

The number of occurrences of patterns in a random tree or forest permutation

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Abstract

The classes of tree permutations and forest permutations were defined by Acan and Hitczenko (2016). We study random permutations of a given length from these classes, and in particular the number of occurrences of a fixed pattern in one of these random permutations. The main results show that the distributions of these numbers are asymptotically normal.

The proof uses representations of random tree and forest permutations that enable us to express the number of occurrences of a pattern by a type of U -statistics; we then use general limit theorems for the latter.

Mathematics Subject Classifications: 05A05, 05C05, 60C05

1 Introduction

A number of authors have studied properties of random permutations drawn uniformly from all permutations of a given (large) length in some given class of permutations. The chosen class of permutations is often a *pattern class*, i.e., is the class of all permutations avoiding a certain set of one or several given patterns; equivalently, the class is closed under taking patterns (subpermutations). (See Section 2 for definitions of various terms used here and below.) Several different properties have been studied; in the present paper we consider the asymptotic distribution of the number of occurrences of some fixed pattern. For this problem (and many others), it seems impossible to give general results valid for all such permutation classes. (See e.g. Garrabrant and Pak [4] for some related impossibility results supporting this.) Therefore, typically these classes are studied one by one, with

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methods depending on the knowledge of some structure theorem for permutations in that particular class. See e.g. [2] and [10] for some results of this type.

The present paper continues this line of research by studying the number of occurrences of a given pattern in a random *tree permutation* or *forest permutation*. These classes of permutations were defined by Acan and Hitczenko [1] as follows.

Definition 1. For a permutation π of $[n]$, its *permutation graph* G_π is the (labelled, undirected) graph with vertex set $[n]$, and an edge ij for every inversion (i, j) in π , i.e., for every pair (i, j) such that $i < j$ and $\pi(i) > \pi(j)$.

A permutation π is a *tree permutation* if G_π is a tree, and a *forest permutation* if the graph G_π is a forest (i.e., acyclic).

Thus, every tree permutation is a forest permutation.

Acan and Hitczenko [1] noted also the following characterization, showing that the forest permutations form a pattern class.

Proposition 2 ([1]). *The forest permutations are precisely the permutations avoiding the patterns 321 and 3412.*

However, the class of tree permutations is *not* a pattern class, since a subpermutation of a tree permutation may be a forest permutation with a disconnected permutation graph. (For example, 312 is a tree permutation, but its subpermutation 12 is not.)

The structures of tree permutations and forest permutations were studied in [1]; see Section 4. Using this, and results on (conditioned) U -statistics, we will show that the number of occurrences of a fixed pattern in a random tree or forest permutation is asymptotically normal, as the length tends to ∞ ; precise results are stated in Section 3, and proved in the remainder of the paper. Section 5 defines the versions of U -statistics that are used in the paper, and cites some results for them from [9] and [11]. Tree and forest permutations are studied in Sections 6–12, leading to a representation of random forest permutations in Section 7 and a, quite different, representation of random tree permutations in Section 10; these representations both enable us to count patterns by U -statistics, which eventually yields proofs of the theorems.

Remark 3. Although we use similar methods for patterns in random tree permutations and in random forest permutations, the details are quite different, and we see no direct relation between the results for the two cases. Note that a random forest permutation is a (random) sum of tree permutations, but most of these are very small (see (7.19) and (7.5)); hence there is no reason to expect a relation between asymptotics for large forest permutations and large tree permutations.

2 Definitions and notation

2.1 Permutations

Let \mathfrak{S}_n be the set of permutations of $[n] := \{1, \dots, n\}$, and $\mathfrak{S}_* := \bigcup_n \mathfrak{S}_n$. Similarly, let \mathfrak{F}_n be the set of all forest permutations of length n and \mathfrak{T}_n the subset of tree permutations, and let $\mathfrak{F}_* := \bigcup_n \mathfrak{F}_n$ and $\mathfrak{T}_* := \bigcup_n \mathfrak{T}_n$. Thus $\mathfrak{T}_n \subseteq \mathfrak{F}_n \subseteq \mathfrak{S}_n$.

We denote the length of a permutation π by $|\pi|$.

2.2 Occurrence of patterns

If $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, then an *occurrence* of σ in π is a subsequence $\pi_{i_1} \cdots \pi_{i_m}$, with $1 \leq i_1 < \cdots < i_m \leq n$, that has the same order as σ , i.e., $\pi_{i_j} < \pi_{i_k} \iff \sigma_j < \sigma_k$ for all $j, k \in [m]$. In this context, σ is often called a *pattern*; we may also say that σ is a *subpermutation* of π . We let $\text{occ}_\sigma(\pi)$ be the number of occurrences of σ in π , and note that

$$\sum_{\sigma \in \mathfrak{S}_m} \text{occ}_\sigma(\pi) = \binom{n}{m}, \quad (2.1)$$

for every $\pi \in \mathfrak{S}_n$ and every m . For example, an inversion is an occurrence of 21, and thus $\text{occ}_{21}(\pi)$ is the number of inversions in π .

We say that a permutation π *avoids* another permutation τ if $\text{occ}_\tau(\pi) = 0$; otherwise, π *contains* τ .

2.3 Sums and decompositions of permutations

If $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$, their (direct) *sum* $\sigma \oplus \tau \in \mathfrak{S}_{m+n}$ is defined by letting τ act on $[m+1, m+n]$ in the natural way; more formally, $\sigma \oplus \tau = \pi \in \mathfrak{S}_{m+n}$ where $\pi_i = \sigma_i$ for $1 \leq i \leq m$, and $\pi_{j+m} = \tau_j + m$ for $1 \leq j \leq n$. It is easily seen that \oplus is an associative operation. We say that a permutation $\pi \in \mathfrak{S}_*$ is *decomposable* if $\pi = \sigma \oplus \tau$ for some $\sigma, \tau \in \mathfrak{S}_*$, and *indecomposable* otherwise; we also call an indecomposable permutation a *block*. See further e.g. [3, Exercise VI.14].

It is easy to see that any permutation $\pi \in \mathfrak{S}_*$ has a unique decomposition $\pi = \pi_1 \oplus \cdots \oplus \pi_\ell$ into indecomposable permutations (blocks) π_1, \dots, π_ℓ (for some, unique, $\ell \geq 1$); we may call these the *blocks of π*

If $i < j < k$ and ik is an edge in the permutation graph G_π (i.e., an inversion), then at least one of ij and jk is also an edge. It follows that the components of the graph G_π are intervals in $[n]$, and then it is easy to see that they correspond to the blocks of π ; in particular, G_π is connected if and only if π is indecomposable.

2.4 Random permutations

τ_n will always denote a uniformly random tree permutation of length n ; similarly, π_n is a uniformly random forest permutation of length n . In other words, these are uniformly random elements of \mathfrak{T}_n and \mathfrak{F}_n , respectively.

$\tilde{\tau}$ denotes a certain random tree permutation of random length defined in Section 7, see (7.5); $\tilde{\tau}_1, \tilde{\tau}_2, \dots$ will denote independent copies of $\tilde{\tau}$. Similarly, τ_m^* is another random tree permutation of random length, defined in Section 10.

2.5 Some further notation

Convergence in distribution is denoted by \xrightarrow{d} , and convergence in probability by \xrightarrow{p} . We let $\stackrel{d}{=}$ denote equality in distribution.

Given sequences of random variables X_n and constants $a_n > 0$, and a fixed exponent $q > 0$, we let $X_n = O_{L^q}(a_n)$ mean $\mathbb{E}|X_n/a_n|^q = O(1)$. Moreover, we write $X_n = O_{L^*}(a_n)$ if $X_n = O_{L^q}(a_n)$ for every $q < \infty$.

By “convergence of all moments” we mean both ordinary and absolute moments, including centered versions.

We find it convenient to express some explicit constants using

$$\phi := \frac{1 + \sqrt{5}}{2}, \tag{2.2}$$

the golden ratio. Recall that $\phi^2 = \phi + 1$. We will also let $p := \phi^{-2}$, see (7.1)–(7.3).

Unspecified limits are as $n \rightarrow \infty$.

3 Main results

Our main results are the following; the proofs are given later. In both cases, note that if σ is not a forest permutation, then $\text{occ}_\sigma(\boldsymbol{\pi}_n) = 0$. Note also that we may assume $|\sigma| \geq 2$, since the case $\sigma = 1$ is utterly trivial with $\text{occ}_1(\pi) = n$ for every $\pi \in \Sigma_n$. Moreover, if $\tau \in \mathfrak{T}_n$ is a tree permutation, then $\text{occ}_{21}(\tau) = n - 1$, since the number of inversions equals the number of edges in the tree G_τ .

Theorem 4. *Let $\boldsymbol{\tau}_n$ be a uniformly random tree permutation of length n , and let σ be a fixed forest permutation with block decomposition $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_d$. Then, as $n \rightarrow \infty$, for some $\gamma^2 = \gamma_\sigma^2 \geq 0$,*

$$\frac{\text{occ}_\sigma(\boldsymbol{\tau}_n) - n^d/d!}{n^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \gamma^2), \tag{3.1}$$

with convergence of all moments. Moreover, $\gamma^2 > 0$ unless $|\sigma_i| \leq 2$ for every i , i.e., unless each block σ_i is either 1 or 21.

We state the special case $d = 1$ separately.

Corollary 5. *Let $\boldsymbol{\tau}_n$ be a uniformly random tree permutation of length n , and let σ be a fixed tree permutation. Then, as $n \rightarrow \infty$, for some $\gamma^2 = \gamma_\sigma^2 \geq 0$,*

$$\frac{\text{occ}_\sigma(\boldsymbol{\tau}_n) - n}{\sqrt{n}} \xrightarrow{d} \mathbf{N}(0, \gamma^2), \tag{3.2}$$

with convergence of all moments. Moreover, $\gamma^2 > 0$ except in the trivial cases $|\sigma| \leq 2$, when $\text{occ}_\sigma(\boldsymbol{\tau}_n)$ is deterministic (n or $n - 1$).

Furthermore, when σ is a tree permutation, we give an exact formula for $\mathbb{E} \text{occ}_\sigma(\boldsymbol{\tau}_n)$ in Theorem 29; this expectation depends on n and $|\sigma|$ only.

The asymptotic variance γ_σ^2 in Theorem 4 and Corollary 5 can be found from our proof, but we do not know any explicit formula; we evaluate it for some simple cases in Example 44. Note that Example 44 shows that γ_σ^2 in Corollary 5 really depends on σ , and, moreover, that it is not simply a function of $|\sigma|$.

Remark 6. If σ is a forest permutation with $d \geq 2$ blocks σ_i , all of lengths $|\sigma_i| \leq 2$, then $\gamma^2 = 0$ in (3.1), but $\text{occ}_\sigma(\tau_n)$ is, in general, not deterministic. We conjecture that $\text{occ}_\sigma(\tau_n)$ is asymptotically normal in this case too, with a variance of smaller order than in Theorem 4, but we have not pursued this and leave it as an open problem. (Cf. Theorem 9 below for random forest permutations π_n .)

Problem 7. Find a combinatorial explanation for the surprising fact that the asymptotic expectation n in (3.2) is the same for all tree permutations σ . (We will see in the proof that this is equivalent to the fact that the expectation in (11.23) is the same for all tree permutations σ .)

More generally, find a combinatorial explanation for the fact that the asymptotic expectation $n^d/d!$ (or, equivalently, $\binom{n}{d}$) in (3.1) depends only on the the number of blocks d in σ .

Moreover, as just mentioned, Theorem 29 shows that for two tree permutations σ_1 and σ_2 of the same length, the expectations $\mathbb{E} \text{occ}_{\sigma_1}(\tau_n)$ and $\mathbb{E} \text{occ}_{\sigma_2}(\tau_n)$ are equal for every n . (This obviously requires $|\sigma_1| = |\sigma_2|$, since $\text{occ}_\sigma(\tau_n) = 0$ if $n < |\sigma|$.) Again, we do not know a simple proof of this fact, although the proof of Theorem 29 gives a kind of combinatorial reason. Also, we do not know whether the equality extends to two forest permutations with the same length and the same number of blocks.

We turn to patterns in a random forest permutation.

Theorem 8. Let π_n be a uniformly random forest permutation of length n , and let σ be a fixed forest permutation with block decomposition $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_d$. Let λ be the number of blocks σ_i of length $|\sigma_i| = 1$, and let

$$\tilde{\mu}_\sigma := \frac{1}{d!} (\phi + 2)^{\lambda-d} \phi^{4d-3\lambda-|\sigma|} = \frac{1}{d!} 5^{-(d-\lambda)/2} \phi^{3d-2\lambda-|\sigma|}. \quad (3.3)$$

Then, for some $\gamma_\sigma^2 \geq 0$,

$$\frac{\text{occ}_\sigma(\pi_n) - \tilde{\mu}_\sigma n^d}{n^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \gamma_\sigma^2), \quad (3.4)$$

with convergence of all moments.

Furthermore, $\gamma_\sigma^2 > 0$ except in the case $\sigma = 1 \cdots d$ (the identity permutation with every $|\sigma_i| = 1$).

Again, the asymptotic variance γ_σ^2 can in principle be found from our proof, but we do not know any explicit formula; see Remark 33 and Example 34.

In the exceptional case $\sigma = 1 \cdots d$, the limit in (3.4) is 0, and a different normalization is required.

