On the Spectral Radius of Minimally 2-(Edge)-Connected Graphs with Given Size

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Abstract

A graph is minimally k-connected (k-edge-connected) if it is k-connected (k-edge-connected) and deleting any arbitrary chosen edge always leaves a graph which is not k-connected (k-edge-connected). Let $m = \binom{d}{2} + t$, $1 \leqslant t \leqslant d$ and G_m be the graph obtained from the complete graph K_d by adding one new vertex of degree t. Let H_m be the graph obtained from $K_d \backslash \{e\}$ by adding one new vertex adjacent to precisely two vertices of degree $d-1$ in $K_d \backslash \{e\}$. Rowlinson [Linear Algebra Appl., 110 (1988) 43–53.] showed that G_m attains the maximum spectral radius among all graphs of size m. This classic result indicates that G_m attains the maximum spectral radius among all 2-(edge)-connected graphs of size $m = \binom{d}{2} + t$ except $t = 1$. The next year, Rowlinson [Europ. J. Combin., 10 (1989) 489–497] proved

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that H_m attains the maximum spectral radius among all 2-connected graphs of size $m = \binom{d}{2} + 1$ ($d \geqslant 5$), this also indicates H_m is the unique extremal graph among all 2-connected graphs of size $m = \binom{d}{2} + 1$ ($d \geqslant 5$). Observe that neither G_m nor H_m are minimally 2-(edge)-connected graphs. In this paper, we determine the maximum spectral radius for the minimally 2-connected (2-edge-connected) graphs of given size; moreover, the corresponding extremal graphs are also characterized.

Mathematics Subject Classifications: 05C50, 05C75

1 Introduction

A graph is said to be *connected* if for every pair of vertices there is a path joining them. The *connectivity* (or *vertex-connectivity*) $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or in the trivial graph. The *edge-connectivity* $\kappa'(G)$ is defined analogously, only instead of vertices we remove edges. A graph is *k-connected* if its connectivity is at least k and *k-edge-connected* if its edge-connectivity is at least k. It is almost as simple to check that the minimal degree $\delta(G)$, the connectivity and edge-connectivity satisfy the following inequality:

$$\delta(G) \geqslant \kappa'(G) \geqslant \kappa(G).$$

One of the most important task for characterization of k-connected graphs is to give certain operation such that they can be produced from simple k-connected graphs by repeatedly applying this operation[1]. This goal has accomplished by Tutte [24] for 3-connected graphs, by Dirac [8] and Plummer [18] for 2-connected graphs and by Slater [21] for 4-connected graphs. A graph is said to be *minimally k-connected* if it is k-connected but omitting any of edges the resulting graph is no longer k-connected. Clearly, a k-connected graph whose every edge is incident with one vertex of degree k is minimally k-connected, especially a k-regular and k-connected graph is minimally k-connected.

Questions in extremal graph theory ask to maximize or minimize a graph invariant over a fixed family of graphs. A classic result of minimally k-connected graph is given by Mader who determined the extremal size of a minimally k-connected graph of high order in [13]. We use $A(G)$ to denote the adjacency matrix of a graph G. The largest eigenvalue of $A(G)$ is called the spectral radius of G, denoted by $\rho(G)$. Giving a graph family $\mathcal{G}$ to study the bounds of spectral radius of graphs in $\mathcal{G}$ and to characterize the extremal graphs that achieves the bound is a famous problem in the spectral extremal graph theory [4], which attracts some scholars and have produced many interesting results published in various magazines [17, 14, 12, 23, 27].

In the origin of researches, $\mathcal{G}$ is restricted to the graphs of order n or size m. For giving order of a graph, Nikiforov proposed a spectral Turán problem which asks to determine the maximum spectral radius of an F-free graph with n vertices. This can be viewed as the spectral analogue of Turán type problem. For more results in this direction, readers are referred to a survey by Nikiforov [16]. For giving size of a graph, Brualdi and Hoffman [4] gave an upper bound on spectral radius: if $m \leqslant \binom{k}{2}$ for some integer $k \geqslant 1$ then $\rho(G) \leqslant k - 1$, with equality if and only if G consists of a k-clique and isolated vertices.

Extending this result, Stanley [22] showed that $\rho(G) \leqslant \frac{\sqrt{1+8m}-1}{2}$. In particular, Nosal [17] in 1970 proved that if G is a triangle-free graph with m edges then $\rho(G) \leqslant \sqrt{m}$.

In the subsequent study, $\mathcal{G}$ is restricted to the graphs that have some combinatorial structure. For examples, in 2002, Nikiforov [14, 15] showed that $\rho(G) \leqslant \sqrt{2m(1-\frac{1}{r})}$ for a graph G with m edges, where r is the clique number of G. Bollobás, Lee and Letzter studied the maximizing spectral radius of subgraphs of the hypercube for giving size m [2]. Very recently, Lin, Ning and Wu [12] proved that $\rho(G) \leqslant \sqrt{m-1}$ when G is non-bipartite and triangle-free graph on m edges. Zhai, Lin and Shu [27] obtained that if G contains no pentagon or hexagon of size m, then $\rho(G) \leqslant \frac{1}{2}+\sqrt{m-\frac{3}{4}}$, with equality holds if and only if G is a book.

In the recent works, some authors restrict $\mathcal{G}$ to the graphs that have fixed connectivity. For examples, Chen and Guo showed that $K_{2,n-2}$ attained the maximal spectral radius among all n-vertices minimally 2-(edge)-connected graphs [5]. Fan, Goryainov and Lin proved that $K_{3,n-3}$ has the largest spectral radius over all minimally 3-connected graphs of order n [9]. All the above studies indicate that the spectral radius of a graph are related with the parameters of graphs (such as order n and size m), structure of graphs (such as forbidding subgraphs) and vertex or edge connectivity of graphs.

Let $m = \binom{d}{2} + t$, $1 \leqslant t \leqslant d$ and G_m be the graph obtained from the complete graph K_d by adding one new vertex of degree t. Let H_m be the graph obtained from $K_d \backslash \{e\}$ by adding one new vertex adjacent to precisely two vertices of degree $d-1$ in $K_d \backslash \{e\}$. In 1988, Rowlinson [19] determined that G_m attains the maximum spectral radius among all graphs of size m. This classic result indicates that G_m attains the maximum spectral radius among all 2-(edge)-connected graphs of size $m = \binom{d}{2} + t$ except $t = 1$. The next year, Rowlinson [20] proved that H_m was the unique 2-(edge)-connected graphs with $m = \binom{d}{2} + 1$ ($d \geqslant 5$) edges and the maximal spectral radius. However, we find that both G_m and H_m are not minimally 2-(edge)-connected graphs. Motivated this, our paper is to study the spectral extremal problem of minimally 2-(edge)-connected graphs under edge-condition restrictions.

Denote by $SK_{2,\frac{m-1}{2}}$ the graph obtained from the complete bipartite graph $K_{2,\frac{m-1}{2}}$ by subdividing an edge once. Let $K_{2,\frac{m-3}{2}} * K_3$ be the graph obtained by identifying a vertex of maximum degree of $K_{2,\frac{m-3}{2}}$ and a vertex of K_3 from the disjoint union of $K_{2,\frac{m-3}{2}}$ and K_3. *A friend graph*, denoted by F_t, is a graph obtained from t triangles by sharing a vertex. In this paper, we determine the maximum spectral radius for the minimally 2-(edge)-connected graphs of given size m, moreover, the corresponding extremal graphs are completely characterized as the following two theorems.

Theorem 1. *Let G be a minimally 2-connected graph of size m.*
(i) *If m is even, then $\rho(G) \leqslant \sqrt{m}$, the equality holds if and only if $G \cong K_{2,\frac{m}{2}}$.*
(ii) *If m is odd and $m \geqslant 9$, then $\rho(G) \leqslant \rho_1^*(m)$, where $\rho_1^*(m)$ is the largest root of $x^3 - x^2 - (m-2)x + m - 3 = 0$, the equality holds if and only if $G \cong SK_{2,\frac{m-1}{2}}$.*

Theorem 2. *Let G be a minimally 2-edge-connected graph of size m.*
(i) *If m is even, then $\rho(G) \leqslant \sqrt{m}$, the equality holds if and only if $G \cong K_{2,\frac{m}{2}}$.*

(ii) *If* $m \geqslant 11$ *is odd and* $m \neq 15$, *then* $\rho(G) \leqslant \rho_2^*(m)$, *where* $\rho_2^*(m)$ *is the largest root of* $x^4 - x^3 + (1-m)x^2 + (m-3)x + m - 3 = 0$, *the equality holds if and only if* $G \cong K_{2,\frac{m-3}{2}} * K_3$. *If* $m = 15$, *then* $\rho(G) \leqslant \frac{1+\sqrt{41}}{2}$, *the equality holds if and only if* $G \cong F_5$.

