Combinatorial constructions of generating functions of cylindric partitions with small profiles into unrestricted or distinct parts

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Abstract

In this paper, cylindric partitions into profiles $c = (1, 1)$ and $c = (2, 0)$ are considered. The generating functions into unrestricted cylindric partitions and cylindric partitions into distinct parts with these profiles are constructed. The constructions are combinatorial and they connect the cylindric partitions with ordinary partitions.

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1 Introduction

Cylindric partitions were introduced by Gessel and Krattenthaler [8].

Definition 1. Let k and ℓ be positive integers. Let $c = (c_1, c_2, \ldots, c_k)$ be a composition, where $c_1 + c_2 + \cdots + c_k = \ell$. A *cylindric partition with profile* c is a vector partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$, where each $\lambda^{(i)} = \lambda_1^{(i)} + \lambda_2^{(i)} + \cdots + \lambda_{s_i}^{(i)}$ is a partition, such that for all i and j,

$$\lambda_j^{(i)} \geqslant \lambda_{j+c_{i+1}}^{(i+1)} \quad \text{and} \quad \lambda_j^{(k)} \geqslant \lambda_{j+c_1}^{(1)}.$$

For example, the sequence $\Lambda = ((6, 5, 4, 4), (8, 8, 5, 3), (7, 6, 4, 2))$ is a cylindric partition with profile $(1, 2, 0)$. One can check that for all j, $\lambda_j^{(1)} \geqslant \lambda_{j+2}^{(2)}$, $\lambda_j^{(2)} \geqslant \lambda_j^{(3)}$ and $\lambda_j^{(3)} \geqslant \lambda_{j+1}^{(1)}$.

We can visualize the required inequalities by writing the partitions in subsequent rows repeating the first row below the last one, and shifting the rows below as much as necessary to the left. Thus, the inequalities become the weakly decreasing of the parts to the right in each row, and downward in each column.

$$
\begin{array}{cccc}
6 & 5 & 4 & 4 \\
8 & 8 & 5 & 3 \\
7 & 6 & 4 & 2 \\
\color{gray}6 & \color{gray}5 & \color{gray}4 & \color{gray}4
\end{array}
$$

The repeated first row is shown in gray.

The size $|\Lambda|$ of a cylindric partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ is defined to be the sum of all the parts in the partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$. The largest part of a cylindric partition Λ is defined to be the maximum part among all the partitions in Λ, and it is denoted by $\max(\Lambda)$. The following generating function

$$
F_c(z, q) := \sum_{\Lambda \in \mathcal{P}_c} z^{\max(\Lambda)} q^{|\Lambda|}
$$

is the generating function for cylindric partitions, where $\mathcal{P}_c$ denotes the set of all cylindric partitions with profile c.

In 2007, Borodin [3] showed that when one sets $z = 1$ to this generating function, it turns out to be a very nice infinite product.

Theorem 2 (Borodin, 2007). *Let k and ℓ be positive integers, and let $c = (c_1, c_2, \ldots, c_k)$ be a composition of ℓ. Define $t := k + \ell$ and $s(i, j) := c_i + c_{i+1} + \cdots + c_j$. Then,*

$$
F_c(1, q) = \frac{1}{(q^t; q^t)_\infty} \prod_{i=1}^{k} \prod_{j=i}^{k} \prod_{m=1}^{c_i} \frac{1}{(q^{m+j-i+s(i+1,j)}; q^t)_\infty} \prod_{i=2}^{k} \prod_{j=2}^{i} \prod_{m=1}^{c_i} \frac{1}{(q^{t-m+j-i-s(j,i-1)}; q^t)_\infty}. \tag{1}
$$

The identity (1) is a very strong tool to find product representation of generating functions of cylindric partitions with a given profile explicitly.

Cylindric partitions have been studied intensively since their introduction [8]. Prominent examples are constructing Andrews-Gordon [1] type evidently positive multiple series companions to some cases in Borodin's theorem [5, 6, 7], or even connections with theoretical physics [9].

The purpose of this paper is to construct generating functions of cylindric partitions with small profiles into unrestricted or distinct parts. In Section 2, we combinatorially reprove generating functions for cylindric partitions with profiles $c = (1, 1)$ and $c = (2, 0)$. The construction is based on the fact that if we have a cylindric partition with profile $c = (1, 1)$ or $c = (2, 0)$, then it can be decomposed into a pair of partitions (μ, β) by a series of combinatorial moves. The results in Section 2 are limiting cases, therefore corollaries, of [10, eq. (7.25)]. The proof techniques are different, though. The approach in Section 2 seems to apply in [10, eq. (7.25)] for $k = 1$ and $s = 1$ or 2. In Section

3, we consider cylindric partitions with small profiles into distinct parts. We construct generating functions for such partitions with profiles $c = (1, 1)$ or $c = (2, 0)$, which turn out to be combinations of infinite products. We refer the reader to [4], where cylindric partitions into distinct parts are also studied. We conclude by constructing an evidently positive series generating function for cylindric partitions with small profiles into odd parts.

2   Generating Functions of Cylindric Partitions With Profiles $c = (1, 1)$ and $c = (2, 0)$

By using (1), one can easily show that

$$F_c(1, q) = \frac{(-q; q^2)_\infty}{(q; q)_\infty}, \tag{2}$$

where $c = (1, 1)$.

In the following theorem, we will give a combinatorial proof of identity (2).

