in $\mathbb{R}^2$, we determine the unique graph attaining the maximum spectral radius among all connected minimally rigid graphs of order n in $\mathbb{R}^2$.

Theorem 4. *Let G be a connected minimally rigid graph of order $n \geqslant 3$. Then $\rho(G) \leqslant \rho(K_2\nabla(n-2)K_1)$, with equality if and only if $G \cong K_2\nabla(n-2)K_1$.*

2 Preliminaries

In this section, we list some basic concepts and lemmas which will be used later.

Let M be a real $n \times n$ matrix, and let $X = \{1, 2, \ldots, n\}$. Given a partition $\pi : X = X_1 \cup X_2 \cup \cdots \cup X_k$, the matrix M can be correspondingly partitioned as

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{pmatrix}.$$

The *quotient matrix* of M with respect to π is defined as the $k \times k$ matrix $B_\pi = (b_{i,j})_{i,j=1}^k$ where $b_{i,j}$ is the average value of all row sums of $M_{i,j}$. The partition π is *equitable* if each block $M_{i,j}$ of M has constant row sum $b_{i,j}$. In this situation, the corresponding quotient matrix B_π is also called *equitable*.

Lemma 5. *(Brouwer and Haemers [2]; Godsil and Royle[7]) Let M be a real symmetric matrix, and let $\lambda_1(M)$ be the largest eigenvalue of M. If B_π is an equitable quotient matrix of M, then the eigenvalues of B_π are also eigenvalues of M. Furthermore, if M is nonnegative and irreducible, then $\lambda_1(M) = \lambda_1(B_\pi)$.*

Recall that B^i_{n,n_1} denotes the graph obtained from $K_{n_1} \cup K_{n-n_1}$ by adding i independent edges between K_{n_1} and K_{n-n_1}.

Lemma 6. *Let $i \geqslant 1$, $a \geqslant i+1$ and $n \geqslant 2a+2$. Then*

$$\rho(B^i_{n,a+1}) < \rho(B^i_{n,a}).$$

Proof. Since $B^i_{n,a}$ contains K_{n-a} as a proper subgraph, we have $\rho(B^i_{n,a}) > \rho(K_{n-a}) = n-a-1$. Note that $A(B^i_{n,a})$ has the equitable quotient matrix

$$C^a_\pi = \begin{bmatrix} i-1 & a-i & 1 & 0 \\ i & a-(i+1) & 0 & 0 \\ 1 & 0 & i-1 & n-(a+i) \\ 0 & 0 & i & n-(a+i+1) \end{bmatrix}.$$

By a simple calculation, the characteristic polynomial of C^a_π is

$$\varphi(C^a_\pi, x) = x^4 + (4-n)x^3 + (an-a^2-3n+5)x^2 + 2(an-a^2-i-n+1)x - i^2 + in - 2i.$$

Also note that $A(B^i_{n,a+1})$ has the equitable quotient matrix C^{a+1}_π, which is obtained by replacing a with $a+1$ in C^a_π. As $n \geqslant 2a+2$, we have

$$\varphi(C^{a+1}_\pi, x) - \varphi(C^a_\pi, x) = x(x+2)(n-(2a+1)) > 0$$

for all $x \geqslant n-a-1$. This implies that $\lambda_1(C^{a+1}_\pi) < \lambda_1(C^a_\pi)$. Therefore, by Lemma 5, we have $\rho(B^i_{n,a+1}) < \rho(B^i_{n,a})$, and the result follows. □

Lemma 7. *(See [14, 22]) Let G be a graph on n vertices and m edges with minimum degree $\delta \geqslant 1$. Then*

$$\rho(G) \leqslant \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}},$$

with equality if and only if G is either a δ-regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Lemma 8. *(See [14, 22]) For nonnegative integers p and q with $2q \leqslant p(p-1)$ and $0 \leqslant x \leqslant p-1$, the function $f(x) = (x-1)/2 + \sqrt{2q - px + (1+x)^2/4}$ is decreasing with respect to x.*

Lemma 9. *Let a and b be two positive integers. If $a \geqslant b$, then*

$$\binom{a}{2} + \binom{b}{2} < \binom{a+1}{2} + \binom{b-1}{2}.$$

Proof. Note that $a \geqslant b$. Then

$$\binom{a+1}{2} + \binom{b-1}{2} - \binom{a}{2} - \binom{b}{2} = a - b + 1 > 0.$$

Thus the result follows. □

For $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X, and let $e_G(X)$ be the number of edges in $G[X]$. Particularly, let $e(G) = e_G(V(G))$ denote the number of edges of G. For $X, Y \subseteq V(G)$, we denote by $E_G(X,Y)$ the set of edges with one endpoint in X and one endpoint in Y, and $e_G(X,Y) = |E_G(X,Y)|$. In particular, let $\partial_G(X) = E_G(X, V(G) - X)$.