A famous sharp lower bound of spectral radius given by Collatz and Sinogowitz in [6] is $\rho(G) \geqslant \frac{2m}{n}$, equality holds if and only if G is a regular graph. Thus, the m-cycle attains the minimum spectral radius among all minimally 2-(edge)-connected graphs of size m. Notice that $\rho_2^*(m) \leqslant \sqrt{m}$ for $m \geqslant 5$. Therefore, by Theorems 1 and 2, we obtain that the spectral radius of a minimally 2-(edge)-connected graph lies in the interval $[2, \sqrt{m}]$ for $m \geqslant 5$. It means that a graph whose spectral radius lies out of $[2, \sqrt{m}]$ will not be minimally 2-(edge)-connected. Theorems 1 and 2 indeed indicate the relationship between spectral radius and connectivity.

The rest of the paper is organized as follows. In the next section, we will give some lemmas and some properties of a minimally 2-(edge)-connected graph. In Sections 3 and 4, we will give the proofs of Theorems 1 and 2, respectively.

2 Preliminaries

In this section, we firstly list some symbols and then write some properties of minimally 2-(edge)-connected graphs and some useful lemmas.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. For $v \in V(G)$, $A \subset V(G)$, denote by $N(v)$ and $d(v)$ the neighborhood and the degree of the vertex v in G, and denote $N_A(v) = N(v) \cap A$, $d_A(v) = |N_A(v)|$. The adjacent matrix of a graph G is defined as the $n \times n$ square matrix $A(G) = (a_{ij})$ whose entries are 1 if $v_i v_j \in E(G)$, otherwise 0. The spectral radius of G, denote by $\rho(G)$, is defined to be the largest eigenvalue of $A(G)$. A *chord* of a graph is an edge between two vertices of a cycle that is not an edge on the cycle. If a cycle has at least one chord, then it is called a *chorded cycle*.

A graph is 2-(edge)-connected graph if it contains a 2-vertex (edge) cut set. A graph is minimally 2-edge-connected, introduced in [1], if it is a 2-edge-connected but omitting any edge the resulting graph is no longer 2-edge-connected. By definition, a 2-connected graph is also 2-edge-connected, but the reverse is not true. However there exists a minimally 2-connected graph that is not minimally 2-edge-connected, for example the graph $H(2,2)$ shown as Fig.1. There exists a minimally 2-edge-connected graph that is not minimally 2-connected, for example the graph $C_n * C_m$. Clearly, C_n is both of minimally 2-connected and minimally 2-edge-connected. Furthermore, we will give some the properties of a minimally 2-connected graph.

Lemma 3 ([8]). *A minimally* 2*-connected graphs with more than three vertices contains no triangles.*

Lemma 4 ([1]). *Every cycle of a minimally* 2*-connected graph contains at least two vertices of degree two.*

Lemma 5 ([18]). *G is a minimally 2-connected graph if and only if no cycle of G has a chord.*

Lemma 6 ([3]). *If G is a minimally 2-(edge)-connected graph, then $\delta(G) = 2$.*

Lemma 7. *A 2-edge-connected subgraph of a minimally 2-edge-connected graph is also minimally 2-edge-connected.*

Proof. Let G be a minimally 2-edge-connected graph, and H be a 2-edge-connected subgraph of G. By contrary that H is not minimal, then there exists an edge $uv \in E(H)$ such that $H - uv$ is 2-edge-connected. Since G is a minimally 2-edge-connected graph, we get that $G - uv$ is 1-edge-connected. Thus $G - uv$ has a cut edge, say xy, which divides $V(G - uv)$ into two vertices sets U and V such that $x \in U$ and $y \in V$, and so $e_{G-uv}(U, V) = 1$. Also we have $u \in U$ and $v \in V$ since G is a 2-edge-connected graph. Noticed that $H - uv$ is a subgraph of $G - uv$ that is assumed to be 2-edge connected, we claim that $H - uv$ has a cycle C connecting u and v which must be contained in $G - uv$. It is a contradiction. Thus, H is a minimally 2-edge-connected graph. $\square$

Lemma 8. *If G is a minimally 2-edge-connected graph, then no cycle of G has a chord.*

Proof. Suppose by contrary that the cycle of the minimally 2-edge-connected graph G has a chord. Then G contains a chorded cycle, which is not minimally 2-edge-connected. This is a contradiction from Lemma 7. $\square$

Recall that $\kappa(G)$ ($\kappa'(G)$) denoted the vertex-connectivity (edge-connectivity) of G.

Lemma 9. *If G is a minimally 2-edge-connected graph with no cut vertex, then G is minimally 2-connected.*

Proof. Since G is a 2-edge-connected graph and G has no cut vertex, we have $2 = \kappa'(G) \geqslant \kappa(G) \geqslant 2$, and so $\kappa(G) = 2$. Note that G is minimally 2-edge connected. We have $\kappa(G - e) \leqslant \kappa'(G - e) \leqslant 1$ for any $e \in E(G)$. Thus G is minimally 2-connected. $\square$

Lemma 10 ([7]). *Let G be a graph with adjacency matrix $A(G)$, and let π be an equitable partition of G with quotient matrix B_π. Then $\det(xI - B_\pi) \mid \det(xI - A(G))$. Furthermore, the largest eigenvalue of B_π is just the spectral radius of G.*

By Lemma 10, we can give the bound of the spectral radius of $SK_{2,\frac{m-1}{2}}$.

Lemma 11. *For odd number $m > 5$, we have $\rho(SK_{2,\frac{m-1}{2}})$ is the largest root of $x^3 - x^2 - (m-2)x + m - 3 = 0$ and $\sqrt{m-2} < \rho(SK_{2,\frac{m-1}{2}})) < \sqrt{m-1}$.*

Proof. The vertices set of $SK_{2,\frac{m-1}{2}}$ has an equitable partition and the quotient matrix is

$$B_\pi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & \frac{m-3}{2} \\ 0 & 2 & 0 \end{pmatrix}.$$

We have $f(x) = \det(xI_3 - B_\pi) = x^3 - x^2 - (m-2)x + m - 3$. By Lemma 10, $\rho(SK_{2,\frac{m-1}{2}})$ is the largest root of $f(x) = 0$. Moreover, one can verify that $f(\sqrt{m-2}) < 0$, and so $\rho(SK_{2,\frac{m-1}{2}}) > \sqrt{m-2}$. Also, we have $f(\sqrt{m-1}) = \sqrt{m-1} - 2 > 0$ for $m \geqslant 6$ and $f'(x) = 3x^2 - 2x - (m-2) > 0$ for $x \geqslant \sqrt{m-1}$. Thus, $\rho(SK_{2,\frac{m-1}{2}}) < \sqrt{m-1}$. □

Notice that if each edge of a k-connected graph is incident with at least one vertex of degree k then the graph is minimally k-(edge)-connected. Clearly, $SK_{2,\frac{m-1}{2}}$ is the minimally 2-(edge)-connected. Moreover, we have the following lemma.

Lemma 12. *Let G^* attain the maximum spectral radius among all minimally 2-edge-connected graphs of size $m \geqslant 6$. Then $\rho(G^*) > \sqrt{m-2}$.*

Proof. If m is even, then $\rho(G^*) \geqslant \rho(K_{2,\frac{m}{2}}) = \sqrt{m} > \sqrt{m-2}$ since $K_{2,\frac{m}{2}}$ is minimally 2-edge-connected. If m is odd, then $\rho(G^*) \geqslant \rho(SK_{2,\frac{m-1}{2}}) > \sqrt{m-2}$ from Lemma 11. □

Lemma 13 ([10, 11])**.** *Let G and H be two graphs, and let $P(G, x)$ be the characteristic polynomial of G.*
(i) *If H is a proper subgraph of G, then $\rho(H) < \rho(G)$.*
(ii) *If $P(H, \lambda) > P(G, \lambda)$ for $\lambda \geqslant \rho(G)$, then $\rho(H) < \rho(G)$.*

Lemma 14 ([14, 15])**.** *Let G be a C_3-free graph of size m. Then $\rho(G) \leqslant \sqrt{m}$, the equality holds if and only if $G \cong K_{a,b}$, where $ab = m$.*

Lemma 15 ([26])**.** *Let u, v be two distinct vertices in a connected graph G, $\{v_i \mid i = 1, 2, \ldots, s\} \subseteq N_G(v) \setminus N_G(u)$. $X = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of G, where x_i is corresponding to v_i ($1 \leqslant i \leqslant n$). Let $G' = G - \{vv_i \mid 1 \leqslant i \leqslant s\} + \{uv_i \mid 1 \leqslant i \leqslant s\}$. If $x_u \geqslant x_v$, then $\rho(G) < \rho(G')$.*

Lemma 16 ([7])**.** *Let G be a connected graph with Perron vector $X = (x_1, x_2, \ldots, x_n)^T$. Let U, V and W be three disjoint subsets of $V(G)$ such that there are no edges between V and W. Let G' be the graph obtained from G by deleting the edges between V and U, and adding all the edges between V and W. If $\sum_{u \in U} x_u \leqslant \sum_{w \in W} x_w$, then $\rho(G) < \rho(G')$.*

Lemma 17 ([25])**.** *Let (H, v) and (K, w) be two connected rooted graphs. Then*

$$\rho((H, v) * (K, w)) \leqslant \sqrt{\rho^2(H) + \rho^2(K)},$$

the equality holds if and only if both H and K are stars, where $(H, v) * (K, w)$ is obtained by identifying v and w from disjoint union of H and K.

