Enumeration of Perfect Matchings of the Cartesian Products of Graphs

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Abstract

A subgraph H of a graph G is nice if $G - V(H)$ has a perfect matching. An even cycle C in an oriented graph is oddly oriented if for either choice of direction of traversal around C, the number of edges of C directed along the traversal is odd. An orientation D of a graph G with an even number of vertices is Pfaffian if every nice cycle of G is oddly oriented in D. Let P_n denote a path on n vertices. The Pfaffian graph $G \times P_{2n}$ was determined by Lu and Zhang [The Pfaffian property of Cartesian products of graphs, J. Comb. Optim. 27 (2014) 530–540]. In this paper, we characterize the Pfaffian graph $G \times P_{2n+1}$ with respect to the forbidden subgraphs of G. We first give sufficient and necessary conditions under which $G \times P_{2n+1}$ ($n \geqslant 2$) is Pfaffian. Then we characterize the Pfaffian graph $G \times P_3$ when G is a bipartite graph, and we generalize this result to the the case G contains exactly one odd

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cycle. Following these results, we enumerate the number of perfect matchings of the Pfaffian graph $G \times P_n$ in terms of the eigenvalues of the orientation graph of G, and we also count perfect matchings of some Pfaffian graph $G \times P_n$ by the eigenvalues of G.

Mathematics Subject Classifications: 05C30, 05C70, 05C75

1 Introduction

The graphs considered in this paper are finite and simple unless otherwise indicated. For terminology and notation not defined here, we refer the reader to [24]. Let $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G. The degree of a vertex v, denoted by $d(v)$, is the number of edges incident with v. An *n-multiple edge* consists of n edges with the same pair of ends. A *perfect matching* of G is a set of independent edges covering all the vertices of G. The number of perfect matchings of G is denoted by $\Phi(G)$. Let H be a subgraph of G and let $G - V(H)$ denote the subgraph obtained from G by deleting the vertices of H and the edges that are incident with the vertices in $V(H)$. A subgraph H of G is *nice* (or *central*) if $G - V(H)$ has a perfect matching. A cycle in a graph contains at least three vertices. An *even cycle* (resp. *odd cycle*) is a cycle on an even (resp. odd) number of vertices. An even cycle C in an oriented graph is *oddly oriented* if for either choice of the direction of traversal around C, the number of edges of C directed in the direction of traversal is odd. An orientation of a graph G is an assignment of directions to each edge of G. Suppose that G is a graph with an even number of vertices. Then an orientation D of G is a *Pfaffian orientation* if every nice cycle C of G is oddly oriented in D. A graph G is said to be Pfaffian if it admits a Pfaffian orientation. Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{w_1, w_2, \ldots, w_m\}$. The *Cartesian product* of G and H, denoted by $G \times H$, is the graph with $V(G \times H) = \{v_j^i : \ u_i \in V(G), \ w_j \in V(H)\}$ and $E(G \times H) = \{v_i^j v_s^t : \ u_i u_s \in E(G) \text{ if } j = t, \text{ or } w_j w_t \in E(H) \text{ if } i = s\}$.

Pfaffian orientations were first applied by the physicists M. E. Fisher, P. W. Kasteleyn and H. N. V. Temperley to enumerate the number of the perfect matchings in a graph [5, 6, 21]. The perfect matchings of a chemical graph correspond to "Kekulé structures" in quantum Chemistry and corresponds to "close-packed dimers" in statistical physics, and the more perfect matchings a polyhex graph possesses the more stable is the corresponding benzenoid molecule. The number of perfect matchings is an important topological index for estimation of total π-electron energy and resonant energy. Valiant [23] proved that counting the number of the perfect matchings in a general graph is #P-complete. The significance of Pfaffian orientations is that if a graph G has a Pfaffian orientation, the number of perfect matchings of G can be evaluated by the determinant, and it can be counted in polynomial time.

Theorem 1 ([8, 12]). *Let G^σ be a Pfaffian orientation of a graph G. Then*

$$\Phi^2(G) = |\det A(G^\sigma)|,$$

where $A(G^\sigma)$ is the skew-adjacency matrix of G^σ.

The following is a classical theorem given by Kasteleyn [8].

Theorem 2 (Kasteleyn [8]). *Every planar graph is Pfaffian.*

Little [11] characterized the bipartite graph that is Pfaffian in terms of forbidden subgraphs.

Theorem 3 (Little [11]). *A bipartite graph is Pfaffian if and only if it contains no even subdivision of $K_{3,3}$ as its nice subgraph.*

Robertson et al. [20] and McCuaig [19] independently gave polynomial-time algorithms to determine whether a given bipartite graph has a Pfaffian orientation. However, for non-bipartite graphs, there is no efficient way to identify if it is Pfaffian. For other investigations on Pfaffian graphs, see [3, 18, 22]. With the help of Pfaffian graphs, many dimer statistics of lattices have been solved [2, 14, 15, 16, 17, 25, 26, 27]. For example, the quadratic lattice [5, 7], 8.8.4 lattice [27] and hexagonal lattice [9, 27]. Kasteleyn [7] and independently Fisher [5] had discussed the number of perfect matchings of the quadrilateral lattice on the plane and deduced an explicit expression. In the following, we use P_n to denote a path on n vertices.

Theorem 4 ([7]). *The number of perfect matchings of a plane quadrilateral lattice $P_m \times P_n$ is*

$$\Phi(P_m \times P_n) = \prod_{k=1}^{\frac{m}{2}} \prod_{l=1}^{n} 2[cos^2(\frac{k\pi}{m+1}) + cos^2(\frac{l\pi}{n+1})]^{\frac{1}{2}}.$$

As a generalization of this result, Yan and Zhang in 2004 [25] considered the enumeration of perfect matchings of $G \times P_2$, and they express the number of perfect matchings in terms of the eigenvalues of G. Here, the eigenvalues of the adjacency matrix of G is termed as the *eigenvalues of* G.

Theorem 5 ([25]). *If G is a bipartite graph without cycles of length $4s$, $s \in \{1, 2, \ldots\}$, then*

$$\Phi(G \times P_2) = \prod (1 + \lambda^2)^{m_\lambda},$$

where the product ranges over all the non-negative eigenvalues λ of G, and m_λ denotes the multiplicity of the eigenvalue λ.

In 2006 Yan and Zhang [26] derived the expression of counting perfect matchings of the graphs $T \times P_3$ and $T \times P_4$ (T is a tree) with respect to all the non-negative eigenvalues of T. There is a natural question "for some families of graphs G, can we enumerate the number of perfect matchings of $G \times P_n$ by the eigenvalues of G?" The answer is "yes". We will prove the following result.

Theorem 6. *Let G be a bipartite graph containing no cycle of length divisible by four. Then*

$$\Phi(G \times P_n) = \prod_{\alpha} \prod_{k=1}^{n} |(4cos^2 \frac{\pi k}{n+1} + \alpha^2)|^{\frac{m_\alpha}{2}},$$

where the first product ranges over all the positive eigenvalues α of G, and m_α is the multiplicity of the eigenvalue α.

Except representing the number of perfect matchings in terms of the eigenvalues of an undirected graph, Yan et.al [26] considered enumerating the number of perfect matchings of $G \times P_2$ by the eigenvalues of an orientation graph (The *eigenvalues of an orientation graph* G^σ is the eigenvalues of the skew-adjacency matrix of G^σ).

Theorem 7 ([26])**.** *Let G be a graph contains no subgraph which is, after contracting at most one cycle of odd length, an even subdivision of $K_{2,3}$. Let G^σ be an orientation of G such that all the cycles are oddly oriented. Then*

$$\Phi(G \times P_2) = \prod_{\lambda}(1 - \lambda^2),$$

where the product ranges over all the non-negative imaginary part eigenvalues λ of the skew-adjacency matrix of G^σ.

Following this research, Lin and Zhang [10] paid attention to the Cartesian product of a non-bipartite graph and a path.

Theorem 8 ([10])**.** *Let G be a non-bipartite graph with a unique cycle, and G^σ an arbitrary orientation of G. Then*

$$\Phi(G \times P_4) = \prod_{\lambda}(1 - 3\lambda^2 + \lambda^4)^{m_\lambda}.$$

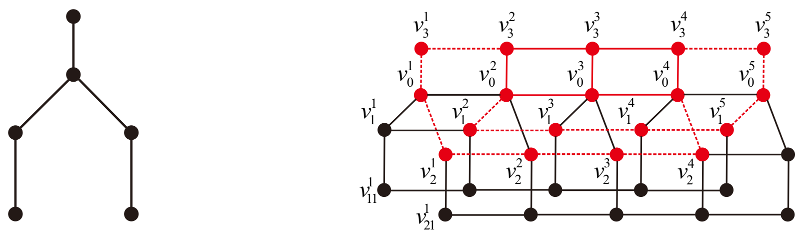
Moreover, if G has a perfect matching, then

$$\Phi(G \times P_3) = \prod_{\lambda}(2 - \lambda^2)^{m_\lambda},$$

where the products range over all the non-negative imaginary part eigenvalues λ of the skew-adjacency matrix of G^σ.

Lu and Zhang [13] have established the Pfaffian property of $G \times P_{2n}$ with respect to the excluded subgraphs of G (see Theorem 9). In this paper, we aim first to characterize the Pfaffian graph $G \times P_{2n+1}$ in terms of the forbidden subgraph of G. Based on these characterizations, we generalize the results in Theorems 1.7 and 1.8. We expresses the number of perfect matchings of the Pfaffian graph $G \times P_n$ in terms of the eigenvalues of an orientation of G. The result is exhibited in Theorem 38.

The rest of the paper is organized as follows. In Section 2, we characterize the Pfaffian graph $G \times P_{2n+1}$ ($n \geqslant 2$) by the excluded subgraphs of G. In Section 3, we provide a necessary condition for the Pfaffian graph $G \times P_3$. In Section 4,we establish the characterization of the Pfaffian graph $G \times P_3$ for the case G is bipartite, and generalize this result to the case that G is a non-bipartite graph with exactly one odd cycle. According to these characterizations of Pfaffian graph $G \times P_n$, Section 5 shows that the number of perfect matchings of the Pfaffian graph $G \times P_n$ can be evaluated by the eigenvalues of some orientation graph of G. Moreover, for some bipartite graph G, the enumeration of perfect matchings of $G \times P_n$ can be estimated by the eigenvalues of G.



(a) H_Y (b) $H_Y \times P_5$

Figure 1: The forbidden subgraph H_Y.

2 A characterization of Pfaffian graph $G \times P_{2n+1}$ with $n \geqslant 2$

Lu and Zhang [13] characterized the Pfaffian property of Cartesian products $G \times P_{2n}$ in terms of forbidden subgraphs (see Theorem 9). In this section we will determine the Pfaffian graphs $G \times P_5$ and $G \times P_{2n+1}$ ($n \geqslant 3$) by the forbidden subgraphs of G, respectively. We begin with some definitions and terminology.

The *Y-tree* is a graph obtained from $K_{1,3}$ by connecting a vertex not in $K_{1,3}$ to a vertex of degree one in $K_{1,3}$. The *Q-graph* is obtained from the cycle C_4 by connecting a vertex not in C_4 to a vertex of degree two in C_4.

Theorem 9 (Lu et al.[13]). *Let G be a connected graph. Then*
(1) $G \times P_2$ is Pfaffian if and only if G contains no subgraph which is, after contracting at most one cycle of odd length, an even subdivision of $K_{2,3}$;
(2) $G \times P_4$ is Pfaffian if and only if G contains neither an even subdivision of Q-graph nor two edge-disjoint odd cycles as its subgraph;
(3) $G \times P_{2n}$ ($n \geqslant 3$) is Pfaffian if and only if G contains no Y-tree as its subgraph.

Let H_Y denote the graph obtained from $K_{1,3}$ by attaching two appending edges to two vertices of degree one in $K_{1,3}$, respectively. See Figure 1(a).

Before presenting the main theorem of this section, we prove first the following lemma.

Lemma 10. *The graph $H_Y \times P_5$ is not Pfaffian.*

Proof. As shown in Figure 1(b), let $V(H_Y \times P_5) = \{v_0^i, v_1^i, v_2^i, v_3^i, v_{11}^i, v_{21}^i : i = 1, 2, \ldots, 5\}$ and $E(H_Y \times P_5) = \{v_0^iv_1^i, v_0^iv_2^i, v_0^iv_3^i, v_1^iv_{11}^i, v_2^iv_{21}^i : i = 1, 2, \cdots, 5\} \cup \{v_s^jv_s^{j+1} : v_s \in V(H_Y)\}$. Since H_Y is a tree, $H_Y \times P_5$ is a bipartite graph. The subgraph H of $H_Y \times P_5$ induced by $\{v_0^2, v_0^3, v_0^4, v_3^2, v_3^3, v_3^4\}$ together with two paths $v_3^2v_3^1v_0^1v_2^1v_2^2v_2^3v_2^4v_0^4$ and $v_3^4v_3^5v_0^5v_1^5v_1^4v_1^3v_1^2v_0^2$ is an even subdivision of $K_{3,3}$. Since $H_Y \times P_5 - H$ has a perfect matching, $H_Y \times P_5$ contains an even subdivision of $K_{3,3}$ as a nice subgraph and hence it is not Pfaffian by Theorem 3. $\square$

For any graph G, the graph $G \times P_{2n}$ has perfect matchings whenever G has perfect matchings or not. Moreover, for any subgraph H of G, the graph $H \times P_{2n}$ is a nice subgraph of $G \times P_{2n}$. These two properties do not hold for the graph $G \times P_{2n+1}$. Thus when we try to characterize the Pfaffian graph $G \times P_{2n+1}$, we suppose that G has at least one perfect matching.

In the following paragraphs, an *odd path* (resp. *even path*) is a path on an odd (resp. even) number of vertices.

Theorem 11. *Let G be a connected graph with a perfect matching. Then $G \times P_5$ is Pfaffian if and only if G contains neither an H_Y as its nice subgraph nor edge-disjoint odd cycles as its subgraph.*

Proof. Suppose that $G \times P_5$ is Pfaffian. Assume, to the contrary, that G contains H_Y as its nice subgraph. Then $G \times P_5$ contains a nice subgraph $H_Y \times P_5$ which is not Pfaffian by Lemma 10 and so $G \times P_5$ is not Pfaffian, a contradiction. If G contains edge-disjoint odd cycles as its subgraph, then $G \times P_4$ is not Pfaffian by Theorem 9. Since G is a graph with a perfect matching, $G \times P_4$ is a nice subgraph of $G \times P_5$. It follows that $G \times P_5$ is not Pfaffian. This is a contradiction. Therefore, if $G \times P_5$ is Pfaffian, then G contains neither H_Y as its nice subgraph nor edge-disjoint odd cycles as its subgraph.

Now we prove the sufficiency. We first consider the case that $|V(G)| \leqslant 4$. In this case, G contains no Y-tree as its subgraph. Then $G \times P_6$ is Pfaffian by Theorem 9. It follows that there is a Pfaffian orientation of $G \times P_6$ under which each nice cycle is oddly oriented. Since G has a perfect matching, each nice cycle in $G \times P_5$ is a nice cycle of $G \times P_6$. Hence we can obtain a Pfaffian orientation of $G \times P_5$ from a Pfaffian orientation of $G \times P_6$.

Consider the case that $|V(G)| \geqslant 6$. If the degree of each vertex in G is at most two, G contains no Y-tree as its subgraph. Then the same analysis as above leads to that $G \times P_5$ is Pfaffian.