Lemma 10. *(See [12]) Let G be a graph with minimum degree δ and U be a non-empty proper subset of $V(G)$. If $|\partial_G(U)| \leqslant \delta - 1$, then $|U| \geqslant \delta + 1$.*

For any partition π of $V(G)$, let $E_G(\pi)$ denote the set of edges in G whose endpoints lie in different parts of π, and let $e_G(\pi) = |E_G(\pi)|$. A part is *trivial* if it contains a single vertex. Let $Z \subset V(G)$, and let π be a partition of $V(G-Z)$ with n_0 trivial parts $v_1, v_2, \ldots, v_{n_0}$. Denote by $n_Z(\pi) = \sum_{1 \leqslant i \leqslant n_0} |Z_i|$, where Z_i is the set of vertices in Z that are adjacent to v_i for $1 \leqslant i \leqslant n_0$.

The following three lemmas about rigid graphs will play crucial roles in the proof of our main theorems.

Lemma 11. *(See [11]) A graph G contains k edge-disjoint spanning rigid subgraphs if for every $Z \subset V(G)$ and every partition π of $V(G-Z)$ with n_0 trivial parts and n_0' nontrivial parts,*

$$e_{G-Z}(\pi) \geqslant k(3 - |Z|)n_0' + 2kn_0 - 3k - n_Z(\pi).$$

Lemma 12. *(See [4, 16]) Let G be a graph. Then G is globally rigid if and only if either G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.*

Lemma 13. *(See [20]) A graph G is a minimally rigid on n vertices and m edges if and only if $m = 2n - 3$ and*

$$e_G(X) \leqslant 2|X| - 3$$

for $X \subseteq V(G)$ with $|X| \geqslant 2$.

3 Proof of the main theorems

In this section, we shall give the proofs of Theorems 2–4.

Proof of Theorem 2. Assume to the contrary that G is not rigid. Then G contains no spanning rigid subgraphs. By Lemma 11, there exist a subset Z of $V(G)$ and a partition π of $V(G-Z)$ with n_0 trivial parts $v_1, v_2, \dots, v_{n_0}$ and n_0' nontrivial parts $V_1, V_2, \dots, V_{n_0'}$ such that

$$e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi), \tag{1}$$

where $n_Z(\pi) = \sum_{1 \leqslant i \leqslant n_0} |Z_i|$, and Z_i is the set of vertices in Z that are adjacent to v_i for $1 \leqslant i \leqslant n_0$. Note that $d_{G-Z}(v_i) \geqslant \delta - |Z_i|$. Then

$$\begin{aligned} e_{G-Z}(\pi) &= \frac{1}{2}\left(\sum_{1 \leqslant i \leqslant n_0'} |\partial_{G-Z}(V_i)| + \sum_{1 \leqslant j \leqslant n_0} d_{G-Z}(v_j)\right) \\ &\geqslant \frac{1}{2}\left(\sum_{1 \leqslant i \leqslant n_0'} |\partial_{G-Z}(V_i)| + \delta n_0 - \sum_{1 \leqslant j \leqslant n_0} |Z_j|\right) \\ &\geqslant \frac{1}{2}\left(\sum_{1 \leqslant i \leqslant n_0'} |\partial_{G-Z}(V_i)| + 6n_0 - n_Z(\pi)\right) \text{ (since } \delta \geqslant 6), \end{aligned} \tag{2}$$

and therefore,

$$e_{G-Z}(\pi) \geqslant 3n_0 - \frac{1}{2}n_Z(\pi). \tag{3}$$

We have the following two claims.

Claim 1. $|Z| \leqslant 2$.

Otherwise, $|Z| \geqslant 3$. By (1),

$$e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi) \leqslant 2n_0 - 4 - n_Z(\pi).$$

Combining this with (3) yields that $n_0 + 4 + \frac{1}{2}n_Z(\pi) \leqslant 0$, which is impossible because $n_0 \geqslant 0$ and $n_Z(\pi) \geqslant 0$.

Claim 2. $n_0' \geqslant 2$.

Otherwise, $n_0' \leqslant 1$. By Claim 1, $0 \leqslant |Z| \leqslant 2$, and it follows from (1) that

$$e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi) \leqslant 2n_0 - 1 - n_Z(\pi).$$

Combining this with (3), we have $n_0 + 1 + \frac{1}{2} n_Z(\pi) \leqslant 0$, which is also impossible.