Theorem 3. *Let $c = (1, 1)$. Then the generating function of cylindric partitions with profile c is given by*

$$F_c(1, q) = \frac{(-q; q^2)_\infty}{(q; q)_\infty}.$$

Proof. We will show that each cylindric partition λ with profile $c = (1, 1)$ corresponds to a unique pair of partitions (μ, β), where μ is an ordinary partition and β is a partition with distinct odd parts. Conversely, we will show that each pair of partitions (μ, β) will correspond to a unique cylindric partition with profile $c = (1, 1)$, where μ is an ordinary partition and β is a partition into distinct odd parts. In this way, we will get the desired generating function for cylindric partitions with profile $c = (1, 1)$.

$$F_c(1, q) = \sum_\lambda q^{|\lambda|} = \sum_{(\mu, \beta)} q^{|\mu| + |\beta|} = \left(\sum_\mu q^{|\mu|} \right) \left(\sum_\beta q^{|\beta|} \right) = \frac{1}{(q; q)_\infty} (-q; q^2)_\infty,$$

where λ, μ, and β are as described above. The first identity is the definition of $F_c(1, q)$. The second identity will be proven below. The third follows from the fact that μ and β are independent, and the last one because unrestricted partitions and partitions into distinct odd parts have the displayed infinite product generating functions [2].

Let λ be a cylindric partition with profile $c = (1, 1)$. Then λ has the following form:

$$
\begin{array}{ccccccc}
 & a_1 & a_2 & a_3 & \dots & a_{r-1} & a_r \\
b_1 & b_2 & b_3 & \dots & b_s & & \\
a_1 & a_2 & a_3 & \dots & a_r & & \\
\end{array}
,
$$

where $r - 1 \leqslant s \leqslant r + 1$. The last line is a repetition of the first one, and the parts are weakly decreasing from left to right and downward.

If we allow zeros at the end of partitions, we can take $s = r$. Namely, if $s = r - 1$, then we append $b_r = 0$; and if $s = r + 1$, then we both append $a_{r+1} = 0$ and update $r + 1$ as r. So, without loss of generality, our cylindric partition with profile $c = (1, 1)$ looks like

$$\lambda = \begin{matrix} & a_1 & a_2 & a_3 & \ldots & a_{r-1} & a_r \\ b_1 & b_2 & b_3 & \ldots & b_{r-1} & b_r & \\ a_1 & a_2 & a_3 & \ldots & a_{r-1} & a_r & \end{matrix} \quad .$$

At this point, only a_r or b_r may be zero, but not both. Therefore either all parts or all parts but one in λ are positive. During the process of obtaining μ and β, some or all parts of λ may become zero. It is possible that μ is a partition consisting entirely of zeros, i.e., the empty partition. But that does not create a problem because r is determined at the beginning, and it is fixed.

Our goal is to transform λ into another cylindric partition $\widetilde{\lambda}$ of the same profile

$$\widetilde{\lambda} = \begin{matrix} & \widetilde{a}_1 & \widetilde{a}_2 & \widetilde{a}_3 & \ldots & \widetilde{a}_{r-1} & \widetilde{a}_r \\ \widetilde{b}_1 & \widetilde{b}_2 & \widetilde{b}_3 & \ldots & \widetilde{b}_{r-1} & \widetilde{b}_r & \\ \widetilde{a}_1 & \widetilde{a}_2 & \widetilde{a}_3 & \ldots & \widetilde{a}_{r-1} & \widetilde{a}_r & \end{matrix} \tag{3}$$

with the additional property that $\widetilde{b}_j \geqslant \widetilde{a}_j$ for all $j = 1, 2, \ldots, r$, allowing zeros at the end. Then, parts of $\widetilde{\lambda}$ can be listed as

$$\mu = (\widetilde{b}_1, \widetilde{a}_1, \widetilde{b}_2, \widetilde{a}_2, \ldots, \widetilde{b}_r, \widetilde{a}_r)$$

to obtain the promised unrestricted partition μ. The remaining inequalities $\widetilde{a}_j \geqslant \widetilde{b}_{j+1}$ for $j = 1, 2, \ldots, (r - 1)$ are ensured by the fact that $\widetilde{\lambda}$ is a cylindric partition with profile $c = (1, 1)$.

We will do this by a series of transformations on λ which will be recorded as a partition β into distinct odd parts. We will then argue that $|\lambda| = |\mu| + |\beta|$.

We read the parts of the cylindric partition

$$\lambda = \begin{matrix} & a_1 & a_2 & a_3 & \ldots & a_{r-1} & a_r \\ b_1 & b_2 & b_3 & \ldots & b_{r-1} & b_r & \\ a_1 & a_2 & a_3 & \ldots & a_{r-1} & a_r & \end{matrix}$$
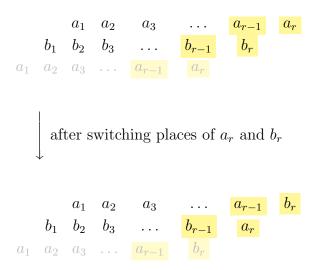
as the pairs: $[b_1, a_1], [b_2, a_2], [b_3, a_3], \ldots, [b_r, a_r]$. We start with the rightmost pair $[b_r, a_r]$. If $b_r \geqslant a_r$, there's nothing to do. We simply set $\widetilde{b}_r = b_r$, $\widetilde{a}_r = a_r$, and do not add any parts to β yet.

If $b_r < a_r$, then we

- switch places of a_r and b_r,

- subtract 1 from each of the parts $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots b_{r-1}$,

- set $\widetilde{b}_r = a_r - 1$ and $\widetilde{a}_r = b_r$,

- add the part $(2r - 1)$ to β.