Theorem 9. Let ι_d be the identity permutation $1 \cdots d$ for some $d \geq 2$. Then, for some $\gamma_{\iota_d}^2 > 0$,

$$\frac{\text{occ}_{\iota_d}(\pi_n) - \binom{n}{d} + \frac{5+\sqrt{5}}{10(d-2)!} n^{d-1}}{n^{d-3/2}} \xrightarrow{d} \mathbf{N}(0, \gamma_{\iota_d}^2), \quad (3.5)$$

with convergence of all moments.

Remark 10. If we consider several patterns, (3.1), (3.2), (3.4) and (3.5) extend to joint convergence to a multi-variate normal limit. This follows by the same proof, using Remarks 18 and 24. We omit the details.

4 Preliminaries on tree and forest permutations

We recall some facts from (mainly) [1] (in our notation); for completeness we sometimes sketch the arguments, but we refer to [1] for further details.

Note first that a permutation is determined by its (labelled) permutation graph, in other words, the mapping $\pi \mapsto G_\pi$ is injective. Furthermore, the induced subgraphs of G_π are the inversion graphs of the patterns occurring in π , up to obvious relabelling.

In particular, it is easily seen that the only induced cycles in a permutation graph are C_3 and C_4 (as unlabelled graphs); these are the permutation graphs of 321 and 3412 (and no other permutations), which proves Proposition 2.

Moreover, G_π is a forest if and only if its component are trees, and thus π is a forest permutation if and only if its blocks are tree permutations. In other words,

$$\pi \in \mathfrak{F}_* \iff \pi = \tau_1 \oplus \cdots \oplus \tau_m \quad (4.1)$$

for some (unique) sequence τ_1, \dots, τ_m of tree permutations. (We will find the asymptotic distribution of the number of blocks in a random forest permutation in Theorem 35.)

Let $t_n := |\mathfrak{T}_n|$ be the number of tree permutations of length n . It is shown in [1] that

$$t_n = \begin{cases} 1, & n = 1, \\ 2^{n-2}, & n \geq 2, \end{cases} \quad (4.2)$$

and thus the corresponding generating function $T(z)$ is

$$T(z) := \sum_{n=1}^{\infty} t_n z^n = z + \frac{z^2}{1-2z} = \frac{z-z^2}{1-2z}, \quad |z| < 1/2. \quad (4.3)$$

As a consequence of (4.3) and (4.1), if f_n is the number of forest permutations of length n (with $f_0 := 1$), then the corresponding generating function is

$$F(z) := \sum_{n=0}^{\infty} f_n z^n = \frac{1}{1-T(z)} = \frac{1-2z}{1-3z+z^2}. \quad (4.4)$$

The sequence (f_n) is A001519 in [13] (where many other interpretations are given).

In a permutation π , label the left-to-right maxima by L, and the right-to-left minima by R. Thus, i is labelled L if $\pi(j) < \pi(i)$ for every $j < i$, i.e., if there are no inversions (j, i) with $j < i$. In other words,

$$i \text{ is labelled L} \iff i \text{ is the left endpoint of every adjacent edge in } G_\pi. \quad (4.5)$$

Similarly, i is labelled R if there are no inversions (i, j) with $j > i$, and

$$i \text{ is labelled R} \iff i \text{ is the right endpoint of every adjacent edge in } G_\pi. \quad (4.6)$$

Now, let π be a forest permutation. If $i < j < k$, then ij and jk cannot both be edges in G_π , since otherwise, $\pi(i) > \pi(j) > \pi(k)$, so (i, k) would also be an inversion, and thus G_π would contain a cycle ijk . It follows that every $j \in [n]$ is labelled either L or R, or possibly both.

Moreover, (4.5)–(4.6) imply that i is labelled both L and R if and only if i is isolated in G_π . In a tree permutation π with $|\pi| \geq 2$, this is impossible. Thus, if π is a tree permutation with $|\pi| \geq 2$, then every $i \in [n]$ is labelled L or R, but not both. Each tree permutation τ with $|\tau| = n \geq 2$ may thus be represented by a string Ω_τ of n symbols L or R. (The notation in [1] is different: there W_1 [W_0] denotes the set of i labelled L [R] here.) The first symbol in Ω_τ is always L and the last is R. We let $\Sigma_n := \{\text{L}\{\text{L}, \text{R}\}^{n-2}\text{R}\}$ be the set of such strings, so $\Omega_\tau \in \Sigma_n$. It is shown in [1] that the map $\tau \mapsto \Omega_\tau$ is a bijection between \mathfrak{T}_n and Σ_n , for every $n \geq 2$. (Note that $|\mathfrak{T}_n| = 2^{n-2} = |\Sigma_n|$ by (4.2).) In other words, for $n \geq 2$, the tree permutations in \mathfrak{T}_n can be encoded by the strings in Σ_n . See Figure 1 for an example.

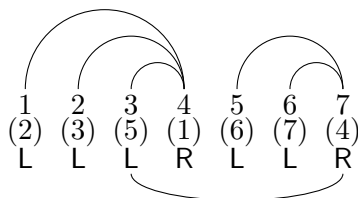


Figure 1: The tree permutation $\tau = 2351674$ with the corresponding permutation graph and the corresponding string $\Omega_{2351674} = \text{LLLRLLR}$. Formally, the vertices are labelled $1, \dots, 7$, but for illustration, the values $\tau(i)$ are added in parentheses.

We follow [1] and define the *blocks* B_1, \dots, B_{2m} of Ω_τ as the successive runs of L and R in Ω_τ . Note that since Ω_τ begins with L and ends with R, there is always an even number of blocks; an odd-numbered block B_{2l-1} is a run of L and an even-numbered block B_{2l} is a run of R. (Note that we also use 'block' in a different sense for the block decomposition of a permutation into its blocks (components) in Section 2.3; there should be no risk of confusion since the two different meanings of 'block' appear in different contexts, and we will not use both at the same time.) [1, Lemma 8] shows how the edges and vertex degrees in G_τ can be found explicitly from the code Ω_τ and the blocks B_i . We summarize this as follows.

Lemma 11 ([1]). *Let τ be a tree permutation with $|\tau| \geq 2$. Then the pairs of symbols in Ω_τ that correspond to edges in G_τ (and thus to inversions in τ) are:*

- (e1) *each L and the nearest following R;*
- (e2) *each R and the nearest preceding L;*

(e3) The last L in a block B_{2k-1} and the first R in B_{2k+2} .

The symbols in Ω_τ that correspond to leaves in G_τ are the following:

(11) every L that is not the last L in its block;

(12) the last but one symbol, if that is L;

(13) every R that is not the first R in its block;

(14) the second symbol, if that is R.

Proof. As said above, this is [1, Lemma 8], in different notation. (The four cases (11)–(14) correspond to parts (a),(c),(d),(f) in that lemma.) \square

If σ is a tree permutation with $|\sigma| \geq 2$ such that its code Ω_σ has $2m$ blocks, we define $b(\sigma) := m$; in other words the code of σ has $b(\sigma)$ L-blocks and $b(\sigma)$ R-blocks. If $|\sigma| = 1$, we do not define any code Ω_σ , but we define (for later convenience) $b(\sigma) := 1$.

5 Preliminaries on U -statistics

A U -statistic is a random variable of the form

$$U_n = U_n(f) = \sum_{1 \leq i_1 < \dots < i_d \leq n} f(X_{i_1}, \dots, X_{i_d}), \quad n \geq 0, \quad (5.1)$$

where X_1, X_2, \dots is an i.i.d. sequence of random variables with values in some measurable space \mathcal{S} , and $f : \mathcal{S}^d \rightarrow \mathbb{R}$ is a given measurable function of $d \geq 1$ variables. (It is often assumed that f is a symmetric function; we do not assume this.) U -statistics were introduced by Hoeffding [6]; we will use versions and results from [9] and [11], see also [10] for similar applications to pattern occurrences in some other pattern classes.

The fundamental central limit theorem for U -statistics, due to Hoeffding [6] in the symmetric case, can in the general (asymmetric) case be stated as follow, see [7, Theorem 11.20] and [9, Corollary 3.5 and (moment convergence) Theorem 3.15]. Assume that the random variables X_i are i.i.d., let X denote a generic X_i , and define (for a given f)

$$\mu := \mathbb{E} f(X_1, \dots, X_d), \quad (5.2)$$

$$f_i(x) := \mathbb{E}[f(X_1, \dots, X_d) \mid X_i = x], \quad (5.3)$$

$$\bar{\sigma}_{ij} := \text{Cov}[f_i(X), f_j(X)], \quad (5.4)$$

$$\bar{\sigma}^2 := \sum_{i,j=1}^d \frac{(i+j-2)!(2d-i-j)!}{(i-1)!(j-1)!(d-i)!(d-j)!(2d-1)!} \bar{\sigma}_{ij}. \quad (5.5)$$

Note that $f_i(x)$ in [9; 11] is $f_i(x) - \mu$ in the present notation. (The notation $\bar{\sigma}$ used here should not be confused with σ used elsewhere for permutations.)

Proposition 12 ([7; 9]). *Suppose that $(X_i)_1^\infty$ are i.i.d. random variables, and that $\mathbb{E}|f(X_1, \dots, X_d)|^2 < \infty$. Then, with the notation in (5.2)–(5.5), as $n \rightarrow \infty$,*

$$\frac{U_n - \binom{n}{d}\mu}{n^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \bar{\sigma}^2). \quad (5.6)$$

Furthermore, $\bar{\sigma}^2 > 0$ unless $f_i(X) = \mu$ a.s. for $i = 1, \dots, d$.

Moreover, if $\mathbb{E}|f(X_1, \dots, X_d)|^p < \infty$ for some $p \geq 2$, the (5.6) holds with convergence of all moments of order $\leq p$. \square

We will need a renewal theory version of Proposition 12. In addition to a sequence $(X_i)_1^\infty$ and a function f as above, let $h : \mathcal{S} \rightarrow \mathbb{R}$ be another measurable function, and assume (for simplicity) that $h(X_i) \geq 0$ a.s. Define

$$\nu := \mathbb{E}h(X_i), \quad (5.7)$$

$$S_n = S_n(h) := \sum_{i=1}^n h(X_i), \quad (5.8)$$

and let for each $x > 0$

$$N(x) := \inf\{N : S_N \geq x\}. \quad (5.9)$$

Remark 13. The definition (5.9) agrees with $N_+(x)$ in [10] but differs slightly from $N_+(x)$ and $N_-(x)$ in [9] and [11]; this does not affect the asymptotic results used here, see [11, Remark 3.19]. (For integer valued h and integer x , as in our application, $N(x) = N_+(x - 1)$.) We will use results from [9] and [11]; note that the event $\{S_k = n \text{ for some } k \geq 0\}$ equals $\{S_{N(n)} = n\}$ in the present notation, and $\{U_{N_-(n)} = n\}$ in the notation of [9] and [11]. (When we condition on this event in propositions below, we tacitly consider only n such that the event has positive probability.)

The following results are special cases of [9, Theorems 3.11, 3.13(iii) and 3.18] (with somewhat different notation).

Proposition 14 ([9]). *Suppose that (X_i) are i.i.d., $\mathbb{E}|f(X_1, \dots, X_d)|^2 < \infty$, and $h(X) \geq 0$ a.s., with $\nu := \mathbb{E}h(X) > 0$ and $\mathbb{E}h(X)^2 < \infty$. Then, with notations as above, as $x \rightarrow \infty$,*

$$\frac{U_{N(x)} - \mu\nu^{-d}d!^{-1}x^d}{x^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \gamma^2), \quad (5.10)$$

where, with $\bar{\sigma}^2$ given by (5.5),

$$\gamma^2 := \nu^{1-2d}\bar{\sigma}^2 - 2\frac{\mu\nu^{-2d}}{(d-1)!d!} \sum_{i=1}^d \text{Cov}[f_i(X), h(X)] + \frac{\mu^2\nu^{-2d-1}}{(d-1)!^2} \text{Var}[h(X)]. \quad (5.11)$$

Moreover, $\gamma^2 > 0$ unless $f_i(X) = \frac{\mu}{\nu}h(X)$ a.s. for $i = 1, \dots, d$. \square

Proposition 15 ([9]). *Suppose in addition to the hypotheses in Proposition 14 that $h(X)$ is integer-valued. Then (5.10) holds also conditioned on $S_{N(x)} = x$ (cf. Remark 13) for integers $x \rightarrow \infty$. \square*

Proposition 16 ([9]). *Suppose in addition to the hypotheses in Proposition 14 or 15 that $\mathbb{E} |f(X_1, \dots, X_d)|^p < \infty$ and $\mathbb{E} |h(X)|^p < \infty$ for every $p < \infty$. Then the conclusion (5.10) holds with convergence of all moments. \square*

Remark 17. In the special case $d = 1$, when the U -statistic (5.1) is a standard single sum, (5.2)–(5.5) and (5.11) simplify to $f_1 = f$, $\bar{\sigma}^2 = \bar{\sigma}_{11} = \text{Var } f(X)$, and

$$\begin{aligned} \gamma^2 &= \frac{1}{\nu} \bar{\sigma}^2 - 2 \frac{\mu}{\nu^2} \text{Cov}[f(X), h(X)] + \frac{\mu^2}{\nu^3} \text{Var } h(X) \\ &= \frac{1}{\nu} \text{Var} \left[f(X) - \frac{\mu}{\nu} h(X) \right]. \end{aligned} \quad (5.12)$$

This special case is classical, see e.g. [5, Theorem 4.2.3].

Remark 18. The results in Propositions 12–16 hold jointly for several f (possibly with different d). This is not stated explicitly in [9] (except for (5.6)), but it follows by the same proofs as in [9] (perhaps, for convenience, using the Skorohod coupling theorem [12, Theorem 4.30] and a.s. convergence in the proofs). See also [11].

