The complexity of the matroid homomorphism problem

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Abstract

We show that for every binary matroid N there is a graph $D(N)$ such that for the graphic matroid $M(G)$ of a graph G, there is a matroid homomorphism from $M(G)$ to N if and only if there is a graph homomorphism from G to $D(N)$. With this we prove a complexity dichotomy for the problem $\mathrm{Hom}_{\mathbb{M}}(\mathrm{N})$ of deciding if a binary matroid M admits a matroid homomorphism to N. The problem is polynomial time solvable if N has a loop or has no circuits of odd length, and is otherwise NP-complete. We also get dichotomies for the list, extension, and retraction versions of the problem.

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1 Introduction

Recall that a binary matroid M is a matroid that is representable by a matrix over GF(2). Its ground set is the set of columns of the matrix, and its circuits are those sets of columns

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that are minimally dependent- in particular the sum of the columns of a circuit over GF(2) is 0. We consider only binary matroids in this paper, so even if it is not mentioned, every matroid is binary. Moreover, for every matroid, we will assume that there are no parallel points, meaning that the columns of the representing matrix are distinct. For two sets A, B, we denote by $A \triangle B$ the symmetric difference $(A - B) \cup (B - A)$ of A and B. For a function $\phi : A \to B$ and a subset X of A, we define $\phi(X) := \triangle_{e \in X}\{\phi(e)\}$. Thus $e' \in \phi(X)$ if and only if $e' = \phi(e)$ for an odd number of edges $e \in X$.

Definition 1. For matroids M and N a *matroid homomorphism* $\phi : M \to N$ is a map $\phi : E(M) \to E(N)$ such that for every circuit C of M, $\phi(C)$ is a disjoint union of circuits of N. We consider the empty set to be the disjoint union of an empty set of circuits. We write $M \to N$ to mean that there exists a matroid homomorphism from M to N.

The main goal of the present paper is to determine the complexity of deciding if there is a matroid homomorphism between given matroids. We define the following decision problem. In all our problems, both of the matroids M and N are given as matrices over GF(2).

Problem. Matroid N-colouring or $\mathrm{Hom}_{\mathbb{M}}(N)$
  Instance: A matroid M.
  Decision: Does $M \to N$?

The problem $\mathrm{Hom}_{\mathbb{M}}(N)$ is in the complexity class NP. Recall that for a basis B of M, one gets a unique circuit $C_e \subseteq B \cup \{e\}$, called a *fundamental circuit*, for each point $e \notin B$, and the set of such circuits, the *fundamental circuits with respect to* B, is a basis of the cycle space of M. Thus to show that $\phi : E(M) \to E(N)$ is a matroid homomorphism, it is enough to check, for every fundamental circuit C with respect to a given basis B of M, that $\phi(C)$ is a disjoint union of circuits. This can be done in polynomial time in the size $|E(M)|$ of the instance, and so the problem $\mathrm{Hom}_{\mathbb{M}}(N)$ is in NP.

Recall that for a graph G, the *graphic matroid* $M(G)$ *of* G, also known as the *cycle matroid of* G, is the matroid with ground set $E(G)$ whose circuits are the cycles of G. A graphic matroid $M(G)$ is binary, represented by the incidence matrix of G. In the case that G has loops, these loops would all be parallel, each being represented by a column of zeros, and so only one exists in $M(G)$. We call a matroid *looped* or *loopless* depending on whether or not it has a loop, and when H is a loopless graph, will write $M_\ell(H)$ for the matroid we get from $M(H)$ by adding a loop.

The problem $\mathrm{Hom}_{\mathbb{M}}(M(H))$, for a graph H, often translates into an interesting edge colouring problem on G for an instance graphic matroid $M(G)$. For example, a matroid homomorphism $\phi : M(G) \to M(K_3)$ must assign colours from $E(K_3) = [3]$ to the edges of G such that for every cycle in G the number of edges of each colour has the same parity.

Though appearing as an edge colouring problem, the problem of finding a matroid homomorphism from $M(G)$ to $M(H)$ is still closely related to graph colouring and graph homomorphism problems. Indeed, it is a trivial exercise to show the following. Recall that a vertex map $\phi : V(G) \to V(H)$ is a *graph homomorphism* $\phi : G \to H$ if it takes edges to edges.

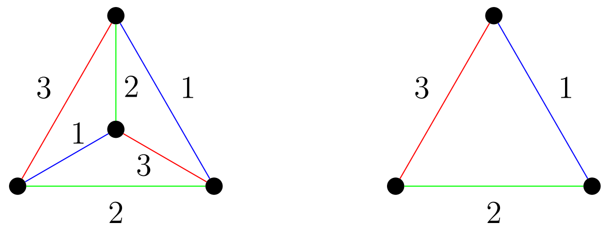

Figure 1: A matroid homomorphism of $M(K_4)$ to $M(K_3)$

Fact 2. *A graph homomorphism $\phi : G \to H$ induces a matroid homomorphism $\phi' : M(G) \to M(H)$ by $\phi'(uv) = \phi(u)\phi(v)$.*

On the other hand, not all matroid homomorphisms are induced by graph homomorphisms in this way. To see this one need only recall that the graphic matroids $M(P_3)$ of the path on three edges, and $M(3K_2)$ of the matching with three edges, are the same matroid. Thus, for example, the identity matroid homomorphism $M(P_3) \to M(3K_2)$ cannot be induced by a graph homomorphism $P_3 \to 3K_2$ in this way. In [7], Whitney showed for 3-connected graphs G and G' that $M(G) \cong M(G')$ implies $G \cong G'$, so perhaps there is a limit to such facetious examples, but there are 3-connected pairs (G, H) of graphs for which $G \to H$ but $M(G) \not\to M(H)$. One basic example was observed in [1] where they show that there is a matroid homomorphism, shown in Figure 1, from $M(K_4)$ to $M(K_3)$.

Starting to address the problem of how different graph homomorphism and matroid homomorphism can be, the authors of [1] went on to show the following. The purpose of the result (not a main result in their paper) was to show that the homorphism (partial) order of matroids, which we discuss in a little more detail in Section 2, has infinite ascending chains.

Proposition 3 ([1])**.** *For all $n \geqslant 3$ there is no matroid homomorphism from $M(K_{\binom{n}{2}+2})$ to $M(K_n)$.*

Our first result, in Section 2 uses a simple version of what we call the cycle gadget construction, to improve on Proposition 3. We get the following.