We need to perform several checks here. First, we will show that at each of the steps listed above, the intermediate cylindric partition satisfies the weakly decreasing condition across rows and down columns. The affected parts are highlighted.

$$
\begin{array}{ccccccc}
 & a_1 & a_2 & a_3 & \ldots & a_{r-1} & a_r \\
 & b_1 & b_2 & b_3 & \ldots & b_{r-1} & b_r \\
a_1 & a_2 & a_3 & \ldots & a_{r-1} & a_r &
\end{array}
$$

$\downarrow$ after switching places of a_r and b_r

$$
\begin{array}{ccccccc}
 & a_1 & a_2 & a_3 & \ldots & a_{r-1} & b_r \\
 & b_1 & b_2 & b_3 & \ldots & b_{r-1} & a_r \\
a_1 & a_2 & a_3 & \ldots & a_{r-1} & b_r &
\end{array}
$$

The inequalities $a_{r-1} \geqslant b_r$ and $a_{r-1} \geqslant a_r$ carry over from the original cylindric partition. The inequalities $b_{r-1} \geqslant a_r$ and $b_{r-1} \geqslant b_r$ are also two of the inequalities implied by the original cylindric partition. All other inequalities are untouched. At this point, we have not altered the weight of the cylindric partition yet.

$\downarrow$ after subtracting 1 from the listed parts

$$
\begin{array}{ccccccc}
 & (a_1 - 1) & (a_2 - 1) & (a_3 - 1) & \ldots & (a_{r-1} - 1) & b_r \\
 & (b_1 - 1) & (b_2 - 1) & (b_3 - 1) & \ldots & (b_{r-1} - 1) & (a_r - 1) \\
(a_1 - 1) & (a_2 - 1) & (a_3 - 1) & \ldots & (a_{r-1} - 1) & b_r &
\end{array}
$$

We argue that this is still a valid cylindric partition. The only inequalities that need to be verified are $a_{r-1} - 1 \geqslant b_r$ and $b_{r-1} - 1 \geqslant b_r$. Because of the original cylindric partition, we have $a_{r-1} \geqslant a_r$ and $b_{r-1} \geqslant a_r$. Because of the case we are examining $a_r > b_r$, so that $a_r - 1 \geqslant b_r$, both being integers. Combining $a_{r-1} - 1 \geqslant a_r - 1$, $b_{r-1} - 1 \geqslant a_r - 1$ and $a_r - 1 \geqslant b_r$ yield the desired inequalities.

$\downarrow$ after relabeling

$$
\begin{array}{ccccccc}
(a_1-1) & (a_2-1) & (a_3-1) & \ldots & (a_{r-1}-1) & \widetilde{a}_r \\
(b_1-1) & (b_2-1) & (b_3-1) & \ldots & (b_{r-1}-1) & \widetilde{b}_r \\
\color{gray}{(a_1-1)} & \color{gray}{(a_2-1)} & \color{gray}{(a_3-1)} & \color{gray}{\ldots} & \color{gray}{(a_{r-1}-1)} & \color{gray}{\widetilde{a}_r}
\end{array}
$$

Now we have $\widetilde{b}_r \geqslant \widetilde{a}_r$ since $a_r - 1 \geqslant b_r$. Also, we subtracted 1 from exactly $2r-1$ parts. We add this $(2r-1)$ as a part in β. At the beginning, β was the empty partition, so it is a partition into distinct odd parts both before and after this transformation. The sum of the weight of β and the weight of the cylindric partition remains constant. It is possible that either or both $\widetilde{a}_r$ and $\widetilde{b}_r$ may be zero, along with some other parts. For example, in the extreme case that $a_1 = a_2 = \cdots = a_r = 1$, $b_1 = b_2 = \cdots = b_{r-1} = 1$ and $b_r = 0$, the cylindric partition becomes the empty partition after the transformation we illustrated.

We should mention that after this point there is no harm in renaming (a_i-1)'s a_i's and (b_i-1)'s b_i's, where applicable. This will lead to the cleaner exposition down below. There is no loss of information, since the subtracted 1's are recorded as a part in β already.

Then, we repeat the following process for $j = (r-1), (r-2), \ldots, 2, 1$ in the given order. At the beginning of the jth step, we have the intermediate cylindric partition

$$
\begin{array}{ccccccccc}
& a_1 & a_2 & \cdots & a_{j-1} & a_j & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_r \\
b_1 & b_2 & \cdots & b_{j-1} & b_j & \widetilde{b}_{j+1} & \cdots & \widetilde{b}_r \\
\color{gray}{a_1} & \color{gray}{a_2} & \color{gray}{\cdots} & \color{gray}{a_{j-1}} & \color{gray}{a_j} & \color{gray}{\widetilde{a}_{j+1}} & \color{gray}{\cdots} & \color{gray}{\widetilde{a}_r}
\end{array} \quad .
$$

The parts weakly decrease from left to right and downward, and the third line is a repetition of the first one. This intermediate cylindric partition satisfies the additional inequalities

$$
\widetilde{b}_{j+1} \geqslant \widetilde{a}_{j+1}, \quad \widetilde{b}_{j+2} \geqslant \widetilde{a}_{j+2}, \quad \cdots \quad \widetilde{b}_r \geqslant \widetilde{a}_r.
$$

Some or all parts in this intermediate partition may be zero.

We focus on the jth pair $[b_j, a_j]$. If $b_j \geqslant a_j$ already, then we do not alter either the intermediate cylindric partition or the partition β into distinct odd parts. We just relabel b_j as $\widetilde{b}_j$, a_j as $\widetilde{a}_j$, and move on to the $(j-1)$th pair.