Note that $\rho(G) \geqslant \rho(B^2_{n,\delta+1}) > \rho(K_{n-\delta-1}) = n - \delta - 2$. By Lemmas 7 and 8,

$$e(G) > \frac{n^2}{2} - \frac{(2\delta+3)n}{2} + (\delta+1)^2. \tag{4}$$

Since G is 2-connected,

$$|\partial_{G-Z}(V_i)| \geqslant 2 - |Z| \tag{5}$$

for $1 \leqslant i \leqslant n'_0$. Recall that $0 \leqslant |Z| \leqslant 2$ and $n'_0 \geqslant 2$. We consider the following two situations.

Case 1. $0 \leqslant |Z| \leqslant 1$.

First suppose that $n'_0 = 2$. Then the partition π consists of two nontrivial parts V_1, V_2 and n_0 trivial parts. Putting (5) into (2), we get

$$e_{G-Z}(\pi) \geqslant \frac{1}{2}(|\partial_{G-Z}(V_1)| + |\partial_{G-Z}(V_2)| + 6n_0 - n_Z(\pi)) \geqslant 2 - |Z| + 3n_0 - \frac{1}{2} n_Z(\pi).$$

Combining this with (1) and $n'_0 = 2$, we have

$$-n_0 - \frac{1}{2} n_Z(\pi) - |Z| \geqslant 0,$$

and hence $n_0 = 0$, $n_Z(\pi) = 0$ and $|Z| = 0$ by the facts that $n_0 \geqslant 0$, $n_Z(\pi) \geqslant 0$ and $|Z| \geqslant 0$. This suggests that the partition π consists of two nontrivial parts V_1, V_2, and $G - Z = G$. Then $V(G) = V_1 \cup V_2$ and $e_G(V_1, V_2) = e_G(\pi) \leqslant 2$ by (1). Note that $e_G(V_1, V_2) = \frac{1}{2}(|\partial_G(V_1)| + |\partial_G(V_2)|) \geqslant 2$ by (5). Thus $e_G(V_1, V_2) = 2$. Let $E_G(V_1, V_2) = \{f_1, f_2\}$. We assert that f_1 and f_2 are two independent edges. If not, suppose that $f_1 \cap f_2 = \{u\}$. Then it is easy to see that u is a cut vertex of G, which is impossible because G is 2-connected. Clearly, G is a spanning subgraph of $B^2_{n,|V_1|}$. Then

$$\rho(G) \leqslant \rho(B^2_{n,|V_1|}), \tag{6}$$

with equality if and only if $G \cong B^2_{n,|V_1|}$. Since $\delta \geqslant 6$ and $|\partial_G(V_1)| = |\partial_G(V_2)| = 2 < \delta - 1$, by Lemma 10, $\min\{|V_1|, |V_2|\} \geqslant \delta + 1$. Combining this with Lemma 6 and (6), we conclude that

$$\rho(G) \leqslant \rho(B^2_{n,\delta+1}),$$

with equality if and only if $G \cong B^2_{n,\delta+1}$. However, this is impossible because $\rho(G) \geqslant \rho(B^2_{n,\delta+1})$ and $G \ncong B^2_{n,\delta+1}$.

Now suppose that $n'_0 \geqslant 3$. Let δ' denote the minimum degree of $G - Z$. Then $\delta' \geqslant \delta - |Z|$. If the partition π contains at most one nontrivial part, say V_j $(1 \leqslant j \leqslant n'_0)$, such that $|\partial_{G-Z}(V_j)| \leqslant \delta' - 1$, then $|\partial_{G-Z}(V_i)| \geqslant \delta'$ for all $i \in \{1, \ldots, n'_0\} \setminus \{j\}$. It follows that

$$\begin{aligned} 2e_{G-Z}(\pi) &= \sum_{1 \leqslant i \leqslant n'_0} |\partial_{G-Z}(V_i)| + \sum_{1 \leqslant j \leqslant n_0} d_{G-Z}(v_j) \\ &\geqslant (n'_0 - 1)\delta' + 2 - |Z| + \delta n_0 - n_Z(\pi) \text{ (since } |\partial_{G-Z}(V_j)| \geqslant 2 - |Z|) \end{aligned}$$

$$\begin{aligned}
&\geqslant (n_0'-1)(\delta-|Z|)+2-|Z|+\delta n_0-n_Z(\pi) \text{ (since } \delta'\geqslant \delta-|Z|)\\
&=2(3-|Z|)n_0'+4n_0-8-2n_Z(\pi)+(\delta-6+|Z|)n_0'+(\delta-4)n_0-\delta+10+n_Z(\pi)\\
&\geqslant 2(3-|Z|)n_0'+4n_0-8-2n_Z(\pi)+2\delta-8+3|Z|+n_Z(\pi) \text{ (since } n_0'\geqslant 3 \text{ and } n_0\geqslant 0)\\
&> 2(3-|Z|)n_0'+4n_0-8-2n_Z(\pi) \text{ (since } \delta\geqslant 6,\ n_Z(\pi)\geqslant 0 \text{ and } 0\leqslant |Z|\leqslant 1),
\end{aligned}$$