Proposition 4. *Let $m, n \geqslant 2$. There is a matroid homomorphism $M(K_m) \to M(K_n)$ if and only if $m = 4$ and $n = 3$, or $m \leqslant n$.*

From the fact that $M(K_4) \to M(K_3)$, and the obvious fact that matroid homomorphisms compose, we have for any 4-chromatic graph G that $M(G) \to M(K_3)$ but $G \not\to K_3$. Indeed, we will see in Section 3 that a graph G satisfies $M(G) \to M(K_3)$ and $G \not\to K_3$ if and only if it is 4-chromatic. To make such a statement for other graphs H in place of K_3 and so to succinctly quantify the difference between graph homomorphisms and matroid homomorphisms, we generalise the cycle gadget construction to build, in Section 3, what we call decision graphs.

A graph $D(N)$ is a *decision graph* for a matroid N if for any graph G we have the following:

$$M(G) \to N \iff G \to D(N).$$

In Section 3 we construct a decision graph $D(N)$ for any matroid N, and observe some of its properties. This is our main tool in determining the computational complexity of the problem $\mathrm{Hom}_{\mathbb{M}}(N)$.

Where $\mathrm{Hom}_{\mathbb{G}}(H)$ for a graph H is the graph H-colouring problem which asks for an instance graph G if there is a graph homomorphism $G \to H$, Hell and Nešetřil proved the following H-colouring dichotomy in [5].

Theorem 5 ([5]). *The problem* $\mathrm{Hom}_{\mathbb{G}}(H)$*, for a graph* H*, is in the complexity class* P *if* H *contains a loop or is bipartite, and is otherwise* NP*-complete.*

Using our decision graphs in Section 4, we are able to quickly reduce the problem of the complexity of $\mathrm{Hom}_{\mathbb{M}}(N)$ to problems covered by the H-colouring dichotomy, and so get the following extension to matroid homomorphisms. A matroid is *bipartite* if every circuit has even cardinality.

Theorem 6. *The problem* $\mathrm{Hom}_{\mathbb{M}}(N)$*, for a matroid* N*, is in* P *if* N *contains a loop or is bipartite, and is otherwise* NP*-complete. Moreover, if* $\mathrm{Hom}_{\mathbb{M}}(N)$ *is* NP*-complete, it is* NP*-complete for graphic instances.*

Having settled the computational complexity of $\mathrm{Hom}_{\mathbb{M}}(N)$ we then look at the following variations, defined for any matroid N, of $\mathrm{Hom}_{\mathbb{M}}(N)$. The graph homomorphism analogues of these problems are well known.

Problem. $\mathrm{ListHom}_{\mathbb{M}}(N)$ or list N-colouring
Instance: A matroid M with list $L(e) \subseteq E(N)$ for each $e \in E(M)$.
Decision: Is there a matroid homomorphism $\phi : M \to N$ with $\phi(e) \in L(e)$ for all e?

An N*-precolouring* of a matroid M is a map $p : S \to E(M)$ for some subset S of $E(N)$. A matroid homomorphism $\phi : M \to N$ *extends* p if it restricts on S to p. The following problem can be viewed as a restriction of $\mathrm{ListHom}_{\mathbb{M}}(N)$ in which the only lists are $E(M)$, for vertices not in S, and lists of cardinality 1, for vertices in S.

Problem. $\mathrm{Ext}_{\mathbb{M}}(N)$ or N-precolour extension
Instance: A matroid M and an N-precolouring p of M.
Decision: Is there a matroid homomorphism $\phi : M \to N$ extending p?

A *submatroid* of a matroid M is any matroid represented by a matrix that consists of some subset of the columns of the matrix representing M. A *retraction* of M to a submatroid N is a matroid homomorphism $\phi : M \to N$ that restricts to the identity on $E(N) \subseteq E(M)$. The following problem can be seen as a restriction of $\mathrm{Ext}_{\mathbb{M}}(N)$ in which the precolouring p is an isomorphism of a submatroid of M to N.

Problem. $\mathrm{Ret}_{\mathbb{M}}(N)$ or N-retraction
Instance: A matroid M containing N as a submatroid.
Decision: Is there a retraction $\phi : M \to N$?

Observing, finally, that an instance M of $\mathrm{Hom}_{\mathbb{M}}(N)$ becomes an instance of $\mathrm{Ret}_{\mathbb{M}}(N)$ by replacing it with the 1-sum (a.k.a. direct sum) $M \oplus_1 N$, we get the following polynomial reductions

$$\mathrm{Hom}_{\mathbb{M}}(N) \leqslant_{\mathrm{P}} \mathrm{Ret}_{\mathbb{M}}(N) \leqslant_{\mathrm{P}} \mathrm{Ext}_{\mathbb{M}}(N) \leqslant_{\mathrm{P}} \mathrm{ListHom}_{\mathbb{M}}(N) \tag{1}$$

where the inequalities mean that there are polynomial time reductions which show that if the problem on the right has polynomial complexity, then so does the problem on the left.

These reductions are standard and are all well known for the graph homomorphism analogues. For graphs, moreover it is known that $\mathrm{Ret}_{\mathbb{G}}(H) =_{\mathrm{P}} \mathrm{Ext}_{\mathbb{G}}(H)$, though the standard reduction for this does not to work for matroid homomorphisms.

Before we state our results with respect to these problems, we recall relevant results about their graph homomorphism analogues. By the analogue of (1) for graphs, all of these problems are NP-complete for loopless non-bipartite graphs.

In [4], Feder, Hell, and Huang showed that for loopless graphs, $\mathrm{ListHom}_{\mathbb{G}}(H)$ is in P if and only if H is bipartite and its complement is the intersection graph of arcs of a circle. In [3], Feder and Hell showed that $\mathrm{ListHom}_{\mathbb{G}}(H)$ is in P for a reflexive graph H (a graph in which every vertex has a loop), if and only if H is an interval graph. In [4], Feder, Hell, and Huang showed that the problem is in P for a general graph H, (in which some subset of the vertices has loops,) if and only if H is a so-called 'bi-arc' graph. By the CSP-dichotomy of Bulatov [2] and Zhuk [8] we know that $\mathrm{Ret}_{\mathbb{G}}(H) = \mathrm{Ext}_{\mathbb{G}}(H)$ is in P if and only if H admits a so-called 'WNU-polymorphism', but there is not a good, full, structural description of the graphs that have such polymorphisms– though several partial results are known.

The picture for the corresponding matroid homomorphism problems is quite different; we point out two big underlying differences.

The first is that for a graph H, the complexity of the problems $\mathrm{Ret}_{\mathbb{G}}(H)$, $\mathrm{Ext}_{\mathbb{G}}(H)$, and $\mathrm{ListHom}_{\mathbb{G}}(H)$, depends significantly on which vertices of H have loops. But $M(H)$ can have at most one loop; it does not matter for the matroid versions of these problems which vertices of H have loops. It only matters if there is one loop or none.