3 Proof of Theorem 1

In this section, we will give the proof of Theorem 1. Let G^* attain maximal spectral radius $\rho^* = \rho(G^*)$ among all minimally 2-connected graphs with size m. Lemma 3 and Lemma 6 indicate G^* has no triangles and $\delta(G^*) = 2$. If m is even, by Lemma 14 we have $G^* \cong K_{2,\frac{m}{2}}$. In what follows, we always assume that m is odd.

Next we will consider the structure of extremal graph G^* for odd size $m \geqslant 9$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G^* with coordinate $x_{u^*} = \max\{x_i \mid i \in V(G^*)\}$. Denote by $A = N(u^*)$ and $B = V(G^*) \setminus (A \cup u^*)$. Since G^* has no triangles, we have $e(A) = 0$, that is, $\sum_{i\in A} d_A(i)x_i = 0$, and thus

$$\begin{aligned}\rho^{*2}x_{u^*} &= \sum_{v\in A}\sum_{u\in N(v)} x_u = d(u^*)x_{u^*} + \sum_{i\in A} d_B(i)x_i + \sum_{i\in B} d_A(i)x_i = d(u^*)x_{u^*} + \sum_{i\in B} d_A(i)x_i \\ &\leqslant (d(u^*) + e(A,B))\, x_{u^*} = (m - e(B))\, x_{u^*}. \qquad (1)\end{aligned}$$

By Lemma 12, $\rho^{*2} > m-2$. Combining with (1), we get $e(B) < 2$. Thus, $e(B) = 0$ or 1.

In the following, we will give four claims to finish our proof.

Claim 18. $d(u^*) \geqslant 3$.

Proof. Otherwise, $d(u^*) \leqslant 2$. Since G^* is minimally 2-connected, we have $d(u^*) \geqslant \delta(G^*) = 2$ by Lemma 6. This induces $d(u^*) = 2$ and so $|A| = 2$. We may assume $A = \{u_1, u_2\}$. Note that $e(B) = 0$ or 1. If $e(B) = 0$ then each vertex in B is adjacent with both of u_1 and u_2 since $\delta(G^*) = 2$. This leads to the size of G^* is even, a contradiction. Thus $e(B) = 1$, and we may assume $B = \{w_1w_2\} \cup I$, where $I = \{v_1, v_2, \ldots, v_t\}$ is an isolated vertices set. Moreover, we see that each v_i is adjacent to u_j for $j = 1, 2$ due to $\delta(G^*) = 2$, and $N_A(w_1) \cap N_A(w_2) = \emptyset$, $d_A(w_1) = d_A(w_2) = 1$ since G has no triangles. Without loss of generality, let $N_A(w_1) = u_1$ and $N_A(w_2) = u_2$. Now G^* is determined and $m = 2t + 5$ in this situation, where $t \geqslant 2$. By the symmetry, we have $x_{u_1} = x_{u_2}$, $x_{w_1} = x_{w_2}$, and $x_{u^*} = x_{v_i}$ for $i = 1, 2, \ldots, t$. Thus from $A(G^*)X = \rho^* X$, we have

$$\begin{cases} \rho^* x_{u^*} &= 2x_{u_1}, \\ \rho^* x_{u_1} &= (t+1)x_{u^*} + x_{w_1}, \\ \rho^* x_{w_1} &= x_{w_2} + x_{u_1} = x_{w_1} + x_{u_1}. \end{cases}$$

Furthermore, we get

$$(\rho^{*2} - 2t - 2)x_{u^*} = \frac{2}{\rho^* - 1} x_{u_1}. \qquad (2)$$

Let $g(\rho^*) = (\rho^{*2} - 2t - 2)(\rho^* - 1) - 2 = \rho^{*3} - \rho^{*2} - (2t+2)\rho^* + 2t$. Then $g(\rho^*) = \rho^{*3} - \rho^{*2} - (m-3)\rho^* + m - 5$. Since $\rho^* > \sqrt{m-2}$ by Lemma 12 and

$$g'(\rho^*) = 3\rho^{*2} - 2\rho^* - (m-3) > g'(\sqrt{m-2}) = 2m - 2\sqrt{m-2} - 3 > 0,$$

we have $g(\rho^*) > g(\sqrt{m-2}) = \sqrt{m-2} - 2 > 0$. Thus $\rho^{*2} - 2t - 2 > \frac{2}{\rho^* - 1}$. Therefore, from (2) we get $x_{u^*} < x_{u_1}$, which contradicts the maximality of x_{u^*}. $\square$

Claim 19. $e(B) = 1$.

Proof. Suppose to the contrary that $e(B) = 0$, that is, B induces an independent set. Recall that A is independent, for $u \in B$, it lies in a 4-cycle $C = uv_1u^*v_2u$ where $v_1, v_2 \in A$. There is at least one of v_1 and v_2 having degree at least three since otherwise u^* will be

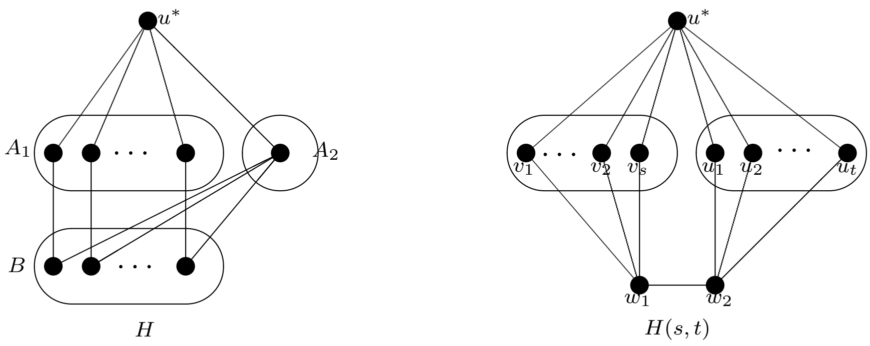

Figure 1: The graphs H and $H(s,t)$, where $s, t \geqslant 1$.

a cut vertex. Also there is at least one of v_1 and v_2 having degree two since otherwise C contains at most one vertex of degree two, which contradicts Lemma 4. It implies that $d(u) = 2$ due to $d(u^*) \geqslant 3$, and then we may assume that $d(v_1) = 2$ and $d(v_2) \geqslant 3$. In addition, the vertex as v_2 in A is unique, because two vertices (in A) of degree greater than 2 must lie in a 4-cycle along with u^* and in this cycle there have been three vertices of degree greater than 2. Therefore, G^* is isomorphic to H showed in Fig.1. By the quotient matrix of H, we have $\rho^* = 1 + \sqrt{\frac{m-1}{3}} < \sqrt{m-2}$ for $m \geqslant 9$, which contradicts Lemma 12. Thus $e(B) = 1$. □

Claim 20. $G^*[B] = K_2$.