In the other case $a_j > b_j$, we

- switch places of a_j and b_j,

- subtract 1 from each of the parts $a_1, a_2, \ldots, a_j, b_1, b_2, \ldots b_{j-1}$,

- set $\widetilde{b}_j = a_j - 1$ and $\widetilde{a}_j = b_j$,

- add the part $(2j-1)$ to β.

We again perform several checks as in the rth case, but this time there are inequalities that involve parts that lie to the right of a_j and b_j. We first show that the listed operations do not violate the weakly decreasing condition on the cylindric partition across rows and

down columns. The affected parts are highlighted. We switch the places of a_j and b_j to obtain

$$
\begin{array}{ccccccc}
a_1 & a_2 & \cdots & a_{j-1} & b_j & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_r \\
b_1 & b_2 & \cdots & b_{j-1} & a_j & \widetilde{b}_{j+1} & \cdots & \widetilde{b}_r \\
a_1 & a_2 & \cdots & a_{j-1} & b_j & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_r
\end{array} .
$$

Each of the required inequalities

$$
a_{j-1} \geqslant b_j \geqslant \widetilde{a}_{j+1}, \quad b_{j-1} \geqslant a_j \geqslant \widetilde{b}_{j+1}, \quad b_{j-1} \geqslant b_j, \quad a_{j-1} \geqslant a_j \geqslant \widetilde{a}_{j+1}, \quad \text{and} \quad b_j \geqslant \widetilde{b}_{j+1}
$$

are already implied in the cylindric partition before the change. The inequalities between the non-highlighted parts carry over. We then subtract one from each of the listed parts. The inequalities we need to verify are

$$
a_{j-1} - 1 \geqslant b_j, \quad a_j - 1 \geqslant \widetilde{b}_{j+1}, \quad b_{j-1} - 1 \geqslant b_j, \quad \text{and} \quad a_j - 1 \geqslant \widetilde{a}_{j+1}.
$$

By the cylindric partition two steps ago, we have

$$
a_{j-1} \geqslant a_j, \quad b_j \geqslant \widetilde{b}_{j+1}, \quad b_{j-1} \geqslant a_j, \text{ and} \quad b_j \geqslant \widetilde{a}_{j+1}.
$$

By the hypothesis, $a_j > b_j$, so $a_j - 1 \geqslant b_j$. This last inequality, combined with the last four displayed inequalities yield the inequalities we wanted. Then we relabel b_j as $\widetilde{a}_j$ and a_j as $\widetilde{b}_j$ in their respective new places. We have $\widetilde{b}_j \geqslant \widetilde{a}_j$, since $a_j - 1 \geqslant b_j$.

On the other hand, we subtracted a total of $(2j-1)$ 1's from the parts of the intermediate cylindric partition, and now we add $(2j-1)$ to β. β still has distinct odd parts, because the smallest part we had added to β must be $\geqslant (2j+1)$ in the previous step. It is also possible that β was empty before adding $(2j-1)$. We should note that $(2j-1)$ is the smallest part in β at the moment. In any case, we have

$$
|\lambda| = |\beta| + \text{the weight of the intermediate cylindric partition.}
$$

λ is the original cylindric partition, before any changes.

Like after the rth step, there is no danger in renaming $(a_i - 1)$'s a_i's and $(b_i - 1)$'s b_i's, where necessary.

Once this process is finished, we have the cylindric partition $\widetilde{\lambda}$ as given in (3), The nonzero parts of $\widetilde{\lambda}$ is listed as parts of the unrestricted partition μ, the alterations in obtaining $\widetilde{\lambda}$ are recorded as parts of the partition β into distinct odd parts, and one direction of the proof is over.

Next; given (μ, β), where μ is an unrestricted partition, and β is a partition into distinct odd parts, we will produce a unique cylindric partition λ with profile $c = (1, 1)$ such that

$$
|\lambda| = |\mu| + |\beta|.
$$

The parts of μ in their respective order are relabeled as:

$$\mu = \mu_1 + \mu_2 + \cdots + \mu_l$$
$$= \widetilde{b}_1 + \widetilde{a}_1 + \cdots + \widetilde{b}_s + \widetilde{a}_s. \tag{4}$$

The relabeling requires an even number of parts, which can be solved by appending a zero at the end of μ if necessary. Then, the $\widetilde{b}$'s and $\widetilde{a}$'s are arranged as the cylindric partition

$$\widetilde{\lambda} = \begin{array}{ccccc} & \widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_s \\ \widetilde{b}_1 & \widetilde{b}_2 & \cdots & \widetilde{b}_s & \\ \widetilde{a}_1 & \widetilde{a}_2 & \cdots & \widetilde{a}_s & \end{array} .$$

All of the required inequalities $\widetilde{a}_j \geqslant \widetilde{a}_{j+1}$, $\widetilde{b}_j \geqslant \widetilde{b}_{j+1}$, $\widetilde{a}_j \geqslant \widetilde{b}_{j+1}$, and $\widetilde{b}_j \geqslant \widetilde{a}_{j+1}$ for $j = 1, 2, \ldots, s-1$ are implied by the inequalities between parts of μ. $\widetilde{\lambda}$ has the additional property that $\widetilde{b}_j \geqslant \widetilde{a}_j$ for $j = 1, 2, \ldots, s$. This is the $\widetilde{\lambda}$ we obtained in the first half of the proof, except for the possibly different number of zeros at the end(s). The positive parts and their positions are the same.

For the smallest part $(2j-1)$ in β, we do the following.