The second big difference is in the list version of the homomorphism problem. For graphs, one can use the lists to restrict the problem to any induced subgraph, and so, show that $\mathrm{ListHom}_{\mathbb{G}}(H)$ is NP-complete by showing that $\mathrm{ListHom}_{\mathbb{G}}(H')$ is NP-complete for any induced subgraph H'. The same works for $\mathrm{ListHom}_{\mathbb{M}}(M)$, but the induced structure is the matroid induced by the points. So for graphic matroids, one gets that $\mathrm{ListHom}_{\mathbb{M}}(M(H))$ is NP-complete if $\mathrm{ListHom}_{\mathbb{M}}(M(H'))$ is NP-complete for any subgraph H', not necessarily induced. More generally if N' is a submatroid of N, then

$$\mathrm{ListHom}_{\mathbb{M}}(N') \leqslant_{\mathrm{P}} \mathrm{ListHom}_{\mathbb{M}}(N). \tag{2}$$

This is a useful convenience in showing that $\mathrm{ListHom}_{\mathbb{M}}(N)$ is NP-complete.

In Section 5 we prove the following using another variation of our cycle gadget construction.

Theorem 7. *The problem* $\mathrm{Ext}_{\mathbb{M}}(N)$ *is in* P *if* N *is the looped projective geometry* $\mathrm{PG}_\ell(n,2)$ *for* $n \geqslant -1$*, or the (loopless) affine geometry* $\mathrm{AG}(n,2)$ *for* $n \geqslant 0$*. Otherwise* $\mathrm{Ext}_{\mathbb{M}}(N)$ *is* NP*-complete, and remains* NP*-complete when restricting only to graphic instances.*

The projective and affine geometries as matroids are described in Section 5, but we note here, for those uncomfortable with a projective geometry with negative parameters, the looped matroids $\mathrm{PG}_\ell(n,2)$ for $n = -1, 0$, and 1 are the looped graphic matroids $M_\ell(K_i)$ for $i = 1,2,3$ respectively, and the matroids $\mathrm{AG}(n,2)$ for $n = 0,1$ and 2 are the graphic matroids $M(K_2)$, $M(2K_2)$ and $M(C_4)$ respectively. These are the only graphic matroids covered by the theorem.

From this, it is a simple task to show that for the list version, any tractable cases with at least 3 points becomes NP-complete for graphic instances. This leaves the following, which we prove in Section 6.

Theorem 8. *The problem* $\mathrm{ListHom}_{\mathbb{M}}(N)$ *is in* P *if* N *is the looped graphic matroid* $M_\ell(K_i)$ *for* $i = 1,2,3$*, or is the graphic matroid* $M(K_2)$ *or* $M(2K_2)$*. Otherwise* $\mathrm{ListHom}_{\mathbb{M}}(N)$ *is* NP*-complete, and remains* NP*-complete when restricting only to graphic instances.*

We finish off by showing the following in Section 7.

Theorem 9. *The problem* $\mathrm{Ret}_{\mathbb{M}}(N)$ *is in* P *if* N *is the looped projective geometry* $\mathrm{PG}_\ell(n,2)$ *for* $n \geqslant -1$*, or the affine geometry* $\mathrm{AG}(n,2)$ *for* $n \geqslant 0$*. Otherwise* $\mathrm{Ret}_{\mathbb{M}}(N)$ *is* NP*-complete.*

Contrary to what happens for $\mathrm{Ext}_{\mathbb{M}}(N)$, the hardness results for $\mathrm{Ret}_{\mathbb{M}}(N)$ do not always persist when we restrict to graphic instances. Indeed, we show in Propositions 29 and 30 of Section 7 that $\mathrm{Ret}_{\mathbb{M}}(N)$ drops to P for graphic instances when N is $M_\ell(2K_2)$ or $M(3K_2)$. Because of this difference, our proof of Theorem 9 requires us to make matroid versions of our gadgets– the graph versions cannot be made to work.

2 Matroid partial order and infinite ascending chains

As the identity map id : E(M) → E(M) is clearly a matroid homomorphism, and matroid homomorphisms $L \to M \to N$ clearly compose to give a matroid homomorphism $L \to N$, the set of matroids is pre-ordered by the relation defined by setting $M \leqslant N$ if $M \to N$. Calling matroids M and N *homomorphically equivalent* if $M \to N$ and $N \to M$, this pre-order induces a partial ordering on classes of homomorphically equivalent matroids. It can be shown that each equivalence class has a unique (up to isomorphism) matroid of minimum size, and this matroid is called the *(matroid) core* of the matroids in the equivalence class.

Properties of the analogously defined partial order of graphs have long been investigated, and it is a trivial exercise to show that there are infinite descending chains (the odd cycles, for example) and infinite ascending chains (cliques, for example) in this partial order.

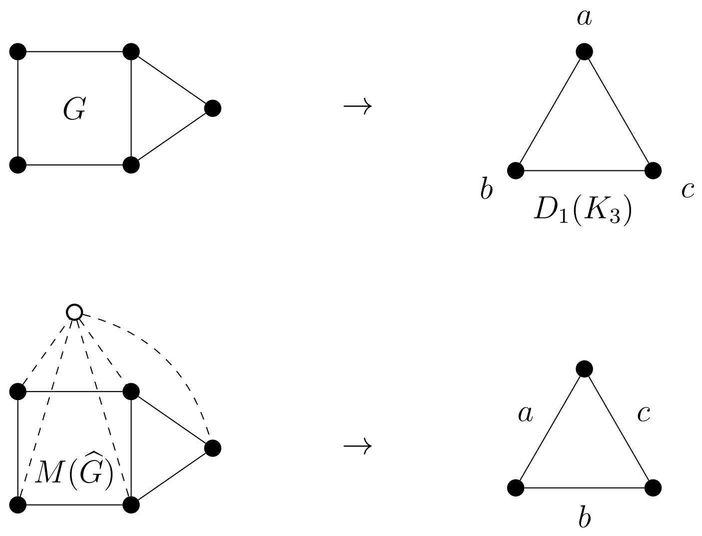

Figure 2: The cycle gadget construction $\widehat{G}$ and $D_1(K_3)$

In [1] it was shown that the graphic matroids of odd cycles are an infinite descending chain in the homomorphism partial order of matroids. The authors then showed that there exist infinite ascending chains in the partial order by showing, for all $n \geqslant 3$, that $M(K_m) \not\to M(K_n)$ if $m \geqslant \binom{n}{2} + 2$.