Proof. Otherwise, by Claim 19, we have $G^*[B] = \{w_1w_2\} \cup I$, where I is a nonempty isolated vertex set. Choosing a vertex $v \in I$. Clearly, $d(v) \geqslant 2$. If $N(v) \subset A_1 = \{v \in A \mid d(v) = 2\}$, then u^* is a cut vertex, a contradiction. If $N(v) \subset A_2 = A \backslash A_1$, then v is included in a 4-cycle which has at most one vertex with degree two, a contradiction. So, $|N_{A_1}(v)| = 1$ and $|N_{A_2}(v)| = 1$. Let $v_1 \in N_{A_1}(v)$, $v_2 \in N_{A_2}(v)$. Then $u^*v_1vv_2u^*$ forms a 4-cycle. Notice that $\min\{|N_A(w_1)|, |N_A(w_2)|\} \geqslant 1$. Choosing two vertices $u_1 \in N(w_1)$ and $u_2 \in N(w_2)$, respectively, then $u^*u_1w_1w_2u_2u^*$ forms a 5-cycle. Moreover, the 4-cycle and 5-cycle either have a common vertex u^* or have a common edge u^*v_2 (u_2 coincides v_2). If the former occurs then u^* is a cut vertex, otherwise G^* contains a chorded cycle, a contradiction. □

By Claims 19 and 20, $G^*[B] = \{w_1w_2\}$. Since G^* has no triangles, $N_A(w_1) \cap N_A(w_2) = \emptyset$. Furthermore, $N_A(w_1) \cup N_A(w_2) = A$. Assume $|N_A(w_1)| = s$, $|N_A(w_2)| = t$. Clearly, $s, t \geqslant 1$ since $\delta(G^*) = 2$. Let $H(s,t)$ be a graph obtained from a double star graph $D_{s,t}$ by joining an isolated vertex u^* to all its leaves vertices (see Fig.1). Now we get that $G^* \cong H(s,t)$ for some s and t satisfying $2s + 2t + 1 = m$.

Claim 21. $G^* \cong H(1, \frac{m-3}{2})$.

Proof. Without loss of generality, we assume $s \leqslant t$. If $s = 1$, then $t = \frac{m-3}{2}$, and so $H(s,t) \cong H(1, \frac{m-3}{2})$. The result holds. Suppose $G^* \cong H(s,t)$ for $s \geqslant 2$. Let Y be the Perron eigenvector of $H(s,t)$ with spectral radius $\rho = \rho(H(s,t))$, and let $N_A(w_1) = \{v_1, \dots, v_s\}$ and $N_A(w_2) = \{u_1, \dots, u_t\}$. By the symmetry, $y_{v_1} = \cdots = y_{v_s}$ and $y_{u_1} = \cdots = y_{u_t}$. We have

$$\begin{cases} \rho y_{v_1} &= y_{u^*} + y_{w_1}, \\ \rho y_{u_1} &= y_{u^*} + y_{w_2}, \end{cases} \quad \begin{cases} \rho y_{w_1} &= s y_{v_1} + y_{w_2}, \\ \rho y_{w_2} &= t y_{u_1} + y_{w_1}. \end{cases}$$

Then $y_{v_1} - y_{u_1} = \frac{1}{\rho}(y_{w_1} - y_{w_2})$ and

$$(\rho+1)(y_{w_1} - y_{w_2}) = s y_{v_1} - t y_{u_1} = \frac{s}{\rho}(y_{w_1} - y_{w_2}) + (s-t) y_{u_1},$$

which indicates that

$$(\rho + 1 - \frac{s}{\rho})(y_{w_1} - y_{w_2}) = (s-t) y_{u_1} \leqslant 0.$$

Clearly, $\rho > \frac{s}{\rho}$ since $H(s,t)$ has $K_{1,s}$ as a subgraph. It follows that $y_{w_1} \leqslant y_{w_2}$. Note that $s \geqslant 2$, obviously, $G' = H(s,t) - v_s w_1 + v_s w_2$ is also a minimally 2-connected graph. By Lemma 15, we have $\rho(G') > \rho(G^*)$, which is a contradiction. $\square$

By the definition of $SK_{2,\frac{m-1}{2}}$ and $H(1, \frac{m-3}{2})$, it is clear that $SK_{2,\frac{m-1}{2}} \cong H(1, \frac{m-3}{2})$. Thus Claim 21 and Lemma 11 imply the Theorem 1.

4 Proof of Theorem 2

In this section, we will give the proof of Theorem 2. As we know, a *block* of a graph is a maximal 2-connected subgraph with respect to vertices. A block of a graph is called *leaf block* if it contains exactly one cut vertex. By Lemma 9, a minimally 2-edge-connected graph G without cut vertex is minimally 2-connected. Otherwise, G is made of some blocks including at least two leaf blocks, in which each block is minimally 2-connected by Lemma 9 and they intersect at cut vertices. In general, we write $G = B(t,k)$ to denote a minimally 2-edge-connected graph with t cut vertices and k blocks. If $t = 0$ then $k = 1$ and $G = B(0,1)$ is a type of minimally 2-connected graph that is considered in Theorem 1. If $t \geqslant 1$ then $k \geqslant 2$ and each block of $G = B(t,k)$ has some cut vertices, in this case $G = B(t,k)$ can be viewed as a tree if each block is regarded as an edge.

Proof of Theorem 2 (i). We may assume that $m \geqslant 4$ and $X = (x_1, \dots, x_n)^T$ is the Perron eigenvector of G.

Case 1. G has no cut vertex.

In this case, G is minimally 2-connected. By Theorem 1, $\rho(G) \leqslant \sqrt{m}$ and the equality holds if and only if $G \cong K_{2,\frac{m}{2}}$, which are just required.

Case 2. G has some cut vertices.

By definition, $G = B(t,k)$ for some $t \geqslant 1$ and $k \geqslant 2$. Let $B_1, \dots, B_k$ be its k blocks and $m(B_i) = m_i$ for $i = 1, 2, \dots, k$. We know that each B_i is minimally 2-connected and $m = m(B(t,k)) = \sum_{i=1}^{k} m_i$. Let $\rho_1^*(m)$ be the largest root of $x^3 - x^2 - (m-2)x + m - 3 = 0$.

By Lemma 11 and simple computation, we have $\rho_1^*(m) = \rho(SK_{2,\frac{m-1}{2}})) < \sqrt{m}$ for any odd m. Thus from Theorem 1, we have $\rho(B_i) \leqslant \sqrt{m_i}$, and the equality holds if and only if $B_i \cong K_{2,\frac{m_i}{2}}$ for $i = 1, 2, \ldots, k$. Notice that each B_i is not a star since B_i is minimally 2-connected. By Lemma 17, we obtain

$$\rho(B(t,k)) < \sqrt{\rho^2(B_1) + \rho^2(B_2) + \cdots + \rho^2(B_k)} \leqslant \sqrt{m_1 + m_2 + \cdots + m_k} = \sqrt{m},$$

as desired. □

In what follows we will show (ii) of Theorem 2, and first we give some lemmas and propositions for the preparations. Clearly, if we transfer a leaf block of $B(t,k)$ to another leaf block, one can simply verify the following result.

Lemma 22. *For a minimally 2-edge-connected graph $G = B(t,k)$, let B_i be a leaf block of G and $u \in B_i$ be a cut vertex. For any $v \in V(G)\backslash\{u\}$, we have $G' = G - \sum_{w\in N(u)\cap B_i} wu + \sum_{w\in N(u)\cap B_i} wv$ is also minimally 2-edge-connected.*

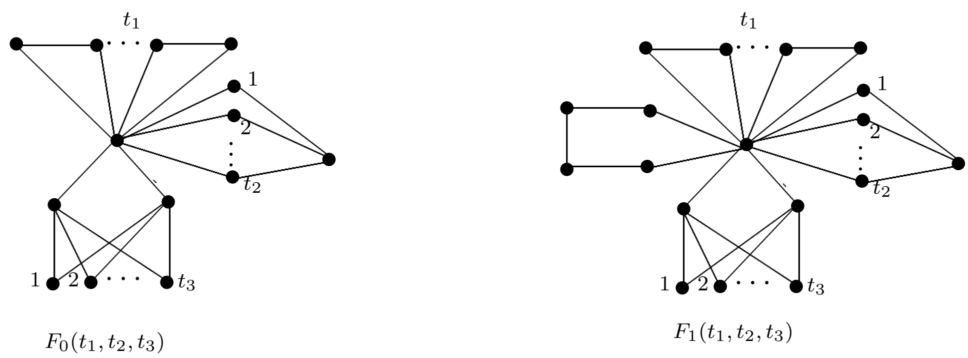

Figure 2: The graphs $F_0(t_1, t_2, t_3)$ and $F_1(t_1, t_2, t_3)$.

Denote by u_1 the maximum degree vertex of the friend graph with t_1 triangles, u_2 a maximum degree vertex of K_{2,t_2}, and u_3 a vertex of K_{2,t_3+1} with degree two. Let $F_0(t_1, t_2, t_3)$ be the graph obtained from the above three graphs by identifying u_1, u_2 and u_3. Denote $F_1(t_1, t_2, t_3)$ the graph by identifying a vertex of C_5 and the maximum degree vertex of $F_0(t_1, t_2, t_3)$ (see Fig.2), where $t_i \geqslant 0$.

In order to give the proof of Theorem 2 (ii), we begin by proving the following two useful propositions.

