The Park–Pham Theorem with Optimal Convergence Rate

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Abstract

Park and Pham's recent proof of the Kahn–Kalai conjecture was a major breakthrough in the field of graph and hypergraph thresholds. Their result gives an upper bound on the threshold at which a probabilistic construction has a $1-\epsilon$ chance of achieving a given monotone property. While their bound in other parameters is optimal up to constant factors for any fixed ϵ, it does not have the optimal dependence on ϵ as $\epsilon \to 0$. In this short paper, we prove a version of the Park–Pham Theorem with optimal ϵ-dependence.

Mathematics Subject Classifications: 05D05, 05D40

1 Introduction

One of the most fundamental tasks in probabilistic combinatorics is finding the thresholds for graph and hypergraph properties. At what p should you expect $G(n,p)$ to be more likely than not to contain a triangle? A Hamiltonian cycle? A common first attempt is to lower bound this p by the first moment method. The Park–Pham Theorem essentially says that applying the first moment method on some structure that is necessary for your desired graph to appear is always within a logarithmic factor of the true threshold.

Let H be a hypergraph on a finite vertex set X. The <u>upward closure</u> of H is

$$\langle H\rangle = \{R \subseteq X : \exists\ S \in H \text{ s.t. } S \subseteq R\},$$

that is, the subsets of X that contain a hyperedge in H. A hypergraph G <u>undercovers</u> H if $H \subseteq \langle G\rangle$, that is, every hyperedge in H contains a hyperedge in G.

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Definition 1 ([5], Section 1)**.** Let $q \in (0,1)$. H is q-small if there is a hypergraph G such that $H \subseteq \langle G \rangle$ and $\sum_{R \in G} q^{|R|} \leqslant \frac{1}{2}$.

Let X_p denote a subset of X where each element is included independently with probability p, let $p_c(H)$ be the probability p_c such that $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_{p_c}) = \frac{1}{2}$ (if H is non-trivial, this must exist and be unique by monotonicity), let $\ell(H)$ be the size of the largest hyperedge of H, and let $q(H)$ be the maximum q such that H is q-small.

To apply the Park–Pham Theorem, X should be the set of objects that are being selected independently at random, for example, the edges of $G(n,p)$ or hyperedges of a random hypergraph, and H is the graph or hypergraph property you want to find the threshold of, for example, hyperedges consisting of minimal edge sets which form a triangle or Hamiltonian cycle. $\ell(H)$ is the maximal number of elements that make up one instance of that property, for example, n for Hamiltonian cycles and 3 for triangles. $q(H)$ is often relatively easy to compute for structured H. Our main goal is to find $p_c(H)$. One motivation for the definition of q-small is that $q(H) \leqslant p_c(H)$ by first moment method on the probability of some $R \in G$ being contained in X_q, a necessary condition for some $S \in H$ to be contained in X_q. The Park–Pham Theorem shows that $q(H)$ gives rise to an upper bound, as well as the lower bound, on $p_c(H)$.

Theorem 2 (Kahn–Kalai Conjecture, now Park–Pham Theorem)**.** *For any hypergraph H,*

$$p_c(H) \leqslant 8q(H)\log(2\ell(H)).$$

In this theorem and throughout the paper, all logarithms are base 2. The Park–Pham Theorem (with arbitrary constant) was conjectured by Kahn and Kalai [5], who called it "extremely strong" and showed many applications of it, and was proven by Park and Pham [7], building off previous work [1, 3]. Theorem 2 will be proven in Section 2. Our proof achieves a significantly lower constant and avoids some of the complications of the original Park–Pham proof. We use similar techniques to Rao [9].