In this section we improve this significantly with the simplest version of our cycle gadget construction.

Definition 10. For a loopless matroid N let $D_1(N)$ be the graph with vertex set $E(N)$ in which vertices e and f are adjacent if they occur together in a 3-circuit of N. For a graph G, $\widehat{G}$ is the graph we get by adding a new vertex v_* adjacent to all vertices.

The idea of the gadget is simple. By adding a dominating vertex v_* to a graph G, we effectively convert vertices of G into edges containing v_* in $\widehat{G}$. Edges $\{u, v\}$ of G become triangles $\{u, v, v^*\}$ in $\widehat{G}$, which being cycles, are used to relate the graph homomorphisms of G to the matroid homomorphisms of $\widehat{G}$.

Lemma 11. *For any loopless matroid N, and any graph G the following holds:*

$$G \to D_1(N) \iff M(\widehat{G}) \to N.$$

Proof. Assume on the one hand that $\phi_{\mathbb{M}} : M(\widehat{G}) \to N$ is a matroid homomorphism. We claim that the map $\phi_{\mathbb{G}} : V(G) \to V(D_1(N))$ defined by $\phi_{\mathbb{G}}(v) := \phi_{\mathbb{M}}(vv_*)$ is a graph homomorphism. Indeed, for any edge uv of G, $\phi_{\mathbb{M}}(\{uv, uv_*, vv_*\})$ is a 3-circuit of N, and so maps uv_* and vv_* to distinct points that occur in a 3-circuit in N. Thus $\phi_{\mathbb{G}}(u) = \phi_{\mathbb{M}}(uv_*)$ is adjacent to $\phi_{\mathbb{G}}(v) = \phi_{\mathbb{M}}(vv_*)$ in $D_1(N)$.

On the other hand, let $\phi_{\mathbb{G}} : G \to D_1(N)$ be a graph homomorphism. Define a map $\phi_{\mathbb{M}} : E(M(\widehat{G})) \to E(N)$ on points containing v_* by $\phi_{\mathbb{M}}(vv_*) = \phi_{\mathbb{G}}(v)$, and on other points

uv corresponding to edges of G by setting $\phi_{\mathbb{M}}(uv)$ to be the third point in the triangle containing $\phi_{\mathbb{G}}(u)$ and $\phi_{\mathbb{G}}(v)$. Such a point exists because $\phi_{\mathbb{G}}(u)$ and $\phi_{\mathbb{G}}(v)$ are adjacent in $D_1(N)$. By definition this map takes circuits of $M(\widehat{G})$ of the form $\{uv, uv_*, vv_*\}$ to circuits of N. Observing that the maximal star centered at v_* in $\widehat{G}$ is a basis of $M(\widehat{G})$, one sees that these circuits make up the set of fundamental circuits with respect to this basis, and so $\phi_{\mathbb{M}}$ is a matroid homomorphism. □

Observe for a graphic matroid $M(H)$ that a clique in $D_1(M(H))$ corresponds to a set of pairwise intersecting edges of H. This can consist only of edges in a triangle, or a star. The largest clique in $D_1(M(K_3))$ is K_3, but the largest clique in $D_1(M(K_n))$ for $n \geqslant 4$ is K_{n-1}. As $K_m \cong \widehat{K}_{m-1}$ we thus get Proposition 4.

3 The decision graph for a matroid

A relation between matroid homomorphisms and the well studied concept of graph homomorphisms is an invaluable tool in understanding matroid homomorphisms. In this section we provide this tool.

Definition 12. A *decision graph* for a matroid N is a graph D such that for all graphs G we have

$$M(G) \to N \iff G \to D.$$

Clearly if there is a decision graph D there is one that is a core with respect to graph homomorphisms, and this must be unique, so we call it *the decision graph* $D(N)$ of N. To construct decision graphs, we start with a simple extension of our cycle gadget construction.

Definition 13. For a matroid N let $D'_k(N)$ be the graph on the k-element multi-sets of $E(N)$ and let two such sets S and T be adjacent if there is some z such that $S \triangle T \triangle \{z\}$ is a disjoint union of circuits of N.

Let $D_k(N)$ be the graph on subsets of $E(N)$ of size up to k having the same parity as k and, again, let two such sets S and T be adjacent if there is some z such that $S \triangle T \triangle \{z\}$ is a disjoint union of circuits of N.

For a graph G let $\widehat{G}_k$ be the graph we get from G by adding a new vertex v_*, and for every vertex v of G a path of k new edges from v_* to v.

The reason we introduce $D'_k(N)$ instead of just $D_k(N)$ is because it makes it clearer that with just obvious changes to the proof of Lemma 11 one can show that for any k, G, and N the following holds:

$$M(\widehat{G}_k) \to N \iff G \to D'_k(N). \tag{3}$$

Observe the following.

Fact 14. *For any matroid N and any $k \geqslant 1$,* *(i) there is a retraction $D_k(N) \to D'_k(N)$ (it follows that $D_k(N)$ and $D'_k(N)$ are graph-homomorphically equivalent,) and*

(ii) there is a graph homomorphism $D_k(N) \to D_{k+1}(N)$.

Proof. We write D_k and D'_k for $D_k(N)$ and $D'_k(N)$. For fixed $e \in E(N)$ the map $\iota_k : V(D_k) \to V(D'_k)$ that adds $k - |S|$ copies of e to the set S, is a graph homomorphism, as $S \triangle T \triangle \{z\}$ is a disjoint union of circuits of N if and only if it is when we add an even number of copies of e. Moreover the map $r_k : D'_k \to D_k$ that reverses this, reducing a multi-set $S \in V(D'_k)$ to the underlying set is also a graph homomorphism, showing that D'_k retracts to D_k. This proves item (i). To see that $D_k \to D_{k+1}$ one can use the initial part of the above argument with the map $\iota : V(D_k) \to V(D_{k+1})$ defined $\iota(S) = S \triangle \{e\}$. □

With (3) and Fact 14 (i) we immediately get the following.

Lemma 15. *For any k, G and N the following holds:*

$$M(\widehat{G}_k) \to N \iff G \to D_k(N)$$

By Fact 14(ii) we have for any matroid N a chain of graph homomorphisms:

$$D_1(N) \to D_2(N) \to \dots.$$

As $|E(N)|$ is finite, there is some $m \leqslant |E(N)|$ such that $D_d(N) \cong D_m(N)$ for all $d \geqslant m$.