Proposition 23. *$\rho(F_1(t_1, t_2, t_3)) < \rho(F_0(t_1+1, t_2+1, t_3))$, where $F_1(t_1, t_2, t_3)$ and $F_0(t_1 + 1, t_2 + 1, t_3)$ have the same number of edges*

$$m = \begin{cases} 3t_1 + 2t_2 + 2t_3 + 7 & \text{for any } t_1 \geqslant 0,\ t_2 = 0 \text{ or} \geqslant 2 \text{ and } t_3 \geqslant 1, \\ 3t_1 + 2t_2 + 5 & \text{for any } t_1 \geqslant 0,\ t_2 = 0 \text{ or} \geqslant 2 \text{ and } t_3 = 0. \end{cases} \quad (3)$$

Proof. Let $G_1 = F_1(t_1, t_2, t_3) \cup K_1$ and let

$$G_1' = G_1 - (w_1w_2 + w_1w_3 + w_2w_4) + (w_3w_4 + u^*u_1' + u_1'v) \cong F_1(t_1 + 1, t_2 + 1, t_3) \cup 2K_1,$$

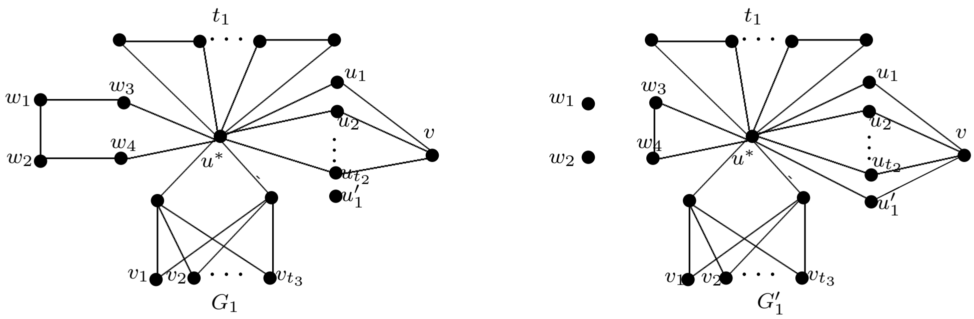

Figure 3: The vertex labels of $G_1 \cong F_1(t_1, t_2, t_3) \cup K_1$ and $G_1' \cong F_0(t_1+1, t_2+1, t_3) \cup 2K_1$.

where the labels of $V(G_1)$ and $V(G_1')$ are shown in Fig.3. Clearly, $\rho(G_1) = \rho(F_0(t_1, t_2, t_3))$, $\rho(G_1') = \rho(F_0(t_1+1, t_2+1, t_3))$ and

$$m(G_1) = m(G_1') = m = \begin{cases} 3t_1 + 2t_2 + 2t_3 + 7 & \text{for any } t_1 \geqslant 0,\ t_2 = 0 \text{ or } \geqslant 2 \text{ and } t_3 \geqslant 1, \\ 3t_1 + 2t_2 + 5 & \text{for any } t_1 \geqslant 0,\ t_2 = 0 \text{ or } \geqslant 2 \text{ and } t_3 = 0. \end{cases}$$

It suffices to show $\rho = \rho(G_1) < \rho' = \rho(G_1')$.

Let $Y = (y_1, \ldots, y_n)$ and $Z = (z_1, \ldots, z_n)$ be the Perron eigenvector of G_1 and G_1', respectively. Then we obtain

$$\begin{aligned} (\rho' - \rho)Y^T Z &= Y^T A(G_1')Z - Y^T A(G_1)Z \\ &= \sum_{ij \in E(G_1')} (y_i z_j + z_i y_j) - \sum_{ij \in E(G_1)} (y_i z_j + z_i y_j) \\ &= (y_{w_3} z_{w_4} + z_{w_3} y_{w_4}) + (y_{u^*} z_{u_1'} + z_{u^*} y_{u_1'}) + (y_{u_1'} z_v + z_{u_1'} y_v) \\ &\quad -[(y_{w_1} z_{w_3} + z_{w_1} y_{w_3}) + (y_{w_2} z_{w_1} + z_{w_2} y_{w_1}) + (y_{w_2} z_{w_4} + z_{w_2} y_{w_4})]. \end{aligned}$$

Notice that $y_{u_1'} = 0$, $y_{w_1} = y_{w_2}$, $y_{w_3} = y_{w_4}$, $z_{w_1} = z_{w_2} = 0$ and $z_{w_3} = z_{w_4}$, we have

$$(\rho' - \rho)Y^T Z = 2(y_{w_3} - y_{w_1})z_{w_3} + y_{u^*} z_{u_1'} + y_v z_{u_1'}. \tag{4}$$

Since C_5 is a proper subgraph of G_1, by Lemma 13, we have $\rho > \rho(C_5) = 2$. By the eigen-equations $\rho y_{w_1} = y_{w_2} + y_{w_3} = y_{w_1} + y_{w_3}$, we have $y_{w_3} = (\rho - 1)y_{w_1} > y_{w_1}$, and so the right of (4) is more than 0. Note that $Y^T Z \geqslant 0$. It follows that $\rho' > \rho$. □

Proposition 24. *$\rho(F_0(t_1, t_2, t_3)) < \rho(F_0(t_1, t_2 + t_3 + 1, 0))$, where $F_0(t_1, t_2, t_3)$ and $F_0(t_1, t_2 + t_3 + 1, 0)$ have the same number of edges $m = 3t_1 + 2t_2 + 2t_3 + 2$, where $t_1 \geqslant 0$, $t_2 = 0$ or $\geqslant 2$, and $t_3 \geqslant 1$.*

Proof. Let $G_2 = F_0(t_1, t_2, t_3) \cup (t_3 + 1)K_1$, its vertices be labelled as in Fig.3 and the isolated set $I_2 = \{u_1', u_2', \ldots, u_{t_3+1}'\}$. Let

$$G_2' = G_2 - (\sum_{i=1}^{2} u^* v_i' + \sum_{i=1}^{2} \sum_{j=1}^{t_3} v_i' v_j) + (\sum_{i=1}^{t_3+1} (u^* u_i' + v u_i')) \cong F_0(t_1, t_2 + t_3 + 1, 0) \cup (t_3 + 2)K_1.$$

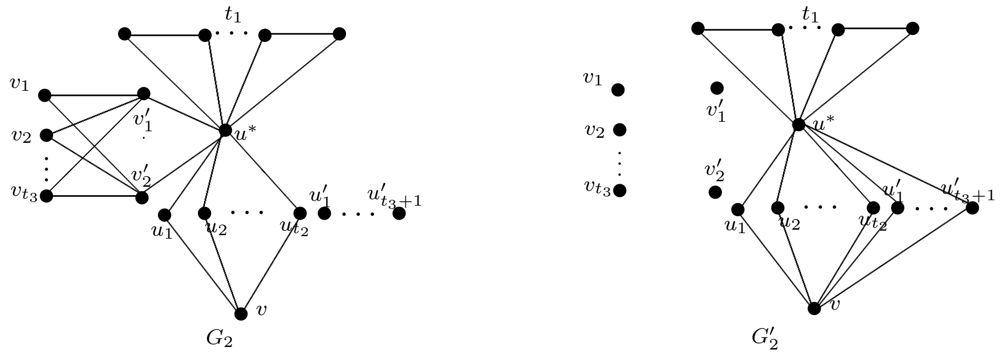

Figure 4: The vertex labels of $G_2 \cong F_0(t_1,t_2,t_3) \cup (t_3+1)K_1$ and $G'_2 \cong F_0(t_1,t_2+t_3+1,0) \cup (t_3+2)K_1$.

Clearly, $\rho(F_0(t_1,t_2,t_3)) = \rho(G_2)$ and $\rho(F_0(t_1,t_2+t_3+1,0)) = \rho(G'_2)$. Also, $F_0(t_1,t_2,t_3)$ and $F_0(t_1,t_2+t_3+1,0)$ have the same number of edges $3t_1+2t_2+2t_3+2=m$. It suffices to show $\rho = \rho(G_2) < \rho' = \rho(G'_2)$.