which contradicts (1). Therefore, the partition π contains at least two nontrivial parts, say V_1, V_2, such that $|\partial_{G-Z}(V_i)| \leqslant \delta'-1$ for $i=1,2$. Furthermore, by Lemma 10, we obtain $|V_i| \geqslant \delta'+1$ for $i=1,2$. We first consider $|Z|=0$. Then $\delta'=\delta$, and $|V_i|\geqslant \delta+1$ for $i=1,2$. If $|V_1|=\max\{|V_1|,|V_2|,\ldots,|V_{n_0'}|\}$ or $|V_2|=\max\{|V_1|,|V_2|,\ldots,|V_{n_0'}|\}$, since $|V_i|\geqslant \delta+1$ and $|V_j|\geqslant 2$ for $i=1,2$ and $3\leqslant j\leqslant n_0'$, by Lemma 9,

$$\sum_{1\leqslant i\leqslant n_0'} e_G(V_i) \leqslant \binom{\delta+1}{2}+\binom{n-\delta-3}{2}+\binom{2}{2}.$$

If there exists a nontrivial part, say V_j, such that $|V_j|=\max\{|V_1|,|V_2|,\ldots,|V_{n_0'}|\}$ for some $3\leqslant j\leqslant n_0'$. Similarly,

$$\sum_{1\leqslant i\leqslant n_0'} e_G(V_i) \leqslant 2\binom{\delta+1}{2}+\binom{n-2\delta-2}{2}.$$

Since $|V_i|\geqslant \delta+1$ for $i=1,2$ and $V_3\geqslant 2$, we have $n_0\leqslant n-\sum_{1\leqslant i\leqslant 3}|V_i|\leqslant n-2\delta-4$ and $n_0'\leqslant \frac{n-(2\delta+4)-n_0}{2}+3$. Note that $G-Z=G$ and $n_Z(\pi)=0$. Then

$$e_G(\pi)\leqslant 3n_0'+2n_0-4\leqslant \frac{3n}{2}-3\delta-1+\frac{n_0}{2}\leqslant 2n-4\delta-3$$

by (1). Thus,

$$\begin{aligned}
e(G) &= \sum_{1\leqslant i\leqslant n_0'} e_G(V_i)+\sum_{1\leqslant i\leqslant n_0} e_G(v_i)+e_G(\pi)\\
&\leqslant \max\left\{\binom{\delta+1}{2}+\binom{n-\delta-3}{2}+\binom{2}{2}, 2\binom{\delta+1}{2}+\binom{n-2\delta-2}{2}\right\}+e_G(\pi)\\
&\leqslant \binom{\delta+1}{2}+\binom{n-\delta-3}{2}+\binom{2}{2}+e_G(\pi) \text{ (since } \delta\geqslant 6 \text{ and } n\geqslant 2\delta+4)\\
&\leqslant \frac{n^2}{2}-\frac{(2\delta+3)n}{2}+\delta^2+4.
\end{aligned}$$

Combining this with (4), we have $\delta<\frac{3}{2}$, which is impossible because $\delta\geqslant 6$. Now assume that $|Z|=1$. Note that $\delta'\geqslant \delta-1$. Then $|V_i|\geqslant \delta'+1\geqslant \delta$ for $i=1,2$. Since $|V_3|\geqslant 2$, we have $n_0'\leqslant \frac{n-|Z|-n_0-\sum_{1\leqslant i\leqslant 3}|V_i|}{2}+3\leqslant \frac{n-n_0-2\delta+3}{2}$. Let $Z=\{w\}$. Then $d_G(w)-n_Z(\pi)\leqslant n-n_0-1$, and it follows from (1) that

$$\begin{aligned}
e_{G-Z}(\pi)+d_G(w) &\leqslant 2n_0'+2n_0-4-n_Z(\pi)+d_G(w)\\
&\leqslant 2n_0'+n_0+n-5\\
&\leqslant 2n-2\delta-2.
\end{aligned}$$

Again by Lemma 9, we obtain

$$\begin{aligned}e(G) &\leqslant \max\left\{\binom{\delta}{2}+\binom{n-|Z|-\delta-2}{2}+\binom{2}{2}, 2\binom{\delta}{2}+\binom{n-|Z|-2\delta}{2}\right\}+e_{G-Z}(\pi)+d_G(w)\\ &\leqslant \binom{\delta}{2}+\binom{n-|Z|-\delta-2}{2}+\binom{2}{2}+e_{G-Z}(\pi)+d_G(w) \text{ (since } \delta\geqslant 6 \text{ and } n\geqslant 2\delta+4)\\ &\leqslant \frac{n^2}{2}-\frac{(2\delta+3)n}{2}+\delta^2+\delta+5.\end{aligned}$$

Combining this with (4), we have $\delta < 4$, which is also impossible.