5.1 Constrained U -statistics

In this subsection we extend some of the results above to constrained U -statistics, defined as follows. We consider here only a case relevant for the application in the present paper; for more general definitions and results, see [11] (with somewhat different notation).

Let, as above, $(X_i)_1^\infty$ be an i.i.d. sequence of random variables in some measurable space \mathcal{S} .

Let $d \geq 1$ and let b_1, \dots, b_d be given non-negative integers. (These are regarded as fixed in this subsection.) Let

$$b'_j := b_j - 1, \quad (5.13)$$

$$D_j := \sum_1^j b_i, \quad 0 \leq j \leq d, \quad (5.14)$$

$$D'_j := \sum_1^j b'_i = D_j - j, \quad 0 \leq j \leq d, \quad (5.15)$$

$$D := D_d = \sum_{i=1}^d b_i = D'_d + d. \quad (5.16)$$

Suppose that $f : \mathcal{S}^D \rightarrow \mathbb{R}$ is a measurable function, and define the *constrained U -statistic*

$$\widehat{U}_n = \widehat{U}_n(f) := \sum_{i_1, \dots, i_d} f((X_{i_1+k})_{k=0}^{b'_1}, (X_{i_2+k})_{k=0}^{b'_2}, \dots, (X_{i_d+k})_{k=0}^{b'_d}) \quad (5.17)$$

summing over all i_1, \dots, i_d such that $i_1 \geq 1$, $i_1 + b'_1 < i_2$, $i_2 + b'_2 < i_3$, \dots , $i_{d-1} + b'_{d-1} < i_d$, and $i_d + b'_d \leq n$. (We have grouped the arguments of f in (5.17), using an obvious notation.) In other words, \widehat{U}_n is defined as U_n in (5.1), with d replaced by D , but only summing over i_1, \dots, i_D such that the b_1 first indices are consecutive, as well as the next b_2 , and so on. In particular, in the special case $b_1 = \dots = b_d = 1$, \widehat{U}_n equals the unconstrained U -statistic U_n in (5.1).

By replacing i_j by $i_j - D'_{j-1}$ in (5.17), we obtain the alternative formula

$$\widehat{U}_n := \sum_{1 \leq i_1 < i_2 < \dots < i_d \leq n - D'_d} f((X_{i_1+k})_{k=0}^{D'_1}, (X_{i_2+k})_{k=D'_1}^{D'_2}, \dots, (X_{i_d+k})_{k=D'_{d-1}}^{D'_d}). \quad (5.18)$$

Define, as in (5.2),

$$\mu = \mu_f := \mathbb{E} f(X_1, \dots, X_D). \quad (5.19)$$

By (5.18), the mean of \widehat{U}_n is

$$\mathbb{E} \widehat{U}_n = \binom{n - D'_d}{d} \mu. \quad (5.20)$$

Proposition 12 extends to constrained U -statistics as follows.

Proposition 19 ([11]). *Let $\widehat{U}_n = \widehat{U}_n(f)$ be a constrained U -statistic defined as above, with $(X_i)_1^\infty$ i.i.d., and assume $\mathbb{E} |f(X_1, \dots, X_D)|^2 < \infty$. Then, with $\mu = \mu_f$ given by (5.19) and some $\bar{\sigma}^2 = \bar{\sigma}_f^2 \geq 0$,*

$$\frac{\widehat{U}_n - \binom{n}{d} \mu}{n^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \bar{\sigma}^2). \quad (5.21)$$

Moreover, if $\mathbb{E} |f(X_1, \dots, X_d)|^p < \infty$ for some $p \geq 2$, the (5.21) holds with convergence of all moments of order $\leq p$.

It does not matter whether we subtract $\mathbb{E} \widehat{U}_n$ or $\binom{n}{d} \mu$ in (5.21), since the difference is $O(n^{d-1}) = o(n^{d-1/2})$ by (5.20).

Proof. This is a special case of [11, Theorems 3.9 and 3.15]. □

The variance $\bar{\sigma}^2$ in (5.21) can be calculated explicitly, see [11, Remark 6.2], but the formulas are a bit complicated, and we omit them. Instead, we give a criterion that often can be used in applications to show that $\bar{\sigma}^2 > 0$. We define, in analogy with (5.3),

$$f_j(x_1, \dots, x_{b_j}) := \mathbb{E}[f(X_1, \dots, X_D) \mid (X_{D_{j-1}+1}, \dots, X_{D_j}) = (x_1, \dots, x_{b_j})]. \quad (5.22)$$

We extend the definition (5.8) to functions $g : \mathcal{S}^b \rightarrow \mathbb{R}$ for any $b \geq 1$ by defining, for such g ,

$$S_n(g) := \sum_{i=1}^n g(X_i, \dots, X_{i+b-1}). \quad (5.23)$$

Proposition 20. *In Proposition 19, the asymptotic variance $\bar{\sigma}_f^2 = 0$ if and only if for every $j \in [d]$, there exists a function $\psi_j : \mathcal{S}^{b_j-1} \rightarrow \mathbb{R}$ such that a.s.*

$$f_j(X_1, \dots, X_{b_j}) - \mu = \psi_j(X_2, \dots, X_{b_j}) - \psi_j(X_1, \dots, X_{b_j-1}), \quad (5.24)$$

and thus a.s., for every $n \geq 1$,

$$S_n(f_j - \mu) = \psi_j(X_{n+1}, \dots, X_{n+b_j-1}) - \psi_j(X_1, \dots, X_{b_j-1}). \quad (5.25)$$

Consequently, if $\bar{\sigma}_f^2 = 0$, then $S_n(f_j)$ is independent of X_{b_j}, \dots, X_n for every $j \in [d]$ and $n \geq b_j$.

Proof. This is essentially a special case of [11, Theorem 8.4]; the difference is mainly notational. The function g_j in [11, Theorem 8.4 and Remark 6.2] is, in our case, given by

$$g_j(x_1, \dots, x_{D'_d+1}) = f_j(x_{D'_{j-1}+1}, \dots, x_{D'_{j-1}+b_j}) - \mu; \quad (5.26)$$

thus g_j is essentially the same as $f_j - \mu$ but contains some redundant variables. [11, Theorem 8.4] says that $\bar{\sigma}_f^2 = 0$ if and only if there exists a function $\varphi_j : \mathcal{S}^{D'_d} \rightarrow \mathbb{R}$ such that a.s.

$$g_j(X_1, \dots, X_{D'_d+1}) = \varphi_j(X_2, \dots, X_{D'_d+1}) - \varphi_j(X_1, \dots, X_{D'_d}). \quad (5.27)$$

This is (5.24), except that we have redundant variables. These may be eliminated one by one. For example, if $D'_{j-1} > 0$, and thus g_j does not depend on x_1 by (5.26), then (5.27) implies that for a.e. fixed $x_1 \in \mathcal{S}$, we have $\varphi_j(X_1, \dots, X_{D'_d}) = \varphi_j(x_1, X_2, \dots, X_{D'_d})$ a.s., and thus a.s.

$$\varphi_j(X_1, \dots, X_{D'_d}) = \varphi'_j(X_2, \dots, X_{D'_d}) \quad (5.28)$$

for some function $\varphi'_j : \mathcal{S}^{D'_d-1} \rightarrow \mathbb{R}$. Continuing in this way, from both ends, we see that a.s.

$$\varphi_j(X_1, \dots, X_{D'_d}) = \psi_j(X_{D'_{j-1}+1}, \dots, X_{D'_{j-1}+b_j-1}) \quad (5.29)$$

for some function ψ_j , and thus (5.27) reduces to (5.24). (Alternatively, one might note that (5.27) implies $\text{Var}[S_n(f_j - \mu)] = \text{Var} S_n(g_j) = O(1)$, and then [8, Theorem 2] yields (5.24) – this essentially repeats part of the argument in [11] yielding (5.27).) Conversely, (5.24) trivially yields (5.27) for a suitable φ_j . \square

We will use a renewal theory version of constrained U -statistics. We assume again that $h : \mathcal{S} \rightarrow \mathbb{R}$ with $h(X_i) \geq 0$ a.s., and use the notation (5.7)–(5.9). The following results are special cases of [11, Theorems 3.20, 8.7, 3.21, and 3.23].

Proposition 21 ([11]). Let $\widehat{U}_n = \widehat{U}_n(f)$ be a constrained U -statistic defined as above, with $(X_i)_{i=1}^\infty$ i.i.d. Suppose that $\mathbb{E}|f(X_1, \dots, X_D)|^2 < \infty$, and that $h(X) \geq 0$ a.s., with $\nu := \mathbb{E}h(X) > 0$ and $\mathbb{E}h(X)^2 < \infty$. Then, with notations as above, as $x \rightarrow \infty$,

$$\frac{\widehat{U}_{N(x)} - \mu\nu^{-d}d!^{-1}x^d}{x^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \gamma^2), \quad (5.30)$$

for some $\gamma^2 \geq 0$. Moreover, $\gamma^2 > 0$ unless, for each $j = 1, \dots, d$, the conditions in Proposition 20 hold with $f - \mu$ replaced by the function $f_j(X_1, \dots, X_{b_j}) - \frac{\mu}{\nu}h(X_1)$.

Proof. The limit (5.30) is a special case of [11, Theorem 3.20]. The only detail that requires a comment is that [11, Theorem 8.7] says that if $\gamma^2 = 0$, then a.s.

$$g_j(X_1, \dots, X_{D'_d+1}) + \mu - \frac{\mu}{\nu}h(X_1) = \varphi_j(X_2, \dots, X_{D'_d+1}) - \varphi_j(X_1, \dots, X_{D'_d}) \quad (5.31)$$

for some function φ , where as above g_j is given by (5.26). If we use (5.26) and define

$$\overline{\varphi}_j(x_1, \dots, x_{D'_d}) := \varphi_j(x_1, \dots, x_{D'_d}) - \sum_{i=1}^{D'_d} \frac{\mu}{\nu}h(x_i), \quad (5.32)$$

then (5.31) is equivalent to

$$f_j(X_{D'_{j-1}+1}, \dots, X_{D'_{j-1}+b_j}) - \frac{\mu}{\nu}h(X_{D'_{j-1}+1}) = \overline{\varphi}_j(X_2, \dots, X_{D'_d+1}) - \overline{\varphi}_j(X_1, \dots, X_{D'_d}). \quad (5.33)$$

The result follows by eliminating redundant variables as in the proof of Proposition 20. \square

Proposition 22 ([11]). Suppose in addition to the hypotheses in Proposition 21 that $h(X)$ is integer-valued. Then (5.30) holds also conditioned on $S_{N(x)} = x$ for integers $x \rightarrow \infty$. \square

Proposition 23 ([11]). Suppose in addition to the hypotheses in Proposition 21 or 22 that $\mathbb{E}|f(X_1, \dots, X_D)|^p < \infty$ and $\mathbb{E}|h(X)|^p < \infty$ for every $p < \infty$. Then the conclusion (5.30) holds with convergence of all moments. \square

Remark 24. Again, the results in Propositions 19 and 21–23 hold jointly for several f (possibly with different d and b_1, \dots, b_d), see [11].

6 Patterns and codes of tree permutations

Consider an occurrence of a tree permutation $\sigma \in \mathfrak{T}_\ell$ in another tree permutation $\tau \in \mathfrak{T}_n$. The occurrence is defined by a subset $I = \{i_1, \dots, i_\ell\}$ of the index set $[n]$. We colour each symbol in the code Ω_τ red if its index belongs to I , and black otherwise. We use also the same colours for the corresponding vertices in G_τ . (All colourings in this paper are in red

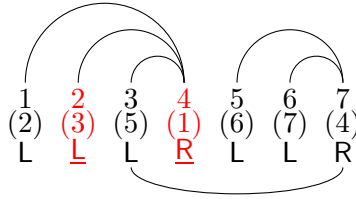


Figure 2: One occurrence of $\sigma = 21$ in $\tau = 2351674$, at indices $\{2, 4\}$, is marked by red, cf. Figure 1. (In this figure, we underline red L and R so that they can be seen also in black-and-white print.) The corresponding coloured version of $\Omega_{2351674}$ is thus LLLRLLR.

and black. We may regard the red symbols or vertices as marked.) See Figure 2 for an example.

Note that in the resulting coloured copy of Ω_τ , the red symbols form the code Ω_σ of σ ; this is a consequence of (4.5)–(4.6) and the fact that the corresponding (red) induced subgraph of G_τ equals G_σ up to an order-preserving relabelling. However, not every subset of ℓ symbols in the right order corresponds to an occurrence of σ . There is a 1–1 correspondence between

1. (nonempty) subsets of $[n]$,
2. (nonempty) subsequences of Ω_τ ,
3. occurrences of some permutation v in τ ,
4. (nonempty) labelled subgraphs of the permutation graph G_τ .

However, the subgraph in (4) is not necessarily a tree, and thus, the permutation v in (3) is not necessarily a tree permutation.

We may characterize the subsets of symbols in Ω_τ that yield occurrences of σ as follows.

Lemma 25. *Let τ and σ be tree permutations with $|\tau| \geq |\sigma| \geq 2$. A colouring of the code Ω_τ corresponds to an occurrence of σ in τ if and only if we may delete the black symbols one by one in some order according to the following rules (always interpreted for the current string) until only red symbols remain, and these form the code Ω_σ . The allowed deletions are (in any order, and possibly repeated):*

- (A1) a black L that is immediately followed by another L;
- (A2) a black L in the last but one position;
- (A3) a black R that is immediately preceded by another R;
- (A4) a black R in position 2.