It remains to consider the case that $|V(G)| \geqslant 6$ and G contains at least one vertex whose degree is lager than two. Let v be such a vertex. Since G has a perfect matching, let M be a prefect matching of G and denote one edge incident with v belonging to M by vx. There exist at least two edges incident with v that are not in M, denoted by vu and vw. If the edge uw does not exist or it exists but is not in M, there exists another two vertices u_1 and w_1 such that the edges u_1u and w_1w lie in M. However, the subgraph induced by $\{vx,\ vu,\ vw,\ u_1u,\ w_1w\}$ is H_Y and it is a nice subgraph of G, a contradiction. Hence the edge uw exists and lies in M.

We assert that all the neighbours of u, v and w belong to $\{u, v, w, x\}$. If u has another neighbours except v, w and x, let u_2 be such a neighbour of u. The edge uu_2 does not lie in M and then there exists a vertex u_3 such that u_2u_3 lies in M. The subgraph induced by $\{uw, uv, vx, uu_2, u_2u_3\}$ is H_Y and is a nice subgraph of G, a contradiction. Hence u does not have another neighbours except v, w and x. Likely we can also deduce that w does not have another neighbours except u, v and x. If v has another neighbours except u, w and x, suppose that one of the neighbours except u, w and x is v_1 and then vv_1 does not lies in M. It follows that there exists another vertex v_2 such that v_1v_2 lie in M. The subgraph induced by $\{vx, vv_1, v_1v_2, vu, uw\}$ is H_Y and is a nice subgraph of G, a contradiction. Hence all the neighbours of v lie in $\{u, v, w, x\}$ and the assertion holds.

If the edge ux exists in G, x can not be adjacent to other vertices except v, u, w. If not, assume that there exists another vertex x_1 that is adjacent to x. The edge xx_1 does

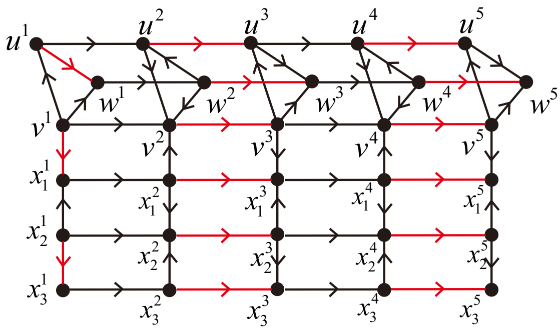


Figure 2: $G \times P_5$.

not lie in M and then there is a vertex x_2 such that x_1x_2 lies in M. The subgraph induced by $\{vx, xx_1, x_1x_2, xu, uw\}$ is H_Y and it is a nice subgraph of G, a contradiction. Then it follows that in this case $V(G) = \{v, x, u, w\}$. This is impossible since $|V(G)| \geqslant 6$. Hence the edge ux does not exist in G. Likewise, we can deduce that w is not adjacent to x. By the above, we can deduce that v is the only vertex whose degree is larger than two in $\{u, v, w, x\}$. From the analysis above, we can conclude the following result. For any vertex in G, if the degree of this vertex is at least three, then we can show that its degree is exactly three and we can find a triangle that contains this vertex. Note that G contains no edge-disjoint odd cycle as its subgraph. Hence the degree of each vertex of G except v is at most two. Besides, the subgraph $G - \{v, u, w\}$ is a path of even length with one end-vertex x, since G has a perfect matching. In other word, G consists of one triangle and a path of odd length. In the following, we will prove that $G \times P_5$ is Pfaffian.

We first give an orientation of $G \times P_5$. Suppose that $V(G) = \{u,\ w,\ v,\ x_1, \ldots,\ x_{2m+1}\}$ and $E(G) = \{uw, vu, vw, vx_1, x_1x_2, \ldots, x_{2m}x_{2m+1}\}$. The graph $G \times P_5$ contains five copies of G, denoted by G_1, G_2, G_3, G_4 and G_5, respectively. Denote the edge set $\{u^iu^{i+1}, w^iw^{i+1}, v^iv^{i+1},\ x_1^ix_1^{i+1}, \ldots, x_{2m+1}^ix_{2m+1}^{i+1} : i = 1, 2, 3, 4\}$ of $G \times P_5$ by E_P. The only perfect matching of G_1 is $M_1 = \{u^1w^1, v^1x_1^1, x_2^1x_3^1, \ldots, x_{2m}^1x_{2m+1}^1\}$. Let $M = M_1 \cup \{u^iu^{i+1}, w^iw^{i+1}, v^iv^{i+1}, x_1^ix_1^{i+1}, \ldots, x_{2m+1}^ix_{2m+1}^{i+1} : i = 2, 4\}$. It is a perfect matching of $G \times P_5$. Let D_1 be any orientation of G_1 and orient $G \times P_5$ in such way: G_1 is oriented as D_1; the directions of edges in G_3 and G_5 are the same as the corresponding edges in G_1 and the directions of edges in G_2 and G_4 are opposite to the corresponding edges in G_1; for $j = 1, 2, 3, 4$, the edges belonging to E_P are all directed from G^j to G^{j+1}. Denote this orientation by D. We will prove that each M-alternating cycles of $G \times P_5$ is oddly oriented in D. Figure 2 shows the orientation D and the perfect matching M of $G \times P_5$ when $m = 1$.

Choose an M-alternating cycle C of $G \times P_5$. If C contains an edge from $E(G^2)$ denoted by $e^2 = s^2t^2$, it must contain two edges s^2s^3 and t^2t^3. If C contains the edge s^3t^3, C is $s^2t^2t^3s^3s^2$ and it is oddly oriented clearly. If C does not contain the edge s^3t^3, replace the path $s^3s^2t^2t^3$ in C by the edge s^3t^3 and then we obtain a new cycle. (This is the *replacement operation.*) According to the orientation of $G \times P_5$, this new cycle is oddly oriented if and only if C is oddly oriented. Likely if C contains an edge in G^4, take the same operation as above. After replacing all such paths in C, we obtain a new cycle, denoted by C_1. We can see that C_1 contains no edge in $E(G^2)$ or $E(G^4)$. Further, C_1 is oddly oriented if and only if C is oddly oriented. Hence, we need to show that C_1 is oddly oriented.

Firstly, we travel along the cycle C_1 and color the edge of C_1 along this travelling by red and color the edges opposite this transversal by blue. Secondly, we contract the edges of $G \times P_5$, which belongs to E_P and the resulting multigraph is denoted by G^*. (This is the *contraction operation*.) After this contraction operation, the cycle C_1 is turned into a closed trial T_r. Each edge in T_r receive the same color and the same direction as the corresponding edge in C_1. Note that G^* can also be obtained from G by replacing each edge in G by a 5-multiple edge. Hence, we suppose that $V(G^*) = V(G)$ and then $V(T_r) \subset V(G)$. It is easy to deduce that the cycle C is oddly oriented if and only if the number of red edges and the number of blue edges in T_r are both odd.

To prove this, we show first that T_r only consists of 2-multiple edges. Since T_r is a closed trail, the degree of each vertex in T_r is even. Since C is an even cycle, T_r contains of an even number of edges (a k-multiple edge contains k edges). Since C_1 contains no edges in G^2 and G^4, T_r only contains single edges, 2-multiple edges or 3-multiple edges. Let i be the maximum number such that $x_i \in V(T_r)$. Since the degree of x_i is even in T_r, $x_{i-1}x_i$ is a 2-multiple edge in T_r. Note that $x_0 = v$. Since the degree of x_{i-1} is even, $x_{i-2}x_{i-1}$ is a 2-multiple edge or does not exist in T_r. Likely, we can deduce that for any $k \in [1, i-1]$, $x_k x_{k+1}$ can not be a 3-multiple edge. Therefore only uw, vw, uv could be 3-multiple edges. No matter which edge is a 3-multiple edge, we can deduce that each of the other two edges is a single or 3-multiple edge since the degrees of u, v and w are even. Then it follows that the number of edges in T_r is odd, a contradiction. Hence uw, vw and uv can not be 3-multiple edges. Likely, we can also deduce that these three edges can not be single edge. Each of them is a 2-multiple edge or does not exist in T_r. Hence T_r only consists of 2-multiple edges.

Now we consider the following two cases that uv and wv lie in T_r or not. Firstly we consider the case that at least one of uv and wv does not lie in $E(T_r)$. Without loss of generality, we assume that $uv \notin E(T_r)$. Then the cycle C is an $M-$alternating cycle in $(G - uv) \times P_5$ whether wv lies in $E(T_r)$ or not. Note that $G - uv$ is a path. By the choice of M, we know that the edges in T_r (omitting the multiple edges) is corresponding to a path of odd length in G. When travel along the closed trail T_r, the two edges in each 2-multiple edge of T_r have different directions and so have different colors. Hence the number of red edges and blue edges in T_r are both odd and then C is oddly oriented.

Next we consider the case that both uv and wv lie in T_r. Let $d_{T_r}(v)$ be the number of multiple edges incident to v in the closed trail T_r. We show that $d_{T_r}(v) = 6$. If $d_{T_r}(v) = 4$, the cycle C does not contain the edge $v^1x_1^1$ since uv and wv are 2-multiple edges. As C is an M-alternating cycle, it contains neither u^1v^1 nor w^1v^1. However, the subgraph of $G \times P_5$ induced by $\{u^i, w^i, v^i : i = 2, 3, 4, 5\}$ can not contribute to M-alternating cycles such that the contraction of E_P leads to T_r. Hence the degree of v is six and C must contain the edge $v^1x_1^1$. In the following, we show that C must contain the edge v^1u^1 or v^1w^1. Suppose to the contrary that C contains none of v^1u^1 and v^1w^1. Then C contains the edge v^1v^2, and so C contains the path $x_1^1v^1v^2v^3$. If C must contain the edge v^3v^4, C contains the path $x_1^1v^1v^2v^3v^4v^5$. In this case, the edges $v^1w^1,\ v^2w^2,\ v^3w^3,\ v^4w^4$ can not be in $E(C)$ and then vw is a single edge in T_r, a contradiction. So we suppose that C contains the edge v^3u^3. Then the edges $v^1w^1,\ v^2w^2,\ v^3w^3$ can not be in $E(C)$. If not, the cycle C can not exit. Since vw is a 2-multiple edge in T_r, C must contain the edges v^4w^4 and v^5w^5. In this case C must contain the edges v^4v^5 and w^4w^5. Then C contains a cycle $v^4w^4w^5v^5v^4$ and the path $x_1^1v^1v^2v^3u^3$ as two components. Clearly, this is impossible since C is a cycle. Thus, C can not contain v^3u^3. Likely we can deduce that C can not contain the edge v^3w^3. Now we remains to consider the case that C contains the edge $v^3x_1^3$. In this case, C can not contain the edges $v^1w^1,\ v^2w^2$ and v^3w^3, since we can deduce a contradiction in a similar approach as above. Hence C must contain one of the edges v^1u^1 and v^1w^1.

Without loss of generality, suppose that C contains the edge v^1u^1. By the choice of M, C contains the path $v^1u^1w^1w^2w^3$. We shall determine which vertex is the other neighbour of w^3 in C. If C contains the edge w^3w^4, then C contains the path $v^1u^1w^1w^2w^3w^4w^5$. It follows that the edges v^1w^1, v^2w^2, v^3w^3 and v^4w^4 can not be in C. Then vw can not be a 2-multiple edge in T_r. This is a contradiction. If C contains the edge w^3u^3, it must contain the path $w^3u^3u^2v^2v^3$. Then the edges v^1w^1, v^2w^2 and v^3w^3 can not be in C. Since vw is a 2-multiple edge in T_r, we can deduce that C contains the cycle $v^4w^4w^5v^5v^4$ and a path as two componnets and likely we get a contradiction. Hence C can not contain the edge w^3u^3. It follows that the other neighbour of w^3 is v^3 and then C contains the path $v^1u^1w^1w^2w^3v^3v^2$. If C contains the edge $v^2x_1^2$, C can not contain the edge $v^3x_1^3$, $v^4x_1^4$ and $v^5x_1^5$, since x_1v is a 2-multiple edge in T_r. In this case, the other path in C from v^1 to v^2 can not contain any copies of u, v and w except v^1 and v^2. Then uv and vw are both single edges in T_r, a contradiction. Hence C must contain the edge v^2u^2 and then it contains the path $v^1u^1w^1w^2w^3v^3v^2u^2u^3u^4u^5$. Since uw is a 2-multiple edge in T_r, C must contain the edge u^5w^5 and then C contains the path $P_1 = v^1u^1w^1w^2w^3v^3v^2u^2u^3u^4u^5w^5w^4v^4v^5$. We can find that no matter which orientation of G is, the numbers of edges of P_1 in two different directions when traveling along C are both even.

We now consider the other path P_2 from v^1 to v^5 in C. Obviously P_2 is an M-alternating path. After the replacement and the contraction operation, P_2 is transformed into a subgraph T_r^* of T_r. In the following, we will show that the subgraph T_r^* consists of an odd number of 2-multiple edges. Firstly, we shall find a subpath P_2' of P_2 such that after the replacement and contraction operations, P_2' is transformed into an odd path without multiple edges and one of its end is v^1.

Suppose that k is the maximum subscript of all the vertices in $V(P_2)$. Search the vertices along P_2 from v^1 to v^5 and denote the first vertex whose subscript is k by x_k^j. We shall prove that $j = 1$ or $j = 2$. If not, we assume that $j = 3, 4$ or 5. If $j = 3$, since x_k^j is the first vertex whose subscript is k, C must contain the edge $x_{k-1}^3x_k^3$. Then the other neighbour of x_k^3 must be x_k^2 and so C contains the path $x_{k-1}^3x_k^3x_k^2x_k^1$. Since k is the maximum subscript of all the vertices in $V(P_2)$, the other neighbour of x_k^1 is x_{k-1}^1. Then P_2 contains a subpath Q_1 from v^1 to x_{k-1}^1. From the structure of $G \times P_5$, we can see that P_2 can not contain a path from x_{k-1}^1 to v^5, which is disjoint from Q_1. This is a contradiction. Hence $j \neq 3$. If $j = 4$, C must contain the edge $x^4_{k-1}x^4_k$. By the choice of M, C contains the path $x^5_{k-1}x^4_{k-1}x^4_kx^5_k$. Since k is the maximum subscript of all vertices in $V(P_2)$, $x^5_kx^5_{k+1}$ can not be in C and then there exists no path from x^5_k to v^5. Hence $j \neq 4$. If $j = 5$, C contains the edge $x^5_{k-1}x^5_k$ and $x^5_kx^4_k$. Note that P_2 contains a subpath from v_1 to x^4_k via x^5_k. In this case, P_2 contains no subpath from x^4_k to v^5, which is disjoint from Q_2. Hence $j \neq 5$. Therefore, we can deduce that $j = 1$ or $j = 2$. By the choice of M, if $j = 2$, P_2 contains the path $x^3_{k-1}x^2_{k-1}x^2_kx^3_kx^4_kx^5_k$, and it is transformed into the path $x^3_{k-1}x^3_kx^4_kx^5_k$ after the replacement operation.