Case 2. $|Z| = 2$.

By (1), we have

$$e_{G-Z}(\pi) \leqslant n_0' + 2n_0 - 4 - n_Z(\pi). \tag{7}$$

If $2 \leqslant n_0' \leqslant 3$, combining (2), (5) and (7), we have

$$0 \leqslant \sum_{1\leqslant i\leqslant n_0'} |\partial_{G-Z}(V_i)| \leqslant 2n_0' - 8 - 2n_0 - n_Z(\pi) \leqslant -2,$$

a contradiction. Thus $n_0' \geqslant 4$. Let δ' denote the minimum degree of $G - Z$. Then $\delta' \geqslant \delta - 2$. If the partition π contains at most one nontrivial part, say V_j $(1 \leqslant j \leqslant n_0')$, such that $|\partial_{G-Z}(V_j)| \leqslant \delta' - 1$, then $|\partial_{G-Z}(V_i)| \geqslant \delta'$ for all $i \in \{1, \ldots, n_0'\} \setminus \{j\}$. It follows that

$$\begin{aligned}2e_{G-Z}(\pi) &= \sum_{1\leqslant i\leqslant n_0'} |\partial_{G-Z}(V_i)| + \sum_{1\leqslant j\leqslant n_0} d_{G-Z}(v_j)\\ &\geqslant (n_0'-1)\delta' + \delta n_0 - n_Z(\pi)\\ &\geqslant (n_0'-1)(\delta-2) + \delta n_0 - n_Z(\pi) \text{ (since } \delta' \geqslant \delta-2)\\ &= (2n_0' + 4n_0 - 8 - 2n_Z(\pi)) + (\delta-4)n_0' - \delta + (\delta-4)n_0 + n_Z(\pi) + 10\\ &\geqslant 2n_0' + 4n_0 - 8 - 2n_Z(\pi) + 3\delta - 6 \text{ (since } n_0' \geqslant 4,\ n_0 \geqslant 0 \text{ and } n_Z(\pi) \geqslant 0)\\ &> 2n_0' + 4n_0 - 8 - 2n_Z(\pi) \text{ (since } \delta \geqslant 6),\end{aligned}$$

contrary to (7). Therefore, the partition π contains at least two nontrivial parts, say V_1, V_2, such that $|\partial_{G-Z}(V_i)| \leqslant \delta' - 1$ for $i = 1, 2$. Furthermore, by Lemma 10, $|V_i| \geqslant \delta' + 1 \geqslant \delta - 1$ for $i = 1, 2$, and hence $n_0' \leqslant \frac{n-|Z|-2(\delta-1)}{2} + 2 = \frac{n}{2} - \delta + 2$. Since $|Z| = 2$, we have $|\partial_G(Z)| + e_G(Z) - n_Z(\pi) \leqslant 2(n - 2 - n_0) + 1$, and it follows from (7) that

$$e_{G-Z}(\pi)+|\partial_G(Z)|+e_G(Z) \leqslant n_0' + 2n - 7 \leqslant \frac{5n}{2} - \delta - 5.$$

Recall that $\delta \geqslant 6$ and $n \geqslant 2\delta + 4$. By Lemma 9,

$$\begin{aligned}e(G) \leqslant \max&\left\{\binom{\delta-1}{2}+2\binom{2}{2}+\binom{n-|Z|-\delta-3}{2}, 2\binom{\delta-1}{2}+\binom{2}{2}+\binom{n-|Z|-2\delta}{2}\right\}\\ &+e_{G-Z}(\pi)+|\partial_G(Z)|+e_G(Z)\end{aligned}$$

$$
\begin{aligned}
&\leqslant \binom{\delta-1}{2}+2\binom{2}{2}+\binom{n-|Z|-\delta-3}{2}+e_{G-Z}(\pi)+|\partial_G(Z)|+e_G(Z)\\
&\leqslant \frac{n^2}{2}-\frac{(2\delta+6)n}{2}+\delta^2+3\delta+13.
\end{aligned}
$$

Combining this with (4), we obtain $n < \frac{2}{3}\delta + 8$, which is impossible because $n \geqslant 2\delta + 4$ and $\delta \geqslant 6$.