The Park–Pham Theorem we prove is equivalent to saying that for $q > q(H)$, $\ell = \ell(H)$, and $p = 8q\log(2\ell(H))$, we have $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_p) > \frac{1}{2}$. But you may want to know more than just when $G(n,p)$ has a 50/50 chance of containing a triangle or Hamiltonian cycle; you may want to know when it has a .999 chance of containing these. So a natural question is to replace $\frac{1}{2}$ with $1-\epsilon$ for any $\epsilon > 0$. Park and Pham also proved an $O(q(H)\log(\ell(H)))$ upper bound for any fixed ϵ, but as it was not their focus, their dependence on ϵ is exponentially worse than the dependence we give. The following theorem is our main result:

Theorem 3. *Let H be a hypergraph that is not q-small and let $\epsilon \in (0,1)$. Let $p = 48q\log\left(\frac{\ell(H)}{\epsilon}\right)$. Then $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_p) > 1-\epsilon$.*

This bound is optimal up to constant factors, that is, has the optimal dependence on all of ℓ, ϵ, and q. We will prove Theorem 3 in Section 3, and then will relate it to other work in Section 4.

2 Proof of the Park–Pham Theorem

Let H be a ℓ-bounded hypergraph, that is, $|S| \leqslant \ell$ for every $S \in H$. Given a set W, and $S \in H$, let $T(S,W)$ be $S' \setminus W$ for $S' = argmin_{S' \in H : S' \subseteq W \cup S} |S' \setminus W|$ (break ties arbitrarily). Note that for a given W, we have that $\{T(S,W) : S \in H\}$ undercovers H, as for every $S \in H$, we have $T(S,W) \subseteq S$. If H is not q-small, then $\{T(S,W) : S \in H\}$ is also not q-small, as any G undercovering $\{T(S,W) : S \in H\}$ also undercovers H.

Proposition 4 ([7], Lemma 2.1)**.** *Let H be any ℓ-bounded hypergraph (which may or may not be q-small!) and $L > 1$. Let $1 \leqslant t \leqslant \ell$ and $\mathcal{U}_t(H,W) = \{T(S,W) : S \in H, |T(S,W)| = t\}$. Let W be chosen uniformly at random from $\binom{X}{Lq|X|}$. Then*

$$\mathbb{E}_W \sum_{U \in \mathcal{U}_t(H,W)} q^t < L^{-t} \binom{\ell}{t}.$$

Proof of Proposition 4. We will follow the proof of Park and Pham [7]. It is equivalent for us to show that $\sum_{W \in \binom{X}{Lq|X|}} |\mathcal{U}_t(H,W)| < \binom{|X|}{Lq|X|} L^{-t} q^{-t} \binom{\ell}{t}$. To achieve an upper bound on the number of $T = T(S,W)$, it suffices to give a procedure for uniquely specifying any valid (W,T) pair (where the T is $T(S,W)$ for that W and some $S \in H$).

First, fix a universal "tiebreaker" function $\chi : \langle H \rangle \to H$ such that $\chi(Y) \subseteq Y$ for all $Y \in \langle H \rangle$.

Now, specify $Z = W \sqcup T$. Note that these two sets are disjoint by definition, so this has size exactly $Lq|X| + t$ and we have at most

$$\binom{|X|}{Lq|X|+t} = \binom{|X|}{Lq|X|} \prod_{i=1}^{t} \frac{|X| - Lq|X| - i + 1}{Lq|X| + i} \leqslant \binom{|X|}{Lq|X|} (Lq)^{-t}$$

valid choices.

Now, we claim that $T \subseteq \chi(Z)$. We know $Z \in \langle H \rangle$ since $S \subseteq Z$, so $\chi(Z) \subseteq Z = W \sqcup T$. If $T \not\subseteq \chi(Z)$, we could not have that T was the minimizer, as we could have instead taken $S' = \chi(Z)$ and then $\chi(Z) \setminus W \subsetneq T$.

We can thus specify T (with $|T| = t$) as a subset of $\chi(Z)$ (with $|\chi(Z)| \leqslant \ell$ since $\chi(Z) \in H$), so there are at most $\binom{\ell}{t}$ choices for T.