Denote the subpath of P_2 from v^1 to x^j_k by P'_2. By the choice of x^j_k, P'_2 has the same vertex set with T^*_r after the replacement and contraction operations. To prove our assertion, we only need to prove that P'_2 is transformed into an odd path without multiple edges, after the replacement and contraction operations. We first prove that the sequence of subscripts of vertices from x^1_1 to x^j_k in P'_2 is monotone increasing. If not, search the vertices along P'_2 from v^1 to x^j_k and denote the first vertex which has a bigger subscript than the vertex after it by x^c_a. Denote the vertex in P'_2 after x^c_a by x^d_b. Note that two adjacent vertices in $P_2 - v^1 - v^5$ have the same subscripts or superscripts. Since $a > b$, it holds that $c = d$ and $b = a - 1$. Then P'_2 contains the edge $x^c_ax^c_{a-1}$. If $c = 1$, P_2 contains a subpath from v^1 to x^1_a via x^2_a. In this case P_2 contains no subpath from x^1_{a-1} to v^5. If $c = 2$, P_2 contains the path $x^3_ax^2_ax^2_{a-1}x^3_{a-1}$. Since x^2_a is the first vertex which has a bigger subscript than the vertex after it, P_2 contains the edge $x^3_{a-1}x^3_a$. In this case P_2 contains a cycle $x^3_ax^2_ax^2_{a-1}x^3_{a-1}x^3_a$. It is a contradiction. If $c = 3$, we will consider two cases depending on which vertex lies before x^3_a in P_2. If P_2 contains the edge $x^2_ax^3_a$, P_2 contains the path $x^1_ax^2_ax^3_ax^3_{a-1}$. By the structure of the perfect matching M, we can find that P_2 contains a path Q_3 from x^3_{a-1} to x^j_k via vertices in G^5. In this case, P_2 contains no subpath from x^j_k to v^5, which is disjoint from Q. It is a contradiction. If P_2 contains the edge $x^4_ax^3_a$, P_2 contains the path $x^5_ax^4_ax^3_ax^3_{a-1}$, which is not an M-alternating path. This is a contradiction. If $c = 4$, P_2 contains a subpath $x^5_ax^4_ax^4_{a-1}x^5_{a-1}$. Since x^4_a is the first vertex which has a bigger subscript than the one after it, P_2 contains the edge $x^5_{a-1}x^5_a$. In this case P_2 contains a cycle $x^5_ax^4_ax^4_{a-1}x^5_{a-1}x^5_a$, a contradiction. If $c = 5$, by a similar approach, we can show that P_2 contains a subpath from v^1 to x^5_{a-1} via x^4_a. In this case P_2 contains no subpath from x^5_{a-1} to x^j_k, a contradiction. In each case, we can deduce a contradiction. Hence such vertex x^c_a does not exist and the sequence of subscripts of vertices from x^1_1 to x^j_k is monotone increasing. In this case, following the replacement and contraction operations, P'_2 is transformed into a path without multiple edge.

To prove that P'_2 is transformed into an odd path, we prove first that there is no internal vertex of P'_2 belonging to $E(G^5)$. Assume, to the contrary, that $x^5_s \in V(P'_2) \cap V(G^5)$. Then P'_2 contains two subpaths, one is from v^1 to x^5_s and the other is from x^5_s to x^j_k. In this case P_2 contains no subpath from x^j_k to v^5, a contradiction. By the choice of M, we can find that there is no internal vertex of P'_2 belonging to $E(G^4)$ in the same approach.

For convenience, let P^0_2 denote the path obtained from P'_2 after the replacement operation. Remove all the isolated vertices in $P^0_2 - E_P$ and the subgraph we obtained is denoted by H. Since the sequence of subscripts of vertices from x^1_1 to x^j_k is monotone increasing, all the edges in P^0_2 are single edges. After the contraction operation, all the vertices of P_2^0 are corresponding to the vertices of T_r^*. To show that T_r^* consists of an odd number of 2-multiple edges, we only need to prove that $|E(H)|$ is odd. Since there is no internal vertex of P_2' belonging to $E(G^4) \cup E(G^5)$ and x_k^j belongs to $E(G^1)$ or $E(G^2)$, H consists of several paths of G^1 and G^3. We call these paths the paths from G^1 and the paths from G^3. By the choice of M, all paths from G^1 are of odd length. Besides, if there exists one path from G^1 except the one whose end is v^1, there exists one odd path from G^3. Hence the number of paths in H from G^1 except the one whose end is v^1 is equal to the number of odd paths in H from G^3. Hence, we can deduce that $|E(H)|$ is odd and then P_2^0 is an odd path. It follows that T_r^* consists of an odd number of 2-multiple edges. In T_r^*, the two edges in each 2-multiple edge have the same directions and then they have different colors. It follows that the number of red edges and blue edges in T_r^* are both odd. Therefore, the number of edges in P_2 along the travelling of C and the number of edges of P_2 opposite this travelling are both odd. Hence C is oddly oriented.

By now, we have deduced that C is oddly oriented and by the arbitrary of C we conclude that D is a Pfaffian orientation of $G \times P_5$. □

Theorem 12. *Let G be a connected graph with a perfect matching. Then $G \times P_{2n+1}$ ($n \geqslant 3$) is Pfaffian if and only if G contains no Y-tree as its subgraph.*

Proof. Suppose that $G \times P_{2n+1}$ is Pfaffian. If G contains a Y-tree T_Y as its subgraph, then $G \times P_{2n+1}$ contains $T_Y \times P_6$ as a nice subgraph. However, $T_Y \times P_6$ is not Pfaffian by Theorem 9 (3). This is a contradiction.

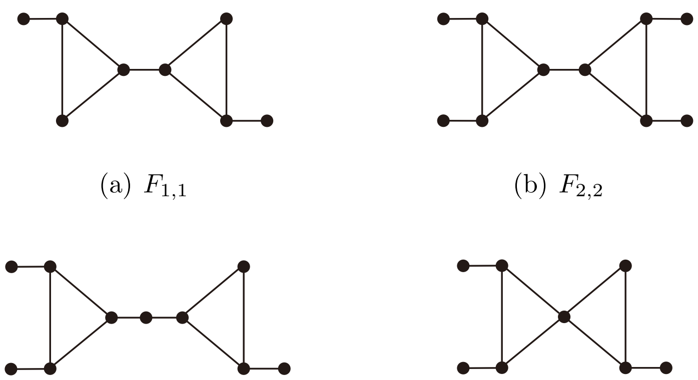
Conversely, if G contains no Y-tree as its subgraph and has a perfect matching, then G is a path, a cycle or $|V(G) \leqslant 4|$. If G is a path or a cycle, $G \times P_{2n+1}$ is a planar graph which is Pfaffian by Theorem 2. If $|V(G) \leqslant 4|$, $G \times P_{2n+2}$ is Pfaffian by Theorem 9 (3). Since G has a perfect matching, a nice cycle in $G \times P_{2n+1}$ is also a nice cycle in $G \times P_{2n+2}$. Hence $G \times P_{2n+1}$ is Pfaffian. □

3 A necessary condition for Pfaffian graph $G \times P_3$

For a graph G with a perfect matching, we will present a necessary condition such that $G \times P_3$ is Pfaffian.

We exhibit four types of forbidden subgraphs of the Pfaffian graph $G \times P_3$. Each of these four subgraphs has one perfect matching and its maximum degree is three.

The graph $F_{1,1}$ consists of two vertex-disjoint triangles which are connected by a single edge, and each of these two triangles is incident with one appending edge as shown in Figure 3(a). For $i, j \in \{0, 2\}$, the graph $F_{i,j}$ consists of two vertex-disjoint triangles which are connected by a single edge, and one triangle is incident with i appending edges and the other triangle is incident with j appending edges (see Figure 3(b) for $F_{2,2}$). For $l \in \{0, 2\}$, the graph $F_{l,1}^1$ consists of two vertex-disjoint triangles which are connected by a path of length two, and one of these two triangles is incident with l appending edges and the other triangle is incident with one appending edge (see Figure 3(c) for $F_{2,1}^1$). For $l \in \{0, 2\}$, the graph $F_{l,1}^2$ consists of two triangles with only one common vertex, and one of these two



(a) $F_{1,1}$ (b) $F_{2,2}$

(c) $F^1_{2,1}$ (d) $F^2_{2,1}$

Figure 3: $F_{1,1}$, $F_{2,2}$, $F^1_{2,1}$ and $F^2_{2,1}$.

triangles is incident with l appending edges and the other triangle is incident with one appending edge (see Figure 3(d) for $F^2_{2,1}$).

Let F be one of the graphs in $\{F_{1,1}, F_{0,0}, F_{2,0}, F_{2,2}, F^1_{0,1}, F^1_{2,1}, F^2_{0,1}, F^2_{2,1}\}$. We define an *even subdivision of* F as the graph obtained from F by replacing each edge in the odd cycles and the path connecting the cycles by a path of odd length. Let G be a graph with a perfect matching. We say that G is an *appending edge expansion* of F if G is obtained from an even subdivision of F by attaching an even number of (probably zero) appending edges to each odd cycles, and make sure that each vertex of the odd cycles is incident with at most one attachment edge. Apparently, F is an appending edge expansion of itself.

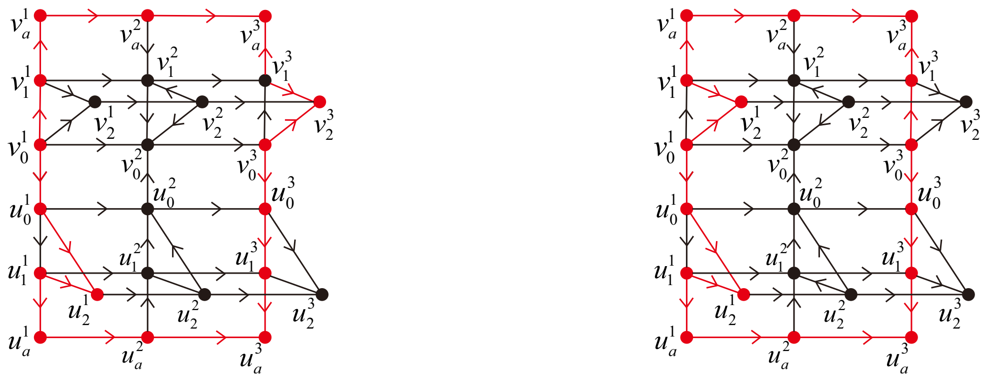
Before showing the main theorem of this section, we introduce the following result given by Norine and Thomas in [18].

Theorem 13 (Norine et al. [18]). *Let G be a connected Pfaffian graph and T a spanning tree of G. Let $e \in G$ be an edge joining two vertices at an even distance in T. Then an arbitrary orientation of $T{+}e$ can be extended to a Pfaffian orientation of G.*

It is clear that if a nice subgraph H of G is not Pfaffian, then $G \times P_3$ is not Pfaffian. Hence we intent to prove that if a graph G is an appending edge expansion of $F_{1,1}$, $F_{i,j}$, $F^1_{l,1}$ and $F^2_{l,1}$, then $G \times P_3$ is not Pfaffian.

Lemma 14. *If G is an appending edge expansion of $F_{1,1}$, then $G \times P_3$ is not Pfaffian.*

Proof. Assume, to the contrary, that $G \times P_3$ is Pfaffian. We first consider the case that the graph G is $F_{1,1}$. Suppose that $V(G) = \{v_0,\ v_1,\ v_2,\ v_a,\ u_0,\ u_1,\ u_2,\ u_a\}$ and $E(G) = \{v_0v_1,\ v_0v_2,\ v_1v_a,\ v_1v_2,\ v_0u_0,\ u_0u_1,\ u_0u_2,\ u_1u_a,\ u_1u_2\}$. The labeling of vertices of $G \times P_3$ is shown in Figure 4(a). Let T_1 be the subgraph of G induced by $\{v_0v_1,\ v_0v_2,\ v_1v_a,\ v_0u_0,\ u_0u_1,\ \ u_0u_2, u_1u_a,\}$. Clearly, T_1 is a spanning tree of G. Let T^1_1, T^2_1 and T^3_1 denote the spanning tree of G^1, G^2 and G^3 corresponding to T_1, respectively. Let T be a spanning tree of $G \times P_3$ such that $T = T^1_1 + T^2_1 + T^3_1 + \{v^1_0v^2_0,\ v^2_0v^3_0\}$.



(a) an orientation of $u_1^1u_2^1$ (b) an evenly oriented nice cycle

Figure 4: $F_{1,1} \times P_3$.

Since $G \times P_3$ is Pfaffian, a Pfaffian orientation of $G \times P_3$ can be obtained by extending an orientation of $T + v_1^1v_2^1$ by Theorem 13. The orientation of $T + v_1^1v_2^1$ is shown in Figure 4(a). The direction of the remaining edges except $u_1^iu_2^i$ $(i = 1,\ 2,\ 3)$ can be determined by the orientation of $T + v_1^1v_2^1$ according to the fact that each nice cycle is oddly oriented. Note that the cycle $C = v_0^1v_1^1v_a^1v_a^2v_a^3v_1^3v_2^3v_0^3u_0^3u_1^3u_a^3u_a^2u_a^1u_1^1u_2^1u_0^1v_0^1$ is a nice cycle of $G \times P_3$. Hence the direction of $u_1^1u_2^1$ is from u_1^1 to u_2^1 and the direction of $u_1^2u_2^2$ and $u_1^3u_2^3$ can be also determined (see Figure 4(b)). However in this orientation, the nice cycle $C' = v_0^1v_2^1v_1^1v_a^1v_a^2v_a^3v_1^3v_0^3u_0^3u_1^3u_a^3u_a^2u_a^1u_1^1u_2^1u_0^1v_0^1$ is not oddly oriented, a contradiction.

Consider the case that G is an even subdivision of $F_{1,1}$. Each edge of G is replaced by a path of odd length. Especially, suppose that the edge v_1v_2 is replaced by the path $v_1s_1\cdots s_{2m}v_2$ and the edge u_1u_2 is replaced by the path $u_1t_1\cdots t_{2n}u_2$. Considering T_1 in the former case, the even subdivision of T_1 together with two paths $s_1\cdots s_{2m}v_2$ and $t_1\cdots t_{2n}u_2$ is a spanning tree of G in the current case. For convenience, we still denote this spanning tree by T_1. Denote the spanning tree corresponding to T_1 in G^1, G^2 and G^3 by T_1^1, T_1^2 and T_1^3, respectively. Denote a spanning tree of $G \times P_3$ by T such that $T = T_1^1 + T_1^2 + T_1^3 + \{v_0^1v_0^2,\ v_0^2v_0^3\}$. Orient $T + v_1^1s_1^1$ in such a way: for each path replacing an edge of $T + v_1^1v_2^1$ in the former case, all the edges on this path have the same direction as the edge which have been replaced; and the direction of edges which have not been replaced stay the same. Since each cycle of length four which does not belong to G^1, G^2 and G^3 is a nice cycle of $G \times P_3$ and oddly oriented, the direction of all edges except $u_1^it_1^i$ $(i = 1,\ 2,\ 3)$ can be determined. Note that the even subdivisions of the cycles C and C' in the former case are still nice cycles of $G \times P_3$ in the current case. Hence we can determine the directions of the edges $u_1^it_1^i$ $(i = 1,\ 2,\ 3)$ and then we can find an evenly oriented cycle similarly as before. This is a contradiction.

If G is an appending edge expansion of $F_{1,1}$ and contains at least two new appending edges, then the even subdivisions of C and C' in the first case are still nice cycles in the current case. Likely we can deduce a contradiction.

In each case we can deduce a contradiction and hence $G \times P_3$ is not Pfaffian. □

(a) orientation of $u_1^1 u_2^1$ (b) an evenly oriented nice cycle

Figure 5: $F_{0,0} \times P_3$.