This completes the proof. □

Recall that, for any partition π of $V(G)$, $E_G(\pi)$ denotes the set of edges in G whose ends lie in different parts of π, and $e_G(\pi) = |E_G(\pi)|$.

Proof of Theorem 3. Assume to the contrary that G is not globally rigid. Since G is a 3-connected graph with minimum degree $\delta \geqslant 6$ and order $n \geqslant 2\delta + 4$, by Lemma 12, we see that G is not redundantly rigid. This suggests that there exists an edge f of G such that $G - f$ is not rigid. Furthermore, by Lemma 11, there exist a subset Z of $V(G)$ and a partition π of $V(G - f - Z)$ with n_0 trivial parts $v_1, v_2, \ldots, v_{n_0}$ and n_0' nontrivial parts $V_1, V_2, \ldots, V_{n_0'}$ such that

$$
e_{G-f-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi). \tag{8}
$$

First we assume that $f \in E_{G-Z}(\pi)$. Then $e_{G-f-Z}(\pi) = e_{G-Z}(\pi) - 1$. By (8),

$$
e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 3 - n_Z(\pi). \tag{9}
$$

Recall that $n_Z(\pi) = \sum_{1 \leqslant i \leqslant n_0} |Z_i|$, where Z_i is the set of vertices in Z that are adjacent to v_i for $1 \leqslant i \leqslant n_0$. Note that $d_{G-Z}(v_i) \geqslant \delta - |Z_i|$. Then

$$
\begin{aligned}
e_{G-Z}(\pi) &= \frac{1}{2}\left(\sum_{1\leqslant i\leqslant n_0'} |\partial_{G-Z}(V_i)| + \sum_{1\leqslant j\leqslant n_0} d_{G-Z}(v_j)\right)\\
&\geqslant \frac{1}{2}\left(\sum_{1\leqslant i\leqslant n_0'} |\partial_{G-Z}(V_i)| + 6n_0 - n_Z(\pi)\right) \text{ (since } \delta \geqslant 6\text{)},
\end{aligned} \tag{10}
$$

and hence

$$
e_{G-Z}(\pi) \geqslant 3n_0 - \frac{1}{2}n_Z(\pi). \tag{11}
$$

We have the following two claims.

Claim 1. $|Z| \leqslant 2$.

Otherwise, $|Z| \geqslant 3$. By (9),

$$
e_{G-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 3 - n_Z(\pi) \leqslant 2n_0 - 3 - n_Z(\pi).
$$

Combining this with (11), we have $n_0 + \frac{1}{2}n_Z(\pi) + 3 \leqslant 0$, which is impossible because $n_0 \geqslant 0$ and $n_Z(\pi) \geqslant 0$.

Claim 2. $n_0' \geqslant 2$.

Otherwise, $n_0' \leqslant 1$. By Claim 1, $0 \leqslant |Z| \leqslant 2$, and it follows from (9) that

$$e_{G-Z}(\pi) \leqslant (3-|Z|)n_0' + 2n_0 - 3 - n_Z(\pi) \leqslant 2n_0 - n_Z(\pi). \tag{12}$$

Combining this with (11), we have

$$n_0 + \frac{1}{2}n_Z(\pi) \leqslant 0.$$

This implies that all equalities hold in (11) and (12), and hence $n_0' = 1$, $n_0 = 0$, $n_Z(\pi) = 0$ and $|Z| = 0$. Then from (8) we deduce that $e_{G-f-Z}(\pi) \leqslant -1$, a contradiction.

Note that $\rho(G) \geqslant \rho(B^3_{n,\delta+1}) > \rho(K_{n-\delta-1}) = n-\delta-2$. By Lemmas 7 and 8,

$$e(G) > \frac{n^2}{2} - \frac{(2\delta+3)n}{2} + (\delta+1)^2. \tag{13}$$

Since G is 3-connected,

$$|\partial_{G-Z}(V_i)| \geqslant 3 - |Z|. \tag{14}$$

Recall that $0 \leqslant |Z| \leqslant 2$ and $n_0' \geqslant 2$. We consider the following two situations.

Case 1. $0 \leqslant |Z| \leqslant 1$.