This process specified T and the disjoint union of T and W, so we have also specified W, and thus have given a way to specify every possible $(W, T(S,W))$ pair, with at most $\binom{|X|}{Lq|X|} (Lq)^{-t} \binom{\ell}{t}$ possible choices. $\square$

Proof of Theorem 2 from Proposition 4. We will iterate the process of replacing each S by $T(S,W)$. Start with $H_0 = H$, $X_0 = X$, and $\ell_0 = \ell(H)$. We will choose W_i to be a uniformly random set in $\binom{X_i}{8q|X_i|}$. Set

$$\mathcal{C}_i = \cup_{t=\lfloor \ell_i/2 \rfloor + 1}^{\ell_i} \mathcal{U}_t(H_i, W_i) \quad \text{and} \quad H_{i+1} = \{T(S,W_i) : S \in H_i, |T(S,W_i)| \leqslant \ell_i/2\}.$$

Set $\ell_{i+1} = \lfloor \ell_i/2 \rfloor$. Note that H_{i+1} is a ℓ_{i+1}-bounded hypergraph on $X_{i+1} = X \setminus \cup_{j=0}^{i} W_j$.

Now, we repeat until we reach an $i = I$ where $\ell_{I+1} < 1$, which then gives $H_{I+1} = \emptyset$ or $H_{I+1} = \{\emptyset\}$. As $\ell_i \leqslant 2^{-i}\ell$, we have $I \leqslant \lfloor \log(\ell) \rfloor$. Let $\mathcal{U} = \cup_{i=0}^{I} \mathcal{C}_i$ and $W = \cup_{i=0}^{I} W_i$. Now, we claim that either there is some $S \in H$ such that $S \subseteq W$ (in which case $H_{I+1} = \{\emptyset\}$ and we have succeeded), or else $H_{I+1} = \emptyset$ and $\mathcal{U}$ undercovers H. This is true because if you trace any $S_0 = S$ through $S_{i+1} = T(S_i, W_i)$, there is some $0 \leqslant i \leqslant I$ at which we have either $T(S_i, W_i) \in \mathcal{C}_i$, which gives that this $\mathcal{C}_i$ undercovers S; or $T(S_i, W_i) = \emptyset$, which means that $S_i \subseteq W_i$, and thus there exists an $S' \in H$ that is in $\cup_{j=0}^{i-1} W_j$, as $S_i = S' \backslash \cup_{j=0}^{i-1} W_j$ for some $S' \in H$.

Therefore, to show that there is a high probability of some $S \in H$ being in W, it suffices to show there is a low probability of $\mathcal{U}$ undercovering H. For each $1 \leqslant t \leqslant \ell$, let $i(t)$ be the highest i such that $\ell_i \geqslant t$. Note that sets of size t are only added to $\mathcal{U}$ at one step of our process, into $\mathcal{C}_{i(t)}$. In other words, $\mathcal{U}_t = \mathcal{U}_t(H_{i(t)}, W_{i(t)})$ and $\mathcal{U} = \cup_{t=1}^{\ell(H)} \mathcal{U}_t$. Then

$$\mathbb{E} \sum_{U \in \mathcal{U}} q^{|U|} = \mathbb{E} \sum_{t=1}^{\ell} \sum_{U \in \mathcal{U}_t} q^t = \sum_{t=1}^{\ell} \mathbb{E} \sum_{U \in \mathcal{U}_t} q^t < \sum_{t=1}^{\ell} 8^{-t} \binom{\ell_{i(t)}}{t} \qquad \text{(Proposition 4 with } L = 8)$$

$$\leqslant \sum_{t=1}^{\ell} 8^{-t} \binom{2t-1}{t} < \sum_{t=1}^{4} 8^{-t} \binom{2t-1}{t} + \sum_{t=5}^{\infty} 8^{-t} 2^{2t-1} = \frac{819}{4096} + \frac{1}{32} < \frac{1}{4}.$$

If $\mathcal{U}$ undercovers H, then $\sum_{U \in \mathcal{U}} q^{|U|} > \frac{1}{2}$. By Markov's Inequality, $\mathbb{P}(\sum_{U \in \mathcal{U}} q^{|U|} > \frac{1}{2}) < \frac{1}{2}$. Using for the first time that H is q-small, this means that with probability more than half, $\mathcal{U}$ does not undercover H and thus some $S \in H$ has $S \subseteq W$.