Suppose to the contrary that $\rho \geqslant \rho'$. Let $Y=(y_1,\ldots,y_n)$ and $Z=(z_1,\ldots,z_n)$ be the Perron eigenvector of G_2 and G'_2, respectively. By symmetry, we have $y_{u'_1}=0$, and $z_{v_1}=z_{v'_1}=0$. Thus we obtain

$$(\rho'-\rho)Y^TZ = Y^TA(G'_2)Z - Y^TA(G_2)Z = \sum_{ij\in E(G'_2)}(y_iz_j+z_iy_j) - \sum_{ij\in E(G_2)}(y_iz_j+z_iy_j)$$

$$\begin{aligned} &= (t_3+1)(y_{u^*}z_{u'_1}+z_{u^*}y_{u'_1}+y_vz_{u'_1}+z_vy_{u'_1}) - 2(y_{u^*}z_{v'_1}+z_{u^*}y_{v'_1}) - 2t_3(y_{v_1}z_{v'_1}+z_{v_1}y_{v'_1}) \\ &= (t_3+1)(y_{u^*}z_{u'_1}+y_vz_{u'_1}) - 2y_{v'_1}z_{u^*}. \end{aligned} \tag{5}$$

By eign-equation of $A(G_2)$ and $A(G'_2)$, we have

$$\begin{cases} \rho y_v = t_2 y_{u_1}, \\ \rho y_{u_1} = y_{u^*}+y_v. \end{cases} \quad \begin{cases} \rho y_{v'_1} = y_{u^*}+t_3y_{v_1}, \\ \rho y_{v_1} = 2y_{v'_1}. \end{cases} \quad \begin{cases} \rho' z_{u'_1} = z_{u^*}+z_v, \\ \rho' z_v = (t_3+t_2+1)z_{u'_1}. \end{cases}$$

Then we obtain

$$y_v = \frac{t_2}{\rho^2-t_2}y_{u^*}, \quad y_{v'_1} = \frac{\rho}{\rho^2-2t_3}y_{u^*} \quad \text{and} \quad z_{u'_1} = \frac{\rho'}{\rho'^2-(t_3+t_2+1)}z_{u^*}.$$

From (5), we get

$$(\rho'-\rho)Y^TZ = \left(\frac{(t_3+1)\rho'}{\rho'^2-(t_3+t_2+1)}\frac{\rho^2}{\rho^2-t_2} - \frac{2\rho}{\rho^2-2t_3}\right)y_{u^*}z_{u^*}. \tag{6}$$

Since $\frac{(t_3+1)\rho'}{\rho'^2-(t_3+t_2+1)}$ monotonically decreases with respect to ρ' and $\rho \geqslant \rho'$, from (6) we get

$$\begin{aligned} (\rho'-\rho)Y^TZ &\geqslant \left(\frac{(t_3+1)\rho}{\rho^2-(t_3+t_2+1)}\frac{\rho^2}{\rho^2-t_2} - \frac{2\rho}{\rho^2-2t_3}\right)y_{u^*}z_{u^*} \\ &= \left(\frac{t_3+1}{\rho^2-(t_3+t_2+1)}\frac{\rho^2}{\rho^2-t_2} - \frac{2}{\rho^2-2t_3}\right)\rho y_{u^*}z_{u^*} \\ &\geqslant \left(\frac{t_3+1}{\rho^2-(t_3+t_2+1)} - \frac{2}{\rho^2-2t_3}\right)\rho y_{u^*}z_{u^*}. \end{aligned} \tag{7}$$

Let $g(\rho) = (t_3+1)(\rho^2 - 2t_3) - 2(\rho^2 - (t_3+t_2+1)) = (t_3-1)\rho^2 - 2t_3^2 + 2t_2 + 2$. It is clear that $g(\rho) > 0$ for $\rho > \rho(K_{2,t_3+1}) = \sqrt{2t_3+2}$. One can also verify that $\rho^2 > t_2 + t_3 + 1$. Hence,

$$\frac{(t_3+1)}{\rho^2 - (t_3+t_2+1)} - \frac{2}{\rho^2 - 2t_3} > 0.$$

From (7) we have $\rho' > \rho$, a contradiction. Therefore, $\rho < \rho'$. □

Now is the time to prove (ii) of Theorem 2.

Proof of Theorem 2 (ii). Let G^* be the graph with the maximum spectral radius over all minimally 2-edge-connected graphs of odd size $m \geqslant 11$, and let $X = (x_1, \ldots, x_n)^T$ be the Perron eigenvector of G^* with coordinate $x_{u^*} = \max\{x_i \mid i \in V(G^*)\}$. Denote by $\rho^* = \rho(G^*)$, $A = N(u^*)$ and $B = V(G^*) \setminus (A \cup u^*)$. Notice that $\delta(G^*) = 2$. Now we give Claims 25-32 to finish the proof of Theorem 2 (ii).

Claim 25. $G^*[A]$ *is isomorphic to the union of some independent edges and isolated vertices.*

Proof. On the one hand, $G^*[A]$ contains no cycle. Otherwise, we assume that a cycle $C_l \subset G^*[A]$ $(l \geqslant 3)$. Then there exists a wheel W_{l+1} in G^*, it forms a chorded cycle in G^*, which contradicts Lemma 8. On the other hand, $G^*[A]$ contains no P_3. Otherwise, G^* contains a chorded cycle with order 4, a contradiction. □

Let A_1 be the isolated vertex set of $G^*[A]$. Then $A_2 = A \setminus A_1$ consists of some independent edges if $A_2 \neq \emptyset$.

Claim 26. $N_B(u) = \emptyset$ *for any* $u \in A_2$.

Proof. Otherwise, there exists a vertex $v \in B$ that is adjacent to a vertex $u_2 \in A_2$. We may further assume that $u_2 \sim u_2' \in A_2$. If v has no neighbor in B, then there exists a vertex $u \in A$ adjacent to v due to $\delta(G^*) \geqslant 2$. It follows that

$$\begin{cases} C = u^*u_2'u_2vu_1u^* \text{ is a cycle with the chord } u^*u_2 & \text{if } u = u_1 \in A_1 \\ C = u^*u_2'vu_2u^* \text{ is a cycle with the chord } u_2'u_2 & \text{if } u = u_2' \in A_2 \\ C = u^*u_2'u_2vu_3u^* \text{ is a cycle with the chord } u_2^*u_2 & \text{if } u = u_3 \in A_2 \end{cases}$$

It is impossible since any cycle of G^* has no chord. So, $d_B(v) \geqslant 1$. However, in this situation, there exists a path $P := vv_1 \cdots v_t$ in $G^*[B]$ such that v_t is adjacent to some $u' \in A$ since otherwise u_2v will be a cut edge. By regarding u' as the above u, as similar above we can find a chorded cycle in G^*, a contradiction. □

If $A_2 \neq \emptyset$, then, from Claim 25, $\{u^*\} \cup A_2$ induces $t_1 = \frac{|A_2|}{2}$ triangles with a common vertex u^*. Moreover, we see from Claim 26 that each of these triangles must be a leaf block of G^*.

Claim 27. $e(B) = 0$ *or* 1.

Proof. By Claims 25 and 26, we know that A_2 induces some independent edges and $N_B(A_2)=\emptyset$. Recall that A_1 induces some isolated vertices. Clearly, $\sum_{i\in A_1} d_{A_1}(i)x_i=0$. From Lemma 12, $\rho^*>\sqrt{m-2}\geqslant 3$ for $m\geqslant 11$. By symmetry, for any $i,j\in A_2$, we have $x_i=x_j$ and so $\rho^* x_i=x_{u^*}+x_j=x_{u^*}+x_i$, which induces $x_i=\frac{x_{u^*}}{\rho^*-1}<\frac{x_{u^*}}{2}$. Then we have

$$\begin{aligned}
\rho^{*2}x_{u^*} &= d(u^*)x_{u^*}+\sum_{i\in A_1} d_{A_1}(i)x_i+\sum_{i\in A_2} d_{A_2}(i)x_i+\sum_{i\in B} d_A(i)x_i\\
&< d(u^*)x_{u^*}+\frac{x_{u^*}}{2}\sum_{i\in A_2} d_{A_2}(i)+e(A,B)x_{u^*}\\
&= d(u^*)x_{u^*}+\frac{x_{u^*}}{2}\cdot 2e(A_2)+e(A,B)x_{u^*}\\
&= (d(u^*)+e(A)+e(A,B))x_{u^*}\\
&= (m-e(B))x_{u^*}.
\end{aligned}$$

Combining it with $(m-2)x_{u^*}<\rho^{*2}x_{u^*}$, we have $e(B)<2$. It follows the result. ☐

If $e(B)=1$, we may denote $e=w_1^*w_2^*$ the unique edge in $G^*[B]$ in what follows. Without loss of generality, we may assume $d_A(w_1^*)\leqslant d_A(w_2^*)$.

Claim 28. $G^*[\{u^*,w_1^*,w_2^*\}\cup N_{A_1}(\{w_1^*,w_2^*\})]\cong C_5$.