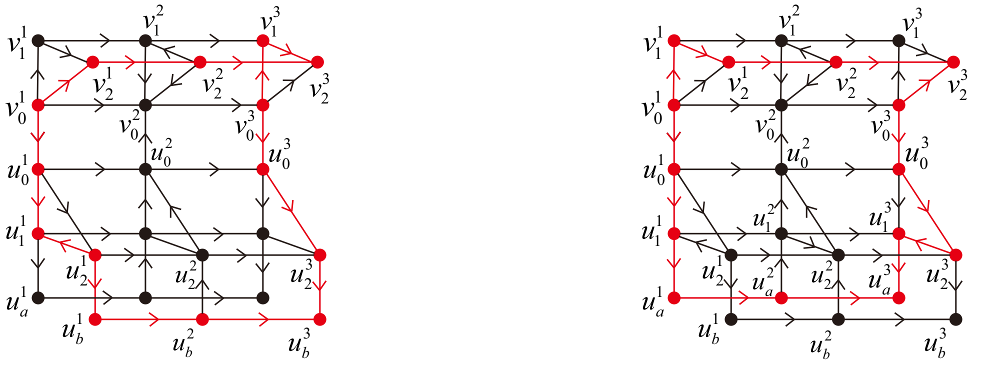
In the proofs of the following lemmas, we consider the case that G is a graph in the set $\{F_{0,0}, F_{2,0}, F_{2,2}, F_{0,1}^1, F_{2,1}^1, F_{0,1}^2, F_{2,1}^1, \}$. For the case of appending edge expansion graph, the proofs are similar as the proof in Lemma 14, and so we omit the proofs of appending edge expansion of these graphs in the following lemmas.

Lemma 15. *If G is an appending edge expansion of $F_{0,0}$, $F_{2,0}$ or $F_{2,2}$, then $G \times P_3$ is not Pfaffian.*

Proof. Assume, to the contrary, that $G \times P_3$ is Pfaffian. We first consider the case that G is $F_{0,0}$. Suppose that $E(F_{0,0}) = \{v_0v_1,\ v_1v_2,\ v_0v_2,\ v_0u_0,\ u_1u_2,\ u_0u_1,\ u_0u_2\}$. Let T be a spanning tree of $K_{0,0} \times P_3$ with $E(T) = \{v_0^i v_1^i,\ v_0^1 v_2^2,\ u_0^i u_1^i,\ u_0^i u_2^i,\ v_0^i u_0^i\ :\ i = 1,\ 2,\ 3\} \cup \{v_0^1 v_0^2,\ v_0^2 v_0^3\}$. A Pfaffian orientation of $F_{0,0} \times P_3$ can be obtained by extending an orientation of $T + v_1^1 v_2^1$ by Theorem 13. The orientation of $T + v_1^1 v_2^1$ is shown in Figure 5(a). Since each cycle of length four is a nice cycle, the orientations of the edges $v_k^i v_k^{i+1}$ and $u_k^i u_k^{i+1}$ for $k = 0, 1, 2$ and $i = 1, 2$ can be determined. Then the orientations of $v_1^2 v_2^2$ and $v_1^3 v_2^3$ can be determined. Now we consider the orientations of $u_1^i u_2^i$ ($i = 1,\ 2,\ 3$). Note that the cycle $v_0^1 v_1^1 v_1^2 v_1^3 v_2^3 v_0^3 u_0^3 u_1^3 u_1^2 u_1^1 u_2^1 u_0^1 v_0^1$ is a nice cycle (see Figure 5(a)). Hence, the direction of $u_1^1 u_2^1$ is from u_1^1 to u_2^1. However, the nice cycle $v_0^1 v_2^1 v_1^1 v_1^2 v_1^3 v_0^3 u_0^3 u_1^3 u_1^2 u_1^1 u_2^1 u_0^1 v_0^1$ is not oddly oriented in this case (see Figure 5(b)), a contradiction. Hence, $F_{0,0} \times P_3$ is not Pfaffian.

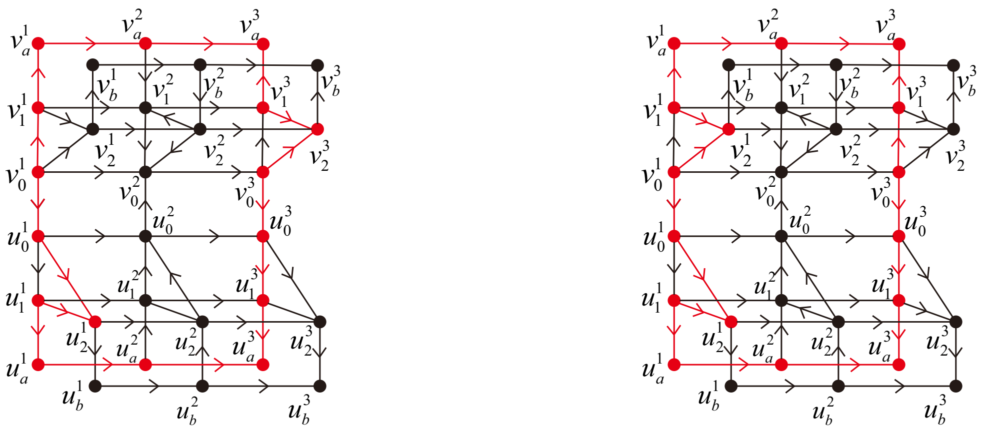
If G is $F_{2,0}$, let T denote the spanning tree of $F_{2,0} \times P_3$ with $E(T) = \{v_0^i v_1^i,\ v_0^i v_2^2,\ v_0^i u_0^i,\ u_0^i u_1^i,\ u_0^i u_2^i,\ u_1^i u_a^i,\ u_2^i u_b^i\ :\ i = 1,\ 2,\ 3\} \cup \{v_0^1 v_0^2,\ v_0^2 v_0^3\}$. See Figure 6. A Pfaffian orientation of $F_{2,0} \times P_3$ can be constructed. The edges except $u_1^i u_2^i$ ($i = 1,\ 2,\ 3$) can be determined by an orientation of $T + v_1^1 v_2^1$. The direction of $u_1^1 u_2^1$ can be determined by the nice cycle $v_0^1 v_2^1 v_2^2 v_2^3 v_1^3 v_0^3 u_0^3 u_2^3 u_b^3 u_b^2 u_b^1 u_2^1 u_1^1 u_0^1 v_0^1$ and it is from u_2^1 to u_1^1. Similarly, the direction of all the other edges can be determined (see Figure 6(a)). However, the nice cycle $v_0^1 v_1^1 v_2^1 v_2^2 v_2^3 v_0^3 u_0^3 u_2^3 u_1^3 u_a^3\ u_a^2 u_a^1 u_1^1 u_0^1 v_0^1$ is not oddly oriented (see Figure 6(b)), a contradiction. Hence $F_{2,0} \times P_3$ is not Pfaffian.

If G is $F_{2,2}$, similarly we can find a spanning tree of $F_{2,2} \times P_3$. By the orientation of this spanning tree with an edge joining two vertices at an even distance, we can determine the direction of all the edges except $u_1^i u_2^i$ ($i = 1,\ 2,\ 3$). The direction of $u_1^1 u_2^1$ can be determined by the nice cycle $v_0^1 v_1^1 v_a^1 v_a^2 v_a^3 v_1^3 v_2^3 v_0^3 v_0^3 u_0^3 u_1^3 u_a^3 u_a^2 u_a^1 u_1^1 u_2^1 u_0^1 v_0^1$ and it is from u_1^1 to u_2^1.



(a) orientation of $u_1^1 u_2^1$ (b) an evenly oriented nice cycle

Figure 6: $F_{2,0} \times P_3$.



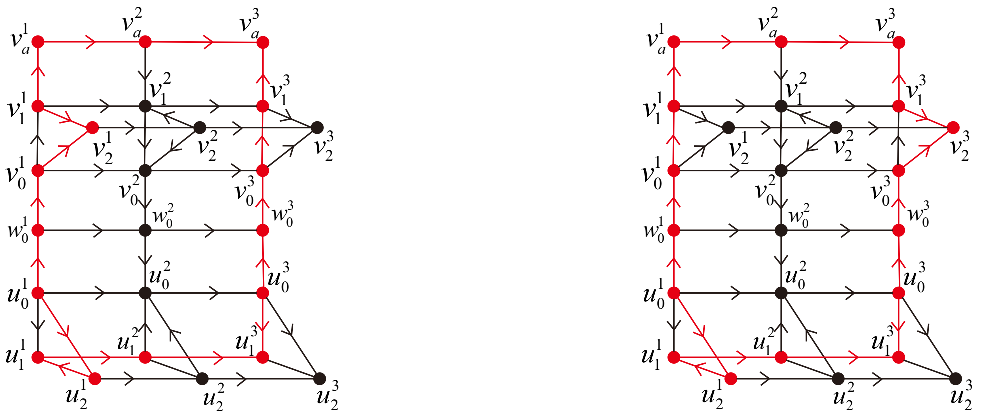
(a) orientation of $u_1^1 u_2^1$ (b) an evenly oriented nice cycle

Figure 7: $F_{2,2} \times P_3$.

The orientation of $F_{2,2} \times P_3$ is shown in Figure 7(a). By Theorem 13, such an orientation is a Pfaffian orientation. However, the nice cycle $v_0^1 v_2^1 v_1^1 v_a^1 v_a^2 v_a^3 v_1^3 v_0^3 u_0^3 u_1^3 u_a^3 u_a^2 u_a^1 u_1^1 u_2^1 u_0^1 v_0^1$ is not oddly oriented (see Figure 7(b)) in this case, a contradiction. Hence $F_{2,2} \times P_3$ is not Pfaffian. $\square$

Lemma 16. *If G is an appending edge expansion of $F^1_{0,1}$ or $F^1_{2,1}$, then $G \times P_3$ is not Pfaffian.*

Proof. Assume, to the contrary, that $G \times P_3$ is Pfaffian. We first consider the case that G is $F^1_{0,1}$. Let T be a spanning tree of $F^1_{0,1} \times P_3$ with $E(T) = \{v_0^i v_1^i,\ v_0^i v_2^i,\ v_1^i v_a^i,\ v_0^i w_0^i,\ w_0^i u_0^i,\ u_0^i u_1^i,\ u_0^i u_2^i :\ i = 1,\ 2,\ 3\} \cup \{v_0^1 v_0^2,\ v_0^2 v_0^3\}$ as shown in Figure 8. Theorem 13 implies that the direction of all the edges except $u_1^i u_2^i$ ($i = 1,\ 2,\ 3$) can be determined according to an orientation of $T + v_1^1 v_2^1$ by Theorem 13. The direction of $u_1^1 u_2^1$ can be determined since the cycle $v_0^1 v_2^1 v_1^1 v_a^1 v_a^2 v_a^3 v_1^3 v_0^3 w_0^3 u_0^3 u_1^3 u_1^2 u_1^1 u_2^1 u_0^1 w_0^1 v_0^1$ is a nice cycle. The orientation of $F^1_{0,1} \times P_3$



(a) orientation of $u_1^1u_2^1$

(b) an evenly oriented nice cycle

Figure 8: $F^1_{0,1}\times P_3$.

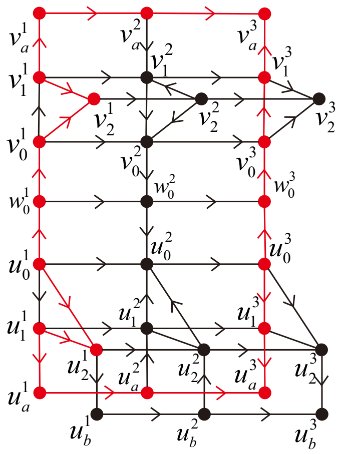
is shown in Figure 8(a). However, the nice cycle $v_0^1v_1^1v_a^1v_a^2v_a^3v_1^3v_2^3v_0^3w_0^3u_0^3u_1^3u_1^2u_1^1u_2^1u_0^1w_0^1v_0^1$ is not oddly oriented (see Figure 8(b)), a contradiction.

If G is $F^1_{2,1}$, denote the two appending edges of $F^1_{2,1}$ by u_1u_a and u_2u_b. Similarly, we can find a spanning tree of $F^1_{2,1}\times P_3$. Let T be a spanning tree with $E(T)=\{v_0^iv_1^i,\ v_0^iv_2^i,\ v_1^iv_a^i,\ v_0^iw_0^i,\ w_0^iu_0^i,\ u_0^iu_1^i,\ u_0^iu_2^i,\ u_1^iu_a^i,\ u_2^iu_b^i\ :\ i=1,\ 2,\ 3\}\cup\{v_0^1v_0^2,\ v_0^2v_0^3\}$. We can give a Pfaffian orientation of $F^1_{2,1}\times P_3$ according to the orientation of $T+v_1^1v_2^1$. We can find that the cycle $v_0^1v_2^1v_1^1v_a^1v_a^2v_a^3v_1^3v_0^3w_0^3u_0^3u_1^3u_a^3u_a^2u_a^1u_1^1u_2^1u_0^1w_0^1v_0^1$ is a nice cycle (see Figure 9(a)). However, the nice cycle $v_0^1v_1^1v_a^1v_a^2v_a^3v_1^3v_2^3v_0^3w_0^3u_0^3u_1^3u_a^3u_a^2u_a^1u_1^1u_2^1u_0^1w_0^1v_0^1$ is not oddly oriented (see Figure 9(b)). A contradiction occurs. □

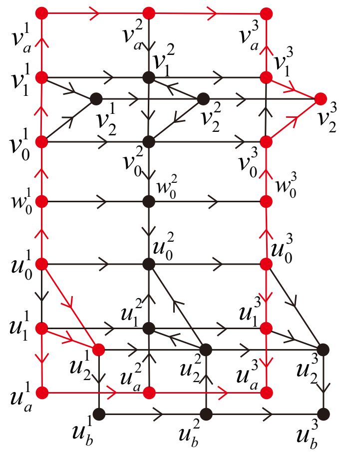
Lemma 17. *If G is an appending edge expansion of $F^2_{0,1}$ or $F^2_{2,1}$, then $G\times P_3$ is not Pfaffian.*

Proof. Assume, to the contrary, that $G\times P_3$ is Pfaffian. If G is $F^2_{0,1}$, suppose that $E(F^2_{0,1}\times P_3)=\{v_0^iv_1^i,\ v_0^iv_2^i,\ v_0^iu_1^i,\ v_0^iu_2^i,\ v_1^iv_a^i,\ v_1^iv_2^i,\ u_1^iu_2^i\ :\ i=1,\ 2,\ 3\}\cup\{v_j^iv_j^{i+1},\ v_a^iv_a^{i+1},\ u_k^iu_k^{i+1}\ :\ i=1,\ 2,\ j=0,\ 1,\ 2,\ k=1,\ 2\}$. From a spanning tree T of $F^2_{0,1}\times P_3$, an edge joining two vertices at an even distance of T and the nice cycle $v_0^1v_2^1v_1^1v_a^1v_a^2v_a^3v_1^3v_0^3u_1^3u_1^2u_1^1u_2^1v_0^1$ (see Figure 10(a)), we can orient the edges of $F^2_{0,1}\times P_3$ such that it is a Pfaffian orientation by Theorem 13. However, the nice cycle $v_0^1v_1^1v_a^1v_a^2v_a^3v_1^3v_2^3v_0^3u_1^3u_1^2u_1^1u_2^1v_0^1$ is not oddly oriented in this orientation (see Figure 10(b)), a contradiction. Hence $F^2_{0,1}\times P_3$ is not Pfaffian.