Proof. Consider first the case of deleting one vertex $i \in [n]$ from the tree G_τ , i.e., restricting the permutation τ to $[n] \setminus \{i\}$ and then relabelling to get a permutation τ_1 in \mathfrak{S}_{n-1} .

The permutation graph G_{τ_1} is an induced subgraph of G_τ , and is thus always a forest; it is a tree if and only if it is connected, which is the case exactly when i is leaf in G_τ . By Lemma 11, the black vertices that may be deleted leaving a tree correspond precisely to the symbols listed in (A1)–(A4).

Thus, to repeatedly remove black symbols according to the rules in the lemma, is equivalent to repeatedly removing black leaves of G_τ , leaving a red subtree; if the resulting red code is Ω_σ , then this yields an occurrence of σ .

Conversely, if the colouring of Ω_τ corresponds to an occurrence of σ in τ , then the red vertices form a red subtree in G_τ , and we may remove the black vertices of G_τ in some order such that we always remove a black leaf of the current tree; this means that we may remove the black symbols in some order such that the rules (A1)–(A4) are followed. \square

We may invert the deletions in Lemma 25, and instead insert black symbols into Ω_σ .

Lemma 26. *Let τ and σ be tree permutations with $|\tau| \geq |\sigma| \geq 2$. A colouring of the code Ω_τ corresponds to an occurrence of σ in τ if and only if we may obtain it by from a red code Ω_σ by inserting black symbols one by one according to the following rules (always interpreted for the current string). The allowed insertions are (in any order, and possibly repeated):*

- (B1) a black L immediately to the left of any L;
- (B2) a black L immediately to the left of the last symbol;
- (B3) a black R immediately to the right of any R;
- (B4) a black R immediately to the right of the first symbol.

Proof. Immediate from Lemma 25. \square

We have so far considered deleting or inserting one symbol at a time. Since only the end result matters, the following version is more convenient for our purposes. (Recall that the first red symbol always is L, and the last is R.)

Lemma 27. *Let σ be a tree permutation with $|\sigma| \geq 2$. A coloured code Ω corresponds to a marked (red) occurrence of σ in some tree permutation τ if and only if we may obtain Ω from a red code Ω_σ by inserting black symbols as follows (the strings may be empty):*

- (C1) a string of black L immediately to the left of each red L except the first;
- (C2) a string of black L immediately to the left of the last red R;
- (C3) a string of black R immediately to the right of each red R except the last;
- (C4) a string of black R immediately to the right of the first red L;
- (C5) any black string that is empty or begins with L before the first red symbol;
- (C6) any black string that is empty or ends with R after the last red symbol.

Proof. It is easily seen that if we take any coloured code obtained by these rules, and insert another black symbol according to the rules in Lemma 26, then the result is also described by (C1)–(C6). Hence, by induction, all possible coloured codes are given by the insertions (C1)–(C6).

Conversely, suppose that Ω is obtained from a red Ω_σ by (C1)–(C6); we have to show that it also can be obtained by repeating (B1)–(B4) in some order. Evidently, (C1)–(C4) can be obtained by repeating (B1)–(B4), so it remains only to show that we may add an arbitrary black string beginning with L before the first red symbol, and an arbitrary black string ending with R after the last red symbol. To see this, note that we may first add a black L to the left by (B1). Then, when the code begins with a black L, we may by either add a black R as the second symbol by (B4), or a black L as the first symbol by (B1), but the latter gives the same result as adding a black L as the second symbol. Hence, we may add an arbitrary black symbol immediately after the first one, and by repeating this we may obtain any black string beginning with L, verifying (C5). The argument for the right side is symmetric. \square

Lemma 28. Fix a tree permutation σ with $|\sigma| \geq 2$. For every n , let $a_{n;\sigma}$ be the number of pairs (τ, σ') of a tree permutation τ of length $|\tau| = n$ together with a marked occurrence σ' of the pattern σ . Define also the generating function

$$A_\sigma(z) := \sum_{n \geq |\sigma|} a_{n;\sigma} z^n. \quad (6.1)$$

Then,

$$A_\sigma(z) = \frac{z^{|\sigma|}}{(1-z)^{|\sigma|-2}(1-2z)^2}. \quad (6.2)$$

Proof. By Lemma 27, $a_{n;\sigma}$ equals the number of coloured codes of length n that can be obtained from a red Ω_σ by the rules (C1)–(C6). These insertions are independent of each other, so they correspond to multiplying factors in the generating function $A_\sigma(z)$.

Each possible application of (C1)–(C4) yields a factor $\sum_{k=0}^{\infty} z^k = (1-z)^{-1}$. There is one possible such application for each symbol in Ω_σ , by (C1) or (C4) for each L, and by (C2) or (C3) for each R. Hence, the total contribution of (C1)–(C4) is $(1-z)^{-|\sigma|}$.

By (C5), we may to the left add a black prefix that is either empty or is an arbitrary string beginning with L, which gives 2^{k-1} possible prefixes of length k for every $k \geq 1$ (and 1 prefix of length 0). This contributes to $A_\sigma(z)$ a factor

$$1 + \sum_{k=1}^{\infty} 2^{k-1} z^k = 1 + \frac{z}{1-2z} = \frac{1-z}{1-2z}. \quad (6.3)$$

Black suffixes by (C6) contribute the same factor. These factors all multiply the term corresponding to the original red symbols Ω_σ , which is $z^{|\sigma|}$. Hence, we obtain

$$A_\sigma(z) = z^{|\sigma|} (1-z)^{-|\sigma|} \left(\frac{1-z}{1-2z} \right)^2, \quad (6.4)$$

which yields (6.2). \square

This yields an exact formula for the expected number of occurrences of σ ; note that the result depends only on $|\sigma|$ and n .

Theorem 29. *Fix a tree permutation σ with $|\sigma| \geq 2$. Then, for $n \geq |\sigma|$,*

$$\begin{aligned} \mathbb{E} \text{occ}_\sigma(\tau_n) &= [z^n](z^{|\sigma|}(2-z)^{2-|\sigma|}(1-z)^{-2}) = [z^{n-|\sigma|}]((2-z)^{2-|\sigma|}(1-z)^{-2}) \\ &= n+3-2|\sigma|+2^{-n} \sum_{i=0}^{|\sigma|-3} (|\sigma|-2-i)2^{|\sigma|-i-1} \binom{n-|\sigma|+i}{i}. \end{aligned} \quad (6.5)$$

Proof. The total number of occurrences of σ in tree permutations of length n is $a_{n;\sigma}$, and the number of such tree permutations is $t_n = 2^{n-2}$ by (4.2). Hence, by (6.1)–(6.2),

$$\begin{aligned} \mathbb{E} \text{occ}_\sigma(\tau_n) &= \frac{a_{n;\sigma}}{2^{n-2}} = [z^n](2^{2-n}A_n(z)) = [z^n](4A_n(z/2)) \\ &= [z^n] \frac{z^{|\sigma|}}{(2-z)^{|\sigma|-2}(1-z)^2}, \end{aligned} \quad (6.6)$$

which gives the first two expressions in (6.5); the explicit formula then follows from the partial fraction expansion, with $m = |\sigma| - 2 \geq 0$,

$$\frac{1}{(2-z)^m(1-z)^2} = \frac{1}{(1-z)^2} - \frac{m}{1-z} + \sum_{j=1}^m \frac{m-j+1}{(2-z)^j}. \quad (6.7)$$

□

7 A random tree permutation of random length

Recall that $T(z)$ is the generating function in (4.3), and let, throughout the paper, p be the (unique) positive root of

$$T(p) = 1. \quad (7.1)$$

By (4.3), this yields $0 < p < 1/2$ and $p - p^2 = 1 - 2p$, or $p^2 - 3p + 1 = 0$, and thus

$$p = \frac{3 - \sqrt{5}}{2} = 0.381966\dots \quad (7.2)$$

Recalling the golden ration ϕ in (2.2), we thus have

$$p = \phi^{-2} = 2 - \phi. \quad (7.3)$$

We note also

$$1 - p = \phi - 1 = \phi^{-1}, \quad 1 - 2p = p(1 - p) = \phi^{-3}. \quad (7.4)$$

We now define a random tree permutation $\tilde{\tau}$ to be a random element of \mathfrak{T}_* with the distribution

$$\mathbb{P}(\tilde{\tau} = \tau) = p^{|\tau|}, \quad \tau \in \mathfrak{T}_*. \quad (7.5)$$

Note that the sum over all $\tau \in \mathfrak{T}_*$ of the probabilities in (7.5) equals $\sum_n t_n p^n = T(p) = 1$, and thus (7.5) really defines a probability distribution.

The random tree permutation $\tilde{\tau}$ thus has random length. It follows from (7.5) that the probability generating function of $|\tilde{\tau}|$ is

$$G_{|\tilde{\tau}|}(z) := \sum_{n=1}^{\infty} t_n p^n z^n = T(pz). \quad (7.6)$$

Lemma 30. *We have*

$$\mathbb{E}|\tilde{\tau}| = \phi + 2 = \frac{5 + \sqrt{5}}{2} = \sqrt{5}\phi \doteq 3.618, \quad (7.7)$$

$$\mathbb{E}|\tilde{\tau}|^2 = 11\phi + 8 = \frac{27 + 11\sqrt{5}}{2} \doteq 25.798, \quad (7.8)$$

$$\text{Var}|\tilde{\tau}| = 6\phi + 3 = 3\phi^3 = 6 + 3\sqrt{5} \doteq 12.708, \quad (7.9)$$

$$\mathbb{E}|\tilde{\tau}|^k < \infty, \quad \forall k < \infty. \quad (7.10)$$

Proof. By (7.6) and straightforward calculations using (7.3)–(7.4),

$$\mathbb{E}|\tilde{\tau}| = G'_{|\tilde{\tau}|}(1) = pT'(p) = \frac{p(1 - 2p + 2p^2)}{(1 - 2p)^2} = \phi^4(\phi^{-3} + 2\phi^{-4}) = \phi + 2. \quad (7.11)$$

Similarly,

$$\mathbb{E}[|\tilde{\tau}|(|\tilde{\tau}| - 1)] = G''_{|\tilde{\tau}|}(1) = p^2 T''(p) = \frac{2p^2}{(1 - 2p)^3} = 2\phi^5 = 10\phi + 6 \quad (7.12)$$

and thus, combining (7.11) and (7.12),

$$\mathbb{E}|\tilde{\tau}|^2 = 11\phi + 8 \quad (7.13)$$

and

$$\text{Var}|\tilde{\tau}| = (11\phi + 8) - (\phi + 2)^2 = 6\phi + 3. \quad (7.14)$$

This shows (7.7)–(7.9).

Finally, (7.10) follows because $G_{|\tilde{\tau}|}(z)$ has radius of convergence greater than 1. (Or directly from (4.2) and (7.5).) \square

7.1 From random trees to random forests

Recall that forest permutations are sums of tree permutations (4.1). Let $\tilde{\tau}_1, \tilde{\tau}_2, \dots$ be an infinite sequence of independent random tree permutations with the distribution (7.5), and let

$$S_m := \sum_{i=1}^m |\tilde{\tau}_i|, \tag{7.15}$$

the total length of the m first of these tree permutations. Thus, for any $m \geq 1$, $\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_m$ is a forest permutation of length S_m , having m blocks.

Suppose that π is a forest permutation with m blocks τ_1, \dots, τ_m . Then, by (7.5),

$$\mathbb{P}(\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_m = \pi) = \mathbb{P}(\tilde{\tau}_i = \tau_i, \forall i \leq m) = \prod_{i=1}^m \mathbb{P}(\tilde{\tau}_i = \tau_i) = \prod_{i=1}^m p^{|\tau_i|} = p^{|\pi|}. \tag{7.16}$$

Note that this depends only on $|\pi|$.

In order to obtain arbitrary forest permutations, we have to consider a random number of blocks. We use a renewal theoretic approach. For any $n \geq 1$, let, as in (5.9),

$$N(n) := \min\{m \geq 1 : S_m \geq n\}. \tag{7.17}$$

Then, $S_{N(n)} \geq n$. Moreover, if $\pi \in \mathfrak{F}_n$ has m blocks π_1, \dots, π_m , then $\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_m = \pi$ entails $S_m = |\pi| = n$, and thus $N(n) = m$. Hence, using also (7.16),

$$\begin{aligned} \mathbb{P}(\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_{N(n)} = \pi) &= \mathbb{P}(N(n) = m \ \& \ \tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_m = \pi) \\ &= \mathbb{P}(\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_m = \pi) = p^{|\pi|} = p^n. \end{aligned} \tag{7.18}$$

This probability is thus the same for all $\pi \in \mathfrak{F}_n$. Consequently, conditioned on $S_{N(n)} = n$, so that $\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_{N(n)} \in \mathfrak{F}_n$, (7.18) implies that $\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_{N(n)}$ has the uniform distribution in \mathfrak{F}_n , and thus

$$\pi_n \stackrel{d}{=} (\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_{N(n)} \mid S_{N(n)} = n). \tag{7.19}$$

In words, we can construct a uniformly random $\pi_n \in \mathfrak{F}_n$ from the infinite sequence $(\tilde{\tau}_i)$ by composing $\tilde{\tau}_1, \tilde{\tau}_2, \dots$ until their total length is at least n , and then condition on the total length being exactly n .