Definition 16. Let $D(N)$ be the graph core of $D_m(N)$ where $m \leqslant |E(N)|$ is the minimum m such that $D_d(N) \cong D_m(N)$ for all $d \geqslant m$.

Example 17. For $N = M(C_5)$ one gets that $D_1(N)$ has no edges, $D_2(N)$ is the Petersen graph, (plus an isolated vertex $\emptyset$), $D_3(N)$ adds 4 more vertices, and $D_4(N)$ adds one more. For higher values of k we have $D_k(N) \cong D_4(N)$. One can check that $D_4(N)$, shown in Figure 3, is a graph core. Indeed, one might recognise it as the Clebsch graph. It admits a matroid homomorphism to C_5 mapping an edge to the point missing from the symmetric difference of its endpoints.

Theorem 18. *For any non-bipartite matroid N, and connected graph G, we have $M(G) \to N$ if and only if $G \to D(N)$. So $D = D(N)$ is the decision graph for N.*

Proof. On the one hand if $G \to D \to D_m(N)$, then $M(\widehat{G}_m) \to N$ by Lemma 15, implying $M(G) \to N$ by Fact 2.

On the other hand, let $M(G) \to N$. As N is non-bipartite, D has at least one edge, and so we have $G \to D$ if G is bipartite. We may therefore assume that G contains an odd cycle. Thus there is some d such that all vertices of G have a walk of length d to some fixed vertex c. This defines a map $\widehat{G}_d \to G$ which induces $M(\widehat{G}_d) \to M(G)$ by Fact 2. Composing this with $M(G) \to N$ gives $M(\widehat{G}_d) \to N$, which by the Lemma 15 implies $G \to D$, as needed. □

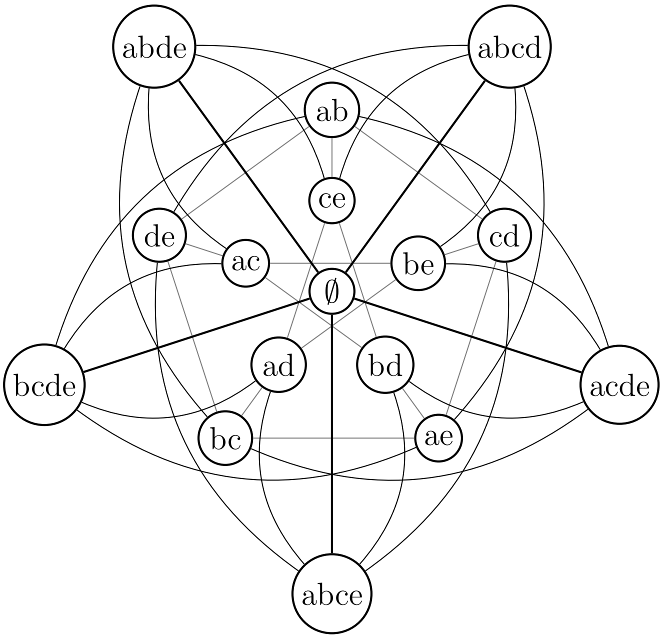

Figure 3: The decision graph $D(M(C_5))$

We finish this section with a couple more observations and remarks about the decision graphs.

Fact 19. *Let H be a graph, and $D = D(M(H))$ be the decision graph of the matroid $M(H)$. We have $M(D) \to M(H) \to M(D)$, so the matroids $M(H)$ and $M(D)$ are matroid-homomorphically equivalent. Moreover, $H \to D$; but these need not be graph-homomorphically equivalent.*

Proof. One gets $M(D) \to M(H)$ by taking $G = D$ in Definition 12, and $H \to D$ by taking $G = H$. This latter implies $M(H) \to M(D)$ by Fact 2. □

As Proposition 4 did in the previous section, Theorem 18 also allows one to translate properties of the homomorphism order of graphs to the homomorphism order of matroids.

Example 20. There are several well known constructions of graphs S_k, for odd $k \geqslant 3$, having odd girth and chromatic number k. For example, one could use a generalised Mycielski construction, or Erdős's sparse incomparability theorem. Such graphs are often given as an example of an infinite antichain in the graph homomorphism order, and an infinite family of their graphic matroids do the same in the matroid homomorphism order. Indeed, to see that $M(S_k)$ and $M(S_\ell)$ are incomparable for $k \geqslant 3$ and $\ell > \chi(D(M(K_k)))$, we observe that $M(S_k) \not\to M(S_\ell)$ as S_k has an odd circuit of cardinality k while the shortest odd circuit in S_ℓ has cardinality ℓ. On the other hand, we have $S_k \to K_k$ and so $M(S_k) \to M(K_k)$, while by Theorem 18, the fact that S_ℓ has chromatic number $\ell > \chi(D(M(K_k)))$ means that $S_\ell \not\to D(M(K_k))$, and so $M(S_\ell) \not\to M(K_k)$. Thus $M(S_\ell) \not\to M(S_k)$.

From the proof of Theorem 18 we see that the integer m for which $D(M(H))$ is the core of $D_m(M(H))$ can further be bounded by the smallest integer d such that $\widehat{H}_d \to H$. This is generally smaller than $|E(M(H))|$. We defined D as the core of D_m, but D_m itself seems often to be a core.

Question 21. Is it true for a matroid N that $D_m(N)$ is a graph core?

4 The Homomorphism Problem for Matroids

In this section we determine the complexity of the problem $\mathrm{Hom}_{\mathbb{M}}(N)$ for a matroid N. As we observed in the introduction, it is in the complexity class NP. Recall that a matroid is bipartite if all of its circuits have even length.

Theorem 6. For a matroid N the problem $\mathrm{Hom}_{\mathbb{M}}(N)$ is polynomial time solvable if N contains a loop or is bipartite, and otherwise is NP-complete. Moreover, if $\mathrm{Hom}_{\mathbb{M}}(N)$ is NP-complete, it is NP-complete for graphic instances.

Proof. If N contains a loop then for an instance M we get a matroid homomorphism by mapping all points to the loop. If N is bipartite, then it is matroid-homomorphically equivalent to its core, which is easily seen to be $M(K_2)$, and so an instance M admits a homomorphism to it if and only if every circuit is even. One can determine this by checking it on a basis of the cycle space; this can be done in polynomial time.