If G is $F^2_{2,1}$, denote the two appending edges of $F^2_{2,1}$ by u_1u_a and u_2u_b. Then we can find a Pfaffian orientation of $F^2_{2,1}\times P_3$ by Theorem 13. The direction of all the edges except $u_1^iu_2^i$ ($i=1,\ 2,\ 3$) can be determined by an orientation of a spanning tree of $F^2_{2,1}\times P_3$ and an extra edge $v_1^1v_1^2$. The direction of $u_1^1u_2^1$ can be determined since $v_0^1v_2^1v_1^1v_a^1v_a^2v_a^3v_1^3v_0^3u_1^3u_a^3u_a^2u_a^1u_1^1u_2^1v_0^1$ is a nice cycle (see Figure 11(a)). In this orientation, the nice cycle $v_0^1v_1^1v_a^1v_a^2v_a^3v_1^3v_2^3v_0^3u_1^3u_a^3u_a^2u_a^1u_1^1u_2^1v_0^1$ is not oddly oriented (see Figure 11(b)). It is a contradiction. Hence $F^1_{1,2}\times P_3$ is not Pfaffian. □

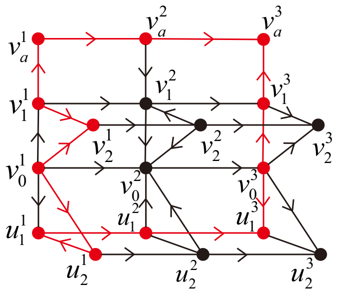


(a) orientation of $u_1^1u_2^1$

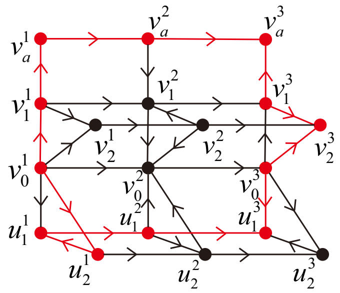


(b) an evenly oriented nice cycle

Figure 9: $F_{2,1}^1 \times P_3$.



(a) orientation of $u_1^1u_2^1$



(b) an evenly oriented nice cycle

Figure 10: $F_{0,1}^2 \times P_3$.

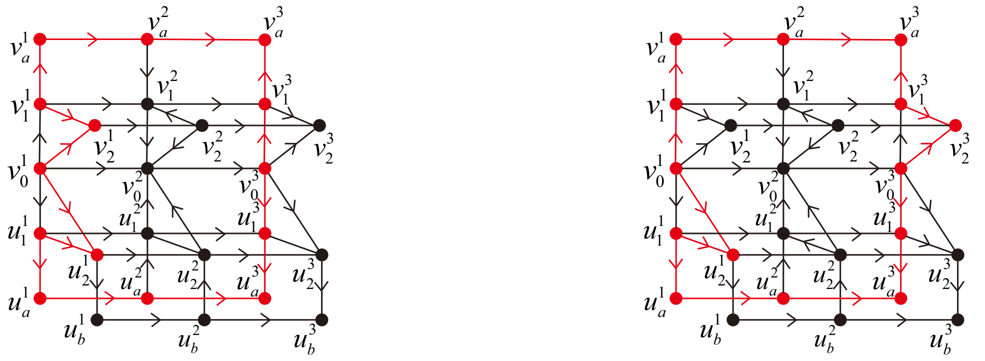
By the above lemmas, it is easy to deduce our main result of this section.

Theorem 18. *Let G be a connected graph with a perfect matching. Let $\mathscr{F} = \{F_{1,1}, F_{0,0}, F_{2,0}, F_{2,2}, F_{0,1}^1, F_{2,1}^1, F_{0,1}^2, F_{2,1}^2\}$. If $G \times P_3$ is Pfaffian, then G contains no appending edge expansion of a graph in $\mathcal{F}$ as its nice subgraph.*

4 A characterization of some Pfaffian graph $G \times P_3$

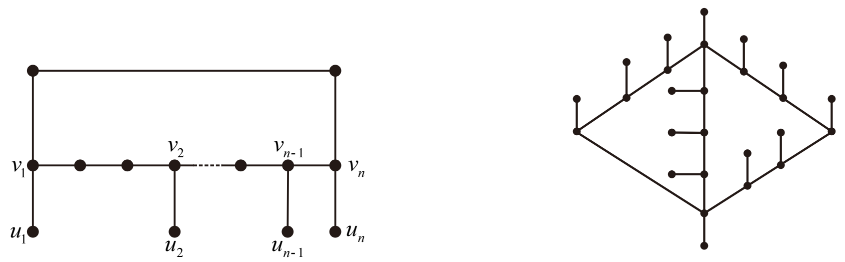
In this section, we first characterize the Pfaffian graph $G \times P_3$ when G is a bipartite graph in terms of the forbidden subgraphs of G. Based on this, we will determine the Pfaffian graph $G \times P_3$ when G is a non-bipartite graph with exactly one odd cycle. Firstly, we exhibit the structure of the forbidden subgraphs of the bipartite graph G such that $G \times P_3$ is Pfaffian.

The graph $H_{m,n}$ consists of an even cycle C_m and n appending edges (n is even and $0 \leqslant n \leqslant m-2$) such that each vertex of C_m is incident with at most one appending edge



(a) orientation of $u_1^1 u_2^1$ (b) an evenly oriented nice cycle

Figure 11: $F_{2,1}^2 \times P_3$.



(a) $H_{m,n}$ (b) H_a

Figure 12: The forbidden graphs $H_{m,n}$ and $K_{2,3}^+$.

and the ends of all the appending edges separate C_m into odd paths. Obviously, the graph $H_{m,n}$ admits a perfect matching (see Figure 12(a)). The graph $K_{2,3}^+$ is obtained from an even subdivision of $K_{2,3}$ by attaching an appending edge to each vertex (see Figure 12(b)).

For a bipartite graph G, we characterize the Pfaffian graph $G \times P_3$ as follows.

Theorem 19. *Let G be a connected bipartite graph with a perfect matching. The graph $G \times P_3$ is Pfaffian if and only if G contains no $H_{m,n}$ or $K_{2,3}^+$ as its nice subgraph.*

To prove Theorem 19, we need the following four lemmas.

Lemma 20. *Neither $H_{m,n} \times P_3$ nor $K_{2,3}^+ \times P_3$ is Pfaffian.*

Proof. The graph $H_{m,n} \times P_3$ is a bipartite graph. We first prove that $H_{4,2} \times P_3$ is not Pfaffian. Suppose that $V(H_{4,2}) = \{v_1,\ u_1,\ v_2,\ u_2,\ v_a,\ v_b\}$. Then $V(H_{4,2} \times P_3) = \{v_1^i,\ u_1^i,\ v_2^i,\ u_2^i,\ v_a^i,\ v_b^i : i = 1,\ 2,\ 3\}$(see Figure 13). The subgraph F of $H_{4,2} \times P_3$ induced by $\{v_2^1,\ v_b^1,\ v_a^1,\ v_a^2,\ v_b^2,\ v_2^2\}$ together with two paths $v_2^1 u_2^1 u_2^2 u_2^3 v_2^3 v_b^3 v_a^3 v_a^2$ and $v_2^2 v_1^2 v_1^1 v_a^1$ is an even subdivison of $K_{3,3}$. Since $H_{4,2} \times P_3 - F$ has a perfect matching, F is a nice subgraph of $H_{4,2} \times P_3$. Therefore, $H_{4,2} \times P_3$ contains an even subdivision of $K_{3,3}$ as a nice subgraph and it is not Pfaffian by Theorem 3.

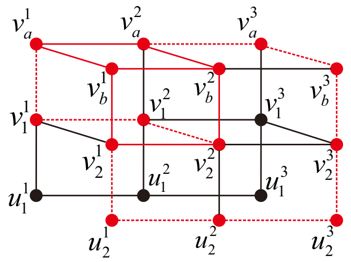


Figure 13: $H_{4,2} \times P_3$.

Now we show that $H_{m,n} \times P_3$ is not Pfaffian when $m > 4$ and $n \geqslant 2$. In this case, $H_{m,n}$ can be obtained from an even subdivision of $H_{4,2}$ by attaching $n-2$ appending edges. Since $n \leqslant m-2$ and $H_{m,n}$ has a perfect matching, there exist two adjacent vertices in $V(C_m)$, denoted by v_a and v_b, such that these two vertices are not incident with any appending edges. Among all the ends of appending edges belonging to $V(C_m)$, denote the one closest to v_a by v_1 and the one closest to v_b by v_2, respectively. Following this labeling of vertices, we can find an even subdivision of F in $H_{m,n} \times P_3$, which is also an even subdivision of $K_{3,3}$ and it is a nice subgraph of $H_{m,n} \times P_3$. Hence $H_{m,n} \times P_3$ contains an even subdivision of $K_{3,3}$ as its nice subgraph and it is not Pfaffian.

Any graph $K_{2,3}^+$ contains an even subdivision of $K_{2,3}$. Hence by Theorem 9, $K_{2,3}^+ \times P_2$ is not Pfaffian. Moreover, $K_{2,3}^+ \times P_2$ is a nice subgraph of $K_{2,3}^+ \times P_3$ and so $K_{2,3}^+ \times P_3$ is not Pfaffian. $\square$

For a cycle C of a graph G, a *chord* of C is an edge e in G such that the end-vertices of e are on C but the edge e is not on C.

Lemma 21. *Let G be a bipartite graph with a perfect matching and without $H_{m,n}$ and $K_{2,3}^+$ as its nice subgraphs. Then for any perfect matching M of G, each cycle of G is an M-alternating cycle or contains no edge in M.*

Proof. Assume, to the contrary, that there is a cycle C' in G, which is not M-alternating and contains some edges in M. Let E' denote a set of edges of $E(G) - E(C')$ such that every edge in E' is in M and has at least one end in $V(C')$. Let H be the subgraph of G induced by $E(C') \cup E'$. Clearly, the graph H is a nice subgraph of G.

We say that two chords of C *cross* each other if their ends alternate on C. We first show that if some chords of C' are in H, then each of these chords does not cross others. If not, suppose that there are two chords crossing each other, denoted by $e_1' = v_1'v_1''$ and $e_2' = v_2'v_2''$. Let P_1' and P_1'' be the two paths of $C' - v_1' - v_1''$ and let P_2' and P_2'' be the two paths of $C' - v_2' - v_2''$. Since e_1' crosses e_2', one of v_2' and v_2'' lies in P_1' and the other lies in P_1''. Suppose that v_2' lies in P_1' and v_2'' lies in P_1''. See Figure 14. If C' contains some chords crossing e_2' whose ends lie in P_1'', among these chords choose one such that the distance between one of its ends and v_2'' is smallest and denote this chord by $e_3' = v_3'v_3''$. The edge e_3' could be e_1' exactly. Then there are two paths in $C' - v_3' - v_3''$. Denote the

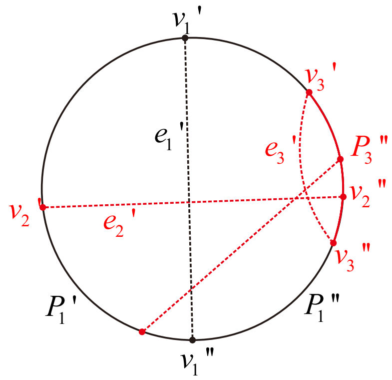


Figure 14: The cycle C' and its chords.

one containing v_2'' by P_3''. The subgraph induced by the cycle $P_3'' + e_3'$ together with the edges in E' that have exactly one end lies in P_3'' is a graph $H_{m,n}$, a contradiction. Hence, C' contains no chords crossing e_2' whose ends lie in P_1''.

Now we show that the number of chords of C' is at most one. Suppose to the contrary that there exist n chords of C' with $n \geqslant 2$. By the previous result, we know that all these chords do not cross each other. Hence C' together with n chords forms $n+1$ induced cycles denoted by $C_1', C_2', \ldots, C_{n+1}'$. For any two cycles C_i' and C_j', if $C_i' \cap C_j' \neq \emptyset$, then $C_i' \cap C_j'$ is a chord of C'. If all these $n+1$ cycles are not incident with any edges in E', then for $i \in [1, n]$, C_{i+1}' contains two edges, and these two edges are incident with C_i'. Then C_i' together with these two edges of C_{i+1}' can be regarded as a graph $H_{m,n}$, which is a nice subgraph of H and so a nice subgraph of G. This is a contradiction. If some of these $n+1$ cycles are incident with edges in E', let C_j' be such a cycle. Then C_j' together with its appending edges in E' forms a graph $H_{m,n}$, which is a nice subgraph of G, a contradiction. Hence we conclude that there exists at most one chord of C' in H.

Let $e_1 = v_1v_2$ and $e_2 = v_3v_4$ be two edges in E' such that $C' - v_1 - v_3$ are two paths of odd lengths and at least one of these two paths is not incident with any other edge in E'. Denote this path by P_1 and the other by P_2. We consider first the case that v_1, v_3 are not adjacent on C' and $e_1 = e_2$. In this case, e_1 is a chord of C'. Since P_1 is not incident with any other edges in E' except e_1, $P_1 + e_1$ is an M-alternating cycle in H. Denote the neighbours of v_1 and v_3 in $V(P_1)$ by v_5 and v_6, respectively. If there is no edge in E' incident with P_2, then $P_2 + e_1 + v_1v_5 + v_3v_6$ is a graph in $H_{m,n}$ and it is a nice subgraph of G. This is a contradiction. If there are some edges in E' incident with P_2, then the subgraph induced by $E(P_2 + e_1)$ and these edges in E' is a graph $H_{m,n}$, which is a nice subgraph of G. For the case that $e_1 \neq e_2$, our proof depends on whether the cycle C' has a chord in H or not. If there is no chord, the graph H is a graph $H_{m,n}$ and it is a nice subgraph of G. If there is a chord e', the graph $C' \cup e'$ contains two cycles containing e'. Denote the one which is incident with e_1 and e_2 by C_1'. Then C_1' together with its appending edges in E' form a nice subgraph $H_{m,n}$ of G. This is a contradiction. $\square$

The following lemma illustrates the relations of the forbidden subgraphs.

Lemma 22. *Let G be a bipartite graph with a perfect matching and without $H_{m,n}$ as its nice subgraph. Then G contains no $K^+_{2,3}$ as its nice subgraph if and only if G contains no even subdivision of $K_{2,3}$ as its subgraph.*

Proof. Since any graph $K^+_{2,3}$ contains an even subdivision of $K_{2,3}$ as its subgraph, the sufficiency is obvious. Conversely, assume, to the contrary, that G contains an even subdivision of $K_{2,3}$ as its subgraph, denoted by H. Let M be any perfect matching of G. Since $|V(H)|$ is odd and G has a perfect matching, there exists an odd number of vertices in $V(H)$ that are incident with some edges attaching to H (these edges are not in H, we call them *attaching edges*) and lying in M. Denote the two even cycles in H by C_1 and C_2, respectively. Suppose that C_1 has an edge belonging to M. Then Lemma 21 implies that both C_1 and C_2 (or C_1 and $C_1 \triangle C_2$)are M-alternating cycles. It contradicts to the truth that H has at least one attaching edge belonging to M. Hence C_1 has no edge belonging to M. Likely, we can deduce that C_2 and $C_1 \triangle C_2$ also have no edge belonging to M. Without loss of generality, we suppose that C_1 is incident with an attaching edge belonging to M. By Lemma 21 and the truth that M covers all vertices in $V(H)$, we can deduce that each vertex in $V(H)$ is incident with an attaching edge in M. However, H together with all the attaching edges form a nice subgraph $K^+_{2,3}$ of G, a contradiction. $\square$

By the result of Lemma 21, we can partition all the cycles in G into two sets $\mathscr{C}_M$ and $\mathscr{C}_{\overline{M}}$, where $\mathscr{C}_M$ is the set of cycles that are M-alternating and $\mathscr{C}_{\overline{M}}$ is the set of cycles containing no edges in M. The following lemma exhibits the kind of cycles sharing common edges.