8 Trees in a random tree permutation $\tilde{\tau}$

The construction (7.19) suggests that it is useful to study the random variable $\text{occ}_\sigma(\tilde{\tau})$, for a given permutation σ . We do this first for a tree permutation σ .

Lemma 31. *Let σ be a tree permutation, and let $\tilde{\tau}$ be random with the distribution (7.5). Then,*

$$\mu_\sigma := \mathbb{E}[\text{occ}_\sigma(\tilde{\tau})] = \begin{cases} \mathbb{E}|\tilde{\tau}| = \phi + 2, & |\sigma| = 1, \\ p^{|\sigma|}(1-p)^{-|\sigma|} \left(\frac{1-p}{1-2p}\right)^2 = p^{|\sigma|/2-2} = \phi^{4-|\sigma|}, & |\sigma| \geq 2. \end{cases} \tag{8.1}$$

$$\mathbb{E}[\text{occ}_\sigma(\tilde{\tau})^k] < \infty, \quad \forall k \geq 1. \tag{8.2}$$

Proof. First, if $|\sigma| = 1$, i.e., $\sigma = 1$, then trivially $\text{occ}_\sigma(\tau) = |\tau|$ for any permutation τ , and thus this case of (8.1) follows from Lemma 30.

Assume now $|\sigma| \geq 2$, and let $a_{n;\sigma}$ and $A_\sigma(z)$ be as in Lemma 28. Then,

$$\sum_{\tau \in \mathfrak{S}_n} \text{occ}_\sigma(\tau) = a_{n;\sigma}, \quad (8.3)$$

and thus it follows from (7.5) that

$$\mathbb{E} \text{occ}_\sigma(\tilde{\tau}) = \sum_{\tau \in \mathfrak{S}_*} \text{occ}_\sigma(\tau) p^{|\tau|} = \sum_{n \geq |\sigma|} p^n \sum_{\tau \in \mathfrak{S}_n} \text{occ}_\sigma(\tau) = \sum_{n \geq |\sigma|} p^n a_{n;\sigma} = A_\sigma(p). \quad (8.4)$$

Consequently, (8.1) follows from (8.4) and (6.4), using (7.4).

Finally, (8.2) follows from (7.10), since $\text{occ}_\sigma(\tau) \leq |\tau|$ for any σ . \square

Example 32. The only tree permutation σ with $|\sigma| = 2$ is 21, and $\text{occ}_{21}(\tau)$ counts the number of inversions in τ , i.e., the number of edges in G_τ . If τ is a tree permutation, we thus have $\text{occ}_{21}(\tau) = |\tau| - 1$. Indeed, Lemma 31 yields $\mathbb{E} \text{occ}_{21}(\tilde{\tau}) = \phi^2$, which equals $\mathbb{E}[|\tilde{\tau}| - 1] = \mathbb{E}|\tilde{\tau}| - 1 = \phi + 1$ given by Lemma 30.

9 Patterns in a random forest permutation

We are now prepared to prove Theorems 8 and 9 on patterns in π_n .

Proof of Theorem 8. Let $\pi \in \mathfrak{S}_n$ have block decomposition $\pi = \pi_1 \oplus \cdots \oplus \pi_N$. If $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_d$ occurs as a pattern in π , then each block σ_j is mapped into some block π_{i_j} , but it is possible that several blocks of σ fit in the same block of π . Let $\text{occ}'_\sigma(\pi)$ be the number of occurrences of σ such that the blocks are mapped to different blocks in π , i.e., where the function $j \mapsto i_j$ is injective, and let $\text{occ}''_\sigma(\pi)$ denote the number of the remaining occurrences.

Let us first consider occ'_σ , which will be the main term. We have

$$\text{occ}'_\sigma(\pi) = \sum_{1 \leq i_1 < \cdots < i_d \leq N} \prod_{j=1}^d \text{occ}_{\sigma_j}(\pi_{i_j}). \quad (9.1)$$

Thus, by (7.19),

$$\text{occ}'_\sigma(\boldsymbol{\pi}_n) \stackrel{d}{=} \left(\sum_{1 \leq i_1 < \cdots < i_d \leq N(n)} \prod_{j=1}^d \text{occ}_{\sigma_j}(\tilde{\tau}_{i_j}) \mid S_{N(n)} = n \right). \quad (9.2)$$

This is a conditioned U -statistic of the type in Proposition 15, based on the i.i.d. sequence $X_i := \tilde{\tau}_i$, with $\mathcal{S} = \mathfrak{S}_*$, the (discrete) space of all permutations, and $h(\tau) := |\tau|$; more precisely, we then have $\text{occ}'_\sigma(\boldsymbol{\pi}_n) \stackrel{d}{=} (U_{N(n)}(f) \mid S_{N(n)} = n)$ with

$$f(\tau_1, \dots, \tau_d) := \prod_{j=1}^d \text{occ}_{\sigma_j}(\tau_j). \quad (9.3)$$

Note that (8.2) and Hölder's inequality imply that $\mathbb{E}[|f(\tilde{\tau}_1, \dots, \tilde{\tau}_d)|^p] < \infty$ for every $p < \infty$. Similarly, $\mathbb{E}[h(\tilde{\tau}_1)^p] < \infty$ by (7.10).

It follows from Proposition 15 that (3.4) holds for occ'_σ , with some $\tilde{\mu}_\sigma$ and γ_σ^2 ; note that in the notation of Section 5, by Lemma 30,

$$\nu := \mathbb{E} h(\tilde{\tau}) = \mathbb{E} |\tilde{\tau}| = \phi + 2, \quad (9.4)$$

and by (5.2), (9.3), the independence of $\tilde{\tau}_i$, and (8.1) in Lemma 31,

$$\mu = \mu_\sigma := \prod_{j=1}^d \mathbb{E}[\text{occ}_{\sigma_j}(\tilde{\tau}_j)] = \prod_{j=1}^d \mu_{\sigma_j} = (\phi + 2)^\lambda \phi^{4(d-\lambda) - (|\sigma| - \lambda)}. \quad (9.5)$$

Thus, by (5.10), $\tilde{\mu}_\sigma$ in (3.4) (so far for occ'_σ) is given by

$$\tilde{\mu}_\sigma = \frac{\mu_\sigma}{\nu^d d!} = \frac{\mu_\sigma}{(\phi + 2)^d d!} = \frac{1}{d!} (\phi + 2)^{\lambda - d} \phi^{4d - 3\lambda - |\sigma|}, \quad (9.6)$$

which yields (3.3).

Similarly, by (5.3),

$$f_i(\tau) = \text{occ}_{\sigma_i}(\tau) \prod_{j \neq i} \mathbb{E} \text{occ}_{\sigma_j}(\tilde{\tau}_j) = \prod_{j \neq i} \mu_{\sigma_j} \cdot \text{occ}_{\sigma_i}(\tau) = \frac{\mu_\sigma}{\mu_{\sigma_i}} \text{occ}_{\sigma_i}(\tau). \quad (9.7)$$

Suppose that $|\sigma_i| > 1$. We may have, with positive probabilities,

1. $|\tilde{\tau}| = 1$, and then $\text{occ}_{\sigma_i}(\tilde{\tau}) = 0$,
2. $\tilde{\tau} = \sigma_i$, and then $\text{occ}_{\sigma_i}(\tilde{\tau}) = 1 > 0$.

Thus it is impossible to have $f_i(\tilde{\tau}) = c|\tilde{\tau}|$ a.s., for any real c . Consequently, Proposition 14 yields $\gamma_\sigma^2 > 0$ if any block σ_i with $|\sigma_i| > 1$ exists.

It remains to show that $\text{occ}''_\sigma(\pi_n)$ is negligible. By grouping the blocks of σ that are mapped into the same block of π , we see that $\text{occ}''_\sigma(\pi)$ can be written as a sum over all decompositions $\sigma = \tilde{\sigma}_1 \oplus \dots \oplus \tilde{\sigma}_k$ with $k < d$, of the number of occurrences with each $\tilde{\sigma}_i$ mapped into a block of π , with these blocks distinct. (Here $\tilde{\sigma}_i$ are necessarily forest permutations.) It follows, using again (7.19), and $N(n) \leq n$, that

$$\begin{aligned} \mathbb{E} \text{occ}''_\sigma(\pi_n) &\leq \frac{1}{\mathbb{P}(S_{N(n)=n})} \mathbb{E} \text{occ}''_\sigma(\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_{N(n)}) \leq C \mathbb{E} \text{occ}''_\sigma(\tilde{\tau}_1 \oplus \dots \oplus \tilde{\tau}_n) \\ &= C \sum_{k=1}^{d-1} \sum_{\tilde{\sigma}_1, \dots, \tilde{\sigma}_k} \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathbb{E} \prod_{j=1}^k \text{occ}_{\tilde{\sigma}_j}(\tilde{\tau}_{i_j}). \end{aligned} \quad (9.8)$$

The number of terms in the multiple sum is $O(n^{d-1})$, and each term is $O(1)$, using independence, the trivial $\text{occ}_{\tilde{\sigma}_j}(\tilde{\tau}) \leq |\tilde{\tau}|^{|\sigma_j|}$, and (7.10). Hence, $\mathbb{E} \text{occ}''_\sigma(\pi_n) = O(n^{d-1})$, and (3.4) follows from the result for $\text{occ}'_\sigma(\pi_n)$.

Moment convergence follows in the same way, using Proposition 16 and Minkowski's inequality; we omit the details. \square

Proof of Theorem 9. We have the trivial identity

$$\sum_{\sigma \in \mathfrak{S}_d} \text{occ}_\sigma(\boldsymbol{\pi}_n) = \binom{n}{d}. \quad (9.9)$$

Furthermore, we only have to consider forest permutations $\sigma \in \mathfrak{F}_d$ in (9.9), since otherwise $\text{occ}_\sigma(\boldsymbol{\pi}_n) = 0$.

Let $\sigma \in \mathfrak{F}_d$, and let d' be its number of blocks. If $\sigma \neq \iota_d$, then $d' < d$. If $d' \leq d - 2$, then (3.4) implies that $\text{occ}_\sigma(\boldsymbol{\pi}_n)/n^{d-3/2} \xrightarrow{P} 0$, so such terms can be ignored.

The remaining terms in (9.9) have $d' = d - 1$, and thus 1 block of length 2 and $d - 2$ blocks of length 1. There are $d - 1$ such permutations; for example, if $d = 4$, they are 2134, 1324 and 1243. For each such σ , we have by (3.3)

$$\tilde{\mu}_\sigma = \frac{1}{(d-1)!} (\phi + 2)^{-1} \phi^{4(d-1)-3(d-2)-d} = \frac{1}{(d-1)!} (\phi + 2)^{-1} \phi^2, \quad (9.10)$$

where, see (7.7),

$$(\phi + 2)^{-1} \phi^2 = \frac{\phi^2}{\sqrt{5}\phi} = \frac{\phi}{\sqrt{5}} = \frac{5 + \sqrt{5}}{10}. \quad (9.11)$$

Hence, Theorem 8 yields

$$\frac{\text{occ}_\sigma(\boldsymbol{\pi}_n) - \frac{5+\sqrt{5}}{10}(d-1)!^{-1}n^{d-1}}{n^{d-3/2}} \xrightarrow{d} \mathbf{N}(0, \gamma_\sigma^2), \quad (9.12)$$

Moreover, the proof of Theorem 8 applies also to the sum $\sum'_\sigma \text{occ}_\sigma$ over these $d - 1$ permutations σ . (Consider the sum of the corresponding functions (9.3). See also Remark 10.) Thus,

$$\frac{\sum'_\sigma \text{occ}_\sigma(\boldsymbol{\pi}_n) - \frac{5+\sqrt{5}}{10}(d-2)!^{-1}n^{d-1}}{n^{d-3/2}} \xrightarrow{d} \mathbf{N}(0, \gamma^2), \quad (9.13)$$

where $\gamma^2 > 0$ by the argument in the proof of Theorem 8.

As said above, we may add all $\sigma \in \mathfrak{F}_k$ with less than $d - 1$ blocks to the sum in (9.13) without changing the limit. The resulting sum is, by (9.9),

$$\sum_{\sigma \in \mathfrak{F}_d \setminus \{\iota_d\}} \text{occ}_\sigma(\boldsymbol{\pi}_n) = \binom{n}{d} - \text{occ}_{\iota_d}(\boldsymbol{\pi}_n), \quad (9.14)$$

and thus (3.5) follows, with $\gamma_{\iota_d}^2 = \gamma^2$ in (9.13),

Moment convergence follows by the same argument. \square

Remark 33. The asymptotic variance γ_σ^2 can by (5.11) and (5.5) be computed from variances and covariances of the $\text{occ}_{\sigma_i}(\tilde{\boldsymbol{\tau}})$ and $|\tilde{\boldsymbol{\tau}}|$. (See also Remark 17 when σ is a tree permutation, so $d = 1$.) We do not know any general formula, but at least for a specific σ , it should be possible to calculate these using methods similar to those in the proof of Lemmas 31 and 28.