We may therefore assume that N has an odd circuit of cardinality $2k+1 \geqslant 3$, and no loop. It is easy to see then that the graph $D_k(N)$ contains a $(2k+1)$-cycle, and no loop as N is loopless, and then so does $D(N)$ using Fact 14. Thus $\mathrm{Hom}_{\mathbb{G}}(D(N))$ is NP-complete by Theorem 5, and as $D(N)$ is the decision graph for N we get that $\mathrm{Hom}_{\mathbb{M}}(N)$ is also NP-complete. In fact, this shows that it is NP-complete for graphic instances. □

5 The Precolouring Extension Problem

In this section we prove Theorem 7, which says that $\mathrm{Ext}_{\mathbb{M}}(N)$ is in P, for a matroid N, if N is $\mathrm{PG}_\ell(r,2)$ or $\mathrm{AG}(r,2)$ for some r, and is otherwise NP-complete.

We will need one more version of our cycle gadget construction.

Definition 22. Given a matroid N, let P be a path, with endpoints u and v, and with an N-precolouring p of P. Let C be the cycle we get from P by adding the new vertex v_* and the edges uv_* and vv_*. Let $D_P(N)$ be the graph with vertex set $E(N)$ such that e and f are adjacent if there is a matroid homomorphism $\phi : M(C) \to N$, extending p, with $\phi(uv_*) = e$ and $\phi(vv_*) = f$.

Given a graph G, let $\widehat{G}_P$ be the graph we get from $\widehat{G}_1$ by subdividing each edge uv of $E(G)$ so that it is the same length as P and has edges precoloured with the same colours as P.

The proof of the following lemma requires only obvious changes to the proof of Lemma 11.

Lemma 23. *For any matroid N, graph G, and precoloured path P*

$$M(\widehat{G}_P) \to N \iff G \to D_P(N).$$

The following is immediate from Lemma 23 by Theorem 5.

Corollary 24. *If $D_P(N)$, for some precoloured path P, contains an odd cycle and no loop, then $\mathrm{Ext}_{\mathbb{M}}(N)$ is NP-complete.*

Before we prove Theorem 7, we recall the definitions of projective and affine geometries as matroids, and observe some of their properties. Recall that the *binary projective geometry* $\mathrm{PG}(n,2)$ of rank $n+1$ is represented by the matrix whose $m = 2^{n+1} - 1$ columns are the non-zero vectors $(x_1, \ldots, x_{n+1})$ in $\mathbb{Z}_2^{n+1}$. Adding the zero vector, the matroid represented by the matrix whose columns consist of all vectors in $\mathbb{Z}_2^{n+1}$ is *looped projective geometry* $\mathrm{PG}_\ell(n,2)$. Removing the columns corresponding to points in the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$ from $\mathrm{PG}(n,2)$ leaves the matrix of 2^n columns of odd support in $\mathbb{Z}_2^{n+1}$. As we get this from $\mathrm{PG}(n,2)$ by removing a hyperplane, it is the *affine geometry* $\mathrm{AG}(n,2)$. Viewing the points of the matroid $\mathrm{PG}(n,2)$ as geometric points, and the rank-2 flats (which are exactly the 3-circuits) as lines, it is clear that this matroids satisfies the defining property of projective geometries: every pair of points lies in a unique line. The given representation of $\mathrm{AG}(n,2)$ makes it just as clear that every set three distinct points lies in a unique 4-circuit. We observe now that this property of a matroid actually characterises finite affine geometries.

Lemma 25. *Let N be a matroid of rank r.*

(i) The matroid N is the looped projective geometry $\mathrm{PG}_\ell(r-1,2)$ if and only if it is looped and every pair of non-loop points are in a common 3-circuit.

(ii) The matroid N is the (loopless) affine geometry $\mathrm{AG}(r-1,2)$ if and only if it is loopless and every set of three points are in a common 4-circuit.

Proof. We prove part (ii); the proof of (i) is similar, and is essentially a fact of finite geometry.

We observed above that $\mathrm{AG}(r-1,2)$ has the required property. On the other hand, let N be a loopless matroid of rank r such that every set of three points is in a 4-circuit; we show it has the representation of $\mathrm{AG}(r-1,2)$ given just above the statement of the lemma. As N is binary it has a matrix representation over $\mathrm{GF}(2)$, and we can assume that the r points of a basis B can be represented by the columns of support 1. As any three of these are in a 4-circuit, the fourth point in the circuit is their sum, and so every column of support 3 must be in N. Continuing in this way, any column of support $2k+1 \leqslant r$ must occur as the fourth point in the 4-circuit containing some point of support $2k-1$ and two points in B. □

With this, the following two propositions serve as the proof of Theorem 7.

Proposition 26. *The problem $\mathrm{Ext}_{\mathbb{M}}(N)$ is* NP*-complete if N is any of the following.*

(i) Loopless and non-bipartite.

(ii) Loopless and contains 3 *points not all together in a* 4*-circuit.*

(iii) Looped and contains 2 *non-loop points not both together in a* 3*-circuit.*

Proof. Part (i) is immediate from Theorem 6 and (1).

For part (ii), assume that N contains three points labelled $1, 2$ and 3 that do not occur together in a 4-circuit. Let P be a 4-path with one precoloured edge of each of the colours $1, 2$ and 3 (and one uncoloured edge).

We claim that $D_P(N)$ is loopless and contains a triangle, giving the result by Corollary 24. Indeed, the cycle C of Definition 22 is a 6-cycle with three coloured edges, each getting a distinct colour from the set $\{1, 2, 3\}$. As the remaining three edges of C can be coloured with the colours $1, 2$ and 3 in any order, $D_P(N)$ contains a triangle on the vertices $1, 2$ and 3. We need just verify that it is loopless. If it has a loop, then there is some matroid homomorphism $\phi : M(C) \to N$ such that $\phi(uv_*) = \phi(vv_*)$, where, recall, uv_* and vv_* are edges of C that were not pre-coloured. Thus $C - \{uv_*, vv_*\}$, which contains the edges precoloured $1, 2$ and 3 maps to a 4-circuit, which contradicts our assumption.

For part (iii), one need only make small obvious changes to the proof of part (ii). □

Proposition 27. *The problem* $\mathrm{Ext}_{\mathbb{M}}(N)$ *is in* P *if* N *satisfies either of the following.*

(i) N *is looped and every set of two non-loop points is in a* 3*-circuit.*

(ii) N *is loopless and bipartite and every set of three points is in a* 4*-circuit.*

Proof. We prove part (ii), the proof of part (i) is similar but simpler.