Lemma 23. *Let G be a graph with a perfect matching and it contains no $H_{m,n}$ and $K^+_{2,3}$ as nice subgraphs. Let M be a perfect matching of G. If the intersection of two cycles is a path, then these two cycles lie in $\mathscr{C}_{\overline{M}}$.*

Proof. Let C' and C'' be two cycles in G such that $C' \cap C'' = P$, where P is a path. By Lemma 22, G contains no even subdivision of $K_{2,3}$ as its subgraph. Then P contains an odd number of common edges. It follows from Lemma 21 that every cycle is M-alternating or contains no edges in M.

Assume, to the contrary, that C' is M-alternating. In this case, we can deduce that C'' can not be M-alternating. If not, assume that C'' is M-alternating. Consider first the case that P contains at least three edges. Then P is an M-alternating path with two ends v_1 and u_1. Let v_2 and u_2 be the neighbours of v_1 and u_1 in P, respectively. The edges v_1v_2 and u_1u_2 must be in M, otherwise v_1 will be incident with two edges in M. Then the cycle $C' \triangle C''$ is not M-alternating and contains the edges in M. This contradicts to Lemma 21. If P contains exactly one edge, then this edge must be in M. Similarly, the cycle $C' \triangle C''$ is still not M-alternating, but contains the edges in M, a contradiction. Hence C'' can not be M-alternating. It follows that C'' contains no edges in M and then each vertex of C'' is incident with an edge in M.

If P contains at least three edges, P is an M-alternating path since C' is M-alternating. This is a contradiction to the truth that C'' can not be M-alternating. If P contains exactly one edge, this edge can not be in M. Hence $C' \triangle C''$ together with edges in M, which are attached to C'', contains a nice subgraph $H_{m,n}$ of G. This is a contradiction. Hence C' does not contain any edge in M, so does C''. Therefore, both C' and C'' lie in $\mathscr{C}_{\overline{M}}$. $\square$

It can be found that the following two propositions hold for the graph $G \times P_3$.

Proposition 24. *A graph G is a bipartite graph if and only if $G \times P_3$ is a bipartite graph.*

Proposition 25. *If H is a nice subgraph of G, then $H \times P_3$ is a nice subgraph of $G \times P_3$.*

To construct a Pfaffian orientation of $G \times P_3$, we introduce the following results on orientation graphs. In 2002, Fischer and Little [4] gave a characterization of the existence of an orientation of a graph such that all the even cycles are oddly oriented.

Theorem 26 (Fischer et al. [4])**.** *A graph has an orientation under which each even cycle is oddly oriented if and only if the graph contains no subgraph which is, after the contraction of at most one odd cycle, an even subdivision of $K_{2,3}$.*

For bipartite graphs, we have the following immediate corollary.

Corollary 27. *There exists an orientation of a bipartite graph G such that all the cycles of G are oddly oriented if and only if G contains no even subdivision of $K_{2,3}$.*

There are several equivalent conditions under which a graph is Pfaffian [12].

Theorem 28 (Lovász et al. [12])**.** *Let G be a graph with an even number of vertices and D an orientation of G. Then the following statements are equivalent.*
(1) D is a Pfaffian orientation of G.
(2) Every nice cycle of G is oddly oriented in D.
(3) If G has a perfect matching, then for some perfect matching M, every M-alternating cycle is oddly oriented in D.

Now, we are ready to show the proof of Theorem 19.

Proof of Theorem 19. Since $|V(P_3)| = 3$, $G \times P_3$ contains three copies of G, denoted by G^1, G^2 and G^3. Let $V(G \times P_3) = \{\ v_i^j \in V(G^j)|\ \forall v_i \in G,\ j = 1,\ 2,\ 3\}$ and $E(G \times P_3) = \{v_s^j v_t^j|\ \forall v_s v_t \in E(G),\ j = 1,\ 2,\ 3\} \cup E_P$, $E_P = \{v_s^j v_s^{j+1}|\ \forall v_i \in G,\ j = 1, 2\}$. If G contains a nice subgraph $H_{m,n}$ (resp. $K_{2,3}^+$), then $H_{m,n} \times P_3$ (resp. $K_{2,3}^+ \times P_3$) is a nice subgraph of $G \times P_3$. By Lemma 20, $H_{m,n} \times P_3$ (resp. $K_{2,3}^+ \times P_3$) is not Pfaffian and hence $G \times P_3$ is not Pfaffian. Therefore, if $G \times P_3$ is Pfaffian, then G contains no $H_{m,n}$ and $K_{2,3}^+$ as its nice subgraphs.

Conversely, suppose that G contains no $H_{m,n}$ and $K_{2,3}^+$ as its nice subgraphs. It follows from Lemma 22 that G contains no even subdivision of $K_{2,3}$ as its subgraph. Then Theorem 26 implies that G has a Pfaffian orientation D' under which each even cycle is oddly oriented. Orient $G \times P_3$ in such a way that G^1 and G^3 have the same orientation as D' and the direction of each edge in G^2 is opposite to the corresponding edge in G^1. The direction of the edges in E_P is from v_i^j to v_i^{j+1} for $j = 1, 2$. Such an orientation of $G \times P_3$ is denoted by D. In the following, we prove that D is a Pfaffian orientation of $G \times P_3$.

Since every even cycle of G^1 is oddly oriented, D' is a Pfaffian orientation of G^1. By Theorem 28, there exists a perfect matching M of G^1 such that every M-alternating cycle of G^1 is oddly oriented in D'. It is obvious that it is oddly oriented in D. Let $M'{=}M \cup \{v_s^2 v_s^3 : \ v_s \in V(G)\}$ be a perfect matching of $G \times P_3$. To prove that D is a Pfaffian orientation of $G \times P_3$, it is sufficient to show that every M'-alternating cycle of $G \times P_3$ is oddly oriented under the orientation D. Let C be an M'-alternating cycle of $G \times P_3$. If C contains no edges in $E(G^3)$, then C is an even cycle in G^1 and then C is oddly oriented.

It remains to consider the case that the M'-alternating cycle C contains edges in $E(G^3)$. If C contains the edges in $E(G^2)$, denote one of these edges by $u_1^2 v_1^2$. Since C is M'-alternating and $u_1^2 v_1^2 \notin M'$, it follows that $u_1^2 u_1^3,\ v_1^2 v_1^3 \in V(C) \cap M'$. Recall that the direction of $u_1^2 u_1^3$ is from u_1^2 to u_1^3 and the direction of $v_1^2 v_1^3$ is from v_1^2 to v_1^3. Furthermore, $u_1^2 v_1^2$ and $u_1^3 v_1^3$ are in converse direction. It follows that the cycle obtained from C by replacing the path $v_1^3 v_1^2 u_1^2 u_1^3$ by $v_1^3 u_1^3$ has the same parity of the number of edges of C in direction when traveling along the cycle. Hence, replacing all this kind of paths in C in this way, we obtain a new cycle C_1. We can find that the cycle C_1 contains no edge in $E(G^2)$. Particularly, if G contains no edges in $E(G^2)$, then $C_1 = C$. In conclusion, the cycle C is oddly oriented if and only if C_1 is oddly oriented.

If C_1 contains no edges in $E(G^1)$, then C_1 is an even cycle in G^3 and obviously is oddly oriented in D. Consider that case that C_1 contains edges in G^1 and G^3. Traveling C along on direction, we color the edges of C_1 along this direction by red and color the edges of C_1 opposite to this direction by blue. We will show that the number of blue edges and the number of red edges in C_1 are both odd.

Recall that C_1 is an even cycle and the direction of edges in E_P is from v_i^j to v_i^{j+1}. It follows that the number of red edges and the number of blue edges in E_P are equal and both even. We contract all the edges of E_P in $G \times P_3$. The resulting graph with multiple edges (without loops)is denoted by G'. Note that G' is obtained from G by replacing each edge of G by three multiple edges. After this contraction operation, C_1 is transformed into a closed trail in G' denoted by T_r. Moreover, the edges in T_r receive the same color as the corresponding edges in C_1. It is easy to deduce that the maximum degree of vertices in T_r is at most four and the degree of each vertex in T_r is even. Furthermore, the number of red (resp. blue) edges in C_1 has the same parity with the number of red (resp. blue) edges in T_r. Therefore, we only need to show that the number of blue edges and the number of red edges in T_r are both odd. In the remaining proof, when we say the cycles in T_r, we only consider the cycles containing at least three vertices. We say an edge uv of T_r (single or multiple) is in M if the corresponding edge uv of G is in M. We will prove the following claims, which are related to cycles and multiple edges in the closed trail T_r. These claims will help us to figure out the structure of T_r.

Claim 29. *Let uv be an multiple edge in T_r. If uv lies in M, then the M'-alternating cycle C contains the edge u^1v^1 and the path $u^2u^3v^3v^2$ in $G \times P_3$.*

Proof. Since uv is an multiple edge in T_r and it belongs to M, the edge u^1v^1 in G^1 lies in C clearly. Assume, to the contrary, that C dose not contain the path $u^3u^2v^2v^3$. Since uv is a multiple edge in T_r, the cycle C contains at least one of the edges u^2v^2 and u^3v^3. If C contains u^3v^3, then C must contain u^2u^3 and v^2v^3 since C is M'-alternating. Thus C contains the path $u^2u^3v^3v^2$. If C does not include u^3v^3, then u^2v^2 lies in C and so the path $u^3u^2v^2v^3$ lies in C. Then C either contains a a path from u^1 to u^3 which does not include v^1 and v^3 (we denote this path by $P_{u^1u^3}$), or contains a path from u^1 to v^3 and this path does not include v^1 and u^3 (denote this path by $P_{u^1v^3}$). Consider first the case that the path $P_{u^1u^3}$ lies in C. After the contraction operation defined above, the path $P_{u^1u^3}$ corresponds to a closed subtrail of T_r, denoted by T_r'. Since u^1v^1 and u^3u^2 are in M', the length of $P_{u^1u^3}$ is odd and so is T_r'. However, G' is a bipartite multigraph in which the length of each closed trail is even. Hence we deduce a contradiction. Now consider the case that the path $P_{u^1v^3}$ lies in C. In this case the edge uv in M is a common edge of two even cycles in G which is impossible by Lemma 23. $\square$

Claim 30. *Any two cycles of T_r are edge-disjoint.*

Proof. Assume, to the contrary, that there are two cycles in T_r having edges in common. Then there exist two cycles C' and C'' in T_r such that $C' \cap C''$ is a path P. Let v_1 and v_2 be the two ends of P. Both C' and C'' are corresponding to cycles in G and both of them contain no edges in M by Lemma 23. Let H_{v_1} be the subgraph of T_r, which is induced by v_1 and the neighbours of v_1 are in T_r. Since $d(v_1) \geqslant 3$ in T_r, the subgraph H_{v_1} contains at least one edge in $E(G^1)$. Among the edges of H_{v_1} belonging to $E(G^1)$, there must exist one edge belonging to M, denoted by $v_1^1u_1^1$. The edge v_1u_1 in T_r does not belong to $E(C') \cap E(C'')$. Since C is M'-alternating, the cycle C can not contain the path $v_1^1v_1^2v_1^3$. Otherwise, $d(v_1) = 2$ in T_r. Then H_{v_1} contains another edge of $E(G^1)$ not in M, denoted by $v_1^1u_2^1$. Since $d(v_1) \geqslant 3$ and $d(v_1)$ is even, there are another two edges in C incident with v_1^2 and v_1^3. One of them is incident with v_1^2, denoted by $v_1^2u_3^2$. The other edge incident with v_1^3 is denoted by $v_1^3u_4^3$. Moreover, the edge $v_1^2v_1^3$ is also in C. The edges v_1u_2, v_1u_3 and v_1u_4 in T_r are corresponding to $v_1^1u_2^1$, $v_1^2u_3^2$ and $v_1^3u_4^3$ in G^1,G^2 and G^3, respectively. These edges lie in $E(C') \cup E(C'')$. Up to now, we have found all the edges incident with v_1^1, v_1^2 and v_1^3 in C and all the edges incident with v_1 in T_r.

Consider the walk of T_r starting with v_1. Since T_r is a close trail, following the edge v_1u_1, there must be a vertex in $V(C') \cup V(C'')$, which is adjacent to u_1. We deduce that this vertex must be v_1. If not, there exists an even cycle containing v_1u_1 in T_r. It corresponds to a cycle in G. Lemma 21 implies that this cycle in G is an M-alternating cycle. This M-alternating cycle has at least two successive common edges with $C' \triangle C''$, which contains no edges in M. This contradicts to Lemma 23. Hence, v_1 lies in another closed subtrail of T_r except C' and C''. This menas that $d(v_1) > 4$ in T_r. This is not possible since the maximum degree of vertices in T_r is at most four. $\square$

Claim 31. *For a multiple edge st in T_r with two ends s and t, if the edge st in G is not in M, then the M'-alternating cycle C contains the edge s^1t^1 and the path $s^3s^2t^2t^3$ in $G \times P_3$.*

Proof. It is easy to deduce that s^1t^1 lies in $E(C)$. Since st is a multiple edge, we show that C contains the path $s^3s^2t^2t^3$. Assume, to the contrary, that G contains the path $s^2s^3t^3t^2$. If C contains a path from s^1 to s^2, which does not contain the vertices t^1 and t^2, we denote such a path by $P_{s^1s^2}$. The length of $P_{s^1s^2}$ is odd. After contracting the edges in E_P, $P_{s^1s^2}$ is converted into a closed trail of odd length in T_r. This is impossible since T_r is a bipartite multigraph. If C contains a path from s^1 to t^2 which does not contain t^1 and s^2, then it will contain a path from t^1 to s^2 which does not contain s^1 and t^2. Then the edge st is a common edge of two cycles in T_r. It contradicts to Claim 30. □

Claim 32. *Any cycle in T_r contains no multiple edge.*

Proof. Suppose to the contrary that there exists a cycle C' in T_r containing a multiple edge denoted by uv. Denote the other edge of C' incident with u by uw. Since the degree of u in T_r is even and at most four, uw is a multiple edge or uw is a single edge that is a common edge of two cycles in T_r. The second case is impossible by Claim 30. If uw is a multiple edge, we consider the other edge of C' incident with w. By this way, we can deduce that each edge of C' is a multiple edge. In this case the closed trail T_r is exactly C', since each vertex in C' is of degree four. This is impossible by the choice of M'. □

The perfect matching chosen in G is M. For any cycle C in T_r, we say C is *associated with* $\mathscr{C}_M$ if the corresponding cycle in G is M-alternating and likely we say C is *associated with* $\mathscr{C}_{\overline{M}}$ if the corresponding cycle in G contains no edges in M.