Example 34. Consider the simplest example $\sigma = 21$, where we count the number of inversions in a random forest permutation π_n . In this case, σ is indecomposable, so $d = 1$. Furthermore, by (9.3) and Example 32,

$$f(\tau) = \text{occ}_{21}(\tau) = |\tau| - 1 = h(\tau) - 1, \quad (9.15)$$

and thus, using also (9.4),

$$\mu_{21} = \mathbb{E} f(\tilde{\tau}) = \nu - 1 = \phi + 1 = \phi^2, \quad (9.16)$$

in agreement with (9.5). Hence, by (9.6) (or (3.3)) and (9.11),

$$\tilde{\mu}_{21} = \frac{\mu_{21}}{\nu} = \frac{\phi^2}{\phi + 2} = \frac{5 + \sqrt{5}}{10}. \quad (9.17)$$

Moreover, (5.12) yields, using also (7.9) and (7.7),

$$\gamma_{21}^2 = \frac{1}{\nu} \text{Var} \left[|\tilde{\tau}| - 1 - \frac{\nu - 1}{\nu} |\tilde{\tau}| \right] = \nu^{-3} \text{Var} |\tilde{\tau}| = \frac{3\phi^3}{(\sqrt{5}\phi)^3} = 3 \cdot 5^{-3/2} \doteq 0.268. \quad (9.18)$$

Consequently, Theorem 8 yields

$$\frac{\text{occ}_{21}(\pi_n) - \frac{5+\sqrt{5}}{10}n}{n^{1/2}} \xrightarrow{d} \mathbf{N}(0, 3 \cdot 5^{-3/2}). \quad (9.19)$$

This implies also that for the case $d = 2$ of Theorem 9, we have $\gamma_{12}^2 = \gamma_{21}^2 = 3 \cdot 5^{-3/2}$.

Note that $\text{occ}_{21}(\pi_n)$ equals the number of edges in the forest G_{π_n} , and thus $n - \text{occ}_{21}(\pi_n)$ is the number of components of G_{π_n} , which equals the number of blocks in π_n . Hence, Example 34 implies a central limit theorem for the number of blocks in a random forest permutation:

Theorem 35. *Let $T(\pi_n)$ be number of blocks in a random forest permutation π_n , i.e., the number of tree permutations in a decomposition (4.1) of π_n . Then*

$$\frac{T(\pi_n) - \frac{5-\sqrt{5}}{10}n}{n^{1/2}} \xrightarrow{d} \mathbf{N}(0, 3 \cdot 5^{-3/2}), \quad (9.20)$$

with convergence of all moments. □

10 Random tree permutations from random blocks

In the remaining sections, we study patterns in a random tree permutation τ_n . In analogy with the construction of π_n from random tree permutations $\tilde{\tau}_i$ in Section 7, we may construct the random tree permutation τ_n with given length from a code with blocks of random lengths. There is only one L-block or R-block of each length, and therefore (cf.

(7.5)) we simply let $(L_i)_1^\infty$ and $(R_i)_1^\infty$ be two infinite sequences of random variables, all i.i.d., with the geometric distribution

$$\mathbb{P}(L_i = \ell) = \mathbb{P}(R_i = \ell) = 2^{-\ell}, \quad \ell \geq 1. \quad (10.1)$$

We also define the random vector

$$X_i := (L_i, R_i), \quad (10.2)$$

and, for a vector $x = (\ell, r)$,

$$h(x) := \ell + r. \quad (10.3)$$

We use the notation of Section 5; in particular,

$$S_m := \sum_{i=1}^m h(X_i) = \sum_{i=1}^m (L_i + R_i). \quad (10.4)$$

For $m \geq 1$, let τ_m^* be the random tree permutation that has a code with $2m$ blocks of lengths $L_1, R_1, \dots, L_m, R_m$, and thus (random) length S_m . Then, for every tree permutation τ having a code Ω_τ with $2m$ blocks with lengths $\ell_1, r_1, \dots, \ell_m, r_m$, by independence and (10.1),

$$\begin{aligned} \mathbb{P}(\tau_m^* = \tau) &= \mathbb{P}(L_1 = \ell_1, R_1 = r_1, \dots, L_m = \ell_m, R_m = r_m) \\ &= \prod_{i=1}^m \mathbb{P}(L_1 = \ell_i) \mathbb{P}(R_i = r_i) = \prod_{i=1}^m 2^{-\ell_i} 2^{-r_i} = 2^{-|\tau|}. \end{aligned} \quad (10.5)$$

It follows as in Section 7.1, cf. (7.18)–(7.19), that if τ is a tree permutation of length n that has $2m$ blocks in its code, then

$$\mathbb{P}(\tau_{N(n)}^* = \tau) = \mathbb{P}(N(n) = m \ \& \ \tau_m^* = \tau) = \mathbb{P}(\tau_m^* = \tau) = 2^{-|\tau|}, \quad (10.6)$$

which is the same for all $\tau \in \mathfrak{T}_n$, and thus

$$\tau_n \stackrel{d}{=} (\tau_{N(n)}^* \mid S_{N(n)} = n). \quad (10.7)$$

Note that $X_{N(n)}$ does not have the same distribution as X_i for a fixed i , see e.g. [5, Section 2.6]. We will use a simple (coarse) estimate (valid for much more general X_i and $h(X_i)$). Define for convenience $h(X_i) := 0$ for $i \leq 0$.

Lemma 36. *For any $j \geq 0$, $k \geq 1$ and $n \geq 1$,*

$$\mathbb{P}[h(X_{N(n)-j}) = k] \leq (k + j \mathbb{E} h(X_1)) \mathbb{P}(h(X_1) = k). \quad (10.8)$$

Hence, for any $q > 0$,

$$\mathbb{E}[h(X_{N(n)-j})^q] \leq \mathbb{E}[h(X_1)^{q+1}] + j \mathbb{E} h(X_1) \mathbb{E}[h(X_1)^q]. \quad (10.9)$$

Proof. Write $Y_i := h(X_i)$ and $Z_i := \sum_{s=1}^j Y_{i+s}$. If $Y_{N(n)-j} = k$, then there exists some $m \geq 0$ (viz. $N(n) - j - 1$) such that $S_m < n$, $Y_{m+1} = k$, and $S_m + Y_{m+1} + Z_{m+1} \geq n$. For a given m , S_m , Y_{m+1} and Z_{m+1} are independent, and thus

$$\begin{aligned}
 \mathbb{P}(Y_{N(n)-j} = k) &\leq \sum_{m=0}^{\infty} \sum_{i=0}^{n-1} \mathbb{P}(S_m = i, Y_{m+1} = k, k + Z_{m+1} \geq n - i) \\
 &= \sum_{m=0}^{\infty} \sum_{i=0}^{n-1} \mathbb{P}(S_m = i) \mathbb{P}(Y_{m+1} = k) \mathbb{P}(k + Z_{m+1} \geq n - i) \\
 &= \mathbb{P}(Y_1 = k) \sum_{i=0}^{n-1} \sum_{m=0}^{\infty} \mathbb{P}(S_m = i) \mathbb{P}(k + Z_1 \geq n - i) \\
 &\leq \mathbb{P}(Y_1 = k) \sum_{i=0}^{n-1} \mathbb{P}(k + Z_1 \geq n - i) \\
 &\leq \mathbb{P}(Y_1 = k) \sum_{s=1}^{\infty} \mathbb{P}(k + Z_1 \geq s) = \mathbb{P}(Y_1 = k) \mathbb{E}(k + Z_1) \\
 &= (k + j \mathbb{E} Y_1) \mathbb{P}(Y_1 = k). \tag{10.10}
 \end{aligned}$$

This proves (10.8). We obtain (10.9) by multiplying (10.8) by k^q and summing over k . \square

We record a simple fact.

Lemma 37. *We have $\mathbb{E} L_i = \mathbb{E} R_i = 2$, and thus*

$$\nu := \mathbb{E} h(X_i) = 4. \tag{10.11}$$

Proof. By definition, $L_i \stackrel{d}{=} R_i \sim \text{Ge}(1/2)$, and thus, as is well known, $\mathbb{E} L_i = \mathbb{E} R_i = 2$. (See also (11.16) below.) Hence, (10.11) follows. \square

11 Trees in a given tree permutation

We next express the number of occurrences of a pattern σ in a tree permutation using codes and block lengths. We consider here only the case when σ is a tree permutation.

Lemma 38. *Let σ be a tree permutation with $|\sigma| \geq 3$ having a code with $2b$ blocks of lengths $\ell_1, r_1, \dots, \ell_b, r_b$, and let τ be a tree permutation with $|\tau| \geq 3$ having a code with $2m$ blocks of lengths $\ell'_1, r'_1, \dots, \ell'_m, r'_m$. Then*

$$\text{occ}_{\sigma}(\tau) = \sum_{s=0}^{m-b} \prod_{i=1}^b \alpha_{L,i}(\tilde{\ell}'_{i+s}) \alpha_{R,i}(\tilde{r}'_{i+s}) \tag{11.1}$$

where

$$\tilde{\ell}'_k := \ell'_k - \mathbf{1}\{k = 1, \ell_1 > 1\}, \tag{11.2}$$

$$\tilde{r}'_k := r'_k - \mathbf{1}\{k = m, r_b > 1\}, \quad (11.3)$$

and

$$\alpha_{L,i}(\ell') := \begin{pmatrix} \ell' - 1 + \mathbf{1}\{i = 1, \ell_1 > 1\} + \mathbf{1}\{i = b, r_b = 1\} \\ \ell_i - 1 + \mathbf{1}\{i = b, r_b = 1\} \end{pmatrix}, \quad (11.4)$$

$$\alpha_{R,i}(r') := \begin{pmatrix} r' - 1 + \mathbf{1}\{i = b, r_b > 1\} + \mathbf{1}\{i = 1, \ell_1 = 1\} \\ r_i - 1 + \mathbf{1}\{i = 1, \ell_1 = 1\} \end{pmatrix}. \quad (11.5)$$

Proof. The occurrences of σ in τ are described by colourings of Ω_τ that can be obtained as in Lemma 27. Consider one such colouring, $\widehat{\Omega}_\tau$ say. We find some properties of it.

(i): Consider first the red symbols in $\widehat{\Omega}_\tau$ that correspond to a single block B_j in Ω_σ . These red symbols have the same type (L or R), and there are no other red symbols between them. It follows from Lemma 27 that they have to belong to the same block, B'_k say, in τ , except for the first and last blocks B_1 and B_{2b} . If $|B_1| \geq 2$, it is also possible that the first L in B_1 corresponds to the last in B'_{k-2} , while all others correspond to red L in B'_k (for some odd $k \geq 3$). We have a symmetric situation for the last block B_{2b} if $|B_{2b}| \geq 2$. Write $k = k(j)$ for the index of the block B'_k in $\widehat{\Omega}_\tau$ that corresponds to B_j . (To be precise in all cases, $B'_{k(j)}$ contains the last red L in B_j if j is odd, and the first red R in B_j if j is even.)

(ii): Furthermore, for an L-block B_{2i-1} in Ω_σ , the last L in the corresponding block $B'_{k(2i-1)}$ in $\widehat{\Omega}_\tau$ has to be red, except in the case of the last L-block B_{2b-1} if $|B_{2b}| = 1$; in that exceptional case there is no restriction on the red subset of $B'_{k(2b-1)}$ (except it having the size ℓ_b of B_{2b-1}). For an R-block B_{2i} there is a symmetric condition, unless $i = 1$ and $|B_1| = 1$.

(iii): In all cases, $k = k(j) \equiv j \pmod{2}$. Moreover, no completely black blocks can be inserted between the red symbols in two consecutive blocks of Ω_σ . Hence, $k(j+1) = k(j) + 1$ for every $j < 2b$, and thus there exists $s \in [0, m-b]$ such that $k(j) = j + 2s$ for all i .

Conversely, any choice of red symbols satisfying (i)–(iii) for some $s \in [0, m-b]$ gives a colouring of the code Ω_τ that can be constructed as in Lemma 27, and thus corresponds to an occurrence of σ in τ .

For each choice of s , the choices of red symbols permitted by (i)–(iii) for an L-block B_{2i-1} is $\alpha_{L,i}(\tilde{\ell}'_{s+i})$; note that for $1 < i < b$, this is just $\binom{\ell'_{i+s}-1}{\ell_i-1}$, while for $i = 1$ and b there are (possibly) some adjustments that are taken care of by the indicator functions in (11.2) and (11.4). Similarly, the choices of red symbols for an R-block B_{2i} is $\alpha_{R,i}(\tilde{r}'_{s+i})$. Hence, still for a fixed s , the total number of choices of red symbols in Ω_τ is given by the product in (11.1), because the choices for the different blocks B_1, \dots, B_{2b} can be made independently of each other. Consequently, (11.1) holds. \square

Remark 39. The condition $|\sigma| \geq 3$ in Lemma 38 excludes the two cases $\sigma = 1$ and $\sigma = 21$. Recall that both these cases are trivial, with $\text{occ}_1(\tau) = |\tau|$ and $\text{occ}_{21}(\tau) = |\tau| - 1$ for any tree permutation τ . (The latter because the number of inversions in τ equals the number

of edges in the tree Ω_τ .) Note that 21 has the code LR , so in the notation above, it has $b = 1$ and $\ell_1 = r_1 = 1$; however, (11.1) is not valid in this case.

Recall that b in Lemma 38 is denoted $b(\sigma)$, see Section 4, and that we also have defined $b(1) := 1$ for the case $\sigma = 1$. For any tree permutation σ we define, for $b = b(\sigma)$ vectors $x_j = (\ell'_j, r'_j)$,

$$\bar{f}_\sigma(x_1, \dots, x_b) := \prod_{i=1}^b \alpha_{L,i}(\ell'_i) \alpha_{R,i}(r'_i), \quad \text{if } |\sigma| \geq 3, \quad (11.6)$$

with $\alpha_{L,i}$ and $\alpha_{R,i}$ given by (11.4)–(11.5), and

$$\bar{f}_\sigma(x_1) := \ell'_1 + r'_1 = h(x_1) \quad \text{if } |\sigma| \leq 2. \quad (11.7)$$

(In the exceptional cases 1 and 21 where (11.7) applies, we have $b(\sigma) = 1$.)