Let N be bipartite and be such that every set of three points is in a 4-circuit. Let the matroid M with an N-precolouring p be an instance of $\mathrm{Ext}_{\mathbb{M}}(N)$ and let $S \subseteq E(M)$ be the support of p. If M is non-bipartite, then the instance is a NO instance because N is bipartite. Thus we may assume M is bipartite. If S contains any circuits C such that $p(C)$ is not a disjoint union of circuits of N, then the instance is a NO instance; call such circuits C bad. Taking a basis of S and checking fundamental circuits with respect to this basis, we can check S for bad circuits in polynomial time. If any exist, we are done, so assume there are no bad circuits. We may assume that p is defined on a spanning submatroid of N by arbitrarily colouring points whose inclusion in the support of p creates no new circuit. Let B be some basis of N contained in the support of p. From here, we extend p to a matroid homomorphism $\phi : M \to N$ as follows. While one exists, choose a point e on which ϕ is not defined. It creates a new fundamental circuit C_e with respect to B. As M is bipartite $C_e \setminus \{e\}$ contains an odd number of points. As N is a binary affine geometry by Lemma 25, these points are represented by columns of odd support, so the sum of an odd number of them over GF(2) is a column of odd support, which is in N. Setting $\phi(e)$ to be this colour ensures that $\phi(C_e)$ is a disjoint union of circuits of N. In this way, we can extend p to a homomorphism $\phi : M \to N$. □

6 The List Homomorphism Problem

We observed before that $\mathrm{Ext}_{\mathbb{M}}(N)$ is a restriction of $\mathrm{ListHom}_{\mathbb{M}}(N)$ to lists of size 1 and $|E(N)|$. From this observation we get that the problems are equivalent when $|E(N)| \leqslant 2$, and so from Theorem 7 we get that $\mathrm{ListHom}_{\mathbb{M}}(N)$ is in P when N is $M_\ell(K_1)$, $M_\ell(K_2)$, $M(K_2)$ or $M(2K_2)$. Moreover we also get that $\mathrm{ListHom}_{\mathbb{M}}(N)$ is NP-complete if N is loopless and non-bipartite. This leaves the cases when $|E(N)|$ is at least 3 and is either looped or bipartite, and then the result follows from the observations (1) and (2) that

$$\mathrm{Ext}_{\mathbb{M}}(N') \leqslant_{\mathrm{P}} \mathrm{ListHom}_{\mathbb{M}}(N') \leqslant_{\mathrm{P}} \mathrm{ListHom}_{\mathbb{M}}(N).$$

In the former case the submatroid N' on points $0, 1$ and 2, where 0 is the loop, is isomorphic to $M_\ell(2K_2)$, so $\mathrm{Ext}_{\mathbb{M}}(N')$ is NP-complete. In the latter case the submatroid N' on the non-loops $1, 2$ and 3 is isomorphic to $M(3K_2)$ so again $\mathrm{Ext}_{\mathbb{M}}(N')$ is NP-complete.

Thus without any work we get Theorem 8.

7 The Retraction Problem

The tractability part of Theorem 9 is immediate from Theorem 7 and (1). On the other hand, we have from Theorem 6, and (1), that $\mathrm{Ret}_{\mathbb{M}}(N)$ is NP-complete if the matroid N is loopless and non-bipartite. Thus to complete the proof of Theorem 9 it is enough to prove the following.

Proposition 28. *The problem* $\mathrm{Ret}_{\mathbb{M}}(N)$ *is* NP*-complete if the matroid* N *is either of the following.*

(i) Bipartite but not $\mathrm{AG}(n, 2)$ *for some* $n \geqslant 0$.

(ii) Looped but not $\mathrm{PG}_\ell(n, 2)$ *for some* $n \geqslant -1$.

Proof. We prove part (i) first. Any such matroid N has a triple $\{a, b, c\} \subseteq E(N)$ of points not in a 4-circuit.

Let C be a 6-circuit with edges a, b, and c precoloured with the colours a, b, and c, and let x and y be two of its uncoloured edges. Let H be the graph with vertex set on $E(N)$ in which vertices e and f are adjacent if there is a matroid homomorphism $\phi_{\mathbb{M}} : M(C) \to N$ extending the precolouring, and taking x and y to e and f respectively. Clearly H contains the triangle on the points a, b and c, and as a, b and c do not occur in N in a 4-circuit, H has no loops. Thus $\mathrm{Hom}_{\mathbb{G}}(H)$ is NP-complete. We show that $\mathrm{Hom}_{\mathbb{G}}(H)$ can be encoded in $\mathrm{Ret}_{\mathbb{M}}(N)$ and so get that $\mathrm{Ret}_{\mathbb{M}}(N)$ is NP-complete.

Let G be an instance of $\mathrm{Hom}_{\mathbb{G}}(H)$, and let N' be the matroid with the ground set $V(G)$ and having rank $|V(G)|$ (meaning it has no circuits). Starting from $N \oplus_1 N'$, with a basis B containing $\{a, b, c\} \cup V(G)$, construct an instance M of $\mathrm{Ret}_{\mathbb{M}}(N)$ by adding the new point e_{uv}, for every edge uv of G, whose corresponding vector in the representation over $GF(2)$ is the sum of those of u, v, a, b and c. Doing so adds a new fundamental 6-circuit $C_{uv} = \{a, b, c, u, v, e_{uv}\}$ with respect to B which enforces that, under retraction to N, the points u and v must map to points that are adjacent as vertices of H. It is clear that a retraction $r : M \to N$ therefore defines a graph homomorphism $\phi_{\mathbb{G}} : G \to H$ by setting $\phi_{\mathbb{G}}(v) = r(v)$ for all $v \in V(G)$.

On the other hand, let $\phi_{\mathbb{G}} : G \to H$ be a graph homomorphism. Partially define the map $r : E(M) \to E(N)$ by letting it be identity on N, and setting $r(v) = \phi_{\mathbb{G}}(v)$ for points in $V(G)$. We complete it to a retraction $r : M \to N$ by setting, for every edge uv of G, $r(e_{uv})$ to be the point of N such that $r(C_{uv})$ is a disjoint union of circuits of N. Such a point exists as uv is an edge of H. That this map takes circuits of N to circuits in N is clear as it is the identity on these points; that it takes other circuits to disjoint unions of circuits comes from the fact that the other fundamental circuits with respect to B are the C_{uv}, and that we defined r on the last point of these circuits in such a way as to make the image a union of circuits of N.

Thus $\mathrm{Hom}_{\mathbb{G}}(H)$ is encoded in $\mathrm{Ret}_{\mathbb{M}}(N)$, as needed.