First suppose that $n_0' = 2$. Then the partition π consists of two nontrivial parts V_1, V_2 and n_0 trivial parts. Putting (14) into (10), we get

$$e_{G-Z}(\pi) \geqslant \frac{1}{2}(|\partial_{G-Z}(V_1)| + |\partial_{G-Z}(V_2)| + 6n_0 - n_Z(\pi)) \geqslant 3 - |Z| + 3n_0 - \frac{1}{2}n_Z(\pi).$$

Combining this with (9) and $n_0' = 2$, we have

$$-n_0 - \frac{1}{2}n_Z(\pi) - |Z| \geqslant 0,$$

and hence $n_0 = 0$, $n_Z(\pi) = 0$ and $|Z| = 0$ by the facts $n_0 \geqslant 0$, $n_Z(\pi) \geqslant 0$ and $|Z| \geqslant 0$. This suggests that the partition π consists of two nontrivial parts V_1, V_2, and $G - Z = G$. Then $V(G) = V_1 \cup V_2$ and $e_G(V_1, V_2) = e_G(\pi) \leqslant 3$ by (9). Note that $e_G(V_1, V_2) = \frac{1}{2}(|\partial_G(V_1)| + |\partial_G(V_2)|) \geqslant 3$ by (14). Thus $e_G(V_1, V_2) = 3$. Let $E_G(V_1, V_2) = \{f_1, f_2, f\}$. We assert that f_1, f_2, f are three independent edges. If not, then G cannot be 3-connected, a contradiction. Observe that G is a spanning subgraph of $B^3_{n,|V_1|}$. Then

$$\rho(G) \leqslant \rho(B^3_{n,|V_1|}), \tag{15}$$

with equality if and only if $G \cong B^3_{n,|V_1|}$. Since $\delta \geqslant 6$ and $|\partial_G(V_1)| = |\partial_G(V_2)| = 3 < \delta - 1$, by Lemma 10, $\min\{|V_1|, |V_2|\} \geqslant \delta + 1$. Combining this with Lemma 6 and (15), we have

$$\rho(G) \leqslant \rho(B^3_{n,\delta+1}),$$

with equality if and only if $G \cong B^3_{n,\delta+1}$. However, this is impossible because $\rho(G) \geqslant \rho(B^3_{n,\delta+1})$ and $G \ncong B^3_{n,\delta+1}$. If $n_0' \geqslant 3$, by using (13) and a similar analysis as in Case 1 of Theorem 2, we also can deduce a contradiction.

Case 2. $|Z| = 2$.

In this case, the proof is similar as in Case 2 of Theorem 2, and we omit it.

Now we assume that $f \notin E_{G-Z}(\pi)$. Then

$$e_{G-Z}(\pi) = e_{G-f-Z}(\pi) \leqslant (3 - |Z|)n_0' + 2n_0 - 4 - n_Z(\pi).$$

By similar arguments as above, we also can deduce a contradiction.

This completes the proof. □

Proof of Theorem 4. Suppose that G has the maximum spectral radius among all minimally rigid graphs of order $n \geqslant 3$. By Lemma 13, we have $e(G) = 2n - 3$ and $e_G(X) \leqslant 2|X| - 3$ for all $X \subseteq V(G)$ with $|X| \geqslant 2$. Note that $K_2\nabla(n-2)K_1$ is a minimally rigid graph. Then

$$\rho(G) \geqslant \rho(K_2\nabla(n-2)K_1) = \frac{1 + \sqrt{8n-15}}{2}. \tag{16}$$

Let δ denote the minimum degree of G. We assert that $\delta \geqslant 2$. In fact, if there exists some vertex $u \in V(G)$ such that $d_G(u) = 1$, then $e_G(V(G) \setminus \{u\}) = 2n - 4$. However, since $V(G) \setminus \{u\} = n - 1 \geqslant 2$, we have $e_G(V(G) \setminus \{u\}) \leqslant 2|V(G) \setminus \{u\}| - 3 = 2n - 5$ by the above argument, a contradiction. Then, by Lemmas 7 and 8,

$$\rho(G) \leqslant \frac{1}{2} + \sqrt{2e(G) - 2n + \frac{9}{4}} = \frac{1 + \sqrt{8n-15}}{2}. \tag{17}$$

Thus the equalities hold in (16) and (17). It follows that $\delta = 2$ and G is either a 2-regular graph, or a bidegreed graph in which each vertex is of degree 2 or $n - 1$ by Lemma 7. If $n = 3$, then $G \cong K_3$, as required. Now suppose that $n \geqslant 4$. Let $t = |\{v \in V(G) | \ d_G(v) = n-1\}|$. If $0 \leqslant t \leqslant 1$, then $e(G) < 2n - 3$, and if $t \geqslant 3$ then $e(G) > 2n - 3$, both are impossible. Thus $t = 2$, and $G \cong K_2\nabla(n-2)K_1$.