We compute also some expectations needed later.

Lemma 40. *Let σ be as in Lemma 38 and let $\alpha_{L,i}$ and $\alpha_{R,i}$ be given by (11.4)–(11.5). Let L_i and R_i have the geometric distribution in (10.1). Then,*

$$\mathbb{E} \alpha_{L,i}(L_i) = (1 + \mathbf{1}\{i = 1, \ell_1 > 1\})(1 + \mathbf{1}\{i = b, r_b = 1\}), \quad (11.8)$$

$$\mathbb{E} \alpha_{R,i}(R_i) = (1 + \mathbf{1}\{i = b, r_b > 1\})(1 + \mathbf{1}\{i = 1, \ell_1 = 1\}). \quad (11.9)$$

Proof. In the definition (11.4), there are two special cases: (I) $i = 1$ and $\ell_1 > 1$; (II) $i = b$ and $r_b = 1$. Note that both may occur together, if $b = 1$; thus there are four possible combinations.

Case 1: Neither (I) nor (II). In this case, (11.4) is simply $\binom{\ell'_i - 1}{\ell_i - 1}$, and thus

$$\mathbb{E} \alpha_{L,i}(L_i) = \mathbb{E} \binom{L_i - 1}{\ell_i - 1}. \quad (11.10)$$

To compute this binomial moment, we note that the probability generating function of $L_i - 1$ is, by (10.1),

$$g_{L-1}(z) := \sum_{\ell=1}^{\infty} z^{\ell-1} 2^{-\ell} = \frac{1/2}{1 - z/2} = \frac{1}{2 - z}, \quad (11.11)$$

and thus,

$$\mathbb{E} \binom{L_i - 1}{k} = \frac{1}{k!} \frac{d^k}{dz^k} g_{L-1}(1) = 1, \quad k \geq 0. \quad (11.12)$$

(Alternatively, compute $[z^k]g_{L-1}(1 + z)$.) Hence, in this case,

$$\mathbb{E} \alpha_{L,i}(L_i) = 1. \quad (11.13)$$

Case 2: (I) but not (II). Then, $\ell_1 \geq 2$ and (11.4) yields

$$\mathbb{E} \alpha_{L,i}(L_i) = \mathbb{E} \binom{L_i}{\ell_i - 1}. \quad (11.14)$$

The probability generating function of L_i is, by (11.11),

$$g_L(z) = z g_{L-1}(z) = \frac{z}{2-z} = \frac{2}{2-z} - 1, \quad (11.15)$$

and thus,

$$\mathbb{E} \binom{L_i}{k} = \frac{1}{k!} \frac{d^k}{dz^k} g_L(1) = 2, \quad k \geq 1. \quad (11.16)$$

(Alternatively, use (11.12) and $\binom{L_i}{k} = \binom{L_i-1}{k} + \binom{L_i-1}{k-1}$.) Hence, (11.14) yields

$$\mathbb{E} \alpha_{L,i}(L_i) = 2. \quad (11.17)$$

Case 3: (II) but not (I). Then, (11.4) yields, using (11.16),

$$\mathbb{E} \alpha_{L,i}(L_i) = \mathbb{E} \binom{L_i}{\ell_i} = 2. \quad (11.18)$$

Case 4: Both (I) and (II). Then, $b = i = 1$, $\ell_1 \geq 2$, and (11.4) yields

$$\mathbb{E} \alpha_{L,i}(L_i) = \mathbb{E} \binom{L_i + 1}{\ell_i}. \quad (11.19)$$

The probability generating function of $L_i + 1$ is, by (11.15),

$$g_{L+1}(z) = z g_L(z) = \frac{2z}{2-z} - z = \frac{4}{2-z} - 2 - z, \quad (11.20)$$

and thus,

$$\mathbb{E} \binom{L_i + 1}{k} = \frac{1}{k!} \frac{d^k}{dz^k} g_{L+1}(1) = 4, \quad k \geq 2. \quad (11.21)$$

(Alternatively, use (11.16) and $\binom{L_i+1}{k} = \binom{L_i}{k} + \binom{L_i}{k-1}$.) Hence, by (11.19),

$$\mathbb{E} \alpha_{L,i}(L_i) = 4. \quad (11.22)$$

We may summarize the four cases (11.13), (11.17), (11.18) and (11.22) as (11.8). Similarly, by only notational changes, (11.5) yields (11.9). \square

Lemma 41. *Let σ be any tree permutation, let $b := b(\sigma)$, and let $(X_i)_i$ be the i.i.d. random vectors defined in (10.2). Then*

$$\mathbb{E} \bar{f}_\sigma(X_1, \dots, X_b) = 4. \quad (11.23)$$

Proof. The case $|\sigma| \leq 2$ is immediate by (11.7) and Lemma 37.

Assume thus $|\sigma| \geq 3$. Then, by (11.6), independence, and Lemma 40,

$$\begin{aligned} \mathbb{E} \bar{f}_\sigma(X_1, \dots, X_b) &= \mathbb{E} \prod_{i=1}^b \alpha_{L,i}(L_i) \alpha_{R,i}(R_i) = \prod_{i=1}^b \mathbb{E} \alpha_{L,i}(L_i) \prod_{i=1}^b \mathbb{E} \alpha_{R,i}(R_i) \\ &= (1 + \mathbf{1}\{\ell_1 > 1\})(1 + \mathbf{1}\{r_b = 1\}) \cdot (1 + \mathbf{1}\{r_b > 1\})(1 + \mathbf{1}\{\ell_1 = 1\}) \\ &= (1 + \mathbf{1}\{\ell_1 > 1\})(1 + \mathbf{1}\{\ell_1 = 1\}) \cdot (1 + \mathbf{1}\{r_b = 1\})(1 + \mathbf{1}\{r_b > 1\}) \\ &= (1 + 1)(1 + 1) = 4, \end{aligned} \tag{11.24}$$

which completes the proof. \square

We have no simple explanation for the, perhaps surprising, fact that the expectation (11.23) is the same for every tree permutation σ , cf. Problem 7.

12 Patterns in a random tree permutation of given length

We next consider the occurrences of a pattern σ in a random tree permutation τ_n . We use the construction and notation in Sections 10 and 5. In particular, X_n and S_m are defined by (10.1)–(10.4) and $N(n)$ by (5.9).

We first consider the case of a tree permutation σ . Recall \bar{f}_σ defined by (11.6)–(11.7).

Lemma 42. *Let σ be a tree permutation and let $b := b(\sigma)$. Then*

$$\text{occ}_\sigma(\tau_n) \stackrel{d}{=} \left(\sum_{s=0}^{N(n)-b} \bar{f}_\sigma(X_{s+1}, \dots, X_{s+b}) \mid S_{N(n)} = n \right) + O_{L^*}(1). \tag{12.1}$$

Proof. Assume first $|\sigma| \geq 3$, so \bar{f}_σ is given by (11.6). Recall τ_m^* defined in Section 10, and note that $\tau_{N(n)}^*$ is a tree permutation having a code with $2N(n)$ blocks of lengths $L_1, \dots, R_{N(n)}$. Lemma 38 thus shows that

$$\begin{aligned} \text{occ}_\sigma(\tau_{N(n)}^*) &= \sum_{s=0}^{N(n)-b} \prod_{i=1}^b \alpha_{L,i}(L_{i+s} - \mathbf{1}\{s=0, i=1, \ell_1 > 1\}) \\ &\quad \cdot \alpha_{R,i}(R_{i+s} - \mathbf{1}\{s=N(n)-b, i=b, r_b > 1\}). \end{aligned} \tag{12.2}$$

Except in the extreme cases $s=0$ and $s=N(n)-b$, the product in the sum in (12.2) is

$$\prod_{i=1}^b \alpha_{L,i}(L_{i+s}) \alpha_{R,i}(R_{i+s}) = \bar{f}_\sigma(X_{s+1}, \dots, X_{s+b}). \tag{12.3}$$

In the cases $s=0$ and $s=N(n)-b$, the product might be smaller, but is still ≥ 0 . Hence, (12.2) yields

$$\sum_{s=1}^{N(n)-b-1} \bar{f}_\sigma(X_{s+1}, \dots, X_{s+b}) \leq \text{occ}_\sigma(\tau_{N(n)}^*) \leq \sum_{s=0}^{N(n)-b} \bar{f}_\sigma(X_{s+1}, \dots, X_{s+b}). \tag{12.4}$$

We claim that

$$\bar{f}_\sigma(X_1, \dots, X_b), \bar{f}_\sigma(X_{N(n)-b+1}, \dots, X_{N(n)}) = O_{L^*}(1). \quad (12.5)$$

This implies that the difference of the first and last sums in (12.4) is $O_{L^*}(1)$, and thus

$$\text{occ}_\sigma(\tau_{N(n)}^*) = \sum_{s=0}^{N(n)-b} \bar{f}_\sigma(X_{s+1}, \dots, X_{s+b}) + O_{L^*}(1). \quad (12.6)$$

To show (12.5), note first that (11.6) and (11.4)–(11.5) imply that

$$\bar{f}_\sigma(X_{k+1}, \dots, X_{k+b}) \leq \prod_{i=1}^b (L_{k+i} + R_{k+i})^c = \prod_{i=1}^b h(X_{k+i})^c, \quad (12.7)$$

for some $c < \infty$ depending on σ only. Hence, using Hölder's inequality, (12.5) follows if we show that for every $q < \infty$ and every $j \in [1, b]$,

$$\mathbb{E} h(X_j)^q = O(1), \quad \mathbb{E} h(X_{N(n)-b+j})^q = O(1), \quad (12.8)$$

The first part is trivial, since for any fixed j , we have $\mathbb{E} h(X_j)^q = \mathbb{E} h(X_1)^q < \infty$. The second part follows from Lemma 36.

Hence, (12.6) holds, and (12.1) follows by conditioning on $S_{N(n)} = n$, recalling (10.7). Note that the error term $O_{L^*}(1)$ survives this conditioning, because $\mathbb{P}(S_{N(n)} = n) \rightarrow 1/\mathbb{E} h(X_1) > 0$, see e.g. [5, Theorem 2.4.2], and thus for any $q < \infty$,

$$\mathbb{E}[|O_{L^*}(1)|^q \mid S_{N(n)} = n] \leq \frac{\mathbb{E}[|O_{L^*}(1)|^q]}{\mathbb{P}[S_{N(n)} = n]} = O(1). \quad (12.9)$$

Finally, if $|\sigma| \leq 2$, then $b = 1$ and

$$\sum_{s=0}^{N(n)-b} \bar{f}_\sigma(X_{s+1}) = \sum_{s=0}^{N(n)-1} h(X_{s+1}) = S_{N(n)}. \quad (12.10)$$

Furthermore, $\text{occ}_\sigma(\tau_n) = n$ or $n - 1$, and thus (12.1) is trivial. \square

The sum in (12.1) is a constrained U -statistic of the type in (5.17), with $d = 1$ and $b_1 = b(\sigma)$. We extend Lemma 42 to forest permutations σ .

Lemma 43. *Let σ be a forest permutation with block decomposition $\sigma = \sigma_1 \oplus \dots \oplus \sigma_d$. Let $b_j := b(\sigma_j)$, and define*

$$\bar{f}_\sigma((x_{1i})_{i=1}^{b_1}, \dots, (x_{di})_{i=1}^{b_d}) := \prod_{j=1}^d \bar{f}_{\sigma_j}(x_{j1}, \dots, x_{jb_j}). \quad (12.11)$$

Then

$$\text{occ}_\sigma(\tau_n) \stackrel{d}{=} \left(\widehat{U}_{N(n)}(\bar{f}_\sigma) \mid S_{N(n)} = n \right) + O_{L^*}(n^{d-1}). \quad (12.12)$$

Proof. Recall again τ_m^* from Section 10, and consider first $\text{occ}_\sigma(\tau_m^*)$, for some given m . By definition, τ_m^* has $2m$ blocks, which we denote by B'_1, \dots, B'_{2m} .

As before, we mark an occurrence of σ in $\tau = \tau_m^*$ by colouring the corresponding symbols in the code Ω_τ red (and the remaining ones black). Then each σ_j corresponds to a set of red symbols, A_j say; these sets A_j are subsets of $\{1, \dots, |\tau_m^*|\}$.

As in (5.13), let $b'_j := b_j - 1$. For each σ_j with $|\sigma_j| \geq 3$, the red symbols A_j are as in the proof of Lemma 38, and they lie in some blocks $B'_{2i_j-1}, \dots, B'_{2(i_j+b'_j)}$, possibly also with a red symbol in B'_{2i_j-3} or $B'_{2(i_j+b'_j)+2}$.

If $|\sigma_j| = 2$, so $\sigma_j = 21$, then the red symbols in A_j are an L and an R forming an edge, and thus described by Lemma 11(e1)–(e3); we then define i_j so that the L belong to B'_{2i_j-1} (and thus the R to B'_{2i_j} or B'_{2i_j+2}).

Finally, if $\sigma_j = 1$, A_j is a single red symbol, which can be either L or R; we define i_j such that this symbol belongs to B'_{2i_j-1} or B'_{2i_j} .