For part (*ii*) the proof is essentially the same, replacing the 6-circuit C with a 5-circuit with precoloured edges a and b where a, b are non-loop points of N not in a 3-circuit, and other designated edges x and y. Under retraction to N it is clear that that x and y must go to distinct points, and so the graph H on $V(N)$ of possible targets of these two points is loopless and contains the triangle on a, b and 0, where 0 is the loop of N. ☐

The following propositions show that the complexity of $\mathrm{Ret}_{\mathbb{M}}(N)$ may differ from that of its restriction to graphic instances. Note that we can use Tutte's algorithm [6] for a graphic matroid M given as a binary matrix to construct a graph G such that $M = M(G)$ in polynomial time. We do not need this algorithm to distinguish YES and NO instances; however we need it to give a certificate for YES instances.

Proposition 29. *Let M be a graphic matroid containing a submatroid N isomorphic to $M(3K_2)$. Then there is a retraction $M \to N$ if and only if M is bipartite and no 4-circuit of M contains every point of N. Consequently,* $\mathrm{Ret}_{\mathbb{M}}(M(3K_2))$ *is polynomial time solvable for graphic instances.*

Proof. Let G be a graph such that $M(G) = M$ and let e_i be the precoloured edge of each of the colours $i \in \{1, 2, 3\}$. Our goal is to decide whether or not G has a *good colouring*: a colouring of the edges of G with the colours $\{1, 2, 3\}$, extending the precolouring, such that every cycle has an even number of edges of each colour. First suppose there is a retraction $M \to N$. If G is non-bipartite, or all of e_1, e_2, and e_3 are in a 4-cycle, then a good colouring is impossible.

To show the other direction, assume that G is bipartite and that not all of the three edges e_1, e_2 and e_3 are in a 4-cycle. We claim that there are non-adjacent vertices v_1 and v_2 in G such that each is in exactly one of the edges e_i. Indeed, let H be subgraph of G consisting of the edges e_1, e_2, and e_3 and only the vertices that they contain. If H is not connected, then there exists e_i (say e_1) such that e_1 shares no vertex with e_2 or e_3. Since e_2 and e_3 form a matching or a 2-path, we can let v_2 be a vertex of e_2 having degree 1 in H. As H is bipartite, there is an end of e_1 that is not adjacent to v_2; let this end be v_1. If H is connected, then because it is bipartite, it is either a 3-path or a copy of $K_{1,3}$.

In either case, we can pick any two leaves as v_1 and v_2; if they are adjacent in G then G contains a 3-cycle, in the case that H is $K_{1,3}$, or a 4-cycle containing all of e_1, e_2, e_3 in the case that H is a 3-path.

With v_1 and v_2 as claimed, we may assume that v_1 is in e_1 and v_2 is in e_2. Colour the edges of G incident to v_1 with colour 1, those incident to v_2 with colour 2, and all other edges with colour 3. Any cycle has 0 or 2 edges of colour 1 and colour 2, and has even length so has an even number of edges of colour 3. This is the good colouring we needed, and thus there is a retraction $M \to N$.

For the last part, with our graphic representation of M, we can check if e_1, e_2, and e_3 occur together in a 4-circuit in constant time, and can use a tree search, in linear time (in the number of points of M), to decide if G is bipartite. □

Proposition 30. *Let M be a graphic matroid containing a submatroid N isomorphic to $M_\ell(2K_2)$, and let e_1 and e_2 be the non-loop points of N. Then there is a retraction $M \to N$ if and only if M contains neither a 3-circuit containing e_1 and e_2, nor a $M(K_4)$ containing e_1 and e_2. Consequently, $\mathrm{Ret}_{\mathbb{M}}(M_\ell(2K_2))$ is polynomial time solvable for graphic instances.*

Proof. Let the loop of N be called e_0. Let G be a graph such that $M(G) = M$. Our goal is to decide if we can colour the edges of G with the colours $0, 1$ and 2 in such a way that edge e_i gets colour i for each i, and every cycle has an even number of edges of each of the colours 1 and 2. We call such a colouring of the edges a *good colouring*.

First suppose there is a retraction $M \to N$, and so G has a good colouring. Under a good colouring, a 3-cycle C gets three possible colourings: all three edges are coloured with 0; two edges are coloured 1 and one is coloured 0; or two edges are coloured 2 and one is coloured 0. In particular, a 3-cycle containing the edges of colour 1 and 2 omit a good colouring. We call such cycles *omitting* 3*-cycles*. If M contains a 3-circuit containing e_1 and e_2, these must get colour 1 and 2, so it is a omitting 3-cycle in G. So towards contradiction, suppose there is a subgraph $H \cong K_4$ of G containing edges e_1 and e_2. Note that if any edge of colour 1 in H shares a vertex with an edge of colour 2, then the vertices of these edges induce an omitting 3-cycle in H, and so there is no good colouring. Thus $\{e_1, e_2\}$ is a matching in H, and any other edge of H is coloured with 0. This is not a good colouring on any 3-cycle in H, giving a contradiction.

To show the other direction, assume that G has neither a 3-cycle containing e_1 and e_2 nor a K_4 containing e_1 and e_2. We claim that there are non-adjacent vertices v_1 and v_2 in G such that each is in exactly one of the edges e_1 and e_2. If e_1 and e_2 have a common vertex, then let v_1, v_2 be non-common vertices of e_1, e_2, respectively. They are adjacent by the assumption that G has no 3-cycle containing e_1 and e_2. If e_1 and e_2 have no common vertex, then there is an end v_i of e_i for each $i \in \{1, 2\}$ such that v_1, v_2 are not adjacent; otherwise there is a K_4 containing e_1 and e_2.

With v_1 and v_2 as claimed, we may assume that v_1 is in e_1 and v_2 is in e_2. Colour the edges of G incident to v_1 with colour 1, those incident to v_2 with colour 2, and all other edges with colour 0. Any cycle has 0 or 2 edges of colour 1 and colour 2, so this is the good colouring we needed, and thus there is a retraction $M \to N$.

Again, we can check in linear time whether G has a 3-cycle or a copy of K_4 containing e_1 and e_2. □

8 Concluding Remarks

The hardness proofs for all of our results depend on the H-colouring dichotomy of [5]; but it is not clear that these results require this hammer. We wonder if any of our main results cannot be proved without resorting to this, or stronger, results.

That $\mathrm{Ret}_{\mathbb{M}}(N)$ and $\mathrm{Ext}_{\mathbb{M}}(N)$ have the same complexity dichotomies suggests that, as in the graph homomorphism case, there could be a direct encoding of $\mathrm{Ext}_{\mathbb{M}}(N)$ in $\mathrm{Ret}_{\mathbb{M}}(N)$. The standard graph reduction does not work, nor should it as the complexities of these problems do not agree for graphic instances. Such encodings exists with various strong properties on N, but we were unable to make them general enough to prove Theorem 9. A direct proof of the equivalence of these problems would be interesting.

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