W is then a uniformly random set of size $\sum_{i=0}^{I} 8q|X_i| \leqslant \sum_{i=0}^{\lfloor \log(\ell) \rfloor} 8q|X| = p|X|$. If we make $W = X_p$ rather than a random set of size $p|X|$, Theorem 2 still holds: we can freely add elements to X that are not in any hyperedge in H and take the limit as the number of these points goes to infinity. $\square$

3 Proof of Theorem 3

In this section, we will prove the main theorem of this paper, which we recall below:

Theorem 5 (3)**.** *Let H be a ℓ-bounded hypergraph that is not q-small and let $\epsilon \in (0,1)$. Let $p = 48q \log\left(\frac{\ell}{\epsilon}\right)$. Then $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_p) > 1 - \epsilon$.*

As a warm-up to the proof of Theorem 3, we first note that we can quickly get logarithmic ϵ-dependence if we allow a product of $\log(\ell)$ and $\log(1/\epsilon)$ instead of a sum:

Proposition 6. *Let H be a ℓ-bounded hypergraph that is not q-small and let $\epsilon \in (0,1)$. Let $p = 8q \log(2\ell) \lceil \log\left(\frac{1}{\epsilon}\right) \rceil$. Then $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_p) > 1 - \epsilon$.*

Proof. Note that if some set W does not contain a hyperedge in H, then $H' = \{S \backslash W : S \in H\}$ undercovers H, and is thus is also not q-small. So let $H_0 = H$ and $X_0 = X$. For all $1 \leqslant i \leqslant \lceil \log\left(\frac{1}{\epsilon}\right) \rceil$, we take $W_i = (X_i)_{8q \log(2\ell)}$, take $X_{i+1} = X_i \backslash W_i$, and take $H_{i+1} = \{S \backslash W_i : S \in H_i\}$. At each step, either we have some $S \in H_i$ such that $S \subseteq X_i$ or

H_{i+1} is ℓ-bounded and not q-small. Thus, at each step, if we have not yet succeeded, we have probability $> \frac{1}{2}$ of W_i containing a hyperedge in H_i by Theorem 2. So after $\lceil \log\left(\frac{1}{\epsilon}\right)\rceil$ steps, we have that $W = \cup_{i=1}^{\lceil \log(1/\epsilon)\rceil} W_i$ has more than a $1-\epsilon$ chance of containing some hyperedge in H. We have $W \sim X_{1-(1-8q\log(2\ell))^{\lceil \log(1/\epsilon)\rceil}}$ and $1-(1-8q\log(2\ell))^{\lceil \log(1/\epsilon)\rceil} < p$, so this is also true for $W \sim X_p$. □

However, we will see in the next section that Proposition 6 is not the bound we want. What the above proof does give us is the important idea that, in this setting, we can repeat a random trial where success is more likely than failure until it succeeds.

Proof of Theorem 3. As in the proof of Theorem 2, we will iterate the process of replacing each S by $T(S,W)$, starting with $H_0 = H$, $X_0 = X$, and $\ell_0 = \ell(H)$. We choose W_i uniformly at random from $\binom{X_i}{8q|X_i|}$ and let $\mathcal{C}_i = \cup_{t=\lfloor \ell_i/2\rfloor+1}^{\ell_i} \mathcal{U}_t(H_i, W_i)$. The main difference now is that we will have a "success" and a "failure" criteria at each stage. By Proposition 4, we know that

$$\mathbb{E}\sum_{U\in\mathcal{C}_i} q^{|U|} = \mathbb{E}\sum_{t=\lfloor \ell_i/2\rfloor+1}^{\ell_i}\sum_{U\in\mathcal{U}_t} q^t < \sum_{t=\lfloor \ell_i/2\rfloor+1}^{\ell_i} 8^{-t}\binom{\ell_i}{t}.$$

We consider step i a "failure" if

$$\sum_{U\in\mathcal{C}_i} q^{|U|} > 2\left(\sum_{t=\lfloor \ell_i/2\rfloor+1}^{\ell_i} 8^{-t}\binom{\ell_i}{t}\right).$$

By Markov's inequality, success is more likely than failure at every step. We always set $X_{i+1} = X_i \setminus W_i$. If step i fails, then we keep $H_{i+1} = \{S\setminus W_i : S \in H_i\}$ (or for that matter, $H_{i+1} = \{T(S,W_i) : S \in H_i\}$) and $\ell_{i+1} = \ell_i$, essentially keeping the same hypergraph and retrying. If step i succeeds, then as before we set $H_{i+1} = \{T(S,W_i) : S \in H_i, |T(S,W_i)| \leqslant \ell_i/2\}$ and $\ell_{i+1} = \lfloor \ell_i/2\rfloor$. In either case, H_{i+1} is a ℓ_{i+1}-bounded hypergraph on $X \setminus \cup_{j=0}^{i} W_j$, so our claim of success on step $i+1$ being more likely than failure still holds. If H_i only contains the empty set, we set $\ell_{i+1} = 0$ and simply do nothing for all remaining i (these steps can be considered successes).