Claim 33. *For any two cycles with common vertices in T_r, the number of common vertices is one. Moreover, one of these two cycles is associated with $\mathscr{C}_M$ and the other is associated with $\mathscr{C}_{\overline{M}}$.*

Proof. In this proof, we neglect the multiples in T_r. If there exist two cycles with more than one common vertex, then there must exist two cycles with common edges which is not possible by Claim 2. Hence for any two cycles in T_r, they share at most one vertex in common. Clearly two cycles with one common vertex in G can not be two M-alternating cycles. To finish our proof, we assume, to the contrary, that C' and C'' are two cycles with one common vertex such that both are associated with $\mathscr{C}_{\overline{M}}$. Denote the common vertex by s and obviously $d(s) \geqslant 4$. However, the four edges incident with s in C' and C'' are neither in M. Hence there exist another edge incident with s in T_r corresponding to an edge of G in M. This is impossible since the maximum degree is four. □

Claim 34. *Let C' be a cycle in T_r associated with $\mathscr{C}_{\overline{M}}$ and C the cycle corresponding to T_r in $G \times P_3$. Let S be a set of edges in T_r such that each edge in S has only one end in C' and the corresponding edge in G lies in M. Then*
(1) if the ends of two edges in S separate C' into two paths, and at least one of these two paths, denoted by P, satisfies that its internal vertices of P are not incident with any edges in S, then P is of odd length;
(2) the number of edges in S is even.

Proof. Let $H_{C'}$ be a subgraph of C such that after the contraction of E_P it is converted into C'. By the discussion in previous pages, we consider the case that $H_{C'}$ only contains edges in $E(G^1)$ and $E(G^3)$. Since C' is associated with $\mathscr{C}_{\overline{M}}$, $H_{C'}$ is incident with several edges in M and so does C'. Since C is M'-alternating, we can find that $S \neq \emptyset$ and S contains at least two edges. Let v_1v_2 and v_3v_4 be the two edges of T_r in M such that v_1 and v_3 belong to C' and $C' - v_1 - v_3$ are two paths P_1 and P_2. Moreover, v_1 and v_3 are chosen such that the path P_1 is not incident with any other edges except v_1v_2 and v_3v_4. Consider the walk of T_r starting with v_1. Since T_r is a close trail, following the edge v_1v_2, there must be a vertex in $V(C')$, which is adjacent to v_2. Hence, v_1 lies in another closed subtrail of T_r except C', denoted by C''. The trail C'' has exactly one common vertex (that is v_1) with C'. This closed trail C'' is corresponding to an even path P' in C. Now we determine the other end of P'. We can find that the edge $v_1^1v_1^2 \notin E(C)$. Otherwise, $d(v) = 2$ in T_r. Moreover, the edge $v_1^2v_1^3 \in E(C)$, since $d(v) \geqslant 3$ in T_r. Hence the path P' is from v_1^1 to v_1^2 or to v_1^3 in C. It is obvious that the length of P' and C'' are both even, and C'' is obtained from P' by contracting edges in E_P. Hence the number of edges in P' belonging to E_P is even. This means that the other end of P' must be v_1^3. Likewise, when we consider the vertex v_3 in C', we obtain a path P'' with two ends v_3^1 and v_3^3.

We now consider the structure of the subgraph F of C which corresponds to P_1. The two ends of P_1 in T_r are v_1 and v_3. By the structure of the perfect matching M', F is a path on C. Since the vertices of the path F lie in $V(G_2) \cup V(G_3)$, we show first that the path F is from v_1^1 to v_3^1 or from v_1^2 to v_3^2. Suppose to the contrary that it is from v_1^1 to v_3^2 or from v_1^2 to v_3^1. Assume first that F is from v_1^1 to v_3^2 and denote the neighbour of v_1^1 in F by u_1^1. Since $v_1^1v_2^1$ is an edge in M, $v_1^1u_1^1$ is not in M. The other edge of C incident with u_1^1 should be in M. However, such an edge does not exist since C' contains no edges in M and the vertices on P_1 are not incident with any other edges in M except v_1v_2 and v_3v_4. Similarly, the path F from v_1^2 to v_3^1 does not exist. Hence, P_1 corresponds to a path from v_1^1 to v_3^1 or from v_1^2 to v_3^2 in C. Following this, it must be a path of odd length. As C' is an even cycle and any path like P_1 is of odd length, it is easy to deduce that the number of edges in S is even. □

The above claims show cycles and multiple edges in T_r. To illustrate the structure of T_r, we need some notations. Let H_1 and H_2 be the two subgraphs of G. Then $H_1 \cup H_2$ denotes the subgraph of G induced by $E(H_1) \cup E(H_2)$. Two graphs H_1 and H_2 are said to be incident if the intersection of H_1 and H_2 is a vertex.

The closed trail T_r can be partitioned into four edge-disjoint subgraphs $H_{C_{\overline{M}}}$, H_{C_M}, H_{Mul} and H'_{Mul}. The subgraph $H_{C_{\overline{M}}}$ consists of all cycles associated with $\mathscr{C}_{\overline{M}}$ in T_r. All these cycles do not share common edges or vertices with each other by Claims 30 and 33. The subgraph H_{C_M} consists of all cycles associated with $\mathscr{C}_M$ in T_r. It contains two types of cycles: the first type contains the cycles that are incident with several cycles in $H_{C_{\overline{M}}}$ and two incident cycles share only one vertex in common; the second type contains the cycles that share no common vertex with any other cycles. All cycles in T_r are in the subgraph $H_{C_{\overline{M}}} \cup H_{C_M}$.

The subgraph H'_{Mul} consists of all the multiple edges in T_r which are corresponding to edges in $\overline{M}$ of G. We call these multiple edges *type I multiple edges*. These type I multiple edges are vertex-disjoint by Claim 31. Each type I multiple edge connects two components of $H_{C_{\overline{M}}} \cup H_{C_M} \cup H_{Mul}$. The subgraph H_{Mul} consists of all the multiple edges in T_r which are corresponding to edges in M of G. It contains two types of multiple edges: one type is the multiple edges that are incident with cycles in $H_{C_{\overline{M}}}$; the other type is the multiple edges that are not incident with cycles in $H_{C_{\overline{M}}}$ (such multiple edges are incident with multiple edges in H'_{Mul}, since C is an M-alternating cycle). We call the first type multiple edges *type II multiple edges* and call the second type multiple edges *type III multiple edges*. Claim 34 implies that each cycle in $H_{C_{\overline{M}}}$ is incident with an even number of type II multiple edges in $H_{Mul} \cup H_{C_M}$. The subgraph $H_{C_{\overline{M}}} \cup H_{C_M} \cup H_{Mul}$ consists of several components. Type III multiple edges are vertex-disjoint by Claim 29.

To prove that the number of red edges in T_r and the number of blue edges in T_r are both odd. We first consider the subgraph H'_{Mul}. For each 2-multiple edge of H'_{Mul}, one edge in this 2-multiple edges is from $E(G^1)$ and the other is from $E(G^3)$. They receive the same orientation. Hence one is red and the other is blue. The same result holds for the multiple edges in H_{Mul}.

Next, we prove that in each component of $H_{C_{\overline{M}}} \cup H_{C_M} \cup H_{Mul}$, the number of red edges and the number of blue edges are both odd. Note that each component consists of cycles associated with $\mathscr{C}_{\overline{M}}$, cycles associated with $\mathscr{C}_M$ and multiple edges. We know that all the edges in T_r are from $E(G^1)$ and $E(G^3)$. In the graph $G \times P_3$, G^1 and G^3 have the same orientation and each even cycle in G^1 or G^3 is oddly oriented under the orientation D. Hence, we can deduce that in each cycle or multiple edge of T_r, the number of red edges and blue edges are both odd. It is sufficient to prove that the number of cycles and multiple edges are odd.

For any component L of $H_{C_{\overline{M}}} \cup H_{C_M} \cup H_{Mul}$, let G_L be a graph such that each vertex of $V(G_L)$ corresponds to a cycle or a multiple edge of L. Two vertices of G_L are adjacent if and only if the corresponding cycles or multiple edges in L are incident. Claim 5 implies that G_L is a tree. We will prove that $|V(G_L)|$ is odd. Color the vertices in G_L with white and black. If a vertex of G_L corresponds to a cycle associated with $\mathscr{C}_{\overline{M}}$, then color it with white; otherwise, color it with black. By the claims above, we can deduce that two vertices in the same color are not adjacent. For each white vertex, it has an even number of neighbours and all its neighbours are black. Choose a black vertex v as a root. we build a rooted tree. In this rooted tree, each vertex except v has one parent vertex and several child vertices. All the parent and child vertices of black vertices are white. All parent and child vertices of white vertices are black. Then for each white vertex, it has an odd number of child vertices and one parent vertex. The leaves of G_L must be black vertices. Otherwise, G_L will be infinity. Hence, the number of vertices in $V(G_L) - v$ is even and then $|V(G_L)|$ is odd. This means that in each component of $H_{C_{\overline{M}}} \cup H_{C_M} \cup H_{Mul}$, the number of red edges and blue edges are both odd.

If there is an odd number of components, the number of type I multiple edges that connects these components is even. We can deduce that the number of blue edges and the number of red edges are both odd. If there is an even number of components, the number of type III multiple edges connecting these component is odd. We can also deduce that the number of blue edges and the number of red edges are both odd. Hence each

M'-alternating cycle in $G \times P_3$ is oddly oriented in D, and so D is a Pfaffian orientation of $G \times P_3$. $\square$

Combining Theorem 19 and Lemma 22, we can deduce the following result.

Corollary 35. *Let G be a bipartite graph with a perfect matching. Then $G \times P_3$ is Pfaffian if and only if G contains no $H_{m,n}$ as its nice subgraph and contains no even subdivision of $K_{2,3}$ as its subgraph.*

We can generalize Theorem 19 to the case that G is a non-bipartite graph with exactly one odd cycle.

Theorem 36. *Let G be a graph with exactly one odd cycle C_0. If $G - e$ has a perfect matching for any edge $e \in E(C_0)$, then $G \times P_3$ is Pfaffian if and only if $G - e$ contains neither $H_{m,n}$ as its nice subgraph nor an even subdivision of $K_{2,3}$ as its subgraph for any edge $e \in E(C_0)$.*

Proof. The necessity is obviously since any nice subgraph of $G - e$ is also a nice subgraph of G. We shall consider the sufficiency. Due to the truth that the number of odd cycles in G is one, C_0 shares no common edge with any other cycles. Since $G - e$ has a perfect matching, the graph G admits a perfect matching, denoted by M_G. Let M_G be a perfect matching of G. It is obvious that M_G is a perfect matching of G^1, and so $M = M_G \cup E_2$ is a perfect matching of $G \times P_3$.

We will establish an orientation of $G \times P_3$, which will be proved to be a Pfaffian orientation of $G \times P_3$. The graph $G - e$ contains no even subdivision of $K_{2,3}$. That is the graph G contains no subgraph which is, after the contraction of at most one odd cycle, an even subdivision of $K_{2,3}$. It follows from Theorem 26 that there is a Pfaffian orientation D^* of G, under which every even cycle of G is oddly oriented. In the graph $G \times P_3$, let G^1 and G^3 have the same orientation D^*, and each edge of G^2 admits the opposite direction as the corresponding edge in G^1. For $i = 1, 2$, the direction of edges in E_i is from v^i to v^{i+1}. Denote this orientation of $G \times P_3$ by D.

To prove that D is a Pfaffian orientation of $G \times P_3$, we need to prove that each M-alternating cycle is oddly oriented under the orientation D. Let C be any M-alternating cycle of $G \times P_3$. If there exists an edge e_0 of C_0 such that C is an M-alternating cycle of $(G - e_0) \times P_3$, then Theorem 19 shows that $(G - e_0) \times P_3$ is Pfaffian. Since $G - e_0$ is a subgraph of G, the orientation D of $G \times P_3$ restricted to $(G - e_0) \times P_3$ is an orientation of $(G - e_0) \times P_3$. By the proof of Theorem 19, this orientation is a Pfaffian orientation of $(G - e_0) \times P_3$. To find such an edge e_0, we need to find an edge e_0 of C_0 such that e_0 does not lie in M_G, and e_0^i does not lie in $E(C)$ for $i = 1, 2, 3$. In this case, M is also a perfect matching of $(G - e_0) \times P_3$. The cycle C is also an M-alternating cycle of $(G - e_0) \times P_3$ and so it is oddly oriented.

Let S be the set collecting edges in $E(C_0)$ such that for each edge in S its three copies do not belong to $E(C)$. We assert that S is not empty. We contract the edges in E_1 and E_2. After the contraction, $G \times P_3$ is transformed into a multiple graph G^* and the cycle C is transformed into a closed trail denoted by T_r. If the assertion is not true, assume that each edge of C_0 has at least one copy in $E(C)$. The cycle C_0 is corresponding to an odd cycle in T_r denoted by C_0^*. Since the cycle C is an even cycle, $|E(T_r)|$ is also even. Since T_r is closed, all the non-multiple edges of T_r belong to some cycles of T_r. We can deduce that all the edges of C_0^* are non-multiple edges. If not, suppose that there exists a multiple edge of C_0^* denoted by uv. Denote the other edge of C_0^* incident with u by uw. Note that each vertex of T_r is of even degree and the maximum degree of a vertex in T_r is at most four. If uw is a non-multiple edge, it belongs to another cycle. In this case C_0 shares a common edge with another cycle. It contradicts to the truth that there is no cycle sharing common edge with C_0. Hence uw is a multiple edge. In the same way, we can prove that all edges of C_0^* are multiple edges. In this case the degree of each vertex of C_0^* is four. It follows that $E(T_r) - E(C_0^*)$ is empty and the closed trail T_r is exactly C_0^*.

Since each edge of T_r is a multiple edge, C must contain the edges from G^1 and G^3. By the choice of M, there exists at least one path of length two in C passing through the three copies of some vertex of G. Suppose that one of these paths is $v_1^1v_1^2v_1^3$. Since $v_1^1v_1^2 \notin M$, there exists another edge $v_0^1v_1^1$ of C incident with v_1^1 and $v_0^1v_1^1 \in M$. Since T_r consists of 2-multiple edges, one of $v_0^2v_1^2$ and $v_0^3v_1^3$ lies in $E(C)$. Since the degree of each vertex of C is two, neither $v_0^2v_1^2$ nor $v_0^3v_1^3$ lies in C. It is a contradiction. Now we have proved that all edges of C_0^* are non-multiple edges. It follows that $|E(T_r) - E(C_0^*)|$ is odd. This is impossible since the remaining edges in $T_r - E(C_0^*)$ belong to even cycles or 2-multiple edges. Hence S is not empty.

Since S is not empty, T_r does not contain the odd cycle. Then the edges in $E(C_0) - S$ do not belong to any cycle in T_r. Since all the non-multiple edges of T_r belong to some cycles in T_r, each edge in $E(C_0) - S$ corresponds to a 2-multiple edge in T_r. We assert that there must exist at least one edge in S that does not belong to M_G. If all the edges in S do not belong to M_G, the assertion holds clearly. Consider the case that there exists an edge $e_1 = v_1u_1$ in S belonging to M_G. We show that there is another edge in S and it does not lie in M_G. Let $e_2 = v_1u_2$ be one of the edges of C_0, which is adjacent to e_1. Clearly, e_2 does not lie in M_G. We shall prove that $e_2 \in S$. If not, suppose that $e_2 \notin S$. Then at least one of its copies, which belongs to G^i $(i = 1, 2, 3)$, lies in $E(C)$. It follows that e_2 is corresponding to a 2-multiple edges in T_r and $u_2^1v_1^1 \in E(C)$. Since C is an M-alternating cycle, for any vertex u_2^1 adjacent to v_1^1 in C, $u_2^1v_1^1 \notin M$. Then the path $u_2^1v_1^1v_1^2v_1^3$ does not exist in C. It contradicts to $u_2^1v_1^1 \in E(C)$. Hence, e_2 lies in S and so there is at least one edge in S that does not belong to M_G. Denote one of these edges by e_0. The M-alternating cycle C of $G \times P_3$ is also an M-alternating cycle of $(G - e_0) \times P_3$. Since $(G - e_0) \times P_3$ is bipartite, Theorem 19 implies that C is oddly oriented with respect to the orientation D.