- delete the part from β,

- add 1 to all parts $a_1, \ldots, \widetilde{a}_{j-1}, b_1, \ldots, \widetilde{b}_j$,

- switch places of $\widetilde{a}_j$ and $(\widetilde{b}_j + 1)$,

- rename $(a_1+1), \ldots, (\widetilde{a}_{j-1}+1), (\widetilde{b}_j+1), (b_1+1), \ldots, (\widetilde{b}_{j-1}+1), \widetilde{a}_j$, in their respective order as $a_1, \ldots, a_{j-1}, a_j, b_1, \ldots, b_{j-1}, b_j$.

We repeat this procedure until β becomes the empty partition, at which time $\widetilde{\lambda}$ has evolved into λ, the cylindric partition with profile $c = (1,1)$ we have been aiming at.

There are a few details to clarify, including the notation. We start by verifying that the inequalities required by the cylindric partition are satisfied at each step. The affected parts are highlighted. We start with the cylindric partition just before the transformations. We add one to each of the listed parts. The required inequalities are naturally satisfied here, because the parts which are supposed to be weakly greater are increased. Then, we switch places of $\widetilde{a}_j$ and $\widetilde{b}_j + 1$.

$$\begin{array}{ccccccc} & (a_1+1) & \cdots & (\widetilde{a}_{j-1}+1) & (\widetilde{b}_j+1) & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_s \\ (b_1+1) & \cdots & (\widetilde{b}_{j-1}+1) & \widetilde{a}_j & \widetilde{b}_{j+1} & \cdots & \widetilde{b}_s & \\ (a_1+1) & \cdots & (\widetilde{a}_{j-1}+1) & (\widetilde{b}_j+1) & \widetilde{a}_{j+1} & \cdots & \widetilde{a}_s & \end{array}$$

Again, the required inequalities are implied by the cylindric partition in the previous step. At the beginning of the first run, we do not have a_1 or b_1 in the cylindric partition, but

rather $\widetilde{a}_1$ or $\widetilde{b}_1$, respectively. However, at the end of each run, the leftmost so many parts in the first and the second rows of the cylindric partition are labeled a_1, b_1, etc.

Because we deleted $(2j-1)$ from β, and we added 1 to exactly $(2j-1)$ of the parts in the intermediate cylindric partition, the sum of weights of β and of the intermediate cylindric partition remains constant. It equals the sum of weights of μ and the original β.

The relabeling of (a_1+1) as a_1 etc. does not interfere with any of the operations before it, and certainly not any of the possible operations that some after it; therefore, it should not cause any confusion.

We tacitly assumed that $j < s$ in the displayed cylindric partition above. This does not have to be the case, as β may have a part greater than the length of μ. The remedy is to append zeros, and increase s as much as necessary. This takes care of the extreme case of $\widetilde{\lambda}$ being the empty partition. All of the arguments above apply for non-negative parts as well as strictly positive parts. We also implicitly assumed that β is nonempty to start with. If β is the empty partition, we do not need the perform any operations $\widetilde{\lambda}$ at all. We simply call $\widetilde{a}_j$'s a_j's, and $\widetilde{b}_j$'s b_j's.

Once all parts of β are exhausted, we clear the trailing pairs of zeros in the cylindric partition at hand, and we declare the obtained cylindric partition λ.

Because the sum of the weights of β and of the intermediate cylindric partition remained constant at each step of the transformation and β is the empty partition at the end, we have

$$|\lambda| = |\mu| + |(\text{the original})\beta|.$$

Except for the relabelings, the adding or subtracting 1's, and adding or deleting parts of β are done in exact reverse order, and they are clearly inverse operations of each other, the process is reversible, and the collection of profile $c = (1,1)$ cylindric partitions λ's are in one-to-one correspondence with the pairs (μ, β) of an unrestricted partition and a partition into distinct odd parts. The relabelings in the two phases of the proof are consistent at the beginning and at the end of the transformation, and between the rounds of operations. This concludes the proof. $\qquad\square$

The following example demonstrates how we construct the pair of partitions (μ, β) if we are given a cylindric partition λ with profile $c = (1,1)$.

Example 4. Let λ be the following cylindric partition with profile $c = (1,1)$:

$$\begin{array}{cccc} 7 & 4 & 4 & 3 \\ 6 & 5 & 4 & \end{array}$$

We read the parts of λ as pairs: $[6,7], [5,4], [4,4]$ and $[0,3]$.

$$\begin{array}{cccc} 7 & 4 & 4 & 3 \\ 6 & 5 & 4 & 0 \end{array}$$

We now start to perform the moves defined in the proof of Theorem 3. We first change the places of 0 and 3 in the rightmost pair and we get the following intermediate partition:

$$\begin{array}{cccc} 7 & 4 & 4 & 0 \\ 6 & 5 & 4 & \boxed{3} \end{array}$$

$\downarrow$ subtract 1 from circled 3 and the parts take place above and on the left of it

$$\begin{array}{cccc} 6 & 3 & 3 & 0 \\ 5 & 4 & 3 & 2 \end{array}$$

We changed the total weight by 7, so we have $\beta_1 = 7$. We do not touch to the pairs $[3,3]$ and $[4,3]$ since $3 \geqslant 3$ and $4 \geqslant 3$. We now correct the places of 6 and 5, then we perform the last possible move:

$$\begin{array}{cccc} & 5 & 3 & 3 & 0 \\ \boxed{6} & 4 & 3 & 2 \end{array}$$

$\downarrow$ subtract 1 from circled 6

$$\begin{array}{cccc} 5 & 3 & 3 & 0 \\ 5 & 4 & 3 & 2 \end{array}$$

We changed the total weight by 1, so we have $\beta_2 = 1$. Therefore, we decomposed λ into the pair of partitions (μ, β), where $\beta = 7 + 1$ and $\mu = 5 + 5 + 4 + 3 + 3 + 3 + 2$.