We repeat this for $I = 6\lfloor \log\left(\frac{\ell}{\epsilon}\right)\rfloor$ steps. Let

$$\mathcal{U} = \bigcup_{1\leqslant i\leqslant I:\text{step } i \text{ succeeded}} \mathcal{C}_i.$$

Again, letting $i(t)$ to be the highest i such that $\ell_i \geqslant t$, we have that $\mathcal{U}_t = \mathcal{U}_t(H_{i(t)}, W_{i(t)})$ and $\mathcal{U} = \cup_{t=\lceil \ell_{I+1}\rceil}^{\ell(H)} \mathcal{U}_t$. Now, by our success criteria and our proof of Theorem 2, we know for sure that

$$\sum_{U\in\mathcal{U}} q^{|U|} \leqslant 2\sum_{t=1}^{\ell} 8^{-t}\binom{\ell_{i(t)}}{t} < \frac{1}{2},$$

so as H is q-small, $\mathcal{U}$ for sure does not undercover H.

If $\ell_{I+1} < 1$, then this means that $\cup_{i=1}^{I} W_i$ contains a hyperedge in H. If we have had at least $\lfloor \log(\ell) \rfloor + 1$ successes, then we do have $\ell_{I+1} < 1$. We have had $6\lfloor \log\left(\frac{\ell}{\epsilon}\right) \rfloor$ steps, each of which had a greater than $\frac{1}{2}$ probability of succeeding (regardless of the success or failure of previous steps). Let X be our number of successes, which we then know is stochastically dominated by $Y \sim Bin(I, \frac{1}{2})$. Standard Chernoff bounds give that

$$\mathbb{P}\left(Y \leqslant (1-\delta)\mathbb{E}Y\right) \leqslant \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mathbb{E}Y}.$$

Note that $\frac{\lfloor \log(\ell) \rfloor}{\mathbb{E}Y} = \frac{\lfloor \log(\ell) \rfloor}{3\lfloor \log(\ell/\epsilon) \rfloor} \leqslant \frac{1}{3}$ as $\epsilon \leqslant 1$. So here,

$$\begin{aligned}\mathbb{P}(\nexists\ S \in H \text{ s.t. } S \subseteq \cup_{i=1}^{I} W_i) &\leqslant \mathbb{P}(\ell_{I+1} \geqslant 1) = \mathbb{P}(X \leqslant \lfloor \log(\ell) \rfloor) \\ &\leqslant \mathbb{P}(Y \leqslant \lfloor \log(\ell) \rfloor) \leqslant \mathbb{P}(Y \leqslant (1-2/3)\mathbb{E}Y) \\ &\leqslant \left(\frac{e^{-2/3}}{(1/3)^{(1/3)}}\right)^{\lfloor \log(\ell/\epsilon) \rfloor} < \left(\frac{1}{2}\right)^{\log(\ell/\epsilon)-1} = \frac{2\epsilon}{\ell} \leqslant \epsilon\end{aligned}$$

for $\ell \geqslant 2$. Then $|\cup_{i=1}^{I} W_i| \leqslant 8Iq|X| \leqslant 48q\log\left(\frac{\ell}{\epsilon}\right)|X|$, and once again we can take $W = X_{48q\log(\ell/\epsilon)}$ instead by adding points to X not in any hyperedge in H. □

4 Why Theorem 3 is the "Optimal Convergence Rate"

Now that we have proven our main theorem, we will explain how it relates to prior work. Park–Pham's paper implied Theorem 3 but with a bound of $p = O(q(\log(\ell) + \epsilon^{-c}))$ for some $c \approx 2$ [7], an exponentially worse ϵ-dependence than our bound of $p = 48q\log\left(\frac{\ell}{\epsilon}\right)$. Our Theorem 3 can be rephrased as follows:

Corollary 7. *Let H be a ℓ-bounded hypergraph that is not q-small and choose any $c \geqslant 1$. Let $p = 48cq\log(2\ell)$. Then $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_p) > 1 - \ell^{-c}$.*

Proof. At the end of the proof of Theorem 3 our failure probability was $\frac{2\epsilon}{\ell}$. Therefore, we can simply set $\epsilon = \frac{1}{2}\ell^{1-c}$, which is at most 1 as required. □

The above corollary is important because it matches the "with high probability" statements of prior work, that is, the probability of X_p containing a hyperedge in H goes to 1 as ℓ goes to infinity. ℓ^{-c} can be thought of as the convergence rate of this probability to 1. The Park–Pham bounds earlier gave for $p = O(q\log(\ell))$ a convergence rate of $(\log(\ell))^{-c}$ for some $c \approx \frac{1}{2}$ [7], so as before this is an exponential improvement.

One motivation for achieving the bound in Theorem 3 was to generalize the similar bound that was achieved under the "fractional expectation-threshold" or "κ-spread" framework by Rao [8], improving the ϵ-dependence of previous work [1, 3].

Definition 8 ([10], Definition 6.6)**.** H is $\underline{\kappa\text{-spread}}$ if for all $Y \subseteq X$,

$$|\{S \in H : S \subseteq Y\}| \leqslant \kappa^{-|Y|}|H|.$$

Theorem 9 ([8], Lemma 4). *Let H be a ℓ-bounded hypergraph that is κ-spread and let $\epsilon \in (0,1)$. There exists a universal constant β such that for $p = \frac{\beta}{\kappa}\log\left(\frac{\ell}{\epsilon}\right)$, we have that $\mathbb{P}(\exists\ S \in H \text{ s.t. } S \subseteq X_p) > 1-\epsilon$.*

Our Theorem 3 is a generalization of Rao's Theorem 9 due to the following basic lemma:

Lemma 10 ([10], Propositions 6.2, 6.7). *If H is κ-spread, H is not $\frac{1}{\kappa}$-small.*

Proof. Let G such that $H \subseteq \langle G\rangle$. For all $r \in \mathbb{N}$, let $n_r = |\{R \in G : |R| = r\}|$ and let c_r be the number of hyperedges $S \in H$ such that $\exists\ R \in G$ s.t. $R \subseteq S$ and $|R| = r$. Because H is κ-spread, $c_r \leqslant n_r|H|\kappa^{-r}$ for all $r \in \mathbb{N}$. However, as every hyperedge in H contains some hyperedge in G, $\sum_{r\in\mathbb{N}} c_r \geqslant |H| \implies \sum_{r\in\mathbb{N}} \frac{c_r}{|H|} \geqslant 1$. Putting these inequalities together, $\sum_{R\in G}(\frac{1}{\kappa})^{|R|} = \sum_{r\in\mathbb{N}} n_r\kappa^{-r} \geqslant 1 > \frac{1}{2}$. As this is true for any G such that $H \subseteq \langle G\rangle$, we must have that H is not $\frac{1}{\kappa}$-small. □

Rao's result, and thus ours, is asymptotically optimal (that is, optimal except for the constant) in ℓ, ϵ, and κ (or q):

Proposition 11 ([2], Lemma 4; adapted from [1]). *Let $\epsilon \in (0, \frac{1}{2}]$ and $\kappa, \ell \in \mathbb{N}$ such that $p = \frac{1}{6\kappa}\log\left(\frac{\ell}{\epsilon}\right)$ has $p \leqslant .7$. There exists a κ-spread hypergraph H such that $\mathbb{P}(\exists\ S \in H$ s.t. $S \subseteq X_p) < 1-\epsilon$.*

In conclusion, this paper has successfully generalized the asymptotically optimal bound from the κ-spread setting to the q-small setting, using a different proof technique from those used to achieve this bound in the κ-spread setting [8, 11, 4, 6].

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