By the arbitrariness of C, each M-alternating cycle of $G \times P_3$ is oddly oriented and hence $G \times P_3$ is Pfaffian. $\square$

5 Enumeration perfect matchings of $G \times P_n$ in terms of eigenvalues

In this section, we will evaluate the number of perfect matchings of $G \times P_n$ in terms of the eigenvalues of G and the eigenvalues of a Pfaffian orientation of G, respectively. We begin with the construction of a Pfaffian orientation of $G \times P_n$.

In the Cartesian product $G \times P_n$, let $G^1, G^2, \ldots, G^n$ denote the n copies of G. For a vertex v in $V(G)$ and an edge $e = uv$ in $E(G)$, v^i denotes the copy of v in $V(G^i)$ and $e^i = u^i v^i$ denotes the copy of e in $E(G^i)$. Let E_i denote the edge set $\{v^i v^{i+1} : \forall v^i \in V(G^i)\}$ for $i = 1, 2, \ldots, n-1$.

Theorem 37. *Let G be a graph such that $G \times P_n$ is Pfafian and G admits an orientation G^σ such that all the even cycles are oddly oriented. Construct an orientation $(G \times P_n)^\sigma$ of $G \times P_n$ as follows:*
(a) G^{2k+1} $(k = 1, 2, \ldots)$ receives the same orientation as G^σ;
(b) G^{2k} $(k = 1, 2, \ldots)$ receives the reverse orientation as G^σ;
(c) any edge $v^i v^{i+1}$ in each E_i is directed from v^i to v^{i+1}.
Then this orientation $(G \times P_n)^\sigma$ is a Pfaffian orientation of $G \times P_n$ when n is even; if G admits a perfect matching, then $(G \times P_n)^\sigma$ is a Pfaffian orientation of $G \times P_n$ when n is odd and $n \neq 3$; if G is bipartite, or G has exactly one odd cycle C_0 such that $G - e$ has a perfect matchings for any edge $e \in E(C_0)$, then $(G \times P_3)^\sigma$ is a Pfaffian orientation of $G \times P_3$.

Proof. For $n = 2$, by the proof of Theorem 11 in [25] and Theorem 8 of [13], we can find that $(G \times P_2)^\sigma$ is a Pfaffian orientation.

For $n = 3$, the proofs of Theorems 19 and 36 show that $(G \times P_3)^\sigma$ is a Pfaffian orientation.

For $n = 4$ and $n = 2k$ $(k \geqslant 3)$, by the the proof of Theorem 8 in [13], $(G \times P_n)^\sigma$ is a Pfaffian orientation.

For $n = 5$, the Pfaffian orientation $(G \times P_6)^\sigma$ of $G \times P_6$ restricted to $G \times P_5$ is a Pfaffian orientation by the proof of Theorem 11. Thus $(G \times P_5)^\sigma$ is a Pfaffian orientation of $G \times P_5$.

For $n = 2k+1$ $(k \geqslant 3)$, the proof of Theorem 12 implies that $(G \times P_{2k+1})^\sigma$ is a Pfaffian orientation of $G \times P_{2k+1}$. □

As a continue of the research in [26], we show that the number of perfect matchings of $G \times P_n$ can be expressed by the eigenvalues of an orientation of G. In the following theorems, we use G^σ to denote the orientation of G such that all the even cycles are oddly oriented.

Theorem 38. *(a) Let G be a graph with a perfect matching. If G is a bipartite graph and contains no $H_{m,n}$ and $K^+_{2,3}$ as its nice subgraphs, or G has exactly one odd cycle C_0 such that $G - e$ has a perfect matching and $G - e$ contains neither $H_{m,n}$ as its nice subgraph* *nor an even subdivision of* $K_{2,3}$ *as its subgraph for any edge* $e \in E(C_0)$. *Then*

$$\Phi(G \times P_3) = \prod_{\lambda} [(2-\lambda^2)|\lambda^2|^{\frac{1}{2}}]^{m_\lambda}, \tag{1}$$

where the product ranges over all the positive imaginary part eigenvalues λ *of* G^σ.

If the graph G *in* (a) *has a unique perfect matching, then*

$$\Phi(G \times P_3) = \prod_{\lambda} (2-\lambda^2)^{m_\lambda}, \tag{2}$$

where the product ranges over all the positive imaginary part eigenvalues λ *of* G^σ.

(b) *Let* G *be the graph containing neither an even subdivision of Q-graph nor two edge-disjoint odd cycles as its subgraph. Then*

$$\Phi(G \times P_4) = \prod_{\lambda} (1-3\lambda^2+\lambda^4)^{m_\lambda}, \tag{3}$$

where the product ranges over all the positive imaginary part eigenvalues λ *of* G^σ.
(c) *Let* G *be a graph with a perfect matching. If* G *contains neither an* H_Y *as its nice subgraph nor edge-disjoint odd cycles as its subgraph, then*

$$\Phi(G \times P_5) = \prod_{\lambda} [(3-4\lambda^2+\lambda^4)|\lambda^2|^{m_\lambda}],$$

where the product ranges over all the positive imaginary part eigenvalues λ *of* G^σ.
(d) *Let* G *be a graph with a perfect matching. If* G *contains no Y-tree as its subgraph, then for* $n \geqslant 6$,

$$\Phi(G \times P_n) = \prod_{\lambda} \prod_{k=1}^{n} |(4cos^2\frac{\pi k}{n+1} - \lambda^2)|^{\frac{m_\lambda}{2}}, \tag{4}$$

where the first product ranges over all the positive imaginary part eigenvalues λ *of* G^σ. *For the case that* n *is even, if* G *does not have a perfect matching, Eq. (4) also holds. For the case that* G *has a unique perfect matching, it holds that*

$$\Phi(G \times P_n) = \prod_{\lambda} \prod_{k=1, k\neq \frac{n+1}{2}}^{n} |(4cos^2\frac{\pi k}{n+1} - \lambda^2)|^{\frac{m_\lambda}{2}}, \tag{5}$$

where the first product ranges over all the positive imaginary part eigenvalues λ *of* G^σ.

Proof. We show first the proof of (d). If the graph G containing no Y-tree as its subgraph is Pfaffian, it follows from the proofs of Theorems 12 and 9 that G is a path, a cycle or $|V(G)| \leqslant 4$ (when n is even, G may also be a star). Hence, G contains no even subdivision of $K_{2,3}$, after the contraction of at most one odd cycle. Theorem 26 implies that G admits an orientation G^σ such that all the even cycles are oddly oriented. Now we construct the Pfaffian orientation $(G \times P_n)^\sigma$ of $G \times P_n$ according to Theorem 37. It follows from Theorem 1 that

$$\Phi^2(G \times P_n) = |\det A((G \times P_n)^\sigma)|.$$

Suppose that $|G| = p$. Let A be the skew-adjacency matrix of G^σ and I the identity matrix of order p.

If n is even, the skew-adjacency matric $A((G \times P_n)^\sigma)$ of $(G \times P_n)^\sigma$ takes on the form below

$$A((G \times P_n)^\sigma) = \begin{pmatrix} A & I & 0 & 0 & \cdots & 0 \\ -I & -A & I & 0 & \cdots & 0 \\ 0 & -I & A & I & \cdots & 0 \\ 0 & 0 & -I & -A & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -I & -A \end{pmatrix}.$$

If n is odd, the matrix $A((G \times P_n)^\sigma)$ is of the form

$$A((G \times P_n)^\sigma) = \begin{pmatrix} A & I & 0 & 0 & \cdots & 0 \\ -I & -A & I & 0 & \cdots & 0 \\ 0 & -I & A & I & \cdots & 0 \\ 0 & 0 & -I & -A & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & -I & A \end{pmatrix}.$$

No matter n is even or odd, taking a series of elementary row operations on the matrix $A((G \times P_n)^\sigma)$, we obtain the matrix

$$\begin{pmatrix} -A & I & 0 & 0 & \cdots & 0 \\ I & -A & I & 0 & \cdots & 0 \\ 0 & I & -A & I & \cdots & 0 \\ 0 & 0 & I & -A & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & I & -A \end{pmatrix}.$$

Thus,

$$|\det A((G \times P_n)^\sigma)| = |\det(-I_n \otimes A + B \otimes I_p)|,$$

where $\otimes$ denotes the Kronecker product of matrices and the matrix B of order n is of the form

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Suppose the eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_p$, and the eigenvalues of B are $\mu_1, \mu_2, \dots, \mu_n$. Then the eigenvalues of $-I_n \otimes A + B \otimes I_p$ are $\lambda_j - \mu_k$ for $1 \leqslant j \leqslant p$ and $1 \leqslant k \leqslant n$.

It is known that the eigenvalues of B are $2cos\frac{k\pi}{n+1}$, $k = 1, 2, \dots, n$. Since A is a skew adjacency matrix, the eigenvalues of A are pure imaginary or zero. Moreover, if λ is an eigenvalue of the real skew symmetric matrix A, so is its conjugate $\overline{\lambda}$. Thus

$$\begin{aligned} \Phi(G \times P_n) &= |\det(-I_n \otimes A + B \otimes I_p)|^{\frac{1}{2}} \qquad (6) \\ &= |\prod_{j=1}^{p} \prod_{k=1}^{n} (2cos\frac{\pi k}{n+1} - \lambda_j)|^{\frac{1}{2}} \\ &= [\prod_{\lambda} \prod_{k=1}^{n} |(2cos\frac{\pi k}{n+1} - \lambda)(2cos\frac{\pi k}{n+1} + \overline{\lambda})|^{\frac{m_\lambda}{2}}][\prod_{k=1}^{n} |2cos\frac{\pi k}{n+1}|^{\frac{m_0}{2}}] \\ &= [\prod_{\lambda} \prod_{k=1}^{n} |(4cos^2\frac{\pi k}{n+1} - \lambda^2)|^{\frac{m_\lambda}{2}}|][\prod_{k=1}^{n} |2cos\frac{\pi k}{n+1}|^{\frac{m_0}{2}}], \end{aligned}$$

where the first product ranges over all the eigenvalues λ of A whose imaginary part are positive, and m_λ is the multiplicity of the eigenvalue λ.

If n is even, the path P_n admits a perfect matching. Then $\prod_{k=1}^{n} |2cos\frac{\pi k}{n+1}| = 1$. If n is odd, we consider the graph G. Since G admits a perfect matching and $|\det(A)| = \Phi^2(G)$, it holds that $\det(A) \neq 0$. Thus A has no zero eigenvalues, and so $m_0 = 0$. This means that $\prod_{k=1}^{n} |2cos\frac{\pi k}{n+1}|^{\frac{m_0}{2}} = 1$. Hence,

$$\Phi(G \times P_n) = \prod_{\lambda} \prod_{k=1}^{n} |(4cos^2\frac{\pi k}{n+1} - \lambda^2)|^{\frac{m_\lambda}{2}},$$

where the first product ranges over all the positive imaginary part eigenvalues λ of G^σ. We have proved that Eq. (4) holds.

Now we prove the result of (c). We use the same method as above. The eigenvalues of P_5 are 0,1,-1,$\sqrt{3}$,-$\sqrt{3}$. Substituting the eigenvalues of P_5 for $2cos\frac{\pi k}{n+1}$ in Equation (5), we obtain that

$$\begin{aligned} \Phi(G \times P_5) &= \prod_{\lambda} \prod_{k=1}^{5} |(4cos^2\frac{\pi k}{6} - \lambda^2)|^{\frac{m_\lambda}{2}} \\ &= \prod_{\lambda} \prod_{k=1}^{5} |(1-\lambda^2)^2(3-\lambda^2)^2\lambda^2|^{\frac{m_\lambda}{2}} \\ &= \prod_{\lambda} [(3 - 4\lambda^2 + \lambda^4)|\lambda^2|^{\frac{1}{2}}]^{m_\lambda}, \end{aligned}$$

where the product ranges over all the positive imaginary part eigenvalues λ of a Pfaffian orientation G^σ of G.

In case (b), the eigenvalues of P_4 are $\pm\sqrt{\frac{3+\sqrt{5}}{2}}$,$\pm\sqrt{\frac{3-\sqrt{5}}{2}}$. Similarly as above procedures, we can find that Eq. (3) holds.

For case (a), the eigenvalues of P_3 are 0,$\sqrt{2}$,-$\sqrt{2}$. We can derive Eq. (1) in a similar approach. If G has a unique perfect matching, then $\prod_{\lambda}(|\lambda^2|^{\frac{1}{2}})^{m_\lambda} = 1$. Hence, Eq. (2) follows in this case. $\square$

Note: Let G be a graph with a perfect matching. If G is a non-bipartite graph with a unique cycle, then Eq. (5) holds by the result in (d).

The number of perfect matchings of $G \times P_n$ can also be expressed by the eigenvalues of G as shown in Theorem 6. Before providing the proof, we introduce some terminology.

Lemma 39. *If a bipartite graph G contains no cycle of length divisible by four, then it contains no even subdivision of $K_{2,3}$.*

Proof. Suppose that G contains a subgraph H which is an even subdivision of $K_{2,3}$. Then H contains two even cycles C_1 and C_2 intersecting along a path P of even length. If one of C_1 and C_2 is of length $4s$ for some positive integer s, then we are done. Thus we may suppose that $|C_1| = 4s_1 + 2$ and $|C_2| = 4s_2 + 2$. We will find that the symmetric difference of C_1 and C_2 is a cycle of length divisible by four. This is a contradiction. Thus G contains no even subdivision of $K_{2,3}$. $\square$

An even cycle C of length $2l$ is said to be *oriented uniformly* if C is oddly oriented relative to G^σ when l is odd, and C is evenly oriented relative to G^σ when l is even.

Theorem 40. *[1] Let G be a bipartite graph and G^σ an orientation graph of G. Then $Sp_s(G^\sigma) = iSp(G)$ if and only if each even cycle is oriented uniformly in G^σ.*

Proof of Theorem 6:

It follows from Lemma 39 that G contains no even subdivision of $K_{2,3}$. Corollary 27 implies that G admits an orientation G^σ such that all the even cycles are oddly oriented. Such an orientation is a Pfaffian orientation. Since G contains no cycle of length $4s$, all the even cycles of G are oriented uniformly. Now we construct the Pfaffian orientation $(G \times P_n)^\sigma$ of $G \times P_n$ according to Theorem 37. It follows from Theorem 1 that

$$\Phi^2(G \times P_n) = |\det A((G \times P_n)^\sigma)|.$$

Let A be the skew-adjacency matrix of G^σ and B the adjacency matrix of P_n, where

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Suppose that G is of order p. Denote the eigenvalues of A by $\lambda_1, \lambda_2, \ldots, \lambda_p$, and denote the eigenvalues of G by $\alpha_1, \alpha_2, \ldots, \alpha_p$. Theorem 40 implies that $\lambda_j = i\alpha_j$ for $1 \leqslant j \leqslant p$. Since G is a bipartite graph, the spectrum of G is symmetric with respect to zero. As G admits a perfect matching, G has no zero eigenvalues. Then by the same analysis as the proof of Theorem 38, we can obtain that Eq. (4) holds. Substituting λ_j by $i\alpha_j$, we get that

$$\Phi(G \times P_n) = \prod_{\alpha} \prod_{k=1}^{n} |(4cos^2 \frac{\pi k}{n+1} + \alpha^2)|^{\frac{m_\alpha}{2}}, \tag{7}$$

where the first product ranges over all the positive eigenvalues α of G, and m_α is the multiplicity of the eigenvalue α. The proof is finished.

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