Proof. Firstly, we will show $N_{A_1}(w_1^*)\cap N_{A_1}(w_2^*)=\emptyset$. Otherwise, let $u_0\in A_1$ be the common vertex. Now, $\{u_0,w_1^*,w_2^*\}$ induces a 3-cycle. If $u_0w_1^*w_2^*$ is a leaf block of G^*, then $G'=G^*-u_0w_1^*-u_0w_2^*+u^*w_1^*+u^*w_2^*$ is minimally 2-edge-connected and $\rho(G')>\rho(G^*)$ by Lemma 15, a contradiction. Therefore, there exists another vertex $u_2\in A_1$ that is adjacent to at least one of $\{w_1^*,w_2^*\}$. It follows that

$$\begin{cases} C=w_2^*u_0u^*u_2w_1^*w_2^* \text{ is a cycle with the chord } u_0w_1^* & \text{if } u_2\sim w_1^*,\\ C=u^*u_0w_1^*w_2^*u_2u^* \text{ is a cycle with the chord } u_0w_2^* & \text{if } u_2\sim w_2^*,\end{cases}$$

which always leads a contradiction.

Secondly, we will show $d_{A_1}(w_1^*)=1$ and $d_{A_1}(w_2^*)\geqslant 1$. In fact, since $\delta(G^*)=2$ and $N_{A_2}(w_1^*)=N_{A_2}(w_2^*)=\emptyset$ by Claim 26, we have $d_{A_1}(w_1^*),d_{A_1}(w_2^*)\geqslant 1$. If $d_{A_1}(w_1^*),d_{A_1}(w_2^*)\geqslant 2$, then G^* contains $H(2,2)$ (see Fig.1) as a subgraph. We see that $H(2,2)$ is not minimally 2-edge-connected, which contradicts Lemma 7.

Combining the above two facts, we have $G^*[\{u^*,w_1^*,w_2^*\}\cup N_{A_1}(\{w_1^*,w_2^*\})]\cong H(1,t_2)$ for some positive $t_2\geqslant 1$, and clearly $H(1,t_2)$ is a leaf block of G^*. Finally, it suffices to show $t_2=1$. Suppose to the contrary that $t_2\geqslant 2$. Now let $N_{A_1}(w_1^*)=\{u_0\}$ and $N_{A_1}(w_2^*)=\{u_1,u_2,\ldots,u_{t_2}\}$. Then $G'=G^*-w_1^*w_2^*+w_1^*u^*$ is a graph obtained from G by replacing the block $H(1,t_2)$ to $K_{2,t_2}*K_3$. Thus, G' is also minimally 2-edge-connected. By Lemma 15, we have $\rho(G')>\rho(G^*)$, a contradiction. ☐

Denote by B_1 the set of all isolated vertices in B. By Claim 27, $B=B_1\cup\{w_1^*w_2^*\}$. By Claim 28, we may assume that $N_{A_1}(w_1^*)=\{v_1^*\}$ and $N_{A_1}(w_2^*)=\{v_2^*\}$ in what follows.

Claim 29. $N_{B_1}(\{v_1^*,v_2^*\})=\emptyset$.

Proof. Suppose to the contrary that there exists a vertex w in B_1 with neighbor v_1^* or v_2^*. Without loos of generality, we assume $N_{A_1}(w) = v_1^*$. Notice that $d(w) \geqslant 2$. We claim that $N_{A_1}(w) = \{v_1^*, v_2^*\}$. Since otherwise, $w \sim v \in A_1 \backslash \{v_1^*, v_2^*\}$, then $u^* v w v_1^* w_1^* w_2^* v_2^* u^*$ is a cycle with the chord $v_1^* u^*$, a contradiction. Thus, $W = \{u^*, w, w_1^*, w_2^*, v_1^*, v_2^*\}$ induces a minimally 2-edge-connected leaf block of G^*. By symmetry, we also have $x_{v_1^*} = x_{v_2^*}$. Let $G' = G^* - w_2^* v_2^* + w_2^* v_1^*$. We see that W also induces a leaf block of G' and so G' is also a minimally 2-edge connected graph. By Lemma 15, we have $\rho(G') > \rho(G^*)$, a contradiction. ☐

Let $A_1' = A_1 \backslash \{v_1^*, v_2^*\}$. By Claim 29, each vertex of B_1 only joins some vertices in A_1'. i.e. $N_{A_1}(w_i) = N_{A_1'}(w_i)$ for any $w_i \in B_1$.

Claim 30. *$|B_1| \geqslant 2$ and $|N_{A_1}(w_1) \cap N_{A_1}(w_2)| = 0$ or 2 for any $w_1 \neq w_2 \in B_1$.*

Proof. Firstly, we show that $|B_1| \geqslant 2$. Otherwise, we may assume $B_1 = \{w_1\}$, then $G^*[N_{A_1}(w_1) \cup \{w_1, u^*\}] \cong K_{2,a_1}$, where $a_1 = |N_{A_1}(w_1)|$. By Claims 26, 28 and 29, we have $G^* \cong F_1(t_1, a_2, 0)$ for some positive $t_1 \geqslant 0$, $a_2 \geqslant 2$ and $3t_1 + 2a_2 + 5 = m(G^*) = m$. By Proposition 23, we know that $\rho(G^*) = \rho(F_1(t_1, a_2, 0)) < \rho(F_0(t_1 + 1, a_2 + 1, 0)$, a contradiction.

Suppose that $|N_{A_1}(w_1) \cap N_{A_1}(w_2)| \geqslant 3$, let $\{v_1, v_2, v_3\} \subseteq N_{A_1}(w_1) \cap N_{A_1}(w_2)$, then G^* contains a 5-cycle $C_2 = v_1 w_1 v_2 w_2 v_3 v_1$ with chord $v_1 w_2$, a contradiction. Next we show that $|N_{A_1}(w_1) \cap N_{A_1}(w_2)| \neq 1$. Otherwise, let $N_{A_1}(w_1) \cap N_{A_1}(w_2) = \{v\}$, then G^* contains a 6-cycle $C_1 = u^* w_1' w_1 v w_2 w_2' u^*$ with chord $u^* v$, where $w_1' \in N_{A_1}(w_1) \backslash \{v\}$ and $w_2' \in N_{A_1}(w_2) \backslash \{v\}$, it is a contradiction. Thus, we have $|N_{A_1}(w_1) \cap N_{A_1}(w_2)| = 0$ or 2. ☐

Notice that $d_{A_1}(w_i) \geqslant 2$ for any $w_i \in B_1$. By Claims 25-30 we can get the structure of G^* shown as Fig.5. In particular, if $e(B) = \emptyset$, then G^* contains no 5-cycle $C = u^* v_1^* w_1^* w_2^* v_2^* u^*$. For two nonnegative integers p, q, $G^*[N_{A_1}(B_1) \cup B_1] \cong \bigcup_{i=1}^{p} K_{1,r_i} \bigcup_{j=1}^{q} K_{2,s_j}$,

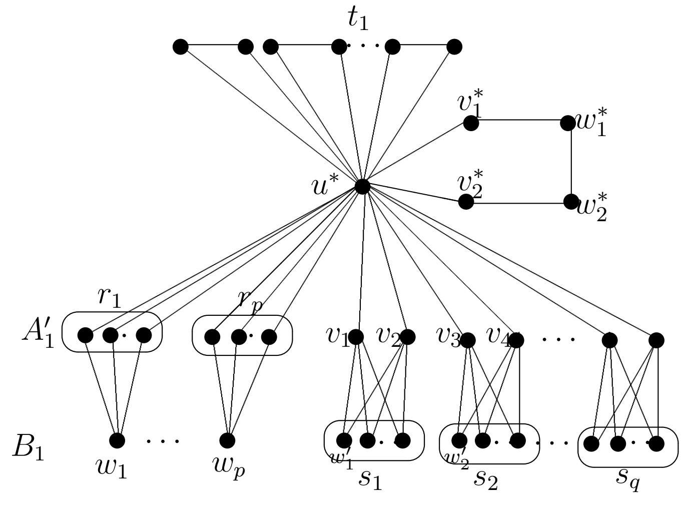

Figure 5: The structure of G^*, where $t_1 \geqslant 0$, $p + \sum_{i=1}^{q} s_i = |B_1|$ and $\sum_{i=1}^{p} r_i + 2q = |A_1'|$.

where $3t_1 + 2\sum_{i=1}^{p} r_i + 2\sum_{i=1}^{q} s_i + 2q + 5 = m$ (r_i, $s_i \geqslant 2$). Furthermore, we will determine the values of p, q.