The sets A_j follow each other in order, and thus we must have $1 \leq i_1 \leq i_2 \leq \dots \leq i_d \leq m$. (Equality is possible, e.g. if $|\sigma_j| = 1$.) Moreover, for a given sequence i_1, \dots, i_d , if all gaps $i_{j+1} - i_j \geq 3$, then the sets A_j can be chosen independently, without interfering with each other (by colliding, having symbols in wrong order, or causing edges between two of them). If furthermore $i_1 > 1$ and $i_d + b'_d < m$, the number of choices for each σ_j with $|\sigma_j| \geq 3$ is $\bar{f}_{\sigma_j}(X_{i_j}, \dots, X_{i_j+b'_j})$ by the proof of Lemma 38. The same holds for $|\sigma_j| \leq 2$ by the definition (11.7): if $\sigma_j = 1$, then A_j is one of the $L_{i_j} + R_{i_j}$ symbols in $B'_{2i_j-1} \cup B'_{2i_j}$; if $\sigma = 21$, then A_j consists of an L in B'_{2i_j-1} and an R in B'_{2i_j} or B'_{2i_j+2} chosen according to one of (e1)–(e3) in Lemma 11, and this too gives $L_{i_j} + R_{i_j}$ choices. (Note that (e1) and (e2) overlap in one possibility.) Hence, for such i_1, \dots, i_d the number of possible choices of A_1, \dots, A_d is

$$\prod_{j=1}^d \bar{f}_{\sigma_j}(X_{i_j}, \dots, X_{i_j+b'_j}) = \bar{f}_\sigma((X_i)_{i=i_1}^{i_1+b'_1}, \dots, (X_i)_{i=i_d}^{i_d+b'_d}). \quad (12.13)$$

If some gap $i_{j+1} - i_j \leq 2$, the number of possibilities may be smaller, but we may conclude that, recalling the definition (5.17),

$$|\text{occ}_\sigma(\tau_m^*) - \widehat{U}_m(\bar{f}_\sigma)| \leq \sum^* \bar{f}_\sigma((X_i)_{i=i_1}^{i_1+b'_1}, \dots, (X_i)_{i=i_d}^{i_d+b'_d}), \quad (12.14)$$

where \sum^* denotes the sum over $i_1, \dots, i_d \in [1, m - b'_d]$ such that either $i_1 = 1$, $i_d = m - b'_d$, or $i_j \leq i_{j+1} \leq i_j + 2$ for some j .

We now take $m = N(n)$, condition on $S_{N(n)} = n$ and use (10.7). It remains only to show that the sum in (12.14) (with $m = N(n)$) is $O_{L^*}(n^{d-1})$; this then survives the conditioning as in (12.9). To see this, consider first the terms with $i_1 = 1$ or $i_j \leq i_{j+1} \leq i_j + 2$ for some i . Since $m = N(n) \leq n$, we may extend the sum to all $i_1, \dots, i_d \in [1, n]$ satisfying one of these conditions. This is a sum of $O(n^{d-1})$ terms, and each term is $O_{L^*}(1)$ by (12.11), (12.7)–(12.8) and Hölder's inequality. Hence the sum of these terms is $O_{L^*}(n^{d-1})$ by Minkowski's inequality.

The remaining sum consists of terms with $i_d = m - b'_d = N(n) - b'_d$, and is thus

$$\leq \bar{f}_{\sigma_d}(X_{N(n)-b'_d}, \dots, X_{N(n)}) \sum_{i_1, \dots, i_{d-1}=1}^n \prod_{j=1}^{d-1} \bar{f}_{\sigma_j}(X_{i_j}, \dots, X_{i_j+b'_j}). \quad (12.15)$$

The first factor is $O_{L^*}(1)$ as shown in (12.7)–(12.8), and the sum is again a sum of $O(n^{d-1})$ terms that are $O_{L^*}(1)$, and thus this sum is $O_{L^*}(n^{d-1})$ by Minkowski's inequality. Hence, (12.15) is $O_{L^*}(n^{d-1})$ by Hölder's inequality, which completes the proof. \square

Proof of Theorem 4. Lemma 43 and Proposition 22 show that

$$\frac{\text{occ}_\sigma(\tau_n) - \mu\nu^{-d}d!^{-1}n^d}{n^{d-1/2}} \xrightarrow{d} \mathbf{N}(0, \gamma^2), \quad (12.16)$$

with convergence of all moments by Proposition 23; note that $\mathbb{E}|\bar{f}(X_1, \dots, X_D)|^p < \infty$ and $\mathbb{E}h(X_1)^p < \infty$ for every $p < \infty$ by (12.11), (12.5), and (12.8). Furthermore, (12.11) and Lemma 41 imply that

$$\mu = \mathbb{E}\bar{f}(X_1, \dots, X_D) = \prod_{j=1}^d \mathbb{E}\bar{f}_{\sigma_j}(X_1, \dots, X_{b_j}) = 4^d, \quad (12.17)$$

while $\nu = 4$ by Lemma 37. Hence, $\mu\nu^{-d} = 1$, and (3.1) follows from (12.16).

To see that $\gamma^2 > 0$ if some $|\sigma_j| \geq 3$, we use the criterion in Proposition 20. By (5.22) and (12.11),

$$\bar{f}_{\sigma_j}(x_1, \dots, x_{b_j}) = \bar{f}_{\sigma_j}(x_1, \dots, x_{b_j}) \prod_{i \neq j} \mathbb{E}\bar{f}_{\sigma_i}(X_1, \dots, X_{b_i}) = c\bar{f}_{\sigma_j}(x_1, \dots, x_{b_j}) \quad (12.18)$$

for some constant $c > 0$. Now suppose that $|\sigma_j| \geq 3$. Then, (12.18), (11.6) and (11.4)–(11.5) show that, with $x_i = (\ell'_i, r'_i)$, $\bar{f}_{\sigma_j}(x_1, \dots, x_{b_j})$ is a polynomial in $\{\ell'_i, r'_i\}$ of total degree

$$\delta_j := \sum_{i=1}^{b_j} (\ell_i - 1 + r_i - 1) + \mathbf{1}\{r_{b_j} = 1\} + \mathbf{1}\{\ell_1 = 1\}. \quad (12.19)$$

We see also that the polynomial has only one term with this degree, and that this term has a positive coefficient. Note further that

$$\delta_j \geq (\ell_1 - 1 + \mathbf{1}\{\ell_1 = 1\}) + (r_{b_j} - 1 + \mathbf{1}\{r_{b_j} = 1\}) \geq 2. \quad (12.20)$$

In particular, if we take $x_1 = \dots = x_{b_j} = (s, s)$, then $\bar{f}_{\sigma_j}(x_1, \dots, x_{b_j})$ is a polynomial in s of degree $\delta_j \geq 2$. Hence, if we fix any $n > 2b_j$, and consider the event (which has positive probability)

$$X_i = (L_i, R_i) = \begin{cases} (s, s), & b_j < i \leq 2b_j, \\ (1, 1), & \text{otherwise} \end{cases} \quad (12.21)$$

for an integer $s \geq 1$, we see that $S_n(\bar{f}_{\sigma,j})$ defined in (5.23) is a polynomial in s of degree $\delta_j \geq 2$. Furthermore, on the same event, $S_n(h)$ is a polynomial in s of degree 1, and thus, $S_n(\bar{f}_{\sigma,j} - \frac{\mu}{\nu}h)$ is a non-constant polynomial in s . Consequently, the condition in Proposition 20 cannot be satisfied for $\bar{f}_{\sigma,j} - \frac{\mu}{\nu}h$, and thus Proposition 21 shows that $\gamma^2 > 0$. \square

We compute the asymptotic variance γ^2 only in a simple special case.

Example 44. Suppose that σ is a tree permutation with $b(\sigma) = 1$; thus its code has only two blocks, of lengths $\ell_1 = \ell$ and $r_1 = r$. Then, the U -statistic $\widehat{U}_N(\bar{f}_\sigma)$ in (12.12) is simply a partial sum:

$$\widehat{U}_N(\bar{f}_\sigma) = \sum_{i=1}^N \bar{f}_\sigma(X_i) = S_N(\bar{f}_\sigma). \quad (12.22)$$

This is the special case $d = 1$ of an unconstrained U -statistic discussed in Remark 17, and (5.12) yields, since $\mu = \nu = 4$ by (12.17) and (10.11),

$$\begin{aligned} \gamma^2 &= \frac{1}{4} \operatorname{Var}[\bar{f}_\sigma(X) - h(X)] \\ &= \frac{1}{4} \left(\operatorname{Var}[\bar{f}_\sigma(X)] - 2 \operatorname{Cov}[\bar{f}_\sigma(X), h(X)] + \operatorname{Var}[h(X)] \right), \end{aligned} \quad (12.23)$$

where $X = (L, R)$ with independent $L, R \sim \operatorname{Ge}(1/2)$ as in (10.1)–(10.2). We recall that $\mathbb{E}L = \mathbb{E}R = 2$. A simple calculation, for example using (11.16), yields $\operatorname{Var}L = \operatorname{Var}R = 2$ and thus $\operatorname{Var}h(X) = \operatorname{Var}(L + R) = 4$.

We consider several cases.

Case 1: $\ell = r = 1$. This means $\Omega_\sigma = \operatorname{LR}$, and thus $\sigma = 21$. As we have seen earlier, this case is trivial and $\operatorname{occ}_\sigma(\tau_n)$ is deterministic. Indeed, we have $\bar{f}_\sigma = h$ and thus (12.23) yields $\gamma^2 = 0$.

Case 2: $\ell > 1, r = 1$. This means that σ is the permutation $23 \cdots (\ell + 1)1$.

By (11.6) and (11.4)–(11.5),

$$\bar{f}_\sigma(L, R) = \alpha_{L,1}(L)\alpha_{R,1}(R) = \binom{L+1}{\ell} \binom{R-1}{0} = \binom{L+1}{\ell}. \quad (12.24)$$

We have, using (11.21),

$$\mathbb{E} \left[L \binom{L+1}{\ell} \right] = (\ell - 1) \mathbb{E} \binom{L+1}{\ell} + (\ell + 1) \mathbb{E} \binom{L+1}{\ell+1} = 8\ell, \quad \ell \geq 2, \quad (12.25)$$

and thus (12.23) yields, using also $\mathbb{E}L^2 = 6$,

$$\gamma^2 = \frac{1}{4} \operatorname{Var} \left[\binom{L+1}{\ell} - (L + R) \right] = \frac{1}{4} \left(\operatorname{Var} \left[\binom{L+1}{\ell} - L \right] + 2 \right)$$

$$= \frac{1}{4} \left(\mathbb{E} \left[\left(\binom{L+1}{\ell} - L \right)^2 \right] - 2 \right) = \frac{1}{4} \mathbb{E} \left[\binom{L+1}{\ell}^2 \right] - 4\ell + 1. \quad (12.26)$$

This can easily be evaluated for any $\ell \geq 2$, although we do not know a closed formula.

Case 3: $\ell = 1, r > 1$. This means that σ is the permutation $(r+1)1 \cdots r \in \mathfrak{S}_{r+1}$. This case is the same as the preceding one, if we exchange $\ell \leftrightarrow r$ and $L \leftrightarrow R$.

Case 4: $\ell > 1, r > 1$. This means that $\sigma = 2 \cdots \ell(\ell+r)1(\ell+1) \cdots (\ell+r-1)$. By (11.6) and (11.4)–(11.5),

$$\bar{f}_\sigma(L, R) = \alpha_{L,1}(L)\alpha_{R,1}(R) = \binom{L}{\ell-1} \binom{R}{r-1}. \quad (12.27)$$

We have, using (11.16),

$$\mathbb{E} \left[L \binom{L}{\ell-1} \right] = (\ell-1) \mathbb{E} \binom{L}{\ell-1} + \ell \mathbb{E} \binom{L}{\ell} = 4\ell - 2, \quad \ell \geq 2, \quad (12.28)$$

and thus (12.23) yields

$$\begin{aligned} \gamma^2 &= \frac{1}{4} \mathbb{E} \left[\left(\binom{L}{\ell-1} \binom{R}{r-1} - L - R \right)^2 \right] \\ &= \frac{1}{4} \mathbb{E} \left[\binom{L}{\ell-1}^2 \right] \mathbb{E} \left[\binom{R}{r-1}^2 \right] - \frac{1}{2} \mathbb{E} \left[\binom{L}{\ell-1} \binom{R}{r-1} (L+R) \right] + \frac{1}{4} \mathbb{E} (L+R)^2 \\ &= \frac{1}{4} \mathbb{E} \left[\binom{L}{\ell-1}^2 \right] \mathbb{E} \left[\binom{R}{r-1}^2 \right] - 4(\ell+r) + 9. \end{aligned} \quad (12.29)$$

Again, this is easily evaluated for any $\ell, r \geq 2$.

$\ell \backslash r$	1	2	3	4	5
1	0	6	52	306	1664
2	6	2	28	174	944
3	52	28	154	800	4150
4	306	174	800	3946	20196
5	1664	944	4150	20196	103010

Table 1: Some numerical values of $\gamma^2 = \gamma^2(\ell, r)$ in Example 44.

Some numerical values for small ℓ and r are given in Table 1. These values are integers (but they do not seem to correspond to any integer sequence in [13]); we conjecture that $\gamma^2(\ell, r)$ is an integer for all $\ell, r \geq 1$, but we have no proof.

Note that $\gamma^2(1, 3) \neq \gamma^2(2, 2)$, which verifies our claim after Corollary 5 that γ_σ^2 can differ for different tree permutations σ , even if they have the same length.

Problem 45. In Example 44, is γ^2 an integer for every $\ell, r \geq 1$?

Problem 46. Is γ_σ^2 an integer for every tree permutation σ ? For every forest permutation σ ?

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