The following example demonstrates how we construct a unique cylindric partition λ with profile $c = (1, 1)$, if we are given a pair of partitions (μ, β) which is described as in the proof of Theorem 3.

Example 5. Let $\mu = 6 + 5 + 5 + 3 + 1$ and $\beta = 9 + 7 + 3$. We read the parts of μ as follows:

$$\begin{array}{ccccc} 5 & 3 & 0 & 0 & 0 \\ 6 & 5 & 1 & 0 & 0 \end{array}$$

The first part of β is 9. Since we want to increase the weight by 9, we add 0's as many as we need when we construct the pairs.

$$\begin{array}{ccccc} 5 & 3 & 0 & 0 & 0 \\ 6 & 5 & 1 & 0 & \boxed{0} \end{array}$$

$\downarrow$ increase by 1 circled 0 and all corresponding parts

$$\begin{array}{ccccc} 6 & 4 & 1 & 1 & 0 \\ 7 & 6 & 2 & 1 & 1 \end{array}$$

$\downarrow$ correct the places of parts in the last pair

$$\begin{array}{ccccc} 6 & 4 & 1 & 1 & 1 \\ 7 & 6 & 2 & \boxed{1} & 0 \end{array}$$

$$\downarrow \text{increase by 1 circled 0 and all corresponding parts}$$

$$\begin{array}{ccccc} 7 & 5 & 2 & 1 & 1 \\ 8 & 7 & 3 & 2 & 0 \end{array}$$

$$\downarrow \text{correct the places of parts in the second pair from the right}$$

$$\begin{array}{ccccc} 7 & 5 & 2 & 2 & 1 \\ 8 & \boxed{7} & 3 & 1 & 0 \end{array}$$

$$\downarrow \text{increase by 1 circled 7 and all corresponding parts}$$

$$\begin{array}{ccccc} 8 & 5 & 2 & 2 & 1 \\ 9 & 8 & 3 & 1 & 0 \end{array}$$

$$\downarrow \text{correct the places of parts in the fourth pair from the right}$$

$$\lambda = \quad \begin{array}{ccccc} 8 & 8 & 2 & 2 & 1 \\ 9 & 5 & 3 & 1 & 0 \end{array}$$

λ is the unique cylindric partition with profile $c = (1,1)$ corresponding to the pair of partitions (μ, β).

Theorem 6. *Let $c = (1,1)$. Then the generating function of cylindric partitions with profile c is given by*

$$F_c(z, q) = \frac{(-zq; q^2)_\infty}{(zq; q)_\infty}.$$

where the exponent of variable z keeps track of the largest part of the cylindric partitions.

Proof. In the proof of Theorem 3, we show that there is a one-to-one correspondence between the cylindric partitions with profile $c = (1,1)$ and the pairs of partitions (μ, β) such that μ is an ordinary partition and β is a partition into distinct odd parts. For the proof, we will use this correspondence. If we take a pair of partitions (μ, β), then during the construction of λ, each part of β increases the largest part of μ by 1. Hence, when the whole procedure is done, the largest part of μ is increased by the number of parts in β. Because of that fact, we write the generating function of β by keeping track of the number of parts, which gives $(-zq; q^2)_\infty$.

The partition μ is an ordinary partition and the generating function of ordinary partitions such that the largest part is M is given by

$$\frac{q^M}{(1-q)\dots(1-q^M)}.$$

If we take sum over all M by keeping track of the largest parts with the exponent of z, we get

$$\sum_{M \geqslant 0} \frac{z^M q^M}{(1-q)\dots(1-q^M)} = \sum_{M \geqslant 0} \frac{(zq)^M}{(q;q)_M} = \frac{1}{(zq;q)_\infty}.$$

The second identity follows from Euler's identity [2]. There is a one-to-one correspondence between the partitions with exactly k parts and the partitions with largest part equals to k via conjugation [2]. Thus, the latter generating function can also be considered as the generating function of ordinary partitions, where the exponent of z keeps track of the number of parts. Finally, since μ and β are two independent partitions, we get the desired generating function.

$\square$

Theorem 7. *Let $c = (2,0)$. Then the generating function of cylindric partitions with profile c is given by*

$$F_c(1,q) = \frac{(-q^2;q^2)_\infty}{(q;q)_\infty}.$$

Proof. The proof is very similar to the proof of Theorem 3. We read the parts of the cylindric partition

$$\lambda = \begin{matrix} & a_0 & a_1 & a_2 & \dots & a_{r-1} & a_r \\ & & b_1 & b_2 & \dots & b_{r-1} & b_r \\ a_0 & a_1 & a_2 & a_3 & \dots & a_r & \end{matrix}.$$

as the pairs: $[b_1, a_1], [b_2, a_2], [b_3, a_3], \dots, [b_r, a_r]$. We note that the largest part of the cylindric partition λ, namely, a_0 is not contained in any pairs. We consider it as a single part.

a_0 is not switched with any part, but it is increased or decreased accordingly when we construct or incorporate β as in the proof of Theorem 3. Thus, β consists of distinct even parts, as opposed to distinct odd parts. $\square$

If we construct the generating function of cylindric partitions with profile $c = (2,0)$ by using (1), we get

$$F_c(1,q) = \frac{1}{(q;q)_\infty (q^2;q^4)_\infty}.$$

If we compare that generating function with the generating function in Theorem 7, we see that they are equal. Both generating functions have the factor $(q;q)_\infty$ in the denominators. If we cancel that factor, we should check whether

$$\frac{1}{(q^2;q^4)_\infty} = (-q^2;q^2)_\infty \tag{5}$$

or not. This identity holds due to the beautiful identity of Euler which states that the number of partitions of a non-negative integer n into odd parts is equal to the number of partitions of n into distinct parts. To obtain (5), we make the substitution $q^2 \to q$ in Euler's identity.