Claim 31. *$p \leqslant 1$ and $q \leqslant 1$.*

Proof. We firstly show $p \leqslant 1$. Suppose $p \geqslant 2$, then there exists two vertices, say w_1, w_2 in B_1 with $G^*[N_{A_1'}(w_i) \cup \{w_i\}] = K_{1,r_i}$ for $i = 1, 2$. Without loss of generality, we may assume that $x_{w_1} \geqslant x_{w_2}$. Denote by

$$G' = G^* - \sum_{v \in N_{A_1}(w_2)} vw_2 + \sum_{v \in N_{A_1}(w_2)} vw_1.$$

By Lemma 15, we have $\rho(G') > \rho^*$. Clearly w_2 is an isolated vertex of G'. Set $G'' = G' - \{w_2\}$. Then G'' is also a minimally 2-edge-connected graph since $N_{A_1}(w_1) \cup N_{A_1}(w_2) \cup \{w_1, u^*\}$ induces a block K_{2,r_1+r_2} in G''. However $\rho(G'') = \rho(G') > \rho^*$, a contradiction.

Now we will show $q \leqslant 1$. Otherwise, $q \geqslant 2$. Then $G^*[N_{A_1}(B_1) \cup B_1]$ contains K_{2,s_1}, K_{2,s_2} ($s_1, s_2 \geqslant 2$) as induced subgraphs. Denote by $w_i' \in V(K_{2,s_i}) \cap B_1$ for $i = 1, 2$. Set $N_{A_1}(w_1') = \{v_1, v_2\}$ and $N_{A_1}(w_2') = \{v_3, v_4\}$. Let X be the Perron vector of G^* whose entry x_v is labelled by vertex v. By the symmetry, $x_{v_1} = x_{v_2}$ and $x_{v_3} = x_{v_4}$. Without loss of generality, we may assume that $x_{v_1} \geqslant x_{v_3}$. Then $x_{v_1} + x_{v_2} \geqslant x_{v_3} + x_{v_4}$. Let $G''' = G^* - w_2'v_3 - w_2'v_4 + w_2'v_1 + w_2'v_2$, and $\rho''' = \rho(G''')$. Clearly, G''' is minimally 2-edge-connected. By Lemma 16, we get $\rho''' > \rho^*$, which contradicts with the maximality of ρ^*. $\square$

By comparing Fig. 2 with Fig. 5, we have $t_2 = r_1$ and $t_3 = s_1$. From Claims 25-31, we know that G^* has two forms: $F_0(t_1, t_2, t_3)$ or $F_1(t_1', t_2', t_3')$, where $t_1, t_3, t_1', t_3' \geqslant 0$, $t_2, t_2' = 0$ or $t_2, t_2' \geqslant 2$ and satisfy

$$m = \begin{cases} 3t_1 + 2t_2 & \text{if } t_3 = 0 \\ 3t_1 + 2t_2 + 2t_3 + 2 & \text{if } t_3 \geqslant 1 \\ 3t_1' + 2t_2' + 5 & \text{if } t_3' = 0 \\ 3t_1' + 2t_2' + 2t_3' + 2 + 5 & \text{if } t_3' \geqslant 1. \end{cases}$$

Clearly, $t_1 \geqslant 1$ since otherwise m is even. Suppose that $G^* \cong F_1(t_1', t_2', t_3')$, by Proposition 23, we have $\rho(F_1(t_1', t_2', t_3')) < \rho(F_0(t_1' + 1, t_2' + 1, t_3'))$, which contradicts the maximality of $\rho(G^*)$. Thus $G^* \cong F_0(t_1, t_2, t_3)$, where $t_3 \geqslant 0$, $t_2 = 0$ or $\geqslant 2$ and

$$m = \begin{cases} 3t_1 + 2t_2 & \text{if } t_3 = 0 \\ 3t_1 + 2t_2 + 2t_3 + 2 & \text{if } t_3 \geqslant 1. \end{cases}$$

If $t_3 = 0$, then $t_2 = \frac{m-3t_1}{2}$, and thus $G^* \cong F_0(t_1, t_2, 0) = F_0(t_1, \frac{m-3t_1}{2}, 0))$. If $t_3 \geqslant 1$, then $t_2 + t_3 + 1 = \frac{m-3t_1}{2}$. By Proposition 24, we have

$$\rho(F_0(t_1, t_2, t_3)) < \rho(F_0(t_1, t_2 + t_3 + 1, 0)) = \rho(F_0(t_1, \frac{m - 3t_1}{2}, 0)).$$

By the maximality of $\rho(G^*)$ again, we get $G^* \cong F_0(t_1, \frac{m-3t_1}{2}, 0)$ for some $t_1 \geqslant 1$ and $\frac{m-3t_1}{2} = 0$ or $\geqslant 2$. At last, we will show $t_1 = 1$.

Claim 32. *$G^* \cong F_0(1, \frac{m-3}{2}, 0)$ for $m \geqslant 11$ and $m \neq 15$, and $G^* \cong F_5$ for $m = 15$.*

Proof. Denote by $f(G,x)$ the characteristic polynomial of quotient matrix of $A(G)$. Let

$$f_1(x) = f(F_0(t_1, \frac{m-3t_1}{2}, 0), x) = x^4 - x^3 + (t_1 - m)x^2 + (m - 3t_1)x - 3{t_1}^2 + mt_1,$$

where $t_1 \geqslant 2$, and let $f_2(x) = f(F_0(1, \frac{m-3}{2}, 0), x) = x^4 - x^3 + (1-m)x^2 + (m-3)x + m - 3$. Then

$$f_1(x) - f_2(x) = (t_1 - 1)x^2 + (3 - 3t_1)x - 3{t_1}^2 + 3 + (t_1 - 1)m = (t_1 - 1)g(x), \qquad (8)$$

where $g(x) = x^2 - x + m - 3(t_1 + 1)$.

If $\frac{m-3t_1}{2} \geqslant 2$, then $2 \leqslant t_1 \leqslant \frac{m-4}{3}$. Note that $m \geqslant 11$. One can verify that $g(x) > 0$ for $x > \sqrt{m-2}$. From (8), we have $f_1(x) - f_2(x) > 0$ for $x > \sqrt{m-2}$. Notice that $f_2(\sqrt{m-2}) = -\sqrt{m-2} - 1 < 0$. By Lemma 13 (ii), we have $\rho(F_0(t_1, \frac{m-3t_1}{2}, 0)) < \rho(F_0(1, \frac{m-3}{2}, 0))$ for any $t_1 \geqslant 2$. Thus, $G^* \cong F_0(1, \frac{m-3}{2}, 0)$.

If $\frac{m-3t_1}{2} = 0$, then $t_1 = \frac{m}{3}$, i.e. $G^* \cong F_0(\frac{m}{3}, 0, 0)$, where $3 \mid m$ and $m \geqslant 11$. By the computation,

$$\rho(F_0(\frac{m}{3}, 0, 0)) = \frac{1 + \sqrt{1 + \frac{8m}{3}}}{2} < \sqrt{m-2} < \rho(F_0(1, \frac{m-3}{2}, 0))$$

for $m \geqslant 19$. For $11 \leqslant m \leqslant 17$, i.e. $m = 15$, we have $\frac{1+\sqrt{41}}{2} = \rho(F_5) > \rho(F_0(1,6,0))$. By the above arguments, $G^* \cong F_5$ for $m = 15$, and $G^* \cong F_0(1, \frac{m-3}{2}, 0)$ for $m \geqslant 11$ and $m \neq 15$. □

Notice that $F_0(1, \frac{m-3}{2}, 0) \cong K_{2,\frac{m-3}{2}} * K_3$ and $\rho(F_0(1, \frac{m-3}{2}, 0))$ is the largest root of $x^4 - x^3 + (1-m)x^2 + (m-3)x + m - 3 = 0$. It completes the proof of Theorem 2 (ii). □

Remark 33. For odd $m < 11$, by Claims 25-31, we get that the minimally 2-edge connected graphs of size $m = 3, 5, 7, 9$ and their extremal graphs are given by Table.1.

m	Minimally 2-edge-connected graph	G^*	$\rho(G^*)$
3	C_3	C_3	2
5	C_5	C_5	2
7	C_7, $SK_{2,3}$, $C_3 * C_4$	$C_3 * C_4$	2.5035
9	C_9, $SK_{2,4}$, $C_3 * C_6$, $C_3 * K_{2,3}$, $C_4 * C_5$, F_3	F_3	3

Table 1: The extremal graphs of minimally 2-edge-connected graph for $m = 3, 5, 7, 9$.

Remark 34. Nosal in 1970 stated that if a graph G of size m is triangle free, then its spectral radius $\rho(G) \leqslant \sqrt{m}$, which is called a spectral Mantel's theorem. A natural question: how large can a graph family be such that $\rho(G) \leqslant \sqrt{m}$? If G is a minimally 2-edge-connected graph of size m, from Theorem 2 we know that $\rho(G) \leqslant \sqrt{m}$. A minimally 2-edge-connected graph may contain a triangle. Thus the minimally 2-edge-connected graph set is a new class of graphs with their spectral radii no more than $\sqrt{m}$.

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