This completes the proof. □

4 Concluding remarks

In this paper, we provide a spectral radius condition for the rigidity (resp., globally rigidity) of 2-connected (resp., 3-connected) graphs with given minimum degree in $\mathbb{R}^2$. In particular, we give the answers to Problem 1 for $k = 2, 3$. Note that every 6-connected graph is rigid (resp., globally rigid). Thus, the Problem 1 becomes more involved for $k = 4, 5$. When $k = 4, 5$, by using similar analysis as Theorems 2 and 3, we can obtain that a k-connected graph G is rigid (resp., globally rigid) if $\rho(G) > \rho(B^k_{n,\delta+1})$. As $B^k_{n,\delta+1}$ is both rigid and globally rigid for $k = 4, 5$, we end the paper by proposing the following problem for further research.

Problem 14. Let $k \in \{4, 5\}$, and let G be a k-connected graph with minimum degree $\delta \geqslant 6$ and order $n \geqslant 2\delta + 4$. Is it true that G is rigid (resp. globally rigid) when $\rho(G) \geqslant \rho(B^k_{n,\delta+1})$?

Acknowledgments

The authors would like to thank the anonymous referee for his/her valuable comments and helpful suggestions.

References

[1] L. Asimow, B. Roth. The rigidity of graphs. *Trans. Amer. Math. Soc.*, 245:279–289, 1978.

[2] A. E. Brouwer, W. H. Haemers. Spectra of Graphs. Springer, Berlin, 2011.

[3] S. M. Cioabă, S. Dewar, X. Gu. Spectral conditions for graph rigidity in the Euclidean plane. *Discrete Math.*, 344(10):112527, 2021.

[4] R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33(4):549–563, 2005.

[5] M. Cucuringu, A. Singer, D. Cowburn. Eigenvector synchronization, graph rigidity and the molecule problem. *Inf. Inference*, 1(1):21–67, 2012.

[6] M. Gáspár, P. Csermely. Rigidity and flexibility of biological networks. *Brief. Funct. Genomics*, 11(6):443–456, 2012.

[7] C. Godsil, G. Royle. Algebraic Graph Theory. Graduate Texts in Mathematices, 207, Springer-Verlag, New York, 2001.

[8] S. J. Gortler, A. Healy, D. Thurston. Characterizing generic global rigidity. *Am. J. Math.*, 132(4):897–939, 2010.

[9] J. Graver, B. Servatius, H. Servatius. Combinatorial Rigidity. Graduate Studies in Mathematics, vol. 2, American Mathematical Society, 1993.

[10] X. Gu, W. Meng, M. Rolek, Y. Wang, G. Yu. Sufficient conditions for 2-dimensional global rigidity. *SIAM J. Discrete Math.*, 35(4):2520–2534, 2021.

[11] X. Gu. Spanning rigid subgraph packing and sparse subgraph covering. *SIAM J. Discrete Math.*, 32(2):1305–1313, 2018.

[12] X. Gu, H. Lai, P. Li, S. Yao. Edge-disjoint spanning trees, edge connectivity, and eigenvalues in graphs. *J. Graph Theory*, 81(1):16–29, 2016.

[13] B. Hendrickson. Conditions for unique graph realizations. *SIAM J. Comput.*, 21(1): 65–84, 1992.

[14] Y. Hong, J. Shu, K. Fang. A sharp upper bound of the spectral radius of graphs. *J. Combin. Theory Ser. B*, 81:177–183, 2001.

[15] B. Jackson, T. Jordán. Connected rigidity matroids and unique realizations of graphs. *J. Combin. Theory Ser. B*, 94(1):1–29, 2005.

[16] B. Jackson, T. Jordán. A sufficient connectivity condition for generic rigidity in the plane. *Discrete Appl. Math.*, 157(8):1965–1968, 2009.

[17] B. Jackson, B. Servatius, H. Servatius. The 2-dimensional rigidity of certain families of graphs. *J. Graph Theory*, 54(2):154–166, 2007.

[18] T. Jordán. Combinatorial Rigidity: Graphs and Matroids in the Theory of Rigid Frameworks. *Discrete Geometric Analysis*, 34:33–112, 2016.

[19] H. Lai, Y. Shao, H. Wu, J. Zhou. Every 3-connected, essentially 11-connected line graph is Hamiltonian. *J. Combin. Theory Ser. B*, 96(4):571–576, 2006.

[20] G. Laman. On graphs and rigidity of plane skeletal structures. *J. Engrg. Math.*, 4:331–340, 1970.

[21] L. Lovász, Y. Yemini. On generic rigidity in the plane. *SIAM J. Algebraic Discrete Methods*, 3(1):91–98, 1982.

[22] V. Nikiforov. Some inequalities for the largest eigenvalue of a graph. *Combin. Probab. Comput.*, 11(2):179–189, 2002.