3   Cylindric partitions into distinct parts

If all parts in a cylindric partition with profile $c = (1,1)$ are distinct, then the inequalities between parts are strict. Given such a partition, if we label the parts in the top row as $a_1, a_2, \ldots$, and the parts in the bottom row as $b_1, b_2, \ldots$,

$$
\begin{array}{ccccccc}
a_1 & a_2 & \cdots & a_{r-1} & a_r & \cdots & a_n \\
b_1 & b_2 & \cdots & b_{r-1} & b_r & \cdots & b_n \\
a_1 & a_2 & \cdots & a_{r-1} & a_r & \cdots & a_n
\end{array}
\qquad , \tag{6}
$$

we have the inequalities

$$
a_r > a_{r+1}, \qquad b_r > b_{r+1}, \qquad a_r > b_{r+1}, \qquad \text{and} \qquad b_r > a_{r+1}
$$

for $r = 1, 2, \ldots, n-1$. In particular,

$$
\min\{a_r, b_r\} > \max\{a_{r+1}, b_{r+1}\} \tag{7}
$$

for $r = 1, 2, \ldots, n-1$. As in the proof of Theorem 3, we lose no generality by assuming that the top row and the bottom row have equal number of parts. We achieve this by allowing one of a_n or b_n to be zero. The inequality (7) ensures that we can switch the places of a_r and b_r without violating the condition for cylindric partition with profile $c = (1,1)$ for $r = 1, 2, \ldots, n$. There are 2^n ways to do this.

Therefore, given a cylindric partition into $2n$ distinct parts with profile $c = (1,1)$, we can switch places of a_r and b_r to make $b_r > a_r$ for $r = 1, 2, \ldots, n$ so that

$$
b_1 > a_1 > b_2 > a_2 > \cdots > b_n > a_n, \tag{8}
$$

where a_n is possibly zero. In other words, we obtain a partition into $2n$ distinct parts in which the smallest part is allowed to be zero.

Conversely, if we start with a partition into $2n$ distinct parts in which the smallest part can be zero, we can label the parts as in (8), then place them as in (6), and allow switching places of a_r and b_r for $r = 1, 2, \ldots, n$; then we will have generated a cylindric partition into $2n$ distinct parts with profile $c = (1,1)$, where one of the parts is allowed to be zero.

It is clear that any such cylindric partition corresponds to a unique partition into an even number of distinct parts, and any partition into $2n$ distinct parts gives rise to 2^n cylindric partitions. We have almost proved the following lemma.

Lemma 8. *Let $d_{(1,1)}(m,n)$ denote the number of cylindric partitions of n into distinct parts with profile $c = (1,1)$ and the largest part equal to m. Then,*

$$D_{(1,1)}(t,q) = \sum_{n,m \geqslant 0} d_{(1,1)}(m,n) t^m q^n = \sum_{n \geqslant 0} \frac{q^{\binom{2n}{2}} t^{2n-1} 2^n}{(tq;q)_{2n}}.$$

Proof. We build the proof on the discussion preceding the statement of the lemma. A partition into $2n$ distinct parts is generated by

$$\frac{q^{\binom{2n}{2}}}{(q;q)_{2n}},$$

where the smallest part is allowed to be zero. When we want to keep track of the largest part, we start by the minimal partition into $2n$ distinct parts

$$(2n-1),(2n-2),\ldots,1,0,$$

hence the t^{2n-1} in the numerator in the rightmost sum in the Lemma. Then, for $j = 1,2,\ldots,2n$, the factor $(1 - tq^j)$ in the denominator contributes to the j largest parts in the partition into distinct parts. $\square$

A similar discussion ensues for cylindric partitions into distinct parts with profile $c = (2,0)$. The generic cylindric partition is

$$\begin{array}{cccccccc} a_0 & a_1 & a_2 & \cdots & a_{r-1} & a_r & \cdots & a_n \\ b_1 & b_2 & \cdots & b_{r-1} & b_r & \cdots & b_n \\ a_0 & a_1 & a_2 & \cdots & a_{r-1} & a_r & \cdots & a_n \end{array}, \tag{9}$$

where the top row contains $n+1$ parts, the bottom row contains n parts, and one of a_n or b_n is allowed to be zero. We still have the inequalities (7). In addition, a_0 is the absolute largest part. Needless to say that a_0 is zero if and only if we have the empty cylindric partition. a_r and b_r can switch places for $r = 1,2,\ldots,n$ to obtain the augmented chain of inequalities

$$a_0 > b_1 > a_1 > b_2 > a_2 > \cdots > b_n > a_n,$$

to get a partition into $(2n+1)$ distinct parts, where the smallest part is allowed to be zero. Conversely, any partition into $(2n+1)$ distinct parts in which the smallest part is allowed to be zero gives rise to exactly 2^n cylindric partitions into distinct parts with profile $c = (2,0)$. We have again almost proved the following lemma.

Lemma 9. *Let $d_{(2,0)}(m,n)$ denote the number of cylindric partitions of n into distinct parts with profile $c = (2,0)$ and the largest part equal to m. Then,*

$$D_{(2,0)}(t,q) = \sum_{n,m \geqslant 0} d_{(2,0)}(m,n) t^m q^n = \sum_{n \geqslant 0} \frac{q^{\binom{2n+1}{2}} t^{2n} 2^n}{(tq;q)_{2n+1}}.$$

The proof is almost the same as that of Lemma 8, and is skipped.

Next, we dissect one of Euler's q-series identities [2]

$$\sum_{n \geqslant 0} \frac{q^{\binom{n}{2}} a^n}{(q;q)_n} = (-a;q)_\infty$$

to separate odd and even powers of a.

$$\sum_{n \geqslant 0} \frac{q^{\binom{2n}{2}} (a^2)^n}{(q;q)_{2n}} = \frac{(-a;q)_\infty + (a;q)_\infty}{2}, \qquad a \sum_{n \geqslant 0} \frac{q^{\binom{2n+1}{2}} (a^2)^n}{(q;q)_{2n+1}} = \frac{(-a;q)_\infty - (a;q)_\infty}{2}.$$

If we plug in $t = 1$ in Lemmas 8 and 9, and $a = \sqrt{2}$ in the above formulas, we obtain the following theorem. We repeat the descriptions of the partition enumerants for ease of reference.

Theorem 10. *Let* $d_{(1,1)}(m,n)$ *and* $d_{(2,0)}(m,n)$ *be the number of cylindric partitions of* n *into distinct parts where the largest part is* m *with profiles* $c = (1,1)$ *and* $c = (2,0)$, *respectively. Let*

$$D_{(1,1)}(t,q) = \sum_{m,n \geqslant 0} d_{(1,1)}(m,n) t^m q^n, \quad \text{and} \quad D_{(2,0)}(t,q) = \sum_{m,n \geqslant 0} d_{(2,0)}(m,n) t^m q^n$$

be the respective generating functions. Then,

$$D_{(1,1)}(1,q) = \frac{(-\sqrt{2};q)_\infty + (\sqrt{2};q)_\infty}{2},$$

$$D_{(2,0)}(1,q) = \frac{(-\sqrt{2};q)_\infty - (\sqrt{2};q)_\infty}{2\sqrt{2}},$$

$$D_{(1,1)}(1,q) + \sqrt{2}\, D_{(2,0)}(1,q) = (-\sqrt{2};q)_\infty.$$

4 Discussion

In Theorem 3, to construct the desired generating function for cylindric partitions with profile $c = (1,1)$, we decompose each cylindric partition λ with profile $c = (1,1)$ into a pair of partitions (μ, β), where μ is an ordinary partition and β is a partition with distinct odd parts. Conversely, if a pair (μ, β) is given, then we find a unique cylindric partition λ with profile $c = (1,1)$. In that way, we find a one-to-one correspondence between λ's and (μ, β)'s. Here, the partitions μ and β are two independent partitions. In the following theorem, we consider the pair of partitions (μ, β) such that μ and β are dependent partitions and we construct a family of cylindric partitions with profile $c = (1,1)$ corresponding to the pairs (μ, β).

Theorem 11. *Let $O_c(1, q)$ be the generating function of cylindric partitions with profile $c = (1, 1)$ such that all parts are odd. Then,*

$$O_c(1, q) = \sum_{k \geqslant 0} \frac{q^{2k}}{(q^2; q^2)_{2k}} \cdot (-q^2; q^4)_k + \sum_{k \geqslant 0} \frac{q^{2k+1}}{(q^2; q^2)_{2k+1}} \cdot (-q^2; q^4)_{k+1} \qquad (10)$$

$$= \sum_{k \geqslant 0} \frac{q^{2k}(-q^2; q^4)_k(1 + q - q^{4k+2} + q^{4k+3})}{(q^2; q^2)_{2k+1}}.$$

Proof. We will construct a one-to-one correspondence between the cylindric partitions with profile $c = (1, 1)$ such that all parts are odd and the pairs of partitions (μ, β), where μ is a partition into odd parts, β is a partition into distinct odd parts such that each part is counted twice. Moreover, μ and β are dependent on each other with respect to the number of parts in μ and the largest odd part in β as follows:

(a) if μ has $2k$ parts, the largest odd part in β is $2k - 1$,

(b) if μ has $2k + 1$ parts, the largest odd part in β is $2k + 1$.

By using exactly the same construction in the proof of Theorem 3, whenever a pair of partitions (μ, β) as in case (a) or (b), we may construct a unique cylindric partition λ with profile $c = (1, 1)$ such that all parts are odd. The only change in the construction is that we increase/decrease the weight of each part in the intermediate cylindric partition by 2 instead of 1. It is clear that the parts of the cylindric partition have to be odd, since all parts of the partition μ are odd and the parts of β are counted twice, i.e., we do not change the parity of the parts during the transformations. Conversely, if we are given a cylindric partition λ with profile $c = (1, 1)$ such that all parts are odd, then we may find a unique pair of partitions (μ, β) just described as above. The first term in the sum in (10) is the generating function of pairs (μ, β) having the property (a) and the latter term in the summation is the generating function of pairs (μ, β) having the property (b). $\square$

A natural question is to ask if similar constructions to the proof of Theorems 3 and 7 could be done for cylindric partitions with larger profiles. Another natural question is to ask if similar infinite product generating functions to Theorem 10 could be discovered for cylindric partitions with larger profiles into distinct parts.

Acknowledgements

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